area, volume, common height & base

Mr Jo's notes

2

School of Science and Technology, Singapore
Mathematics Department

Secondary 2 - Congruence and Similarity Notes

Name:________________________________ (   )   Class: S2-0 __

Unit Enduring Understanding
1. **Diagrams** of figures help us to visualise their congruence or similarity.
2. Two figures are congruent if and only if their sides and angles remain **invariant** under translation, rotation and reflection.
3. Two similar figures have corresponding sides that are **proportional**.

Unit Essential Questions
1. How do diagrams facilitate problem solving?
2. How do properties of congruent figures remain invariant under transformations?
3. How does proportionality undergird the concept of similarity?

Unit Key words:
Ratio, proportion, corresponding, scale, enlargement, reduction, similarity, and congruency

Knowledge & Skills (from O Level Syllabus)

G2. Congruence and similarity

2.1. congruent figures
2.2. similar figures
2.3. properties of similar triangles and polygons:
- corresponding angles are equal
- corresponding sides are proportional

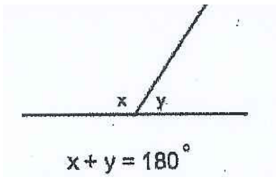
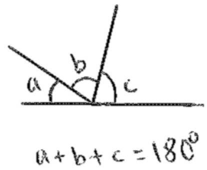
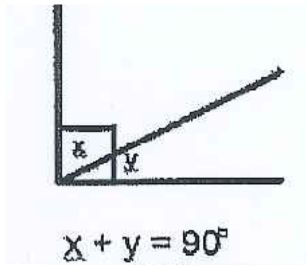
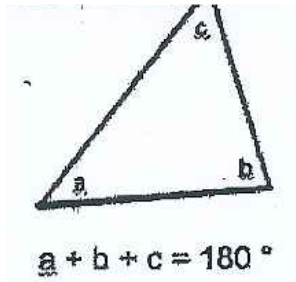
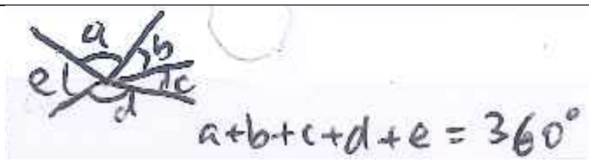
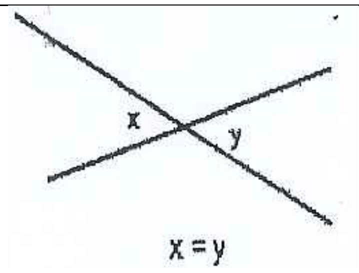
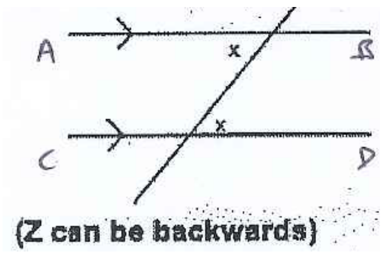
2.4. enlargement and reduction of a plane figure
2.5. scale drawings
2.6. solving simple problems involving congruence and similarity
2.7. determining whether two triangles are:
- congruent
- similar

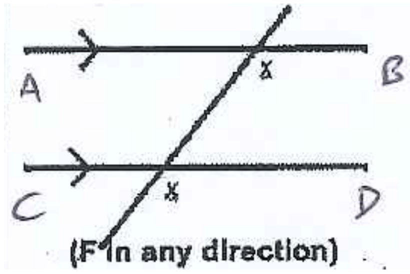
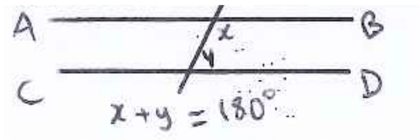
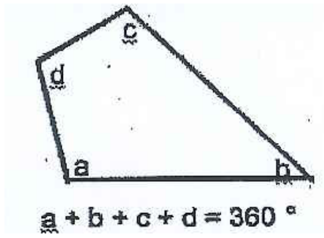
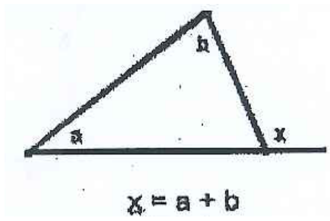
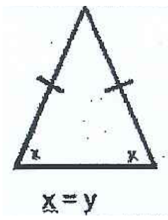
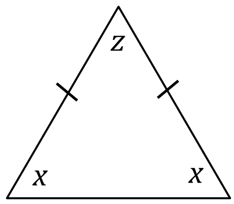
2.8. ratio of areas of similar plane figures
2.9. ratio of volumes of similar solids

