School of Science and Technology, Singapore
Mathematics Department

SCHOOL OF SCIENCE AND TECHNOLOGY, SINGAPORE

A

2023 Secondary 3 ADDITIONAMATHEMATICS
Unit 01. Surds

Name Solutions ( ) Class S3–0

Enduring Understanding

Students will understand the following:

- The mathematical system is constructed from common and special notions called axioms and postulates, e.g. rules of surds derived from rules of indices.
- Surds remain **equivalent** when they are simplified using rules of surds.
- The use of algebraic identities helps to rationalize the denominator of fractions containing surds and result in simpler but **equivalent** fractions.

Essential Questions

- How are the rules of surds derived from the rules of indices?
- How is equilibrium maintained in manipulating equations involving surds?

Big Ideas

- Notations: Notations are symbols and conventions of writing used to represent mathematical objects, and their operations and relationships in a concise and precise manner. Examples in this unit include the use of the radical sign to denote the *n*th root.
- Equivalence: Equivalence is a relationship that expresses the 'equality' of two mathematical objects that may be represented in two different forms based on a criterion. Transformation or conversion of an expression or equation from one form to another equivalent form is the basis of many manipulations for analyzing and comparing them and algorithms for finding solutions.

Learning Objectives

At the end of the unit, students should be able to

- Simplify expressions involving surds
- Make use of conjugate surds to rationalize the denominator of an expression containing a surd
- Solve equations involving surds

Lesson sequence in the unit								
Student Learning Outcomes	Dimensions (Please tick the appropriate boxes)							
	FUNCTIONS **F**	INVARIANCE **I**	NOTATIONS **N**	DIAGRAMS **D**	MEASURES **M**	EQUIVA-LENCE **E**	PROPOR TIONALITY **P**	MODELS **M**
Surds			✓			✓		
Simplifying expressions involving surds			✓			✓		
Solving equations involving surds			✓			✓		

1.1 Surds

An irrational number that involving a root is called a **surd**.
For example, $\sqrt{2}$, $\sqrt[3]{11}$, $3\sqrt{5}$, $1+\sqrt{7}$ are suds, but $\sqrt{9}=3$ or $\sqrt[3]{8}=2$ are not surds. *WHY?*

In this unit, we will only deal with surds involving square roots.

Activity 1 **[Inquiry]**
Learning objective: To explore the laws of surds to facilitate the operations on expressions in surds.

Is $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$? Is $\sqrt{a\times b}=\sqrt{a}\times\sqrt{b}$?

1. With the help of a calculator (where necessary), answer the following questions.

(a) Is $\sqrt{16+9}=\sqrt{16}+\sqrt{9}$?

(b) Is $\sqrt{16-9}=\sqrt{16}-\sqrt{9}$?

(c) Is $\sqrt{16\times 9}=\sqrt{16}\times\sqrt{9}$?

(d) Is $\sqrt{\dfrac{9}{16}}=\dfrac{\sqrt{9}}{\sqrt{16}}$?

2. From **Question 1**, we can express the true statements in the general form as follows:

(i) $\sqrt{a\times b}=\sqrt{a}\times\sqrt{b}$ where $a,\ b\geq 0$

(ii) $\sqrt{\dfrac{a}{b}}=\dfrac{\sqrt{a}}{\sqrt{b}}$ where $a,\ b\geq 0$

We have learnt that $a^{\frac{1}{n}}=\sqrt[n]{a}$, where $a\geq 0$ and n is a positive integer.
In other words, $\sqrt{a}=a^{\frac{1}{2}}$, where $a\geq 0$.

Using the Laws of Indices, prove statements **(i)** and **(ii)**.

(i) $\sqrt{a\times b} = (ab)^{\frac{1}{2}} = a^{\frac{1}{2}} b^{\frac{1}{2}} = \sqrt{a}\sqrt{b}$

(ii) $\sqrt{\dfrac{a}{b}} = \left(\dfrac{a}{b}\right)^{\frac{1}{2}} = \dfrac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} = \dfrac{\sqrt{a}}{\sqrt{b}}$

I wonder...

What if $b=a$?
i.e. what is $\sqrt{a}\times\sqrt{a}$?

3. Using the Laws of Indices, simplify $\sqrt{a} \times \sqrt{a}$.

$\sqrt{a} \times \sqrt{a} = (a^{\frac{1}{2}})(a^{\frac{1}{2}}) = a^{\frac{1}{2}+\frac{1}{2}} = a$

Conclusion

If $a > 0$ and $b > 0$,

Law 1 of Surds: $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$

Law 2 of Surds: $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Law 3 of Surds: $\sqrt{a} \times \sqrt{a} = a$ [Special Case]

Pause

Think

$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ and $\sqrt{a-b} \neq \sqrt{a} - \sqrt{b}$

How do you remind yourself of these?

