SCHOOL OF SCIENCE AND TECHNOLOGY, SINGAPORE
MATHEMATICS DEPARTMENT 2023
SECONDARY 3 ADDITIONAL MATHEMATICS

A

Name: Solutions ( ) Class: S__-__

A MATH UNIT: Quadratic Functions; Equations and Inequalities

a. ENDURING UNDERSTANDING

Students will understand

- the **equivalent** algebraic forms of a quadratic function are useful in finding the key features of the same graph.
- quadratic functions provides a **model** to represent real-world situations and solve problems.
- the **equivalent** relationship between solving a pair of simultaneous equations and finding the points of intersection of two graphs.
- the regions of the quadratic graphs **(diagrams)** below or above the x-axis represent the solutions of a quadratic inequality.
- the discriminant conveys the nature of the roots, which highlight geometrical features **(diagrams)**.
- finding the intersection between a line and a curve is **equivalent** to finding the roots of a quadratic function.

b. ESSENTIAL QUESTIONS

- How are the different equivalent forms of a quadratic function useful?
- How are quadratic functions used to model real-world situations?
- How are the solutions of a pair of simultaneous equations represented graphically?
- How are the solutions of quadratic inequality represented graphically?
- How do the roots and the coefficients of a quadratic equation related?
- How are the nature of roots of a quadratic equation related to the conditions for the intersection between a line and a curve?

c. KNOWLEDGE & SKILLS (from O Level Syllabus)

A1 Quadratic functions		A2 Equations and inequalities	
1.1	Finding the maximum or minimum value of a quadratic function using the method of completing the square	2.1	Conditions for a quadratic equation to have: • two real roots • two equal roots • no real roots and related conditions for a given line to: • intersect a given curve • be a tangent to a given curve • not intersect a given curve
1.2	Conditions for $y = ax^2 + bx + c$ to be always positive (or always negative)	2.2	Solving simultaneous equations in two variables by substitution, with one of the equations being linear equation
1.3	Using quadratic functions as models	2.3	Solving quadratic inequalities, and representing the solution on the number line

d. RESOURCES

1. Chow, W.K. (2011). "*Discovering Mathematics 3A*". Singapore: Star Publishing Pte Ltd. PP1 – 20, 64 – 71.
2. L. K. Lee (2011). "*Pass with Distinction: Additional Mathematics (By Topic)*". Singapore: Shinglee Publishers Pte Ltd.
3. K.C. Yan, B.K. Chng & N.H. Khor (2020). "*Additional Maths 360 2nd Edition*". Singapore: Marshall Cavendish Education.

e. CONTENTS

1.0 TEACHING TO THE BIG IDEA …

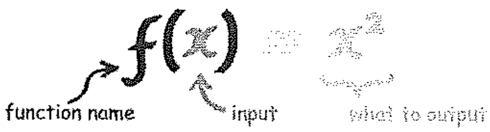
Lesson sequence in the unit								
Student Learning Outcomes	Dimensions (Please tick the appropriate boxes)							
	FUNCTIONS F	INVARIANCE I	NOTATIONS N	DIAGRAMS D	MEASURES M	EQUIVALENCE E	PROPORTIONALITY P	MODELS M
Quadratic functions	√		√	√		√		√
Simultaneous equations				√		√		
Quadratic inequalities				√		√		√
Discriminant and the nature of roots				√		√		
Intersection between a line and a curve				√		√		

1.1 – Functions

A function is a relationship between two sets of objects where each input determines exactly one output according to a rule or operation.

On a Cartesian plane, any point can be seen as a relationship between the x-coordinates and the y-coordinates. When each input x has exactly one output y, the relationship is a function.

We can use the notation $f(x)$ to represent a function. For example, a quadratic function of x can be written as $f(x) = ax^2 + bx + c$, where a, b and c are constants.



We say *"f of x equals x squared"*

Example 1 Given that $f(x) = x^2 - 3x + 7$, find

(a) $f(5)$, (b) $f(2)$, (c) $f(-1)$, (d) $f(0)$.