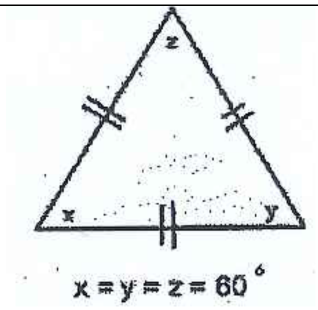
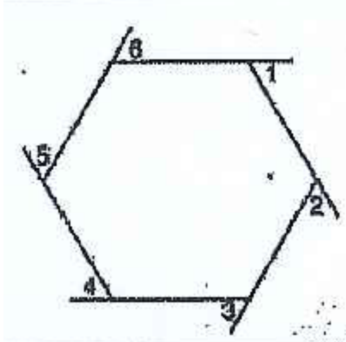
Teaching To The Big Idea …

Lesson sequence in the unit								
Student Learning Outcomes	Dimensions (Please tick the appropriate boxes)							
	FUNCTIONS F	INVARIANCE I	NOTATIONS N	DIAGRAMS D	MEASURES M	EQUIVALENCE E	PROPORTIONALITY P	MODELS M
Congruent Figures, Tests for Congruent Triangles		√		√				
Similarity, Enlargement and Reduction of Plane Figure, Tests for Similar Triangles				√			√	
Ratios and Volumes of Similar Figures				√			√	

Recap: Properties of Angles

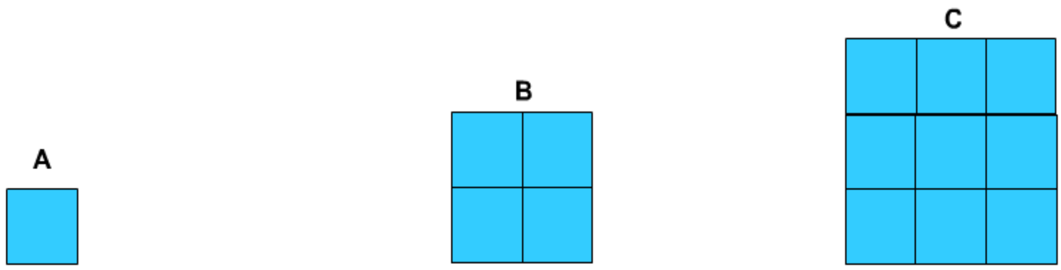
	Property	*Abbreviation*	Diagram (example)
1	Angles that adjacent on a straight line add up to 180°. Note: supplementary angles refer to 2 angles only.	angles on a straight line	 
2	Complementary Angles (Angles that are adjacent on a right angle. Complementary angles add up to 90°)	complementary angles	
3	Angles in a triangle add up to 180°	angle sum of triangle/ sum of angles in a triangle	
4	Angles at a point add up to 360°	angles at a point	
5	Vertically opposite angles are equal	vertically opposite angles	
6	Alternate angles are equal (Look out for "Z" pattern)	alternate angles, *AB* parallel to *CD*	

7	Corresponding angles are equal (Look out for “F” pattern)	corresponding angles, *AB* parallel to *CD*	
8	Interior angles of parallel lines add up to 180°. (Look out for “C” pattern)	interior angles, *AB* parallel to *CD*	
9	Angles in a quadrilateral add up to 360°	angle sum of quadrilateral/ sum of angles in a quadrilateral	
10	Exterior angles of a triangle add up to the sum of two opposite interior angles	exterior angle of triangle = sum of 2 interior opposite angles	
11	In an isosceles triangle, the base angles are equal.	base angles of isosceles triangle	
12	Sum of angles in an isosceles triangle add up to 180°	angle sum of isosceles triangle/ sum of angles in an isosceles triangle	 Angle $x = \dfrac{180° - z}{2}$

13	In an equilateral triangle, all the angles are equal (60°)	angles of equilateral triangle	
14	Sum of interior angles of an n-sided polygon $=\left(n-2\right)\times 180^\circ$ Sum of exterior angles of an n-sided polygon = 360°		

12. Areas of Similar Figures

Consider a square A of side 2 cm. Notice what happens to the area when you double and treble its sides.