$\sqrt{9+4} \neq \sqrt{9} + \sqrt{4}$

$\sqrt{9-4} \neq \sqrt{9} - \sqrt{4}$

Note that surds should be expressed in the simplest form, in other words, in the form $a\sqrt{b}$, where b does not have any factor that is a perfect square.

Worked Example 1
Expressing surds in simplest form

(a) **(i)** Express 48 in terms of its prime factors.

(ii) Hence simplify $\sqrt{48}$.

(b) Express $80^{\frac{1}{2}}$ as a surd in its simplest form.

(a) (i) $48 = 2^4 \times 3$

(ii) $\sqrt{48} = (2^4 \times 3)^{\frac{1}{2}} = 2^2 \times \sqrt{3} = 4\sqrt{3}$

(b) $80^{\frac{1}{2}} = \sqrt{16 \times 5} = \sqrt{16}\sqrt{5} = 4\sqrt{5}$ #

What is your Problem Solving Strategy for such problem? Write it here...

Psst!

Worked Example 2
Expressing surds in simplest form

Simplify each of the following without using a calculator.

(a) $\sqrt{2} \times \sqrt{32}$

(b) $\dfrac{\sqrt{216}}{\sqrt{6}}$

(c) $\dfrac{\sqrt{27} \times \sqrt{2}}{\sqrt{6}}$

(a) $\sqrt{2} \times \sqrt{32} = \sqrt{64} = 8$ #

(b) $\dfrac{\sqrt{216}}{\sqrt{6}} = \sqrt{\dfrac{216}{6}} = \sqrt{36} = 6$ #

(c) $\dfrac{\sqrt{27} \times \sqrt{2}}{\sqrt{6}} = \sqrt{\dfrac{54}{6}} = \sqrt{9} = 3$ #

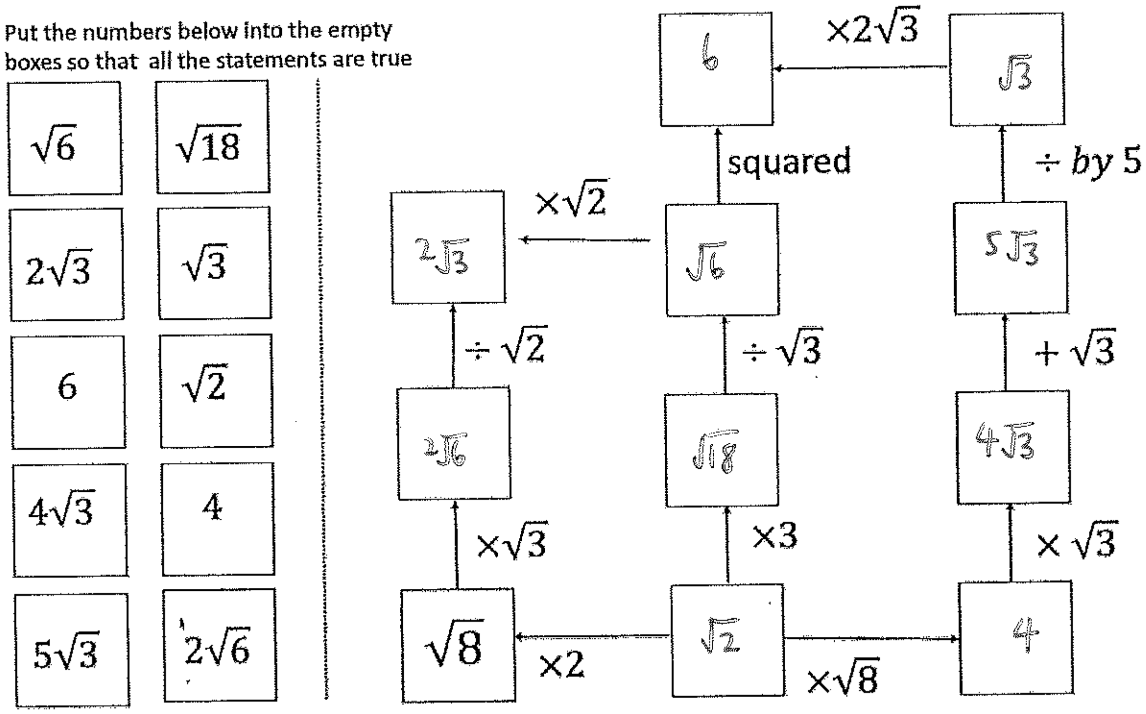
1.2 Simplifying Expressions Involving Surds

We manipulate surds in a similar way as algebraic terms.

