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Mr Jo's notes

School of Science and Technology, Singapore
Mathematics Department

Secondary 2 - Congruence and Similarity Notes

Name:________________________________ (   )   Class: S2-0 __

Unit Enduring Understanding
1. **Diagrams** of figures help us to visualise their congruence or similarity.
2. Two figures are congruent if and only if their sides and angles remain **invariant** under translation, rotation and reflection.
3. Two similar figures have corresponding sides that are **proportional**.

Unit Essential Questions
1. How do diagrams facilitate problem solving?
2. How do properties of congruent figures remain invariant under transformations?
3. How does proportionality undergird the concept of similarity?

Unit Key words:
Ratio, proportion, corresponding, scale, enlargement, reduction, similarity, and congruency

Knowledge & Skills (from O Level Syllabus)

G2. Congruence and similarity

2.1. congruent figures
2.2. similar figures
2.3. properties of similar triangles and polygons:
- corresponding angles are equal
- corresponding sides are proportional

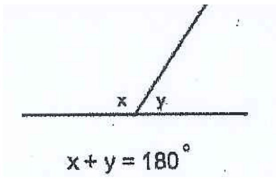
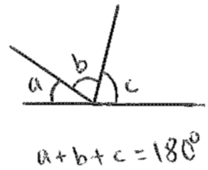
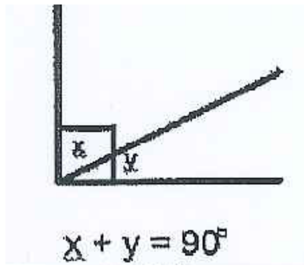
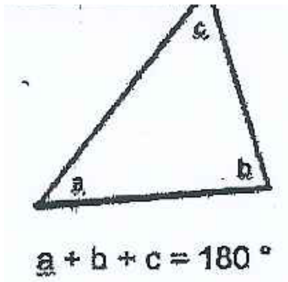
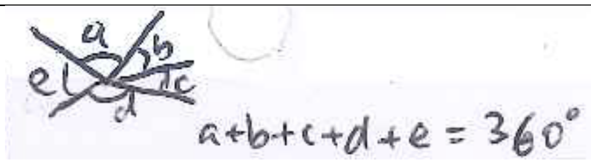
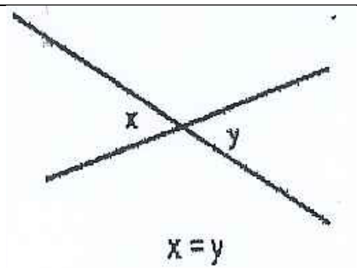
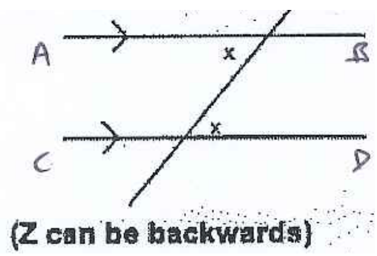
2.4. enlargement and reduction of a plane figure
2.5. scale drawings
2.6. solving simple problems involving congruence and similarity
2.7. determining whether two triangles are:
- congruent
- similar

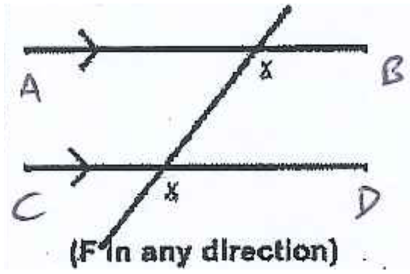
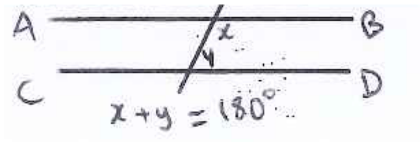
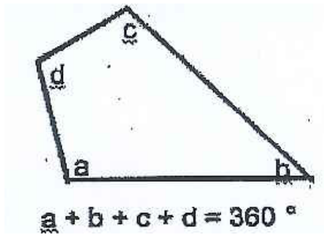
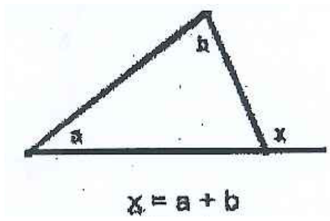
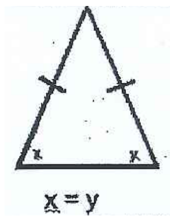
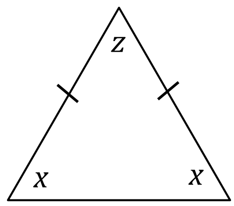
2.8. ratio of areas of similar plane figures
2.9. ratio of volumes of similar solids