Area = 4 cm^2 Area = 16 cm^2 Area = 36 cm^2

Ratio of the sides of squares A and B = $\frac{2}{4}$

Ratio of the areas of squares A and B = $\frac{4}{16}$

$$\frac{4}{16} = \left(\frac{2}{4}\right)^2$$

Ratio of the areas of A and B = (Ratio of their sides)^2

Ratio of the sides of squares A and C = $\frac{2}{6}$

Ratio of the areas of squares A and C = $\frac{4}{36}$

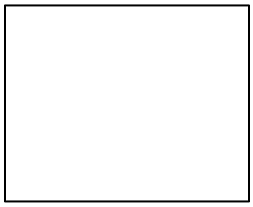
$$\frac{4}{36} = \left(\frac{2}{6}\right)^2$$

Ratio of the areas of A and C = (Ratio of their sides)^2

When two figures are **similar**, the ratio of their areas is equal to the **square** of the ratio of any two **Corresponding sides** of the two figures.

If A_1 and A_2 denote the areas of **similar figures**, and l_1 and l_2 denote their corresponding lengths, we have

dimensions
l_1 by w_1



dimensions
l_2 by w_2

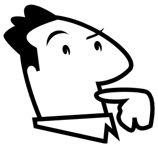
$$\frac{A_1}{A_2} = \frac{l_1 \times w_1}{l_2 \times w_2}$$

since

$$\frac{l_1}{l_2} = \frac{w_1}{w_2}$$

then

$$\frac{A_1}{A_2} = \frac{l_1 \times l_1}{l_2 \times l_2} = \left(\frac{l_1}{l_2}\right)^2$$

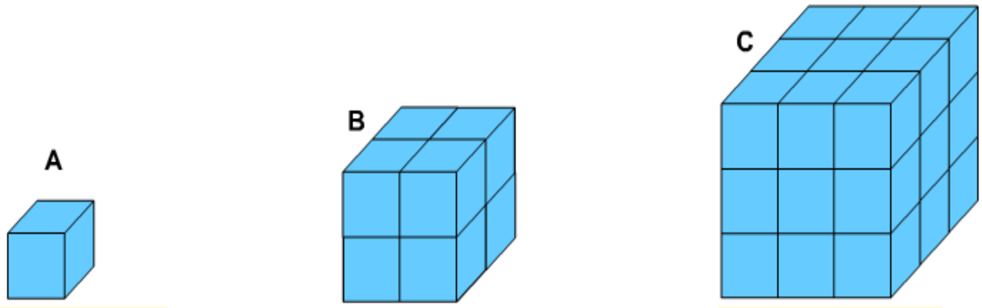


How to test that 2 **circles** are similar?
If they are similar what would be the ratio of their surface area?

13. Volumes of Similar Figures

Use the same concept to derive the relationship between Volumes of similar figures.

Consider a solid A with side 2 cm. Notice what happens to the volume when you double and treble its sides



Volume = 8 cm^3 | Volume = 64 cm^3 | Volume = 216 cm^3

Ratio of the sides of A and B = $\frac{2}{4}$

Ratio of the volumes of A and B = $\frac{8}{64}$

$$\frac{8}{64} = \left(\frac{2}{4}\right)^3$$

Ratio of the volumes of A and B = (Ratio of their sides)^3

Ratio of the sides of A and C = $\frac{2}{6}$

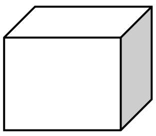
Ratio of the volumes of A and C = $\frac{8}{216}$

$$\frac{8}{216} = \left(\frac{2}{6}\right)^3$$

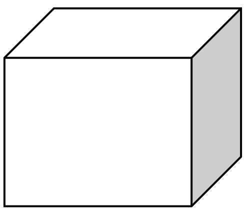
Ratio of the volumes of A and C = (Ratio of their sides)^3

When two figures are **similar**, the ratio of their volumes is equal to the __cube__ of the ratio of any two __Corresponding sides__ of the two figures.