Surds		Algebraic terms
Like surds are surds with the same radicand (the number under the radical sign), e.g. $\sqrt{2}$, $2\sqrt{2}$, $-3\sqrt{2}$		Like terms are terms that contain the same variable which is raised to the same power, e.g. x, $2x$, $-3x$
Add and subtract like surds: $2\sqrt{2}+3\sqrt{2}-\sqrt{2}=(2+3-1)\sqrt{2}$ $=4\sqrt{2}$	analogous to	Add and subtract like terms: $2x+3x-x=(2+3-1)x$ $=x$
Multiply surds: $(3\sqrt{2})(4\sqrt{5})=(3\times4)(\sqrt{2}\times\sqrt{5})$ $=12\sqrt{10}$		Multiply algebraic terms: $(3x)(4y)=(3\cdot4)(x\cdot y)$ $=12xy$

Activity 2 **[Think-Pair-Share]**

Learning objective: To apply laws of surds and simplify surds.



1.2.1 Adding and Subtracting Surds

Worked Example 3
Adding surds

Simplify each of the following without using a calculator.

(a) $\sqrt{32}+\sqrt{50}$

(b) $3\sqrt{75}-\sqrt{12}+\sqrt{27}$

(a) $\sqrt{32}+\sqrt{50} = \sqrt{2}\sqrt{16}+\sqrt{2}\sqrt{25}$

$= 4\sqrt{2}+5\sqrt{2}$

$= 9\sqrt{2}$

(b) $3\sqrt{75}-\sqrt{12}+\sqrt{27}$

$= 3\cdot\sqrt{25}\sqrt{3}-\sqrt{4}\sqrt{3}+\sqrt{3}\sqrt{9}$

$= 15\sqrt{3}-2\sqrt{3}+3\sqrt{3}$

$= 16\sqrt{3}$ #

Worked Example 4
Adding surds

Given that $\sqrt{28}-\sqrt{175}+\sqrt{112}=a\sqrt{7}$, find the value of a.

$\sqrt{28}-\sqrt{175}+\sqrt{112} = \sqrt{4}\sqrt{7}-\sqrt{25}\sqrt{7}+\sqrt{16}\sqrt{7}$

$= 2\sqrt{7}-5\sqrt{7}+4\sqrt{7}$

$= \sqrt{7}$

$a=1$ #

1.2.2 Multiplying Surds

Worked Example 5
Multiplying surds

Simplify each of the following without using a calculator.

(a) $\sqrt{3}(5+\sqrt{2})$

(b) $(6+\sqrt{3})(2-5\sqrt{3})$

(c) $(6+\sqrt{3})^2$

(d) $(2-5\sqrt{3})^2$

(e) $(6+\sqrt{3})(6-\sqrt{3})$

(f) $(2-5\sqrt{3})(2+5\sqrt{3})$

(g) $(2-5\sqrt{3})(-2-5\sqrt{3})$

(a) $\sqrt{3}(5+\sqrt{2}) = 5\sqrt{3}+\sqrt{6}$

(b) $(6+\sqrt{3})(2-5\sqrt{3}) = 6(2)-30\sqrt{3}+2\sqrt{3}-5(3)$
$= -3-28\sqrt{3}$

(c) $(6+\sqrt{3})^2 = 6^2+2(6)\sqrt{3}+(\sqrt{3})^2$
$= 36+12\sqrt{3}+3$
$= 39+12\sqrt{3}$

(d) $(2-5\sqrt{3})^2 = 2^2-4(2)(5\sqrt{3})+(5\sqrt{3})^2$
$= 4-40\sqrt{3}+75$
$= 79-40\sqrt{3}$

(e) $(6+\sqrt{3})(6-\sqrt{3}) = 6^2-6\sqrt{3}+6\sqrt{3}-(\sqrt{3})^2$
$= 36-3$
$= 33$

(f) $(2-5\sqrt{3})(2+5\sqrt{3}) = 2^2+10\sqrt{3}-10\sqrt{3}-(5\sqrt{3})^2$
$= 4-75$
$= -71$

(g) $(2-5\sqrt{3})(-2-5\sqrt{3}) = -4-10\sqrt{3}+10\sqrt{3}+25(3) = 71$

Pause

Think

Do you observe any interesting result(s) in **Worked Example 5**?

Worked Example 6
Simplifying surds involving arithmetic operations

Simplify, without using a calculator, $(2+\sqrt{3})^2-(3-\sqrt{3})^2$.

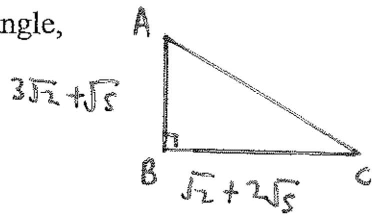
$$(2+\sqrt{3})^2-(3-\sqrt{3})^2 = 2^2+4\sqrt{3}+3-(3^2-6\sqrt{3}+3)$$
$$= 4+4\sqrt{3}+3-9+6\sqrt{3}-3$$
$$= -5+10\sqrt{3}$$

Worked Example 7
Solving problems involving surds

In triangle ABC, angle $ABC=90°$, $AB=3\sqrt{2}+\sqrt{5}$ and $BC=\sqrt{2}+2\sqrt{5}$.
Find the exact value of

(a) the area of the triangle,

(b) AC^2.