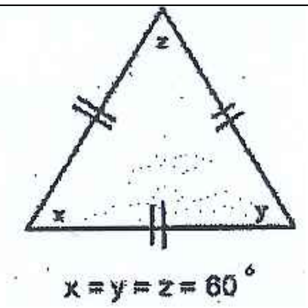
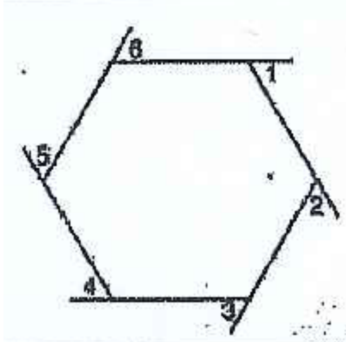
Teaching To The Big Idea …

Lesson sequence in the unit								
Student Learning Outcomes	Dimensions (Please tick the appropriate boxes)							
	FUNCTIONS F	INVARIANCE I	NOTATIONS N	DIAGRAMS D	MEASURES M	EQUIVALENCE E	PROPORTIONALITY P	MODELS M
Congruent Figures, Tests for Congruent Triangles		√		√				
Similarity, Enlargement and Reduction of Plane Figure, Tests for Similar Triangles				√			√	
Ratios and Volumes of Similar Figures				√			√	

Recap: Properties of Angles

	Property	*Abbreviation*	Diagram (example)
1	Angles that are adjacent on a straight line add up to 180°. Note: supplementary angles refer to 2 angles only.	angles on a straight line	 
2	Complementary Angles (Angles that are adjacent on a right angle. Complementary angles add up to 90°)	complementary angles	
3	Angles in a triangle add up to 180°	angle sum of triangle/ sum of angles in a triangle	
4	Angles at a point add up to 360°	angles at a point	
5	Vertically opposite angles are equal	vertically opposite angles	
6	Alternate angles are equal (Look out for "Z" pattern)	alternate angles, *AB* parallel to *CD*	

7	Corresponding angles are equal (Look out for "F" pattern)	corresponding angles, *AB* parallel to *CD*	
8	Interior angles of parallel lines add up to 180°. (Look out for "C" pattern)	interior angles, *AB* parallel to *CD*	
9	Angles in a quadrilateral add up to 360°	angle sum of quadrilateral/ sum of angles in a quadrilateral	
10	Exterior angles of a triangle add up to the sum of two opposite interior angles	exterior angle of triangle = sum of 2 interior opposite angles	
11	In an isosceles triangle, the base angles are equal.	base angles of isosceles triangle	
12	Sum of angles in an isosceles triangle add up to 180°	angle sum of isosceles triangle/ sum of angles in an isosceles triangle	 Angle $x = \dfrac{180° - z}{2}$

13	In an equilateral triangle, all the angles are equal (60°)	angles of equilateral triangle	
14	Sum of interior angles of an n-sided polygon $= (n-2) \times 180°$ Sum of exterior angles of an n-sided polygon = 360°		