If V_1 and V_2 denote the volumes of **similar figures**, and l_1 and l_2 denote their corresponding lengths, we have



dimensions
l_1 by w_1 by h_1



dimensions
l_2 by w_2 by h_2

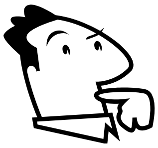
$$\frac{V_1}{V_2} = \frac{l_1 \times w_1 \times h_1}{l_2 \times w_2 \times h_2}$$

since

$$\frac{l_1}{l_2} = \frac{w_1}{w_2} = \frac{h_1}{h_2}$$

then

$$\frac{V_1}{V_2} = \frac{l_1 \times l_1 \times l_1}{l_2 \times l_2 \times l_2} = \left(\frac{l_1}{l_2}\right)^3$$



How to test that 2 **hemispheres** are similar?

If they are similar what would be the ratio of their (i) surface area and (ii) volume ?

Summary: Areas and Volumes of Similar Figures

For two geometrically similar solids,

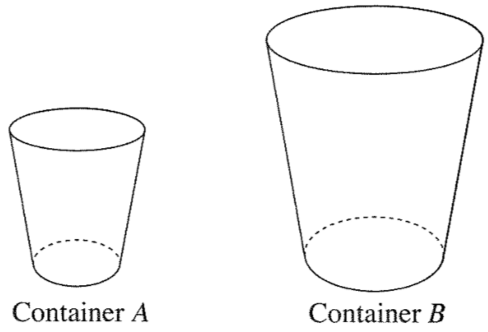
- $\dfrac{length_1}{length_2} = \dfrac{height_1}{height_2}$
- $\dfrac{area_1}{area_2} = \left(\dfrac{length_1}{length_2}\right)^2 = \left(\dfrac{height_1}{height_2}\right)^2$
- $\dfrac{volume_1}{volume_2} = \left(\dfrac{length_1}{length_2}\right)^3 = \left(\dfrac{height_1}{height_2}\right)^3$
- $\dfrac{mass_1}{mass_2} = \dfrac{volume_1}{volume_2} = \left(\dfrac{length_1}{length_2}\right)^3 = \left(\dfrac{height_1}{height_2}\right)^3$, if the two solids have same density.

Note:
Mass = density x volume

Question:
If 2 boxes of different sizes are made up of the same materials, what could you infer?

How to remember:
Hint:
What is the unit for lengths?
what is the unit for areas?
what is the unit for volumes?

Example



The diagram shows two geometrically similar containers A and B. The base areas of the containers A and B are 16 cm^2 and 25 cm^2 respectively.

1. what information provided?
2. what key words used?
3. any assumptions made?

(a) Find the ratio of the heights of container A and container B.

(b) Containers A and B are filled with flour. The mass of flour in container B is 7.5 kg. Find the mass of flour in container A.

(a)

$$\boxed{\frac{h_A}{h_B} = \sqrt{\frac{16}{25}}}$$

What is the concept ?

$$\frac{h_A}{h_B} = \frac{4}{5}$$

Height of container A : Height of container B
= 4 : 5

(b)