(a) Area of triangle $= \frac{1}{2}(3\sqrt{2}+\sqrt{5})(\sqrt{2}+2\sqrt{5})$
$$= \frac{1}{2}(3(2)+6\sqrt{10}+\sqrt{10}+2(5))$$
$$= \frac{1}{2}(16+7\sqrt{10})$$
$$= 8+\frac{7}{2}\sqrt{10} \text{ units}^2$$

(b)
$$AC^2 = (3\sqrt{2}+\sqrt{5})^2+(\sqrt{2}+2\sqrt{5})^2$$
$$= (3\sqrt{2})^2+2(3\sqrt{2})(\sqrt{5})+(\sqrt{5})^2+(\sqrt{2})^2+2(\sqrt{2})(2\sqrt{5})+(2\sqrt{5})^2$$
$$= 18+6\sqrt{10}+5+2+4\sqrt{10}+20$$
$$= 45+10\sqrt{10} \text{ units}^2$$

Checklist for Self-Assessment on Surds		
I am UNSURE how to	I am ABLE to	
☐	☐	simplify surds into its simplest form.
☐	☐	simplify expressions involving surds using various operations.

1.2.3 Conjugate Surds

$(\sqrt{m}+\sqrt{n})$ and $(\sqrt{m}-\sqrt{n})$ are **conjugate surds** of each other.

Observe the structure of the two expressions.

What do you obtain when you multiply a pair of conjugate surds?
i.e. what is $(\sqrt{m}+\sqrt{n})(\sqrt{m}-\sqrt{n})$?

Notice the similarity between $(\sqrt{m}+\sqrt{n})(\sqrt{m}-\sqrt{n})$ and $(x+y)(x-y)$?

Activity 3 [Inquiry]
Learning objective: To explore the product of a pair of conjugate surds.

1. Use the algebraic identify $(x+y)(x-y)=x^2-y^2$ to simplify the following:
   - **(a)** $(\sqrt{m}+\sqrt{n})(\sqrt{m}-\sqrt{n})$
   - **(b)** $(a+b\sqrt{n})(a-b\sqrt{n})$
   - **(c)** $(a\sqrt{m}+b)(a\sqrt{m}-b)$
   - **(d)** $(a\sqrt{m}+b\sqrt{n})(a\sqrt{m}-b\sqrt{n})$

2. What can you conclude from the results in **Question 1**?

1(a) $(\sqrt{m}+\sqrt{n})(\sqrt{m}-\sqrt{n}) = (\sqrt{m})^2-(\sqrt{n})^2 = m-n$

(b) $(a+b\sqrt{n})(a-b\sqrt{n}) = (a)^2-(b\sqrt{n})^2 = a^2-b^2n$

(c) $(a\sqrt{m}+b)(a\sqrt{m}-b) = (a\sqrt{m})^2-(b)^2$
$= a^2m-b^2$

(d) $(a\sqrt{m}+b\sqrt{n})(a\sqrt{m}-b\sqrt{n}) = (a\sqrt{m})^2-(b\sqrt{n})^2$
$= a^2m-b^2n$

2. They are rational numbers.

Conclusion

> The product of conjugate surds, $(\sqrt{m}+\sqrt{n})$ and $(\sqrt{m}-\sqrt{n})$, is a **rational** number.

I wonder...

> What is the conjugate surd of $\sqrt{a}$?

$-\sqrt{a}$

Worked Example 8
Finding conjugate surds

Write a conjugate surd for each of the following.

(a) $2+3\sqrt{5}$

(b) $-2+3\sqrt{5}$

(c) $2\sqrt{7}-3\sqrt{5}$

(d) $2\sqrt{7}+3$

(e) $\sqrt{7}$

(a) $2-3\sqrt{5}$

(b) $-2-3\sqrt{5}$

(c) $2\sqrt{7}+3\sqrt{5}$

(d) $2\sqrt{7}-3$

(e) $-\sqrt{7}$

I wonder...

> How is the fact that product of conjugate surds is rational useful?

1.2.4 Rationalising of Surds

A fraction involving surds is considered simplified when the denominator does not contain any surd. The process of simplifying a fraction with a surd in its denominator into an equivalent form where the denominator is rational is called **rationalising the denominator**.

To **rationalise the denominator**, we multiply the surd in the denominator by its **conjugate surd**.

Worked Example 9
Rationalising the denominator

Simplify each of the following by rationalising the denominator.

