SCHOOL OF SCIENCE AND TECHNOLOGY, SINGAPORE
MATHEMATICS DEPARTMENT 2023
SECONDARY 3

A

Name: Solutions ( ) Class: S3-0__

ADDITIONAL MATHEMATICS: Exponential and Logarithmic Functions

a. ENDURING UNDERSTANDINGS

Students will be able to understand that

- Exponential functions and logarithmic functions are inverses of each other.
- The laws of logarithm can be deduced from the laws of indices
- Logarithmic and exponential notations are interchangeable.
- Bases of logarithmic functions can be changed (if necessary)
- The basic geometrical features of exponential and logarithmic graphs

b. ESSENTIAL QUESTIONS

- How are the laws of logarithm related to the laws of indices?
- How and why can logarithms be used to solve logarithmic and exponential equations?
- How to change the base of logarithms and how to decide which base to change to?
- What are inverse functions and how are exponential and logarithmic functions inverses?

c. KNOWLEDGE & SKILLS

At the end of the units, students will be able to

- Use the equivalence of $y = a^x$ and $x = \log_a y$ to solve equations.
- State that a logarithm $\log_a y$ is defined when $y > 0$, (ii) $a > 0$, $a \neq 1$.
- Apply the results $\log_a 1 = 0$ and $\log_a a = 1$.
- Evaluate common and natural logarithms using a calculator.
- Use laws of logarithms.
- Evaluate a logarithm to any base by first converting it to base 10 or base e
- Apply the property $\log_a M = \log_a N \Leftrightarrow M = N$ to solve logarithmic equations.
- Solve equations of the form $a^x = b$ using logarithms.
- Apply the rules of indices and laws of logarithms to solve real-world problems
- Sketch the graphs of exponential and logarithmic functions.

d. RESOURCES

1. Yeap B. H., J. Yeo, The K. S., Loh C. Y., I. Chow (2013). "New Syllabus Additional Mathematics". 9th Ed. PP 105 – 153. Singapore: Shinglee Publishers Pte Ltd. Reference Text Chapter 4
2. Chow, W.K. (2010). "Discovering Additional Mathematics". Singapore: Star Publishing Pte Ltd. PP 21 – 36
3. Lee, L.K. (2011). "Pass with Distinction: Additional Mathematics (By Topic)". Singapore: Shinglee Publishers Pte Ltd.
4. Sadler, A.J. and Thorning, D.W.S. (1987). "Understanding Pure Mathematics". UK: Oxford University Press.
5. Chow W. K. (2007). "Discovering Mathematics 3". Singapore: Star Publishing Pte Ltd.
6. Ho S. T. and Khor N. H. (2007). "Additional Mathematics". Singapore: Panpac
7. Yan K. C. and Eric C. B. K. (2020). "Additional Mathematics 360". Marshall Cavendish Education

CONCEPT DEVELOPMENT …

Lesson sequence in the unit						
Concept	Student Learning Outcomes	Dimensions (Please tick the appropriate boxes)			
		Secondary 1	Secondary 2	Secondary 3	Secondary 4
5.1	Introduction to Logarithms & Law of Logarithms	Laws of Addition, Subtraction, and Multiplication and Division	Algebraic Manipulation and Fractions	Law of Indices, Logarithmic Functions	Calculus
5.2	Logarithmic Functions	Manipulation of Fractions	Algebraic Manipulation and Fractions	Law of Indices, Logarithmic Functions	Calculus
5.3	Logarithms and Equations in the Form $a^x = b$	Manipulation of Fractions	Algebraic Manipulation and Fractions	Law of Indices, Logarithmic Functions	Calculus
5.4	Graphs and Applications of Logarithmic Functions	Graph Plotting	Graph Plotting	Graphs of Exponential Functions	Calculus

TEACHING TO THE BIG IDEA …

Lesson sequence in the unit								
Student Learning Outcomes	Dimensions (Please tick the appropriate boxes)							
	FUNCTIONS F	INVARIANCE I	NOTATIONS N	DIAGRAMS D	MEASURES M	EQUIVALENCE E	PROPORTIONALITY P	MODELS M
Introduction to Logarithms	✓		✓			✓		
Law of Logarithms	✓					✓		
Logarithmic Functions	✓					✓		
Logarithms and Equations in the Form $a^x = b$	✓					✓		

SECTION 5.1 – INTRODUCTION TO LOGARITHMS

Introduction

- The term logarithm is made of 2 Greek words (*logos* means ratio and *arithmos* number).
- You should have seen by now that not all powers of an expression can be expressed in an exact form.
- Logarithms are another way of expressing the index form expression which we have just learnt.

Expressing an Exponent in Logarithmic form

Given that $10^x = 50$, find the value of x.

Think!
We do know that $10^1 = 10$, and that $10^2 = 100$, so x has to be a value where $1 < x < 2$. So, how do we pinpoint the exact value of x here?

$y = a^x$ (Index form)	$\Leftrightarrow$	$\log_a y = x$ (Logarithmic form)

It is possible to establish a relationship between an exponent and logarithm based on the conversion. The symbol "log" represents a logarithmic expression.

Example 1
Convert the following expressions from index to logarithmic form:

(a) $10^x = 50$ $x = \log_{10} 50 = \lg 50$	(b) $e^x = 30$ $x = \log_e 30 = \ln 30$
(c) $m^x = q$ $x = \log_m q$	(d) $2^5 = 32$ $5 = \log_2 32$

Example 2
Convert the following expressions from logarithmic to index form:

(a) $\log_5 y = 2$ $y = 5^2 = 25$	(b) $\log_2 \sqrt{2} = \frac{1}{2}$ $\sqrt{2} = 2^{\frac{1}{2}}$
(c) $\log_a 4 = 16$ $a^{16} = 4$	(d) $\log_e x = 4$ $x = e^4$

How to Think about Logarithms Intuitively

Convert the following expressions from index to logarithmic form. What do you notice?

(a) $2^1 = 2$ $\log_2 2^1 = 1$	(b) $2^2 = 4$ $\log_2 4 = 2$ $\log_2 2^2 = 2$
(c) $2^3 = 8$ $\log_2 8 = 3$ $\log_2 2^3 = 3$	(d) $2^4 = 16$ $\log_2 16 = 4$ $\log_2 2^4 = 4$

We can interpret $\log_a x$ to be the power of a which results in x.

Example 3

Using this new understanding of logarithms, try to evaluate the following logarithmic expressions

(a) $\log_3 81$ $\log_3 81 = \log_3 3^4$ $= 4$	(b) $\log_3 \frac{1}{81}$ $\log_3 \frac{1}{81} = \log_3 3^{-4}$ $= -4$	(c) $\log_5 125$ $\log_5 125$ $= \log_5 5^3$ $= 3$
(d) $\log_4 8$ $\log_4 8 = \log_4 8^{\frac{2}{3}}$ $= \frac{2}{3}$	(e) $\log_a a^{125}$ $\log_a a^{125} = 125$	(f) $\log_a 1$ $\log_a 1 = \log_a a^0$ $= 0$
(g) $\log_{0.5} 8$ $\log_{0.5} 8 = \log_{0.5} \left(\frac{1}{2}\right)^{-3}$ $= -3$	(h) $\log_5 \sqrt[123]{25}$ $\log_5 \sqrt[123]{25}$ $= \log_5 5^{\frac{2}{123}}$ $= \frac{2}{123}$	(i) $\log_{\sqrt{2}} 8$ $\log_{\sqrt{2}} 8 = \log_{\sqrt{2}} (\sqrt{2})^6$ $= 6$

Checking the validity of a logarithmic expression

Example 4
Determine the values of the expressions below where possible.
For undefined values, explain why. [Hint: Convert the expressions to index form.]