$f(5) = 5^2 - 3(5) + 7$
$= 17$

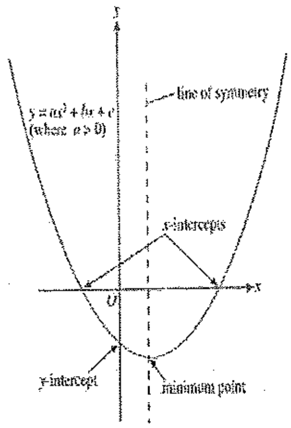
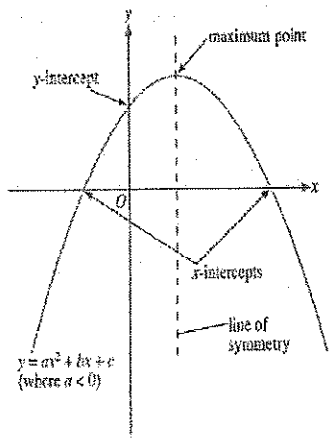
$f(2) = 2^2 - 3(2) + 7$
$= 5$

$f(-1) = (-1)^2 - 3(-1) + 7$
$= 1 + 3 + 7$
$= 11$

$f(0) = 0^2 - 3(0) + 7$
$= 7$

1.2 – Recap: Graphs of Quadratic Functions

- The general form of a quadratic function is $y = f(x) = ax^2 + bx + c$, where a, b, and c are real constants and $a \neq 0$.
- The graph of a quadratic function is called a p<u>arabola</u>. The shape of the graph depends on the value of <u>a</u>, the coefficient of <u>x^2</u>.

Graph of $y = ax^2 + bx + c$, where $a > 0$	Graph of $y = ax^2 + bx + c$, where $a < 0$
1. When $a > 0$, the shape of the graph is called a parabola and it opens upward. 2. The graph has a lowest point. This point is called the minimum point of the parabola. 3. The vertical line through the minimum point is the line of symmetry of the graph. 4. The smaller the numerical value of a, the wider the graph opens. 5. The graph may cut the x-axis at 0, 1 or 2 points. However, it cuts the y-axis at only 1 point. 	1. When $a < 0$, the shape of the graph is a parabola that opens downward. 2. The graph has a highest point. This point is called the maximum point of the parabola. 3. The vertical line through the maximum point is the line of symmetry of the graph. 4. The smaller the numerical value of a, the wider the graph opens. 5. The graph may cut the x-axis at 0, 1, or 2 points. However, it cuts the y-axis at only 1 point. 

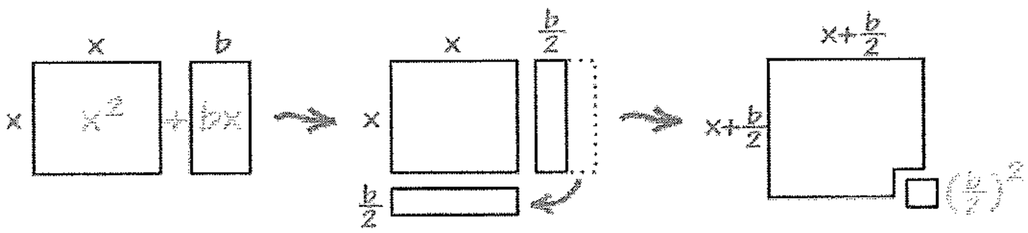
1.3 – Maximum and Minimum Value of a Quadratic Function

Recap: Completing the square

To complete the square is where we take a quadratic polynomial $ax^2 + bx + c$ and convert to the form $a(x - h)^2 + k$, where a, b, c, h and k are real numbers and $a \neq 0$.

Recall completing the square from the Secondary 2 notes:

First, let's see how we can **complete the square** for $x^2 + bx$ using "model method".



As you can see, $x^2 + bx$ can **almost** be rearranged into a square. It requires an additional "piece" of $\left(\frac{b}{2}\right)^2$ to make it complete.

To put it algebraically,

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

By adding $\left(\frac{b}{2}\right)^2$, we can complete the square. On an important note, this method works only when the **coefficient of x^2 is 1.**

Another way that we can look at completing the square:

Part 1

Expansion	**Rearranging…**
$(x+1)^2 = x^2 + 2x + 1$	$x^2 + 2x = (x+1)^2 - 1$
$(x+2)^2 = x^2 + 4x + 4$	$x^2 + 4x = (x+2)^2 - 4$
$(x+3)^2 = x^2 + 6x + 9$	$x^2 + 6x = (x+3)^2 - 9$

Let's try…	
$x^2 + 2x = (x+1)^2 - 1$	$x^2 + 10x =$ $(x+5)^2 - 25$
$x^2 + 4x = (x+2)^2 - 4$	$x^2 + 12x =$ $(x+6)^2 - 36$
$x^2 + 6x = (x+3)^2 - 9$	$x^2 + 3x =$ $(x+\frac{3}{2})^2 - \frac{9}{4}$
$x^2 + 8x =$ $(x+4)^2 - 16$	$x^2 + 11x =$ $(x+\frac{11}{2})^2 - \frac{121}{4}$

Therefore, we have

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2.$$

Part 2

Expansion	**Rearranging…**
$(x-1)^2 = x^2 - 2x + 1$	$x^2 - 2x = (x-1)^2 - 1$
$(x-2)^2 = x^2 - 4x + 4$	$x^2 - 4x = (x-2)^2 - 4$
$(x-3)^2 = x^2 - 6x + 9$	$x^2 - 6x = (x-3)^2 - 9$