(a) $\frac{6}{\sqrt{3}}$ (b) $\frac{\sqrt{5}}{2\sqrt{3}}$ (c) $\frac{2\sqrt{6}}{\sqrt{3}}$

(a) $\frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{\sqrt{3}\sqrt{3}}$

$= 2\sqrt{3}$

(b) $\frac{\sqrt{5}}{2\sqrt{3}} = \frac{\sqrt{5}\sqrt{3}}{2\sqrt{3}\sqrt{3}}$

$= \frac{\sqrt{15}}{6}$

(c) $\frac{2\sqrt{6}}{\sqrt{3}} = \frac{2\sqrt{6}\sqrt{3}}{\sqrt{3}\sqrt{3}}$

$= \frac{6\sqrt{2}}{3}$

$= 2\sqrt{2}$ ✱

Worked Example 10
Rationalising the denominator

Simplify each of the following by rationalising the denominator.

(a) $\frac{6}{2+\sqrt{3}}$

(b) $\frac{6}{2\sqrt{3}-1}$

(c) $\frac{5-\sqrt{3}}{2-\sqrt{3}}$

(d) $\frac{\sqrt{2}+5}{\sqrt{3}-2}$

(e) $\frac{\sqrt{3}-5\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

(a) $\frac{6}{2+\sqrt{3}} = \frac{6(2-\sqrt{3})}{2^2-(\sqrt{3})^2}$

$= \frac{12-6\sqrt{3}}{1}$

$= 12-6\sqrt{3}$

(b) $\frac{6}{2\sqrt{3}-1} = \frac{6(2\sqrt{3}+1)}{(2\sqrt{3}-1)(2\sqrt{3}+1)}$

$= \frac{6(2\sqrt{3}+1)}{(4\times 3-1)}$

$= \frac{6}{11}(2\sqrt{3}+1)$

$= \frac{12}{11}\sqrt{3} + \frac{6}{11}$ ✱

(c) $\frac{5-\sqrt{3}}{2-\sqrt{3}} = \frac{(5-\sqrt{3})(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$

$= \frac{10+5\sqrt{3}-2\sqrt{3}-3}{2^2-(\sqrt{3})^2}$

$= 7+3\sqrt{3}$ ✱

(d) $\frac{\sqrt{2}+5}{\sqrt{3}-2} \times \frac{\sqrt{3}+2}{\sqrt{3}+2}$

$= \frac{\sqrt{6}+5\sqrt{3}+2\sqrt{2}+10}{(\sqrt{3})^2-2^2}$

$= -\sqrt{6}-5\sqrt{3}-2\sqrt{2}-10$ ✱

(e) $\frac{\sqrt{3}-5\sqrt{2}}{\sqrt{3}+\sqrt{2}} \times \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

$= \frac{3-\sqrt{6}-5\sqrt{6}+5(2)}{3-2}$

$= 13-6\sqrt{6}$ ✱

Worked Example 11
Adding/Subtracting fractions involving surds

Express each of the following as a single fraction in its simplest form.

(a) $\dfrac{5}{2-3\sqrt{3}}+\dfrac{7}{3\sqrt{3}+2}$

(b) $\dfrac{5}{\sqrt{5}-2}-7\sqrt{5}$

(c) $\dfrac{1}{\sqrt{2}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{4}}$

(a) $\dfrac{5}{2-3\sqrt{3}}+\dfrac{7}{3\sqrt{3}+2}$

$=\dfrac{5(2+3\sqrt{3})+7(2-3\sqrt{3})}{(2-3\sqrt{3})(2+3\sqrt{3})}$

$=\dfrac{10+15\sqrt{3}+14-21\sqrt{3}}{4-(3\sqrt{3})^2}$

$=\dfrac{24-6\sqrt{3}}{4-27}$

$=\dfrac{24-6\sqrt{3}}{-23}$

(b) $\dfrac{5}{\sqrt{5}-2}-7\sqrt{5}$

$=\dfrac{5-7\sqrt{5}(\sqrt{5}-2)}{\sqrt{5}-2}$

$=\dfrac{5-35+14\sqrt{5}}{\sqrt{5}-2}$

$=\dfrac{-30+14\sqrt{5}}{\sqrt{5}-2}\times\dfrac{\sqrt{5}+2}{\sqrt{5}+2}$

$=\dfrac{-30\sqrt{5}-60+70+28\sqrt{5}}{5-4}$

$=10-2\sqrt{5}$ #

(c) $\dfrac{1}{\sqrt{2}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{4}}$

$=\dfrac{\sqrt{3}-\sqrt{2}}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}+\dfrac{\sqrt{4}-\sqrt{3}}{(\sqrt{4}+\sqrt{3})(\sqrt{4}-\sqrt{3})}$

$=\dfrac{\sqrt{3}-\sqrt{2}}{3-2}+\dfrac{\sqrt{4}-\sqrt{3}}{4-3}$

$=\sqrt{4}-\sqrt{2}$

$=2-\sqrt{2}$ #

Worked Example 12
Solving problems involving division of surds

A rectangle has length $(3+2\sqrt{3})$ cm and an area of $(6+5\sqrt{3})\text{ cm}^2$.