(a) $\log_1 1 =$ undefined. $1^1=1,\ 1^2=1,\ 1^3=1\ldots$	(b) $\log_1 2 =$ undefined. $1^{\square}$ does not result in 2.
(c) $\log_{-3} 4 =$ undefined. Log does not deal with negative bases	(d) $\log_5(-5) =$ undefined. $5^{\square}$ does not result in -5.
(e) $\log_{10} 10 =$ 1	(f) $\log_0 2 =$ undefined. $0^{\square}$ does not result in 2.
(g) $\log_8 2 =$ $\frac{1}{3}$	(h) $\log_5 0 =$ undefined $5^{\square}$ does not result in 0.
(i) $\log_5 5 =$ 1	(j) $\log_{10} 1 =$ 0

Generalisations

For any logarithmic expression $\log_a x = m$ to be valid

- a must be positive $a \neq 1$
- m must be real
- x must be positive

Some observations
- $\log_a a =$ 1
- $\log_a 1 =$ 0

SECTION 5.2 - COMMON AND NATURAL LOGARITHMS

Common Logarithms

- **Common logarithms** are logarithms to base 10.
- They can be summarized as "lg".
  - e.g. $\log_{10} 4$ can be written as $\lg 4$
- Such expressions can be calculated using the $\boxed{\log}$ function on your calculator.

Natural Logarithms

- **Natural logarithms** are logarithms to base e.
- They can be summarized as "ln".
  - e.g. $\log_e 4$ can be written as $\ln 4$.
- Such expressions can be calculated using the $\boxed{\ln}$ function on your calculator.

Hence, by the definition of logarithms,

$\lg a = m$ can be expressed in index form as $10^m = a$
$\ln a = m$ can be expressed in index form as $e^m = a$

SECTION 5.3 - LAWS OF LOGARITHMS

In this section, the proofs of the laws of logarithms are listed for your reference.
For each of the proof, we begin by standardising the relationships $b = a^x, c = a^y$.

Product Law

Proof

$\because b = a^x,\ c = a^y$

$\therefore \log_a b = x, \log_a c = y$

$(bc) = a^x \bullet a^y$

Product Law

$\log_a(bc) =$ $\log_a b + \log_a c$, **where** a, b, c are positive, $a \neq 1$

Example 5

Without using a calculator, simplify the following expressions.

(a) $\log_{10} 5 + \log_{10} 2$ $= \log_{10} 5 \times 2$ $= \log_{10} 10$ $= 1$	(b) $\log_3 3 + \log_3 6$ $= \log_3 3 \times 6$ $= \log_3 18$
(c) $\log_3 40 + \log_3 0.1 + \log_3 0.25$ $= \log_3 (40 \times 0.1 \times 0.25)$ $= \log_3 1$ $= 0$	(d) $\lg p + \lg \frac{1}{2} q + \lg \frac{2}{pq}$ $= \lg\ p(\frac{1}{2}q)(\frac{2}{pq})$ $= \lg 1$ $= 0$
(e) $\log_9 9x + \log_9 3x$ $= \log_9 27x^2$	(f) $\lg 0.125 + \lg 8 + 2\lg 10$ $= \lg 0.125 + \lg 8 + \lg 10 + \lg 10$ $= \lg (0.125 \times 8 \times 10 \times 10)$ $= \lg 100$ $= 2$ #

Quotient Law

Proof

$$\because b = a^x, c = a^y$$
$$\therefore \log_a b = x, \log_a c = y$$
$$\frac{b}{c} = a^x \div a^y$$
$$= a^{x-y}$$
$$\log_a\left(\frac{b}{c}\right) = \log_a a^{(x-y)}$$
$$= x - y$$
$$= \log_a b - \log_a c$$

Quotient Law
$\log_a\left(\frac{b}{c}\right) =$ $\log_a b - \log_a c$, where a, b, c are positive, $a \neq 1$

Example 6

Without using a calculator, simplify the following expressions.

(a) $\log_{10} 30 - \log_{10} 3$ $= \log_{10} \frac{30}{3}$ $= \log_{10} 10$ $= 1$	(b) $-\log_4 2 + \log_4 8$ $= \log_4 \frac{8}{2}$ $= \log_4 4$ $= 1$
(c) $\ln 3e - \ln 3$ $= \ln \frac{3e}{3}$ $= \ln e$ $= 1$	(d) $\log_{10} x^{79} - \log_{10} x^{78}$ $= \log_{10} \frac{x^{79}}{x^{78}}$ $= \log_{10} x$
(e) $\log_b 3\sqrt{b} + \log_b b\sqrt{b} - \log_b 3b$ $= \log_b \left(\frac{3\sqrt{b} \times b\sqrt{b}}{3b}\right)$ $= \log_b b^2$ $= 2$	

Power Law

Proof

$$\because b = a^x$$
$$\therefore \log_a b = x$$
$$\log_a b^n = \log_a \left(a^x\right)^n$$
$$= \log_a \left(a^{nx}\right)$$
$$= nx$$
$$= n\log_a b$$

> **Think!**
>
> Does Power Law apply in this situation?
>
> $\left(\log_a a\right)^3$

Power Law

$\log_a b^m =$ ___$m \log_a b$___, where ___a, b, c are positive, $a \neq 1$___

Example 7

Without using a calculator, simplify the following expressions.

(a) $\log_a a^3$ $= 3\log_a a$ $= 3$	(b) $\log_3 q^{2k} + 2\log_3 q^{2-k}$ $= 2k\log_3 q + 2(2-k)\log_3 q$ $= 2k\log_3 q + 4\log_3 q - 2k\log_3 q$ $= 4\log_3 q$
(c) $\ln\sqrt{e^3}$ $= \ln e^{\frac{3}{2}}$ $= \frac{3}{2}\ln e$ $= \frac{3}{2}$	(d) $2\log_a \frac{1}{p^3} - 3\log_a p^2$ $= 2\log_a p^{-3} - 3\log_a p^2$ $= 2(-3)\log_a p - 6\log_a p$ $= -12\log_a p$
(e) $4\log_3 25 - \log_3 125$ $= 4\log_3 5^2 - \log_3 5^3$ $= 8\log_3 5 - 3\log_3 5$ $= 5\log_3 5$	(f) $4\log_2 0.6 + 2\log_2 \frac{5}{3} - 2\log_2 0.15$ $= 4\log_2 \frac{3}{5} + 2\log_2 \frac{5}{3} - 2\log_2 \frac{3}{20}$ $= \log_2 \left[\frac{\left(\frac{3}{5}\right)^4 \left(\frac{5}{3}\right)^2}{\left(\frac{3}{20}\right)^2}\right]$ $= \log_2 \left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)\left(\frac{5}{3}\right)\left(\frac{5}{3}\right)\left(\frac{20}{3}\right)\left(\frac{20}{3}\right)$ $= \log_2 16$ $= 4$

Change-of-base Law

Proof

Let $\log_a x = y$. Then, we have $a^y = x$.

To change the base, we want to express y in terms of $\log_b x$.

Since $a^y = x$,

$$\log_b x = \log_b a^y$$
$$= y\log_b a$$

So $\log_a x = y$

$$= \frac{\log_b x}{\log_b a}$$

Change of Base

$\log_a x = \frac{\log_b x}{\log_b a}$, **where** $a, b, c > 0$, $a \neq 1$, $b \neq 1$

Example 8

Using a calculator, evaluate the following expressions.

(a) $\log_5 3$	(b) $4(\log_3 5)^2$
$= \frac{\lg 3}{\lg 5}$ $= 0.683$ (3 sf)	$= 4\left(\frac{\lg 5}{\lg 3}\right)^2$ $= 8.58$ (3 sf)

Example 9

Without using a calculator, evaluate the following expressions.

(a) $\log_7 81 \times \log_3 100 \times \lg 49$	(b) $\log_x 16 \times \log_{64} x$
$= \frac{\lg 3^4}{\lg 7} \times \frac{\lg 10^2}{\lg 3} \times \lg 7^2$ $= \frac{4\cancel{\lg 3}}{\cancel{\lg 7}} \times \frac{2}{\cancel{\lg 3}} \times 2\cancel{\lg 7} = 8$	$= \frac{\log_2 2^4}{\log_2 x} \times \frac{\log_2 x}{\log_2 2^6}$ $= \frac{4\log_2 2}{6\log_2 2}$ $= \frac{2}{3}$
(c) $\log_a 9 \times \log_{27} a$	(d) $ln8 \times 5log_2 e$
$= \frac{\log_3 9}{\log_3 a} \times \frac{\log_3 a}{\log_3 27}$ $= \frac{\log_3 3^2}{\log_3 3^3}$ $= \frac{2}{3}$	$= (\ln 2^3)(5)\left(\frac{\ln e}{\ln 2}\right)$ $= \frac{15(\ln 2)}{\ln 2}$ $= 15$

Comparison between the rules of indices and the laws of logathithmns

Laws of Indices / Exponents	Laws of Logarithms
$b^m \times b^n = b^{m+n}$	$\log_b xy = \log_b x + \log_b y$
$b^m \div b^n = b^{m-n}$	$\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
$(b^m)^n = b^{mn}$	$\log_b x^n = n\log_b x$
$b^1 = b$	$\log_a a = 1$
$b^0 = 1$	$\log_b 1 = 0$
	$\log_a b = \dfrac{\log_c b}{\log_c a}$

Notice that the laws of logarithms are quite similar to the laws of indices. In fact, you might think that the laws of logarithms are exactly 'opposite' to the laws of indices. For example, to **add** 2 logarithmic terms of the same base, we **multiply** their results whereas to **multiply** two exponential terms with the same base, we **add** their powers. As we will learn later, this is because logarithms and indices are inverses.