9. Similarity

Hence, two polygons are similar if

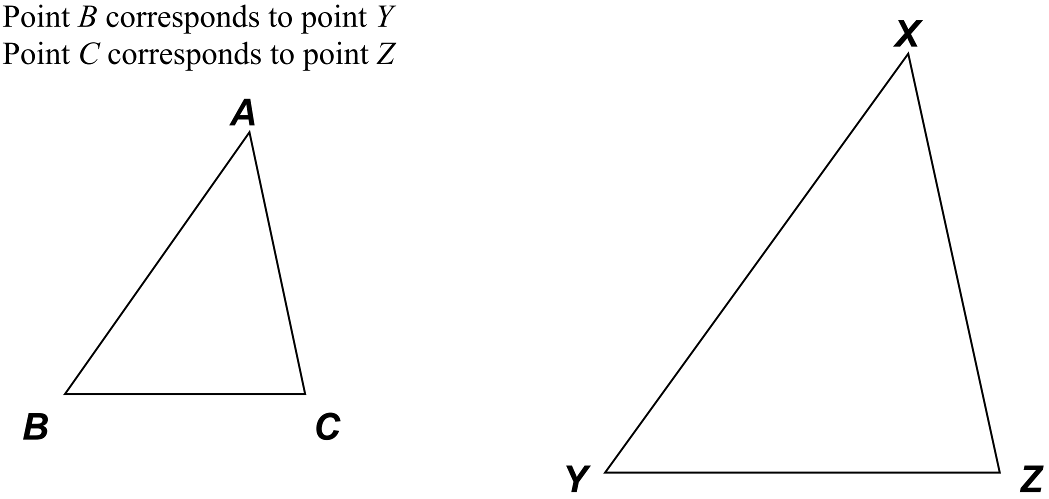
all the corresponding angles are equal, **and**

the ratios of the corresponding sides are equal.

Note: Congruence is a special case of similarity.

Given that triangle *ABC* and triangle *XYZ* are similar, the following can be deduced:

- Point *A* corresponds to point *X*
- Point *B* corresponds to point *Y*
- Point *C* corresponds to point *Z*



$\angle BAC = \angle YXZ$,

$\angle ABC = \angle XYZ$,

$\angle ACB = \angle XZY$,

$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{XZ}$.

Order must correspond strictly

When 2 triangles are similar, we will write it as:

Triangle ***ABC*** is similar to Triangle ***XYZ***

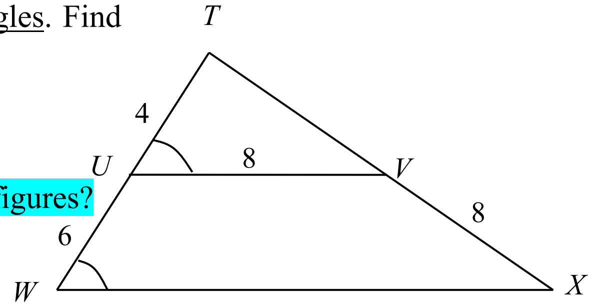
*****The order of the vertices of one triangle must correspond/ match to the vertices of the other triangle.*****

(In the above example, writing "triangle *ABC* is similar to triangle *ZYX*" is incorrect.)

There is **no** standard symbol for similarity.

Example 5

Given that ΔUTV and ΔWTX are similar triangles. Find the length of TV and WX



Thinking process:

1. What key information given?
2. Simplify the diagram(s) to determine similar figures?

Based on the similar triangles, write down the 3 pairs of ratios of ***corresponding sides as fractions****:*

Equation 1

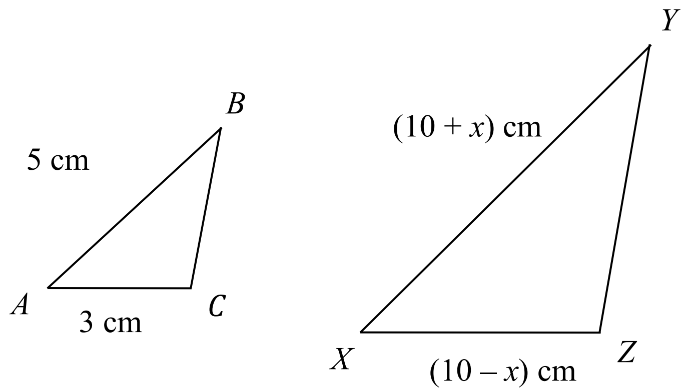
$$\frac{UT}{WT} = \frac{TV}{\quad} = \frac{\quad}{WX}$$

Equation 2

Then select the pairs that help you find the required lengths:

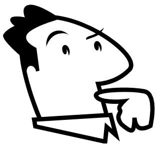
Example 6

ΔABC is similar to ΔXYZ. Given that $AB = 5$ cm, $AC = 3$ cm, $XY = (10 + x)$ cm and $XZ = (10 - x)$ cm, form an equation in x and solve it.



Thinking process:

1. What key information given? Check units and labels used.
2. Find ratios using corresponding sides?
3. Form equation in x.
4. Solve x.
5. Check if x is suitable. Reliability check. Example: is there a unit for x?