Find, **without using a calculator**, the width of the rectangle, in cm, in the form $(a+b\sqrt{3})$, where a and b are integers.

Width $=\dfrac{6+5\sqrt{3}}{3+2\sqrt{3}}\times\dfrac{3-2\sqrt{3}}{3-2\sqrt{3}}$

$=\dfrac{18-12\sqrt{3}+15\sqrt{3}-10\sqrt{3}\sqrt{3}}{9-4(3)}$

$=\dfrac{-12+3\sqrt{3}}{-3}$

$=4-\sqrt{3}$ cm #

Worked Example 13 *[Source: Add Maths 360, Ex. 3.1, Q18]*

Solving problems involving surds in real-world context

Two resistors with resistances R_1 ohms and R_2 ohms are connected in parallel in an electric circuit. The total resistance, R ohms, is given by $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$. The voltage, V volts, across the circuit is given by $V = IR$, where I amperes is the current flowing through the circuit. Given that $R=\frac{1}{10}(6\sqrt{2}+7\sqrt{3})$, $R_1=\sqrt{3}+3\sqrt{2}$ and $I=5\sqrt{6}$, find in the simplest surd form, the value of

(i) V,

(ii) R_2.

(i) $V = IR$

$= 5\sqrt{6}\left(\frac{1}{10}\right)(6\sqrt{2}+7\sqrt{3})$

$= 3\sqrt{6}\sqrt{2} + \frac{7}{2}\sqrt{6}\sqrt{3}$

$= 3\sqrt{12} + \frac{7}{2}\sqrt{18}$

$= 6\sqrt{3} + \frac{21}{2}\sqrt{2}$ #

(ii) $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

$\frac{1}{R_2} = \frac{1}{R} - \frac{1}{R_1}$

$= \frac{R_1 - R}{RR_1}$

$R_2 = \frac{RR_1}{R_1 - R}$

$\therefore R_2 = \frac{\frac{1}{10}(6\sqrt{2}+7\sqrt{3})(\sqrt{3}+3\sqrt{2})}{\sqrt{3}+3\sqrt{2}-\frac{1}{10}(6\sqrt{2}+7\sqrt{3})}$

$= \frac{(6\sqrt{2}+7\sqrt{3})(\sqrt{3}+3\sqrt{2})}{10\sqrt{3}+30\sqrt{2}-6\sqrt{2}-7\sqrt{3}}$

$= \frac{6\sqrt{6}+18(2)+7(3)+21\sqrt{6}}{24\sqrt{2}+3\sqrt{3}}$

$= \frac{57+27\sqrt{6}}{24\sqrt{2}+3\sqrt{3}}$

$= \frac{19+9\sqrt{6}}{8\sqrt{2}+\sqrt{3}} \times \frac{8\sqrt{2}-\sqrt{3}}{8\sqrt{2}-\sqrt{3}}$

$= \frac{(19+9\sqrt{6})(8\sqrt{2}-\sqrt{3})}{128-3}$

$= \frac{152\sqrt{2}-19\sqrt{3}+72\sqrt{12}-9\sqrt{18}}{125}$

$= \frac{152\sqrt{2}-19\sqrt{3}+144\sqrt{3}-27\sqrt{2}}{125}$

$= \frac{125\sqrt{2}+125\sqrt{3}}{125} = \sqrt{2}+\sqrt{3}\ \Omega$ #

Concept Quiz 1 | Assignment 1

Checklist for Self-Assessment on Surds		
I am UNSURE how to	I am ABLE to	
☐	☐	identify a conjugate surd of a given surd.
☐	☐	simplify fraction involving surds by rationalizing denominator.
☐	☐	simplify expressions involving fractions with surds using various operations.
☐	☐	solve problems involving surds.

Activity 4 **[Inquiry]**

Learning objective: To explore the rational and irrational roots of quadratic equations.

Consider the quadratic equation $ax^2+bx+c=0$, where a, b and c are ***rational*** numbers.

1. Solve the following equations by factorization where possible.
If the values of the roots are not exact, leave your answers in surd form.

(a) $2x^2+3x-2=0$ **(b)** $x^2+2x-1=0$

2. **(i)** Which of the above quadratic equations can be solved by factorization? Are its roots rational or irrational?

(ii) Does the other quadratic equation have rational or irrational roots?

(iii) The formula for the general solution is $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.
When are the roots rational and when are they irrational?

3. **(i)** When the roots of a quadratic equation is irrational, are they **conjugate surds**?

(ii) Why the irrational roots conjugate surds? Explain this using the formula for the general solution.