Example 10

Simplify the following expressions as a single logarithm.

(a) $2+\log_6 2$

$= \log_6 36 + \log_6 2$

$= \log_6 72$

(b) $\dfrac{1}{2}\lg 25 - 3$

$= \lg 25^{\frac{1}{2}} - \lg 1000$

$= \lg 5 - \lg 1000$

$= \lg \frac{1}{200}$

(c) $\log_b b^3 + \log_4 3$

$= 3\log_b b + \log_4 3$

$= 3 + \log_4 3$

$= \log_4 64 + \log_4 3$

$= \log_4 192$

(d) $2\log_x 5 - 3\log_x 2 + \log_x 4$

$= \log_x 25 - \log_x 8 + \log_x 4$

$= \log_x \dfrac{25\times 4}{8}$

$= \log_x \left(\frac{25}{2}\right)$

(e) $\lg\left(\dfrac{8}{75}\right) - 2\lg\left(\dfrac{3}{5}\right) + 4\lg\left(\dfrac{3}{2}\right)$

$= \lg 8 - \lg 75 - 2\lg 3 + 2\lg 5 + 4\lg 3 - 4\lg 2$

$= 3\lg 2 - 2\lg 5 - \lg 3 - 2\lg 3 + 2\lg 5 + 4\lg 3 - 4\lg 2$

$= -\lg 2 + \lg 3$

(f) $4 + 2\ln 4e$

$= \ln e^4 + \ln (4e)^2$

$= \ln 16e^6$

Example 11

Given that $\log_a x = k$, express each of the following in terms of *k*.

(a) $(\log_a x)^{33}$ $= k^{33}$	(b) $(\log_a x^2)^3$ $= (2\log_a x)^3$ $= (2k)^3$ $= 8k^3$
(c) $\log_a \frac{a^2}{x^3}$ $= \log_a a^2 - \log_a x^3$ $= 2 - 3\log_a x$ $= 2 - 3k$	(d) $\log_a \frac{\sqrt{x}}{\sqrt[3]{a}}$ $= \log_a \sqrt{x} - \log_a a^{\frac{1}{3}}$ $= \frac{1}{2}\log_a x - \frac{1}{3}$ $= \frac{1}{2}k - \frac{1}{3}$
(e) $\log_{\frac{1}{a}} \sqrt{x}$ $= \frac{\log_a \sqrt{x}}{\log_a \frac{1}{a}}$ $= \frac{\frac{1}{2}k}{-1} = -\frac{k}{2}$ #	(f) $(\log_x a^2)^3$ $= (2\log_x a)^3$ $= \left(\frac{2\log_a a}{\log_a x}\right)^3$ $= \left(\frac{2}{k}\right)^3 = \frac{8}{k^3}$
(g) $\log_{\sqrt{x}} \frac{1}{a}$ $= \frac{\log_a \frac{1}{a}}{\log_a \sqrt{x}}$ $= \frac{-1}{\frac{1}{2}k}$ $= -\frac{2}{k}$ #	(h) $\log_{x^2} ax$ $= \frac{\log_a ax}{\log_a x^2}$ $= \frac{1 + \log_a x}{2\log_a x}$ $= \frac{1+k}{2k}$ #

Example 12*

Given that $\log_6 2 = a$ and $\log_5 3 = b$, express $\log_5 2$ in terms of *a* and *b*.

$\log_6 2$

$= \frac{\log_5 2}{\log_5 6}$

$= \frac{\log_5 2}{\log_5 2 + \log_5 3}$

$= a$

$\frac{\log_5 2}{\log_5 2 + b} = a$

$\log_5 2 = a\log_5 2 + ab$

$\log_5 2(1-a) = ab$

$\log_5 2 = \frac{ab}{1-a}$ #

What is the significance of e ? Also, how can we derive the value of e ?

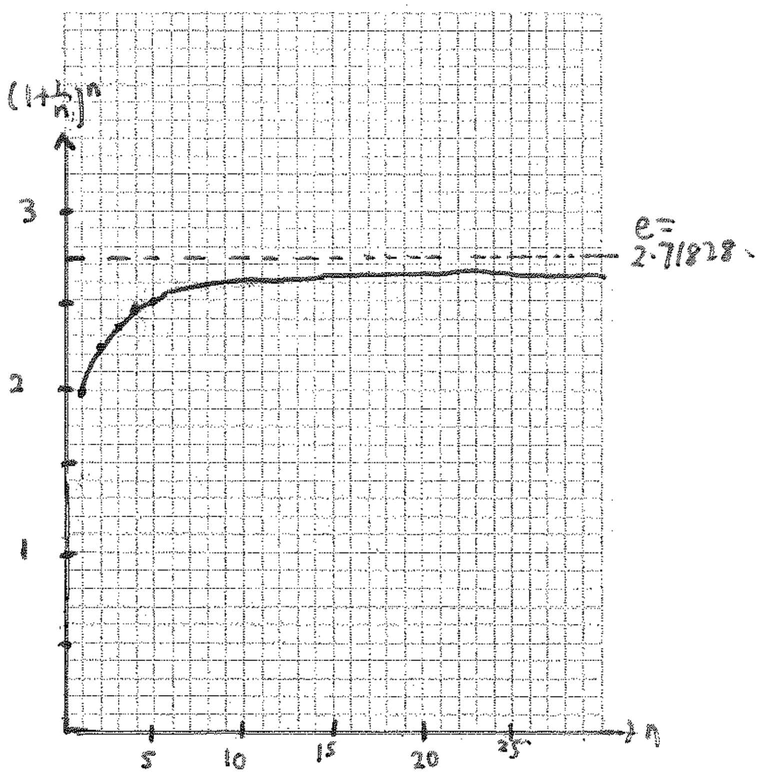
e is an irrational constant approximately equal to 2.718281828. It is sometimes called Euler's number, and the value of e serves as the limit of $(1+\frac{1}{n})^n$, as n approaches infinity. We will now work out the "natural" value of e using the example below:

A bank account has an initial amount of \$1.00. The bank pays a yearly interest of 100% once per year. This simply means that we can expect to collect a whopping *\$2.00* (\$1 initial amount + \$1 of interest) at the end of Year 1.

However, what if the bank were to compound the amount twice yearly, at a frequency of every 6 months? This means that instead of paying out the 100% interest after 1 year, the bank pays out the first 50% of interest after 6 months and the next 50% interest after another 6 months. This arrangement is beneficial to the client because the interest gained in the first 6 months can be used to accumulate more interest in the next 6 months.

It can be shown that the total amount accumulated is given by the formula $(1+\frac{1}{n})^n$. Calculate the total amount accumulated using the table below. What do you notice about the value of the amount accumulated as n continues to increase? What can you comment about the shape of the graph?

n	$(1+\frac{1}{n})^n$
1	2.00
2	2.25
3	2.37
4	2.44
5	2.49
6	2.52
7	2.55
8	2.57
9	2.58
10	2.59
100	2.70
1000	2.72
10000	2.72
100000	2.72



SECTION 5.5 - SOLVING EXPONENTIAL EQUATIONS USING LOGARITHMNS

All simple exponential equations of the form $a^x = b$ can be solved by converting to logarithmic form! Another way to think about conversion to logarithmic form is that it makes the power the subject of the formula.

