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1.) A graph has no restrictions on its structure, while trees can't have cycles and there must be a unique path from one vertex to every other.

2) A graph where only a single vertex is disconnected:
Consider that if there are two components of a graph A, B. Each vertex on A must be disconnected from B (and viceversa) eliminating all edges (a,b) from $V(A) \times V(B)$ The case that minimizes this reduction of edges is having two components, one with a single vertex

Thus, the maximum comes from a graph with $n-1$ vertices fully connected and a single disconn. vertex. That is expressed as the number of (pair) combinations of $n-1$ vertices

$$\binom{n-1}{2} = \frac{(n-1)!}{2!\,(n-3)!} = \frac{(n-1)!}{2\,(n-1)!}(n-2)(n-1) = \frac{(n-2)(n-1)}{2}$$

3) From the statement we know that $\deg(v) = 5$ for every vertex v in the graph formed by the routers network. We can use a property of undirected graphs that states

$$\sum_{v \in V} \deg(v) = 2|E| \quad \text{for a graph } G = (V, E)$$

For this case we know that the sum of degrees is $23 \cdot 5$ (23 vertices with degree 5) $= 115$. Then $2|E| = 115$, but 115 is odd, which is a contradiction to the property. Therefore, such graph does not exist.

4) We know $|U| + |V| = n$, and the size of a bipartite graph is $|U||V|$. Let $k := |U|$, then $n - k = |V|$. Now we want to maximize:

$$\begin{aligned} &\max |U||V| \\ = &\max k(n-k) \\ = &\max kn - k^2 \end{aligned}$$

$n^2 - nk$ has a global maximum and can be found where its slope is 0 (since it's a parabola with a maximum).

$$\frac{d}{dk}(kn - k^2) = 0 \qquad\qquad \max kn - k^2$$

$$n - 2k = 0 \quad \Rightarrow \quad = \left(\frac{n}{2}\right)\cdot n - \left(\frac{n}{2}\right)^2$$

$$k = \frac{n}{2} \qquad\qquad = \frac{2n^2}{4} - \frac{n^2}{4}$$

$$= \frac{n^2}{4}$$

5) If we arrange the graph like a grid with each vertex (v_i, v_j) being in the row i and column j, we'll notice each vertex is connected to every other that is not in the same row or column, since in a complete graph, no vertex is adjacent to itself, so $(v_i, v_i) \notin V(G \times G)$.

Example.