1 (a) $2x^2+3x-2=0$

$x=\dfrac{-3\pm\sqrt{3^2-4(2)(-2)}}{2(2)}$

$=\dfrac{-3\pm\sqrt{25}}{4}$

$=\frac{1}{2}$ or $-\frac{1}{2}$

1 (b) $x^2+2x-1=0$

$x=\dfrac{-2\pm\sqrt{2^2-4(1)(-1)}}{2(1)}$

$=\dfrac{-2\pm\sqrt{8}}{2}$

$=-1\pm\sqrt{2}$

2 (i) For both (a), the roots are rational. For (b), the roots are irrational.

(iii) The roots are rational when b^2-4ac is a square number
The roots are irrational when $b^2-4ac>0$ and it is not a square number.

3 (i) They are.

(ii)

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\dfrac{-b}{2a}+\dfrac{\sqrt{b^2-4ac}}{2a}$ and $-\dfrac{b}{2a}-\dfrac{\sqrt{b^2-4ac}}{2a}$ are conjugate surds.

1.3 Solving Equations Involving Surds

Activity 5 **[Inquiry]**
Learning objective: To explore the equality of surds.

If we are given $a+b\sqrt{2}=3+5\sqrt{2}$, where a and b are rational numbers, what can we say about the values of a and b?

To answer this question, let us arrange the original equation as follows:

$$b\sqrt{2}+5\sqrt{2}=3-a$$
$$(b-5)\sqrt{2}=3-a$$

(i) Is the RHS of the new equation a rational or an irrational number?
(ii) If $b-5\neq 0$, is the LHS of the new equation a rational or an irrational number?
(iii) In order for the equation to hold, what can we conclude about $(b-5)$?
(iv) So what can we conclude about the value of b?
(v) Finally, what can we conclude about the value of a?

In the original equation, a and b are rational numbers. If a and b are irrational numbers, will the results in parts **(iv)** and **(v)** still hold true?

Conclusion

If $a+b\sqrt{k}=p+q\sqrt{k}$, where *a*, *b*, *c* and *d* are *rational numbers*, then $a=p$ and $b=q$

This is the **equality property of surds.**

Worked Example 14
Solving equation involving surds

The solution of the equation $x\sqrt{12}=x\sqrt{7}+\sqrt{3}$ is $\frac{p+\sqrt{q}}{5}$.

Find the values of the integers p and q.

$x\sqrt{12}=x\sqrt{7}+\sqrt{3}$

$2x\sqrt{3}-x\sqrt{7}=\sqrt{3}$

$x(2\sqrt{3}-\sqrt{7})=\sqrt{3}$

$x=\frac{\sqrt{3}}{2\sqrt{3}-\sqrt{7}}\times\frac{2\sqrt{3}+\sqrt{7}}{2\sqrt{3}+\sqrt{7}}$

$=\frac{6+\sqrt{21}}{12-7}$

$=\frac{6+\sqrt{21}}{5}$

$p=6,\ q=21$

Worked Example 15
Solving for unknown(s) using equality of surds

(a) Given that $(4-3\sqrt{3})(5-a\sqrt{3})=b-7\sqrt{3}$, where a and b are integers, find the value of a and of b.

(b) Given that $(p+q\sqrt{5})^2=29-12\sqrt{5}$, where p and q are rational numbers, find the possible pairs of values of p and q.

(a) $(4-3\sqrt{3})(5-a\sqrt{3})=b-7\sqrt{3}$

$20-4a\sqrt{3}-15\sqrt{3}+3a(3)=b-7\sqrt{3}$

$(20+9a)-(4a-15)\sqrt{3}=b-7\sqrt{3}$

$\therefore 4a-15=-7$

$4a=8$

$a=2$ #

$20+9a=b$

$20+9(2)=b$

$b=38$ #

(b) $(p+q\sqrt{5})^2=29-12\sqrt{5}$

$p^2+2pq\sqrt{5}+5q^2=29-12\sqrt{5}$

$p^2+5q^2=29$ —(1)

$2pq=-12$ —(2)

$q=\frac{-12}{2p}=-\frac{6}{p}$ —(3)

subs (3) into (1):

$p^2+5\left(-\frac{6}{p}\right)^2=29$

$p^2+\frac{180}{p^2}=29$

$p^4+180=29p^2$

$p^4-29p^2+180=0$

$(p^2-9)(p^2-20)=0$

$p^2=9$ or $p^2=20$ (rejected because p is rational)

$p=3$ or -3

When $p=3$, $q=-\frac{6}{3}=-2$

When $p=-3$, $q=-\frac{6}{-3}=2$ #

When we have an equation that involves an unknown under the square root sign, we could solve the equation by squaring both sides of the equation, for example,

$$\sqrt{x} = 2$$
$$(\sqrt{x})^2 = 2^2$$

This technique may introduce solutions that do not satisfy the original equation. Hence, we must *check the solutions to ensure that they satisfy the original equation.*

Worked Example 16
Solving equation involving surds that requires squaring

Solve each of the following equations.