Example 13

(a) $4.1^x = \pi$ **[0.811]** $x = \frac{\lg \pi}{\lg 4.1} = 0.811$	(b) $e^{1+x} = 19$ **[1.94]** $1+x = \ln 19$ $x = \ln 19 - 1$ $= 1.94$
(c) $4e^{2y} = 21$ **[0.829]** $e^{2y} = \frac{21}{4}$ $2y = \ln(\frac{21}{4})$ $y = \frac{1}{2}\ln(\frac{21}{4}) = 0.829$	(d) $9 = 10^{x-4}$ **[lg9 + 4]** $x-4 = \lg 9$ $x = \lg 9 + 4$ $= 4.95$

Example 14

(ai) Express $2^x(3^{2x}) = 6^{x+2}$ in the form $a^x = b$.

(aii) Hence, solve $2^x(3^{2x}) = 6^{x+2}$

(i) $2^x(3^{2x}) = 6^{x+2}$

$2^x 3^x 3^x = 6^{x+2}$

$6^x 3^x = 6^x 6^2$

$3^x = 36$ #

(ii) $2^x(3^{2x}) = 6^{x+2}$

$3^x = 36$

$x = \log_3 36$

$= 3.26$ #

(b) Solve $2^{2x+2}3^{x-1}5^{x+1} = 999$

$2^{2x}(4)(3^x)(\frac{1}{3})(5^x)(5) = 999$

$4^x 3^x 5^x = \frac{999 \times 3}{20}$

$60^x = \frac{2997}{20}$

$x = 1.22$ #

(c) Solve $2^{-5x}(7^{2x+1}) = 6^{x+3}$

$\frac{7^{2x}(7)}{2^{5x}6^x(6^3)} = 1 \Rightarrow \frac{49^x}{32^x 6^x} = \frac{216}{7} \Rightarrow \left(\frac{49}{192}\right)^x = \frac{216}{7} \Rightarrow x = -2.51$ #

For more complicated exponential equations, you need to make a suitable substitution.

Example 15

(a) $25^x - 3(5^x) = 0$

$5^{2x} - 3(5^x) = 0$

let $y = 5^x$

$y^2 - 3y = 0$

$y(y-3) = 0$

$y = 0$ or $y = 3$

$5^x = 0$ (NA) or $5^x = 3$

$x = 0.683$

(b) $9^y + 5(3^y - 10) = 0$

$3^{2y} + 5(3^y) - 50 = 0$

let $x = 3^y$

$x^2 + 5x - 50 = 0$

$(x+10)(x-5) = 0$

$x = -10$ or $x = 5$

$3^y = -10$ (NA) or $3^y = 5$

$y = 1.46$

(c) $2^x - 2(4^x) = 0$

$2^x - 2(2^{2x}) = 0$

Let $y = 2^x$

$y - 2y^2 = 0$

$2y^2 - y = 0$

$y(2y-1) = 0$

$y = 0$ or $y = \frac{1}{2}$

$2^x = 0$ (NA) or $2^x = \frac{1}{2}$

$x = -1$

(d) $3e^x - 5 = -2e^{-x}$

Let $y = e^x$

$3y - 5 = -\frac{2}{y}$

$3y^2 - 5y + 2 = 0$

$(3y+1)(y-2) = 0$

$y = -\frac{1}{3}$ or $y = 2$

$e^x = -\frac{1}{3}$ (NA) or $e^x = 2$

$x = \ln 2 = 0.693$

SECTION 5.6 - SOLVING LOGARITHMIC EQUATIONS

There are a few strategies that we can use to solve logarithmic equations.

<u>Strategy 1: Convert from Logarithmic Form to Exponential Form</u>

Example 16

(a) $\lg x = 0.61$ **[4.07]** $x = 10^{0.61}$ $= 4.07$ (3 sf)	(b) $(\ln x)^2 = 3$ **[5.65, 0.177]** $\ln x = \sqrt{3}$ or $\ln x = -\sqrt{3}$ $x = e^{\sqrt{3}}$ or $x = e^{-\sqrt{3}}$ $x = 5.65$ or $x = 0.177$ (3 sf)
(c) $\ln 2 \bullet \ln 4x = 3$ **[18.9]** $\ln 4x = \frac{3}{\ln 2}$ $\ln 4 + \ln x = \frac{3}{\ln 2}$ $\ln x = \frac{3}{\ln 2} - \ln 4$ $= 2.9418$ $x = e^{2.9418}$ $= 18.9$ (3 sf)	**(d)** $\lg(x-2) = (\lg 3)^2$ **[~~3.69~~]** 3.26 $\lg(x-2) = (\lg 3)^2$ $x-2 = e^{(\lg 3)^2}$ $x = 2 + e^{(\lg 3)^2}$ $= 3.26$ (3 sf)
(e) $\ln 4x = \lg 3 \bullet \lg 5$ **[0.349]** $4x = e^{\lg 3 \cdot \lg 5}$ $x = \frac{1}{4} e^{\lg 3 \cdot \lg 5}$ $= 0.349$ (3 sf)	(f) $\lg(x-1) = \ln(e^2 - 1)$ **[72.5]** $x - 1 = 10^{\ln(e^2-1)}$ $x = 1 + 10^{\ln(e^2-1)}$ $= 72.5$ (3 sf)

Strategy 2: Remove the Logarithmic Function

$$\log_a M = \log_a N \Leftrightarrow M = N$$

Note (1): This only works if the bases are the same!

Note (2): Always remember to check the validity of the equation after solving (refer to Page 4), by substituting the values back into the equation, and rejecting whichever solutions are not applicable.

Example 17

(a) $\log_3(x+2)+\log_3(x-2)=\log_3(2x-1)$ **[x=3]**

$\log_3 (x+2)(x-2) = \log_3 (2x-1)$

$(x+2)(x-2) = (2x-1)$

$x^2-4 = 2x-1$

$x^2-2x-3=0$

$(x-3)(x+1)=0$

$x=3$ or $x=-1$ (NA)

(b) $\log_2(x-1)^2 = 2+\log_2(x+2)$ **[x=7, -1]**

$\log_2 (x-1)^2 = \log_2 4 + \log_2 (x+2)$

$\log_2 (x-1)^2 = \log_2 4(x+2)$

$(x-1)^2 = 4(x+2)$

$x^2-2x+1 = 4x+8$

$x^2-6x-7=0$

$(x-7)(x+1)=0$

$x=7$ or $x=-1$ *

(c) $\log_2(x-1)+\log_2(x-4)=\log_2(2x-6)$ **[x=5]**

$\log_2 (x-1)(x-4) = \log_2 (2x-6)$

$(x-1)(x-4) = (2x-6)$

$x^2-5x+4 = 2x-6$

$x^2-7x+10=0$

$(x-5)(x-2)=0$

$x=5$ or $x=2$ (NA)

(d) $3\log_x 2+\log_x 18=2$. **[x=12]**

$3\log_x 2 + \log_x 18 = \log_x x^2$

$\log_x (2^3)(18) = \log_x x^2$

$x^2 = 144$

$x=12$ or -12 (NA)

(e) $2\log_p 8-\log_p 4=2$ **[p=4]**

$\log_p 8^2 - \log_p 4 = \log_p p^2$

$p^2 = \frac{8^2}{4} = 16$

$p=4$ or $p=-4$ (NA)

Strategy 3: Make the Bases Equal (Using the Change of Base Law)

Example 18

(a) Solve $\log_5 x - \log_{25}(x+6) = 0$ **[x=3]**

$\log_5 x - \frac{\log_5(x+6)}{\log_5 25} = 0$

$\log_5 x = \frac{1}{2}\log_5(x+6)$

$\log_5(x+6) = 2\log_5 x = \log_5 x^2$

$x^2 = (x+6)$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3$ or $x = -2$ (NA)

(b) Express y in terms of x if $\log_4 y = \log_2 x + \log_2 9 - \log_2 3$ **[$y=9x^2$]**