To prove that two triangles are similar, do we have to fulfill all corresponding angles are equal and all corresponding sides are proportional?
Higher order question (requires complete proof)

Triangles are special.

To prove that 2 triangles are similar, we need only prove that:

1. all corresponding angles are equal

or

2. all corresponding sides are proportional

or

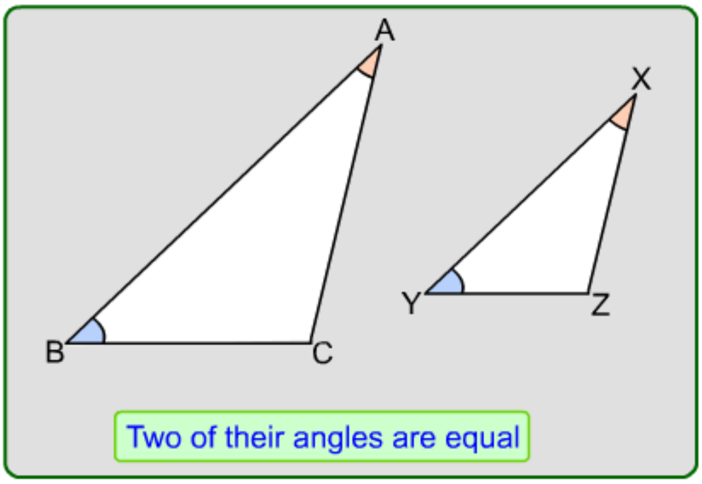
3. two pairs of corresponding sides are proportional and the included angles are equal.

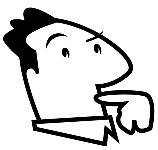
10. Similarity Tests

(1) All the corresponding angles are equal (AA Similarity Test)
Essentially, we only need 2 corresponding pairs of angles. Why?

Statement

If 2 angles of one triangle are equal to the corresponding 2 angles in the other triangle, then all the corresponding angles are equal, as a result of the total sum of interior angles being 180 degrees.

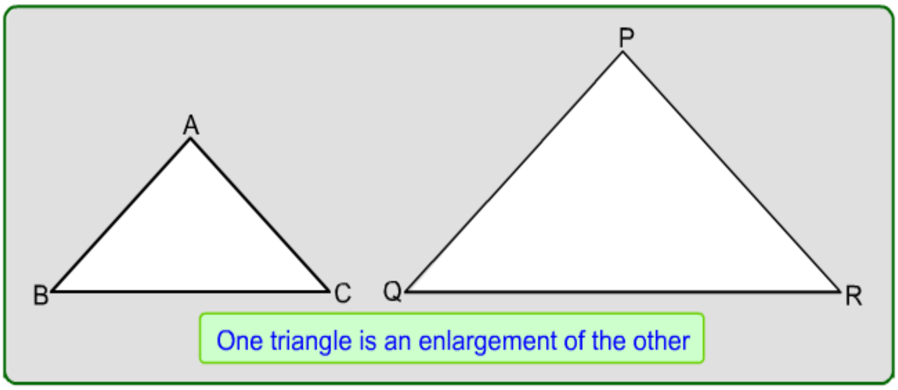




If 2 **quadrilaterals** have all corresponding angles equal, are they necessarily similar?

WHY?

(2) Ratio of all the corresponding sides are proportional (SSS Similarity Test)

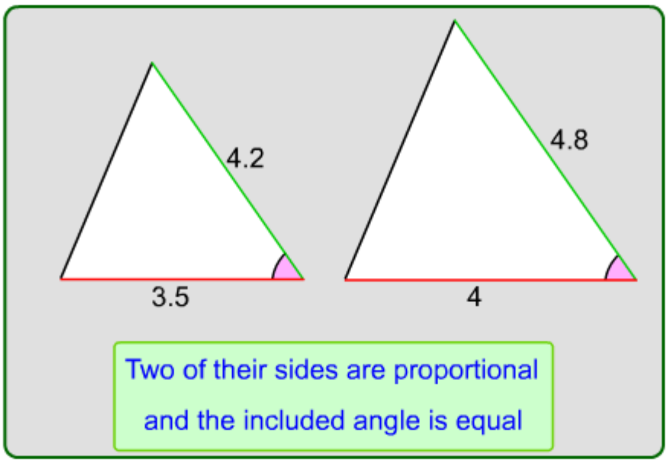


Statement

If the ratios of the corresponding sides are equal, then one triangle is an enlargement of the other. The proportion of the corresponding sides is known as the **enlargement factor**.