$$\boxed{\frac{m_A}{m_B} = \left(\frac{4}{5}\right)^3}$$

$$\frac{m_A}{7.5} = \frac{64}{125}$$

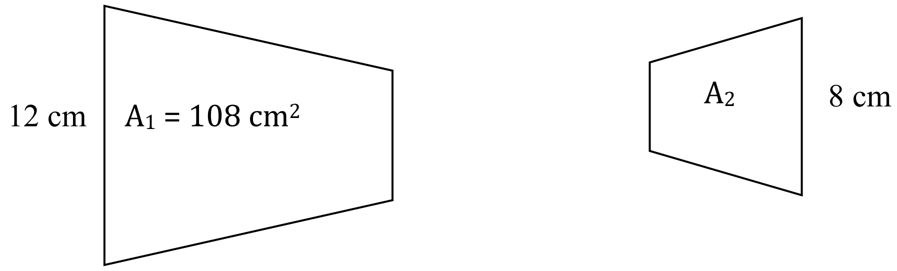
$$m_A = \frac{64}{125} \times 7.5$$

$$m_A = 3.84$$

mass of flour in container A is 3.84 kg

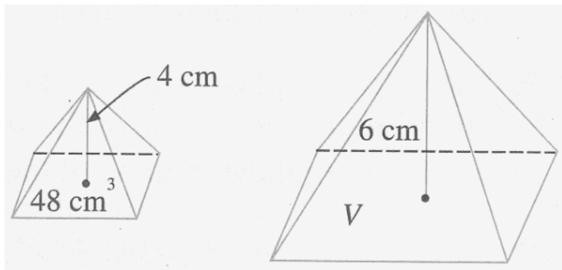
Example 11

Find the unknown area A_2 given that the shapes are similar.



Example 12

Find the unknown volume V the following pairs of similar objects.



Example 13

Two similar cones of the same material have base diameters 24 cm and 16 cm respectively. The volume of the larger cone is 378 cm^3 and the mass of the smaller cone 928 g. Calculate

(a) the volume of the smaller cone,

(b) the mass of the larger cone.

Example 14

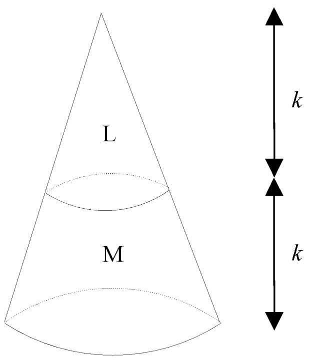
A cylinder K has a volume of 200 cm^3. Calculate the volume of

(a) a cylinder similar to K but with radius twice that of K,

(b) a cylinder with height twice that of K and radius one quarter that of K.

Example 15

A right circular cone is divided into 2 portions, L and M, by a plane parallel to the base as shown in the diagram. The height of each portion is k units. Find the ratio of the volume of L to the volume of M.

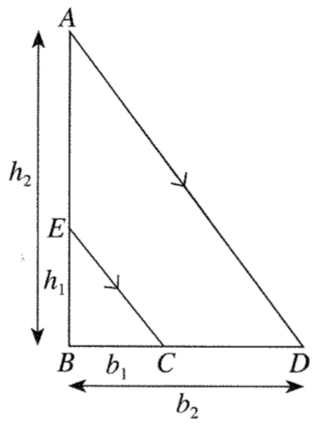


HINT

$$\frac{\text{Volume L}}{\text{Volume L+M}} = \left(\frac{k}{2k}\right)^3$$

14. Finding Areas of Triangles Using Ratios

(I) Similar Triangles



Consider two similar triangles ΔBCE and ΔBDA.

$$\frac{\text{Area of }\Delta BCE}{\text{Area of }\Delta BDA} = \left(\frac{b_1}{b_2}\right)^2$$

OR

$$\frac{\text{Area of }\Delta BCE}{\text{Area of }\Delta BDA} = \left(\frac{h_1}{h_2}\right)^2$$

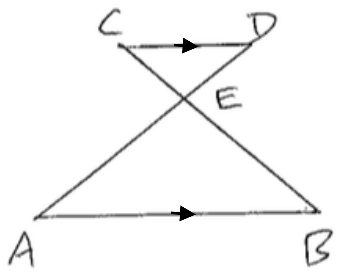
Example

In the diagram, *AED* and *BEC* are straight lines. It is given that *CD* is parallel to *AB* and $CD = \frac{2}{3}AB$.

Find the ratio $\dfrac{area\ of\ \Delta ABE}{area\ of\ \Delta DCE}$.