$\frac{\log_2 y}{\log_2 4} = \log_2 x + \log_2 9 - \log_2 3$

$\frac{1}{2}\log_2 y = \log_2\left(\frac{9x}{3}\right)$

$\log_2 y = 2\log_2\left(\frac{9x}{3}\right) = \log_2(3x)^2$

$y = 9x^2$

(c) Solve $\log_5 x = 4\log_x 5 + 3$. **[x=625 or 0.2]**

$\log_5 x = \frac{4\log_5 5}{\log_5 x} + 3$

Let : $y = \log_5 x$

$y = \frac{4}{y} + 3$

$y^2 = 4 + 3y$

$y^2 - 3y - 4 = 0$

$(y-4)(y+1) = 0$

$\log_5 x = 4$ or $\log_5 x = -1$

$x = 5^4 = 625$

or $x = 5^{-1} = \frac{1}{5}$

Strategy 4: Substitution

Example 19

(a) By using the substitution $a = \log_3 x$ or otherwise, solve the equation $\log_3 x^3 = (\log_3 x)^3$, leaving your answers in exact form where necessary. **[$x=1, 3^{\sqrt{3}}, 3^{-\sqrt{3}}$]**

$\log_3 x^3 = (\log_3 x)^3$

$3a = a^3$

$a^3 - 3a = 0$

$a(a^2 - 3) = 0$

$a = 0$ or $a = \sqrt{3}$ or $a = -\sqrt{3}$

$\log_3 x = 0$ or $\log_3 x = \sqrt{3}$ or $\log_3 x = -\sqrt{3}$

$x = 1$ or $x = 3^{\sqrt{3}}$ or $x = 3^{-\sqrt{3}}$

(b) Solve $(\log_3 y)^2 + \log_3(y^2) = 8$. **[$y=3^{-4}$, 9]**

Let $a = \log_3 y$

$a^2 + 2a = 8$

$a^2 + 2a - 8 = 0$

$(a+4)(a-2) = 0$

$a = -4$ or $a = 2$

$\log_3 y = -4$ or $\log_3 y = 2$

$y = 3^{-4}$ or $y = 3^2 = 9$

SECTION 5.6 - SOLVING SIMULTANEOUS EQUATIONS INVOLVING LOGARITHMIC AND/OR EXPONENTIAL FUNCTIONS

Example 20

Solve, for x and y, in the simultaneous equations given below.

(a) $\lg(15-6x)-\lg y=2\lg 3, \log_2(2x-y+15)=3$

(b) $2\log_x y+2\log_y x=5, \quad xy=8$

(c) $\log_3 y=\log_3 x-\log_3 2, \quad \log_4(x+2)=1+\log_2 y$

(a) $\lg(15-6x)-\lg y=2\lg 3$

$\frac{15-6x}{y}=3^2$

$15-6x=9y$

$6x+9y=15$

$2x+3y=5$ —— (1)

$\log_2(2x-y+15)=3=\log_2 8$

$2x-y+15=8$

$2x-y=-7$ —— (2)

(1)−(2): $4y=12$

$y=3$

subs $y=3$ into (2):

$2x=-4$

$x=-2$

(b) $xy=8 \Rightarrow y=\frac{8}{x}$ —— (1)

$2\log_x y+2\log_y x=5$

$2\log_x y+\frac{2\log_x x}{\log_x y}=5$

$2(\log_x y)^2+2=5(\log_x y)$

$2(\log_x y)^2-5(\log_x y)+2=0$

$(2\log_x y-1)(\log_x y-2)=0$

$\log_x y=\frac{1}{2}$ or $\log_x y=2$

$y=x^{\frac{1}{2}}$ or $y=x^2$

subs (1):

$\frac{8}{x}=x^{\frac{1}{2}}$ or $\frac{8}{x}=x^2$

$x^{\frac{3}{2}}=8$ or $x^3=8$

$x=4$ or $x=2$

(c) $\log_3 y=\log_3 x-\log_3 2$

$\log_3 y=\log_3\frac{x}{2}$

$y=\frac{x}{2}$ —— (1)

$\log_4(x+2)=1+\log_2 y$

$\frac{\log_2(x+2)}{\log_2 4}=\log_2 2+\log_2 y$

$\frac{1}{2}\log_2(x+2)=\log_2 2y$

$\log_2(x+2)=2\log_2(2y)$

$=\log_2(2y)^2$

$4y^2=x+2$ —— (2)

subs (1) into (2):

$4\left(\frac{x}{2}\right)^2=x+2$

$x^2-x-2=0$

$(x-2)(x+1)=0$

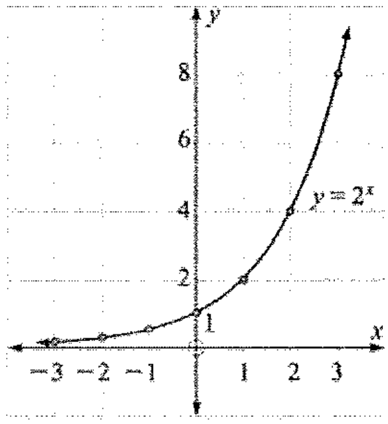
$x=2$ or $x=-1$ (NA)

when $x=2$, $y=1$

SECTION 5.7 - GRAPHS OF EXPONENTIAL FUNCTIONS

The most simplest general form of the exponential function takes the form of $y = b^x$, where $b > 0$ and $b \neq 0$.

For example, $y = 2^x$ is an exponential function.
To we graph the function of $y = 2^x$, we need to construct a table of values:



x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

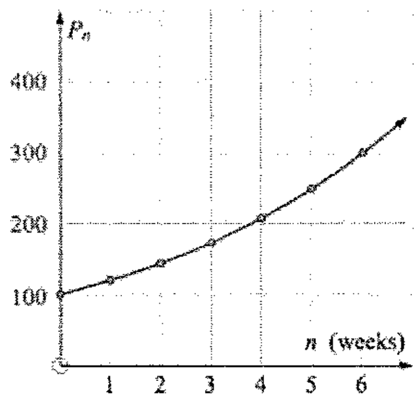
Notice that the gradient of graph of an exponential function gets steeper and steeper as $x \to \infty$.

State the asymptote of the function of $y = 2^x$.

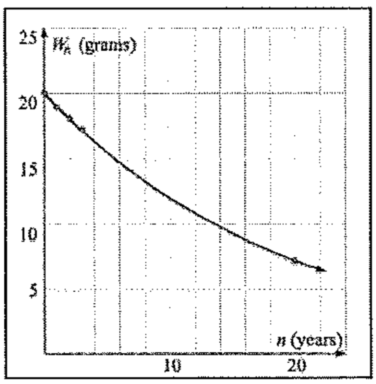
$y = 0$

Real Life Applications
Exponential functions are often used to describe situations in real life where quantities are either increasing or decreasing *exponentially*. These situations are known as growth and decay. Some examples of growth are the populations of animals, people and bacteria. Some examples of decay are radioactive substances or items that depreciate in value.

Example of Growth
Consider a population of 100 mice, which is increasing by 20% each week.
The population function is $P_n = 100 \times (1.2)^n$



Example of Decay
Consider a radioactive substance with an original weight of 20 grams, which decays or reduces by 5% each year. The function for weight is $W_n = 20 \times (0.95)^n$.

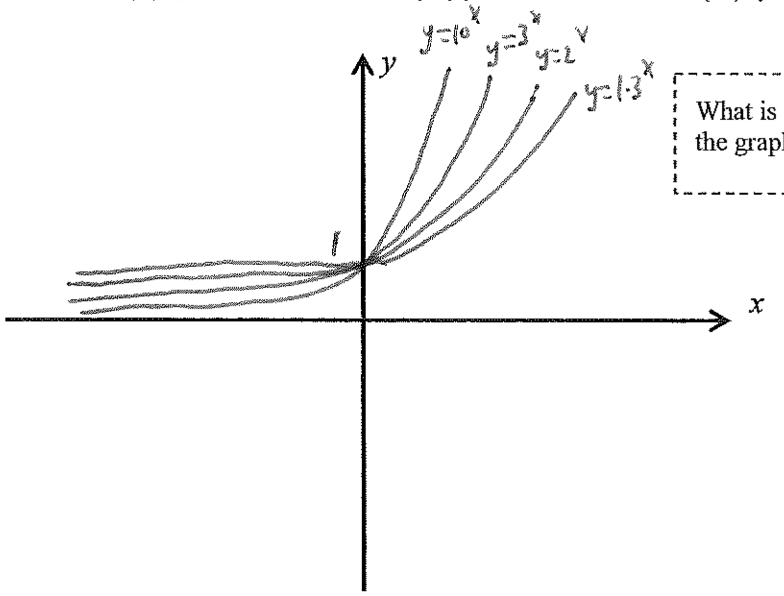


Graphs of the form $y = a^x$

Example 21

On the same set of axes provided below, sketch the graphs of the following functions.