Consequently, the two triangles are **similar**.

(3) Two corresponding sides are proportional and the included angle is equal (SAS Similarity Test)



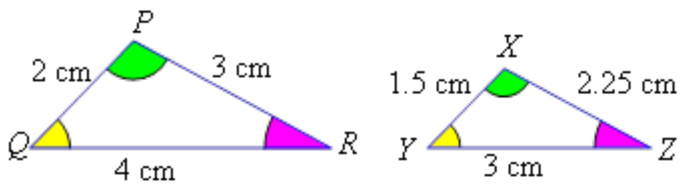
Statement

Two triangles are similar if an angle of one triangle is equal to an angle of the other triangle, and the sides which include the equal angle of both triangles are proportional (ratios of the pairs of corresponding sides are equal).

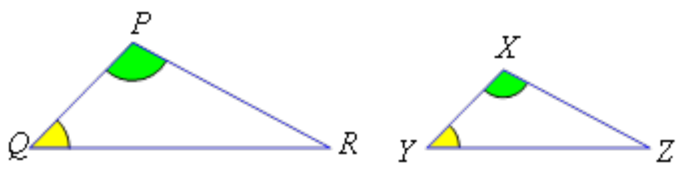
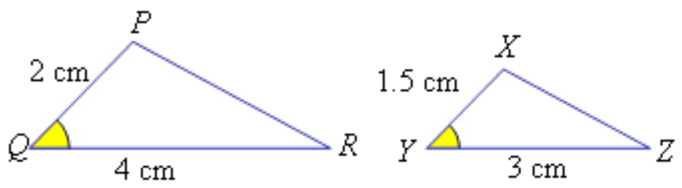
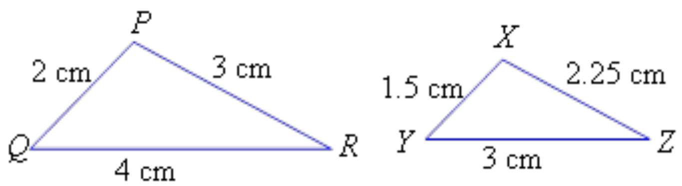
Consequently, the two triangles are similar.

11. Tests for Similar Triangles (Summary)

Summary (Similar Triangles)

For 2 similar triangles, ● their **corresponding angles** are equal ● the **ratios** of their corresponding sides are equal.	 Triangle *PQR* **is similar to** Triangle *XYZ*.

Similarity Test

1. If **two angles of one triangle are equal to the corresponding two angles** in the other triangle, then the two triangles are similar. (AA similarity test)	 $R\hat{P}Q = Z\hat{X}Y$ $P\hat{Q}R = X\hat{Y}Z$ Triangle *PQR* **is similar to** Triangle *XYZ*.
2. If the **ratios of two pairs of corresponding sides** are equal and their included angles are equal, then the two triangles are similar. (SAS similarity test)	 $\frac{PQ}{XY} = \frac{QR}{YZ} = \frac{4}{3}$ $P\hat{Q}R = X\hat{Y}Z$ Triangle *PQR* **is similar to** Triangle *XYZ*.
3. If the **ratios of all pairs of corresponding sides of two triangles are equal**, then the two triangles are similar. (SSS similarity test)	 $\frac{PQ}{XY} = \frac{QR}{YZ} = \frac{RP}{ZX} = \frac{4}{3}$ Triangle *PQR* **is similar to** Triangle *XYZ*.

<u>Tests for Similar Triangles</u>

For 2 similar triangles,

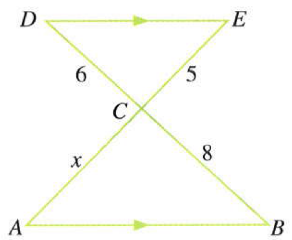
- their corresponding angles are equal
- the ratios of their corresponding sides are equal.

To explain that two triangles are similar, include:
- 3 statements with reasons
- conclusion with test used.