Given

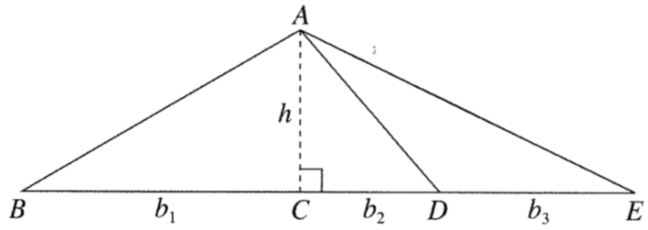
$\frac{CD}{AB} = \frac{2}{3}$



Solution:

(II) Triangles Sharing Same Height/ Base

Ratios of Areas of Triangles with Common Heights

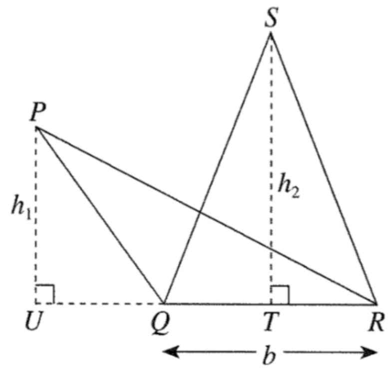


Consider ΔABC, ΔACD, ΔADE, ΔABD, ΔACE and ΔABE with common height, h.

$$\frac{\text{Area of }\Delta ACD}{\text{Area of }\Delta ADE}=\frac{\frac{1}{2}\times b_2\times h}{\frac{1}{2}\times b_3\times h}=\frac{b_2}{b_3}$$

$$\boxed{\frac{\text{Area of }\Delta ABC}{\text{Area of }\Delta ACE}=\frac{\frac{1}{2}\times b_1\times h}{\frac{1}{2}\times (b_2+b_3)\times h}=\frac{b_1}{b_2+b_3}}$$

Ratios of Triangles with Common Base



Consider ΔPQR and ΔSQR with common base, b.

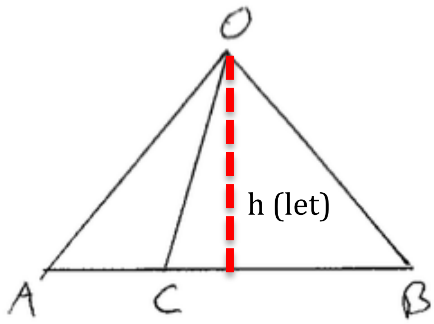
$$\boxed{\frac{\text{Area of }\Delta PQR}{\text{Area of }\Delta SQR}=\frac{\frac{1}{2}\times h_1\times b}{\frac{1}{2}\times h_2\times b}=\frac{h_1}{h_2}}$$

Key concept:
The triangles have a COMMON (1) Height or (2) Base.

<u>Example</u>

In the diagram, OAB is a triangle. C is a point on AB such that $AC=\frac{2}{5}AB$.

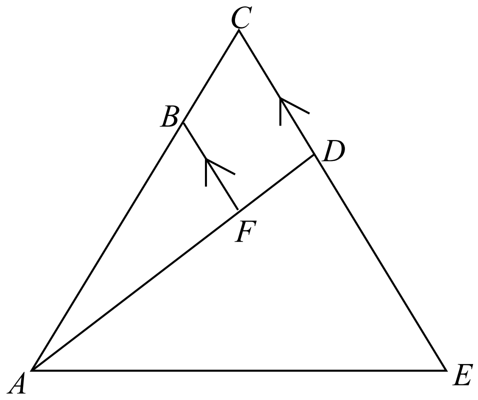
Find the ratio $\frac{area\ of\ \Delta OAC}{area\ of\ \Delta OBC}$.



Solution:

Example 16

In the diagram, *ABC*, *AFD* and *CDE* are straight lines and *BF* is parallel to *CD*.



(a) Show that triangle *ABF* and triangle *ACD* are similar.

(b) Given that $CD = \frac{1}{3}CE$ and $BC = \frac{1}{2}AB$, find

(i) $\dfrac{\text{area of } \Delta ABF}{\text{area of } \Delta ACD}$,

(ii) $\dfrac{\text{area of } \Delta ABF}{\text{area of } \Delta ADE}$.