(i) $y = 2^x$ (ii) $y = 3^x$ (iii) $y = 10^x$ (iv) $y = (1.3)^x$



What is the domain and range for the graphs that you have sketched?

What does the effect does changing the value of b have on the shape of the graph? graph becomes steeper

What is the y-intercept of each graph? 1

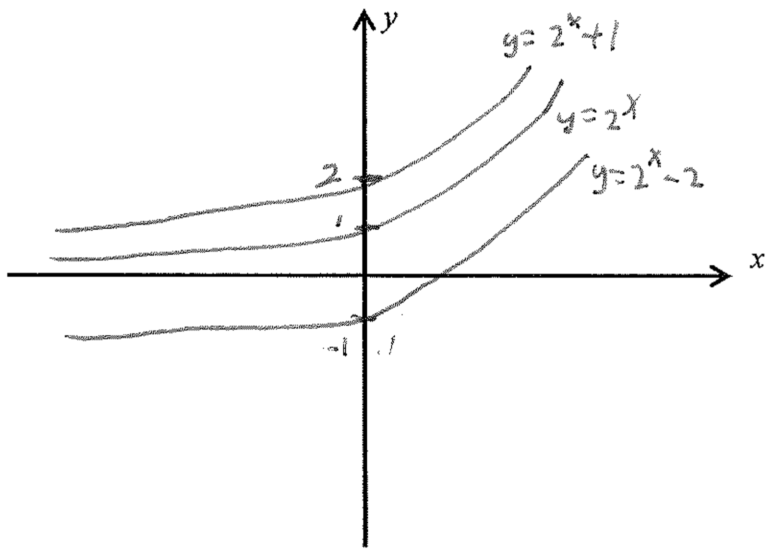
What is the horizontal asymptote of each graph? $y=0$

Graphs of the form $y = a^x + d$

Example 22

On the same set of axes provided below, sketch the graphs of the following functions.

(i) $y = 2^x$ (ii) $y = 2^x + 1$ (iii) $y = 2^x - 2$



Describe the transformation that occurred when we graph $y = 2^x + d$ from $y = 2^x$. Translate the graph d units upwards

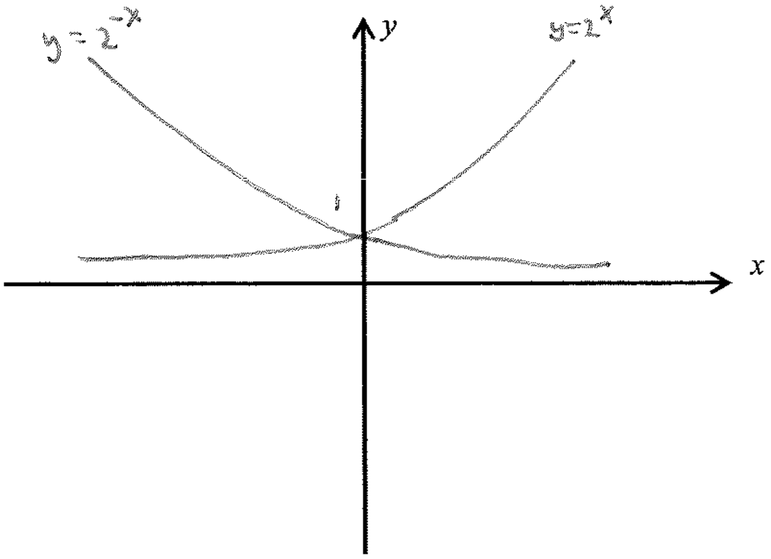
What is the horizontal asymptote of $y = 2^x + d$? $y=d$.

Graphs of the form $y = a^{-x}$

Example 23

On the same set of axes provided below, sketch the graphs of the following functions.

(i) $y = 2^x$ (ii) $y = 2^{-x}$



What is the y-intercept of each graph? 1 What is the horizontal asymptote of each graph? $y=0$ Describe the transformation that occurred when we graph $y = 2^{-x}$ from $y = 2^x$. Reflect $y=2^x$ about the y- axis.

Example 24

Answer the whole of this question on a single sheet of graph paper.

The number of bacteria in a culture is given by the formula $y = 50(2^t)$ where t is the number of hours after the start of the observation.

(a) Find the number of bacteria in the culture
   i. at the start of the observation
   ii. 3 hours later

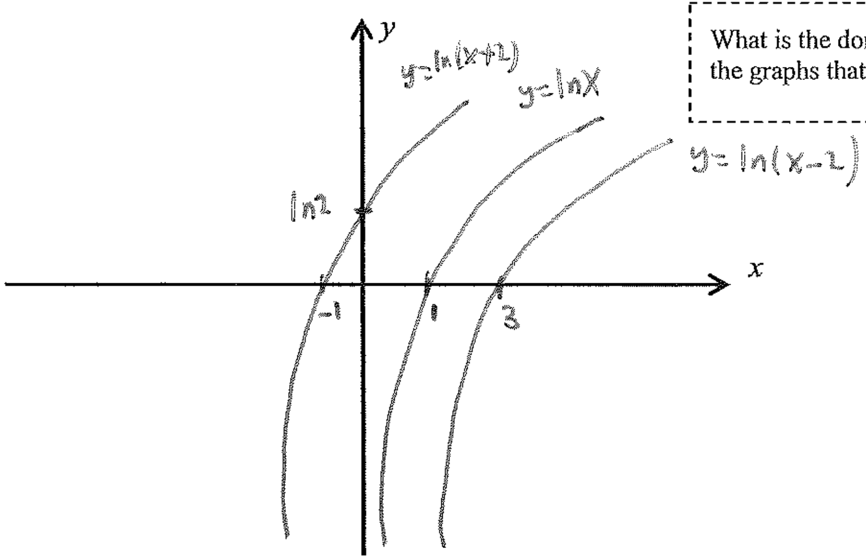
(b) Draw the graph of $y = 50(2^t)$ for $0 \le x \le 4$.

(c) Estimate the time at which there are 300 bacteria.

SECTION 5.8 - GRAPHS OF LOGARITHMIC FUNCTIONS

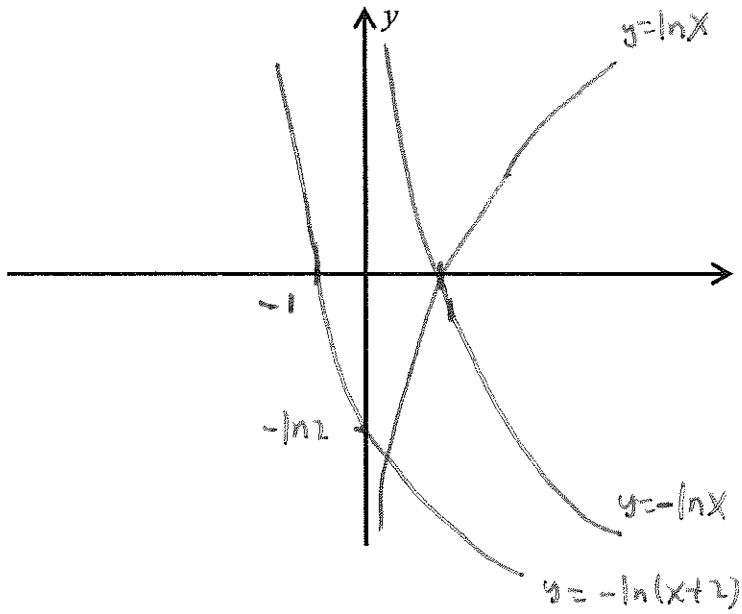
It is given that $y = \ln x$ **or** $y = \log_e x$ is a logarithmic function.
The gradient of graph of a logarithmic function gets gentler and gets gentler as $x \to \infty$.

Example 25
On the same set of axes provided below, sketch the graphs of the following functions.
(i) $y = \ln x$ (ii) $y = \ln(x-2)$ (iii) $y = \ln(x+2)$



What is the domain and range for the graphs that you have sketched?