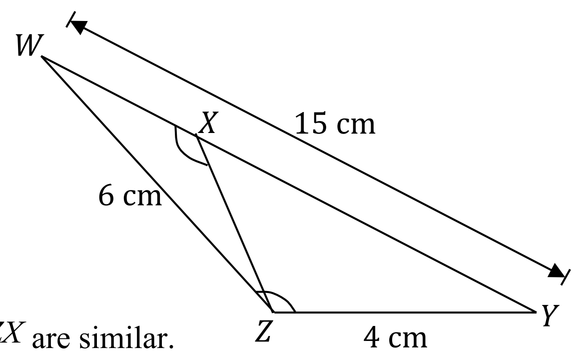
Note that the vertices of each triangle must correspond.

<u>Examples</u>

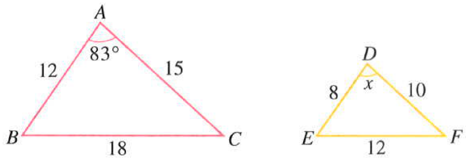
(1a) Similarity Test: AA Similarity Test

Question	In the figure, AB is parallel to DE, $BC = 8$ cm, $CD = 6$ cm and $CE = 5$ cm. **Prove** that triangle ABC and triangle EDC are **similar**. 
Solution	$\angle BAC = \angle DEC$ (alternate angles, AB parallel to DE) $\angle ABC = \angle EDC$ (alternate angles, AB parallel to DE) $\angle ACB = \angle ECD$ (vertically opposite angles) Any 2 of these Conditions for proof. Must be explicit $\therefore$ Triangle ABC is similar to triangle EDC. (AA similarity test) Important final statement

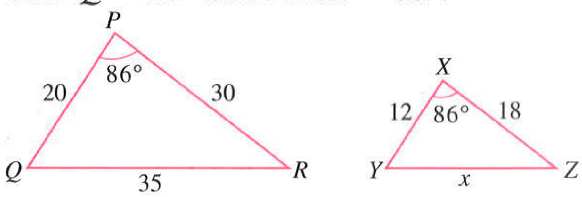
(1b) Similarity Test: AA Similarity Test

Question	In the diagram, <u>WXY is a straight line</u> and $\angle WXZ = \angle WZY$. It is also given that $WY = 15$ cm, $WZ = 6$ cm and $ZY = 4$ cm.  Prove that triangle WYZ and triangle WZX are similar.
Solution	$\angle WZY = \angle WXZ$ (given) $\angle ZWY = \angle XWZ$ (Common angles) Conditions for proof. Must be explicit *[Hence the third angles are equal:* $\angle WYZ = \angle WZX$ *(angles sum of triangle)]* $\therefore$ Triangle WYZ is similar to triangle WZX. (AA similarity test) Important final statement

(2) Similarity Test: SSS Similarity Test

Question	In the diagram, $AB = 12$ cm, $BC = 18$ cm, $CA = 15$ cm, $DE = 8$ cm, $EF = 12$ cm, $FD = 10$ cm and $\angle BAC = 83°$.  Prove that triangle ABC and triangle DEF are similar.
Solution	$\frac{AB}{DE} = \frac{12}{8} = \frac{3}{2}$ $\frac{BC}{EF} = \frac{18}{12} = \frac{3}{2}$ $\frac{CA}{FD} = \frac{15}{10} = \frac{3}{2}$ Using ratio of corresponding sides method. Test ALL corresponding sides. $\therefore \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{3}{2}$ Linking all the ratios to prove similarity $\therefore$ Triangle *ABC* is similar to triangle *DEF*. (SSS similarity test) Concluding statement

(3) Similarity Test: SAS Similarity Test

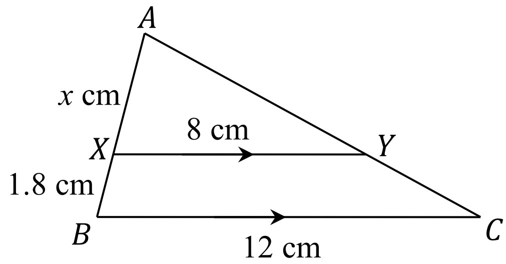
Question	In the diagram, $PQ = 20$ cm, $QR = 35$ cm, $RP = 30$ cm, $XY = 12$ cm, $ZX = 18$ cm, $\angle RPQ = 86°$ and $\angle ZXY = 86°$.  Prove that triangle PQR and triangle XYZ are similar.
Solution	$\angle RPQ = \angle ZXY$ (given) $\frac{PQ}{XY} = \frac{20}{12} = \frac{5}{3}$ $\frac{RP}{ZX} = \frac{30}{18} = \frac{5}{3}$ $\therefore \frac{PQ}{XY} = \frac{RP}{ZX} = \frac{5}{3}$ $\therefore$ Triangle *PQR* is similar to triangle *XYZ*. (SAS similarity test)

Example 7

In the figure, XY is parallel to BC.