(a) $\sqrt{x^2+3} = 2x$

(b) $\sqrt{15-2x} = x$

Solution:

(a)

$$\begin{aligned} & \sqrt{x^2+3} = 2x \\ \text{Squaring both sides,}\quad & x^2+3 = (2x)^2 \\ & 3x^2 = 3 \\ & x^2 = 1 \\ & x = \pm 1 \end{aligned}$$

Check: Sub. $x = 1$ into the original equation:

LHS $= \sqrt{1^2+3} = 2$ and RHS $= 2(1) = 2 =$ LHS

So $x = 1$ is a solution.

Check: Sub. $x = -1$ into the original equation:

RHS $= 2(-1) = -2$ but LHS $= \sqrt{x^2+3}$ cannot be negative.

So $x = -1$ is not a solution.

$\therefore$ the solution is $\underline{\underline{x = 1}}$.

BIG idea
Equivalence

Does square both sides of an equation, for example, $\sqrt{x^2+3}=2x$, produce an equivalent equation?

From the solution of **Worked Example 16(a)**, we see that the solutions of the equation $x^2+3=(2x)^2$ are $x=\pm1$, but the solution of the original equation $\sqrt{x^2+3}=2x$ is only $x=1$. Therefore, squaring both sides of an equation will introduce an *extraneous solution*, so it is important to check the solutions of an equation if we square both sides of it.

Worked Example 17
Solving equation involving surds that requires squaring

Solve each of the following equations.

(a) $2\sqrt{x-1}-\sqrt{6-x}=0$

(b) $\sqrt{5x+1}-\sqrt{x}=2$

Before we solve these equations...

Pause & Think

How are the two equations in **Worked Example 17** different from each other?

(a) $2\sqrt{x-1}-\sqrt{6-x}=0$

$2\sqrt{x-1}=\sqrt{6-x}$

Square both sides.

$4(x-1)=(6-x)$

$4x-4=6-x$

$5x=10$

$x=2$.

Subs $x=2$ into the original equation.

$2\sqrt{2-1}-\sqrt{6-2}=0$

$\therefore x=2$

(b) $\sqrt{5x+1}-\sqrt{x}=2$

$\sqrt{5x+1}=2+\sqrt{x}$

Square both sides.

$5x+1=(2+\sqrt{x})^2$

$5x+1=4+4\sqrt{x}+x$

$4x-3=4\sqrt{x}$

Square both sides.

$(4x-3)^2=16x$

$16x^2-24x+9=16x$

$16x^2-40x+9=0$

$(4x-9)(4x-1)=0$

$x=\frac{9}{4}$ or $x=\frac{1}{4}$.

Check, when $x=\frac{9}{4}$

$\sqrt{5(\frac{9}{4})+1}-\sqrt{\frac{9}{4}}$

$=\sqrt{\frac{49}{4}}-\sqrt{\frac{9}{4}}$

$=\frac{7}{2}-\frac{3}{2}=2$

when $x=\frac{1}{4}$

$\sqrt{5(\frac{1}{4})+1}-\sqrt{\frac{1}{4}}$

$=\frac{3}{2}-\frac{1}{2}$

$=1$ (rejected)

$\therefore x=\frac{9}{4}$

Concept Quiz 2 | Assignment 2

Checklist for Self-Assessment on Surds		
I am UNSURE how to	I am ABLE to	
☐	☐	solve equations involving surds, including checking validity of solutions by substitution.
☐	☐	solve problems in real-world context involving surds.

REVIEW & REFLECT

- What were the mathematical concepts or ideas that you have learnt?
- Reflect on a mistake that you have made or a misconception that you used to have. What do you learn from this mistake or misconception?
- What questions or uncertainties do you still have about… ? If you don't have a question or uncertainty, write a similar problem to a skill you wish to practise more and solve it.

Assignment 1: Simplifying Expressions Involving Surds

Textbook: Additional Maths 360 Volume A (2nd Edition), Marshall Cavendish
Exercise 3.1, pages 57 – 58

- **Tier A**: Questions 2b, 2c, 3b, 3c, 4
- **Tier B**: Questions 6b, 6c, 8b, 8c, 11, 12
- **Tier C**: Questions 16, 17

Assignment 2: Solving Equations Involving Surds

Textbook: Additional Maths 360 Volume A (2nd Edition), Marshall Cavendish
Exercise 3.2, page 61

- **Tier A**: Questions 1c, 1d, 2b, 2c
- **Tier B**: Questions 5, 6b, 7, 9b, 9d
- **Tier C**: Questions 10, 11