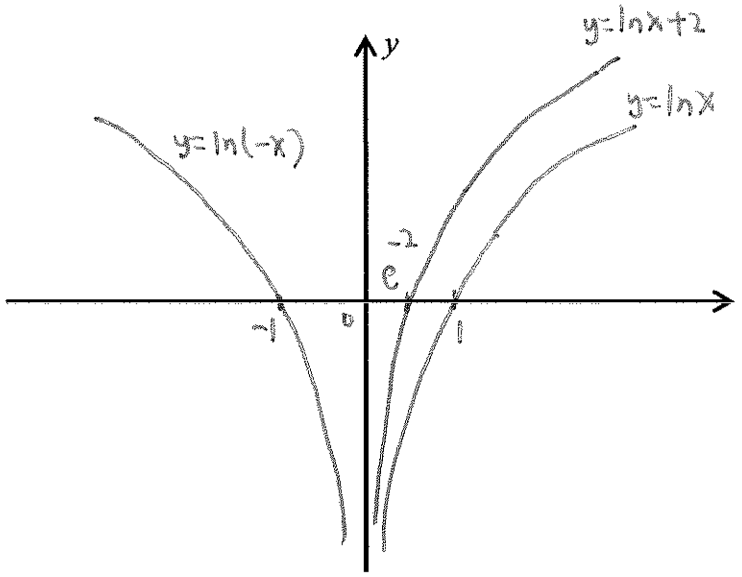
Example 26
On the same set of axes provided below, sketch the graphs of the following functions.
(i) $y = \ln x$ (ii) $y = -\ln x$ (iii) $y = -\ln(x+2)$



Example 27

On the same set of axes provided below, sketch the graphs of the following functions.

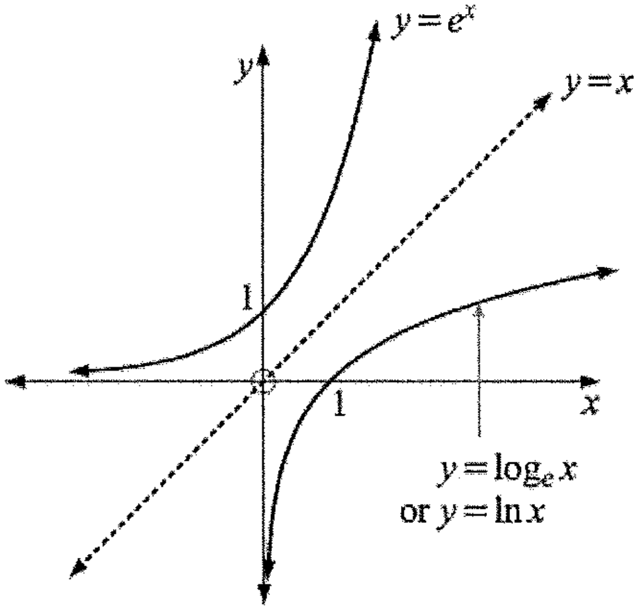
(i) $y = \ln x$ (ii) $y = \ln(-x)$ (iii) $y = \ln x + 2$



SECTION 5.9 - RELATIONSHIP BETWEEN GRAPHS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

The graphs of $y = e^x$ and $y = \ln x$ are shown below.

The graph of $y = e^x$ is an inverse function of $y = \ln x$.
Similarly, the graph of $y = \ln x$ is an inverse function of $y = e^x$.



What do you observe?

graph of $y=e^x$ is a reflection of $y=\ln x$ about the line $y=x$

SECTION 5.10 - REAL-LIFE APPLICATIONS OF EXPONENTIAL AND LOGARITHM FUNCTIONS IN GROWTH AND DECAY

Exponential functions are often used to model growth and decay situations. Logarithms are often used to solve problems involving exponential functions involving growth and decay.

Exponential Growth and Decay Model
The model for growth and decay is given by $y = Ce^{kt}$ where t is time, C is the original amount, y is the amount after time t amd k is a constant determined by the rate of growth.

Bacteria Growth

Example 28

The mass of bacteria, M, in a culture t hours after establishment is given by $M = 25e^{0.2t}$.
Find out the time required for the mass of the culture to reach 80 grams.
Give your answer to the nearest minute.

$80 = 25e^{0.2t}$

$\ln 80 = \ln 25 + \ln e^{0.2t}$

$= \ln 25 + 0.2t$

$t = 5(\ln 80 - \ln 25)$

$= 5.82$

Time = 5.82 min.

Population Growth

Example 29

The population of rabbits, P_t, in a given habitat is given by $P_t = P_0(1+r)^t$ where P_0 is the original population size, r is the rate of increase or decrease of the population and t is the time that has passed. Given that a group of 34 rabbits increase by 40% every year, find out

(a) how many rabbits there are after 3 years, and

(b) how many years will it take for the population of rabbits to double.

(c) After the rabbit population has doubled, how many more years will it take for the population of rabbits to double again?

[(a) 93 ~~years~~ rabbits (b) 2.06 years, (c) 2.06 years]

(a) $P_t = P_0(1+r)^t$

population $= 34(1+0.4)^3$

$= 93$ (nearest whole number)

(b) $\frac{P_t}{P_0} = (1+r)^t = 2$

$1.4^t = 2$

$t = \log_{1.4} 2$

$= 2.06$ years

No of years = 2.06

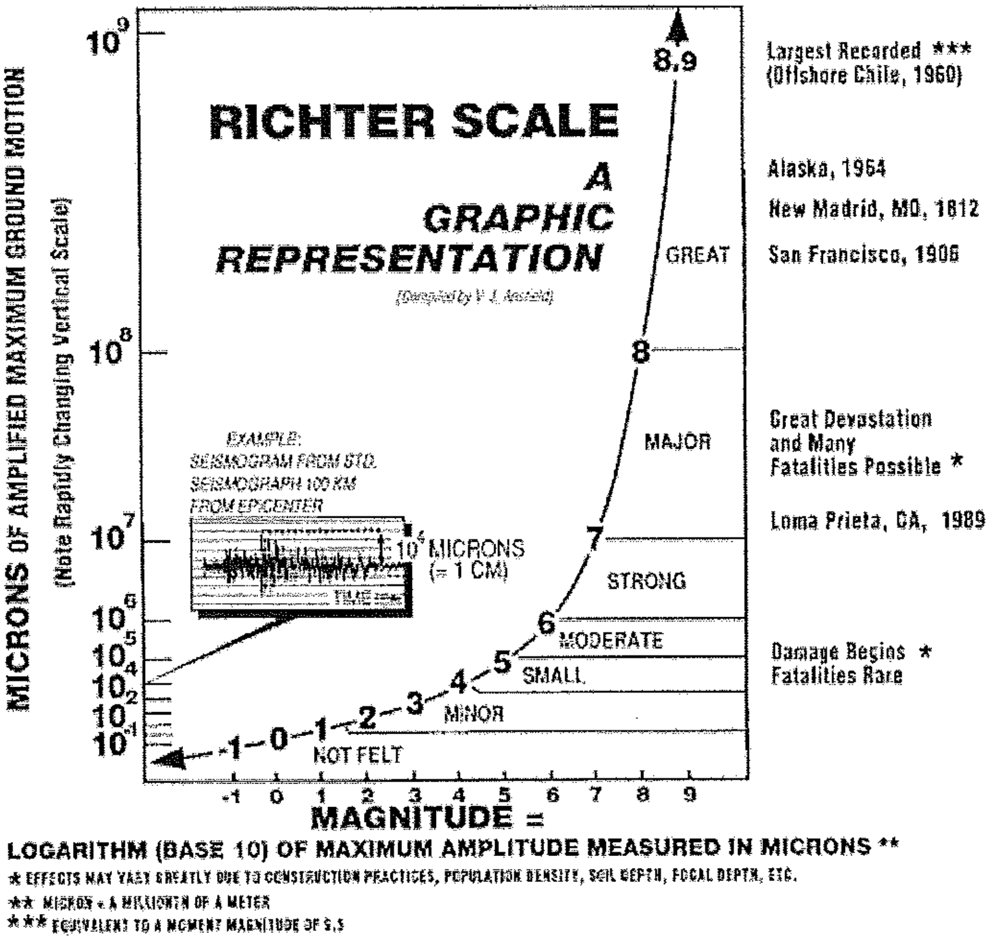
(c) No of years = 2.06

Earthquakes and the Richter Scale

In 1935 Charles Richter defined the magnitude of an earthquake to be where I is the intensity of the earthquake (measured by the amplitude of a seismograph reading taken 100 km from the epicenter of the earthquake) and S is the intensity of a "standard earthquake" (whose amplitude is 1 micron $=10^{-4}$ cm).