(i) **Explain** why ΔABC is similar to ΔAXY.

(ii) **Calculate** the value of x.



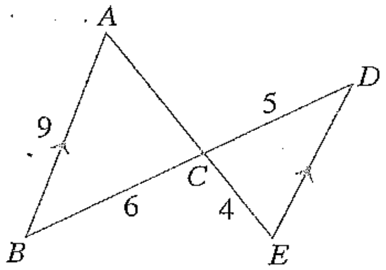
Think

- Explain (proof) with valid reasons using one of the similarity tests
- Calculate – mathematical computation required with working. Check unit.

Kahoot activity

Example 8

In the diagram, $AB = 9$ cm, $BC = 6$cm, $CD = 5$ cm and $CE = 4$ cm. It is also given that BA is parallel to ED.



(a) Name two triangles that are similar.
Thinking process:

note 1: BA is parallel to ED therefore angle BAE = angle AED (alternate angles)

Vertex A correspond to Vertex E
Vertex B correspond to Vertex D
Vertex C correspond to Vertex C

note 2: angle ACB = angle ECD (vertically opposite angles)

Hence,

Solution:

EDC?
DEC? note 3: order is important
Triangle ABC is similar to Triangle CDE? .

(b) Calculate the lengths of AC and of DE.
Solution:

Based on the similar triangles, write the 3 pairs of ratios of corresponding sides as fractions: using Triangle ABC similar to Triangle EDC.

eq1 eq2 eq3

$$\frac{AB}{ED} = \frac{AC}{EC} = \frac{BC}{DC} = \frac{BA}{DE}$$

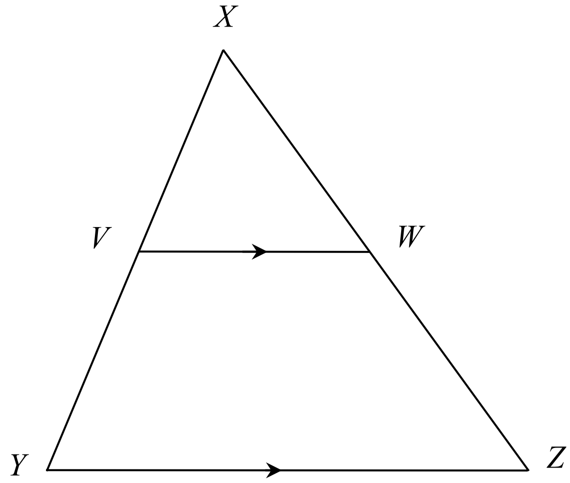
Then select the pairs that help you find the required lengths:

Name: ______________________________ (   ) S2-08

Class Exercise

Example 9

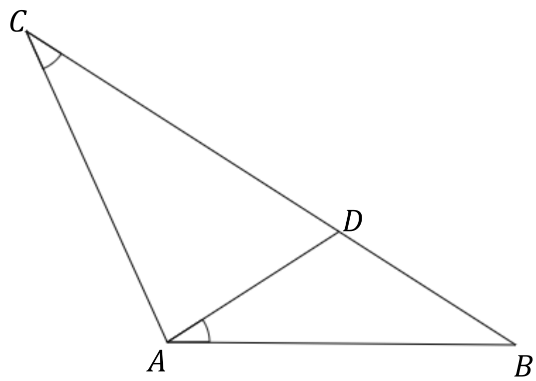
In the diagram, $VX = 12$ cm, $VW = 9$ cm and $YZ = 15$ cm.
Calculate the value of VY.



Example 10

In the diagram, ABC is a triangle.

D is the point on BC such that $\angle BCA = \angle BAD$.



Given that $BD = 4$ m and $DC = 5$ m, find the length of AB.