Richter studied many earthquakes that occurred between 1900 and 1950. The largest had magnitude of 8.9 on the Richter scale, and the smallest had magnitude 0.

The Richter scale provides more manageable numbers to work with. Each number increase on the Richter scale indicates an intensity of ten times stronger.



Source: http://www.sms-tsunami-warning.com/theme/tsunami/img/earthquakes/richter-scale/richter_scale_graphic_representation.gif

An earthquake of magnitude 6 is ten times stronger than an earthquake of magnitude 5.
An earthquake of magnitude 7 is 10 x 10 = 100 times strong than an earthquake of magnitude 5.
An earthquake of magnitude 8 is 10 x 10 x 10 = 1000 times stronger than an earthquake of magnitude 5.

Richter Scale Intensity Model
$M = \log_{10} \frac{I}{S}$ where M is the magnitude of the earthquake, I is the intensity measured by the amplitude of the seismograph 100 km from the epicentre (expressed in cm) and S is a constant and the intensity of a "regular earthquake".

Example 30

The magnitude of an earthquake was defined by Charles Richter in 1935 using the expression

$$M = \log_{10}\frac{I}{S}$$

where I is the intensity of the earthquake and S is the intensity of a "regular earthquake".

The magnitude of a regular earthquake is

$$M = \log_{10}\frac{S}{S}$$

$$M = \log_{10}1$$

$$M = 0$$

(a) If the intensity of an earthquake in location X is three times a regular earthquake, what is its magnitude? [Hint: Let the intensity of the earthquake in X be $I = 3S$] [Ans: 0.477]

$$\text{Magnitude} = \log_{10}\frac{3S}{S}$$

$$= \log_{10}3$$

$$= 0.477 \text{ (3 sf)}$$

(b) In 1906, an earthquake in San Francisco was registered 8.3 on the Richter scale. By what factor of S was the intensity of this earthquake higher than that of a regular earthquake? [Ans: $10^{8.3}$]

$$8.3 = \log_{10}\frac{I}{S}$$

$$\frac{I}{S} = 10^{8.3}$$

(c) The intensity of earthquake A is 6 times the intensity of earthquake B. Given that the magnitude of earthquake B is p, express the magnitude of earthquake A in terms of p.

$$p = \log_{10}\frac{I_B}{S}$$

$$M_A = \log_{10}\frac{I_A}{S} = \log_{10}\frac{6I_B}{S} = \log_{10}6 + \log_{10}\frac{I_B}{S} = \log_{10}6 + p$$

(d) The intensity of earthquake C is 6 times less intense than earthquake B. Given that the magnitude of earthquake B is p, express the magnitude of earthquake C in terms of p.

$$p = \log_{10}\frac{I_B}{S}$$

$$M_C = \log_{10}\frac{I_C}{S} = \log_{10}\frac{I_B}{6S} = \log_{10}\frac{1}{6} + \log_{10}I_B = \log_{10}\frac{1}{6} + p$$

(e) ~~The magnitude of earthquake X is 1.2 more than the magnitude of earthquake Y. How much more intense is earthquake X than earthquake Y?~~

STEM in Exponential and Logarithmic Functions

1. A gas in a cylinder with an initial volume $V_1 = 0.25$ m^3 and pressure $P_1 = 100$ kPa is compressed to a final pressure $P_2 = 450$ kPa. Given that

$$P_1(V_1)^{1.33} = P_2(V_2)^{1.33},$$

determine the final volume V_2.

$$P_1 V_1^{1.33} = P_2 V_2^{1.33}$$

$$(100)\, 0.25^{1.33} = (450)\, V_2^{1.33}$$

$$V_2^{1.33} = 99.591$$

$$V_2 = 99.591^{\frac{1}{1.33}}$$

$$= 31.8 \text{ kPa}$$

2. A pulley is driven by a flat belt with an angle of lap $\theta = \frac{2\pi}{3}$. Given that T_1, T_2 are tensions on each end of the belt and

$$T_2 = T_1 e^{\mu\theta}$$

Determine T_2 for $T_1 = 1000$ N and $\mu = 0.2$ (coefficient of friction)

$$T_2 = 1000\, e^{0.2\left(\frac{2\pi}{3}\right)}$$

$$= 1520 \text{ N} \quad (3 \text{ sf})$$

3. The power gain, G, in neper is defined as

$$G = \frac{1}{2} ln\left(\frac{p_0}{p_1}\right)$$

Where p_o = output power and p_1 = input power. Find G for

   a. $p_o = 1$ W and $p_1 = 0.2$ W
   b. $p_o = 20$ mW and $p_1 = 1$ mW

a. $G = \frac{1}{2}\ln\left(\frac{1}{0.2}\right) \approx 0.805$

b. $G = \frac{1}{2}\ln\left(\frac{20}{1}\right) = 1.50$

4. The voltage, V, across a capacitor is given by

$$V = 9 - 10e^{-0.1t}$$

Find t for $V = 0$.

$V = 9 - 10e^{-0.1t}$

$0 = 9 - 10e^{-0.1t}$

$10e^{-0.1t} = 9$

$e^{-0.1t} = 0.9$

$-0.1t = \ln 0.9$

$t = -\frac{\ln 0.9}{0.1} = 1.05$

5. In a thermodynamic system we have the relationship

$$\log(P) + n\log(V) = \log(C)$$

where P represents pressure, V represents volume, C is a constant and n is an index. Show that

$$PV^n = C$$

$\log P + n \log V = \log C$

$\log PV^n = \log C$

$PV^n = C$

Computational Thinking in Exponential and Logarithmic Functions

1. Write an algorithm / flow chart / program to solve a given logarithmic equation.

A Math Assignment 5A (Introduction & Law of Logarithms)
A Math Textbook: Marshall Cavendish Additional Math 360 Textbook

Tier A: Textbook Exercise 5.3 (pages 116 to 117): Questions 1, 2, 5

Tier B: Textbook Exercise 5.3 (pages 116 to 117): Questions 6, 7, 8b, 8c, 12c, 13

Tier C: Textbook Exercise 5.3 (pages 116 to 117): Questions 14

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A Math Assignment 5B (Solving Exponential Equations using Logarithms)
A Math Textbook: Marshall Cavendish Additional Math 360 Textbook

Tier A: Textbook Exercise 5.1 (page 100 to 101): Questions 3, 4

Tier B: Textbook Exercise 5.1 (page 100 to 101): Questions 7b, 7d, 8b, 9c, 10b, 13i
Textbook Exercise 5.4 (page 123): Questions 20, 21a, 21c

Tier C: Textbook Exercise 5.1 (page 100 to 101): Questions 16
Textbook Exercise 5.4 (page 123): Questions 25, 27

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A Math Assignment 5C (Solving Logarithmic Equations)
A Math Textbook: Marshall Cavendish Additional Math 360 Textbook

Tier A: Textbook Exercise 5.4 (page 122 to 123): Questions 1b, 2b, 2d, 3b, 3d

Tier B: Textbook Exercise 5.4 (page 122 to 123): Questions 6c, 7b, 8b, 10e, 16, 17

Tier C: Textbook Exercise 5.4 (page 122 to 123): Questions 22

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A Math Assignment 5D (Graphs and Application of Exponential Functions and Logarithmic Functions)
A Math Textbook: Marshall Cavendish Additional Math 360 Textbook

Tier A: Textbook Exercise 5.2 (page 106 to 107): Questions 1b, 3, 6
Textbook Exercise 5.5 (page 128 to 129): Questions 1b, 1d, 2

Tier B: Textbook Exercise 5.2 (page 106 to 107): Questions 10, 13, 15
Textbook Exercise 5.5 (page 128 to 129): Questions 5, 8, 11, 13

Tier C: Textbook Exercise 5.2 (page 106 to 107): Questions 17
Textbook Exercise 5.5 (page 128 to 129): Questions 17, 19