Let's try…	
$x^2 - 2x = (x-1)^2 - 1$	$x^2 - 10x =$ $(x-5)^2 - 25$
$x^2 - 4x = (x-2)^2 - 4$	$x^2 - 12x =$ $(x-6)^2 - 36$
$x^2 - 6x = (x-3)^2 - 9$	$x^2 - 3x =$ $(x-\frac{3}{2})^2 - \frac{9}{4}$
$x^2 - 8x =$ $(x-4)^2 - 16$	$x^2 - 11x =$ $(x-\frac{11}{2})^2 - \frac{121}{4}$

Therefore, we have $x^2 - bx = \left(x - \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$.

Summary on Completing the Square

To complete the square, the coefficient of x^2 must be 1.

When the coefficient of x^2 is not 1, we need to factorise the coefficient of the x^2 term from the quadratic expression before we can apply the method of completing the square.

$$x^2 + bx = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

$$x^2 - bx = x^2 - bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 = \left(x - \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

Example 2

Express each of the following in the form $a(x-h)^2 + k$, where a, h and k are real numbers and $a \neq 0$.

(a) $x^2 + 6x - 1$,

(b) $x^2 - x + 2$,

(c) $2x^2 + 6x - 5$,

(d) $-x^2 - 4x - 7$.

(a) $x^2 + 6x - 1$

$= (x+3)^2 - 3^2 - 1$

$= (x+3)^2 - 10$

(b) $x^2 - x + 2$

$= (x - \frac{1}{2})^2 - (\frac{1}{2})^2 + 2$

$= (x - \frac{1}{2})^2 + \frac{7}{4}$

(c) $2x^2 + 6x - 5$

$= 2(x^2 + 3x) - 5$

$= 2[(x+\frac{3}{2})^2 - (\frac{3}{2})^2] - 5$

$= 2(x+\frac{3}{2})^2 - \frac{9}{2} - 5$

$= 2(x+\frac{3}{2})^2 - \frac{19}{2}$

(d) $-x^2 - 4x - 7$

$= -(x^2 + 4x) - 7$

$= -[(x+2)^2 - 2^2] - 7$

$= -(x+2)^2 + 4 - 7$

$= -(x+2)^2 - 3$

Example 3
Consider the following functions. With reference to your answers in Example 2, complete the following table.

Function, f(x)	f(x) = $2x^2 + 6x - 5$	f(x) = $-x^2 - 4x - 7$
Type of Turning Point (Maximum or Minimum) and Coordinates of Turning Point	Minimum point: (-3/2, -19/2)	Maximum point: (-2 , -3)
Maximum/ Minimum Value of $f(x)$	Minimum value of $f(x)$ = -19/2	Maximum value of f(x) = -3
Value of x at which the minimum value/ maximum value occur	x = -3/2	x = -2
Range of values of f(x)	$\geq$ -19/2	$\leq$ -3

Verify your answers using a graphing software.

Exercise 1
Refer to A Math Textbook: Marshall Cavendish Additional Math 360 (2nd Edition) Volume A
Use the space below to write your solutions.

Tier B
- Textbook Exercise 1.1 (page 6): Question 5

Example 4
Explain why the maximum value of $f(x) = -3x^2 + 4x - 2$ is $-\frac{2}{3}$.

$$f(x) = -3x^2 + 4x - 2$$
$$= -3(x^2 - \tfrac{4}{3}x) - 2$$
$$= -3\left[(x - \tfrac{2}{3})^2 - (\tfrac{2}{3})^2\right] - 2$$
$$= -3(x - \tfrac{2}{3})^2 + 3(\tfrac{4}{9}) - 2$$
$$f(x) = -3(x - \tfrac{2}{3})^2 - \tfrac{2}{3}$$

The maximum value of $f(x)$ occurs when $(x - \frac{2}{3})^2 = 0$.

Therefore, maximum value of $f(x) = -3(0)^2 - \frac{2}{3} = -\frac{2}{3}$

Example 5
Show that $x^2 - 4x + 7 \geq 3$ for all real values of x.

$$x^2 - 4x + 7 = (x-2)^2 - 2^2 + 7$$
$$x^2 - 4x + 7 = (x-2)^2 + 3$$
$$x^2 - 4x + 7 \geq 0 + 3$$
$$x^2 - 4x + 7 \geq 3$$ for all real values of x.

Checklist for Self-Assessment on Quadratic Functions, Equations and Inequalities		
I am unsure of…	**I am able to…**	
☐	☐	use completing the square to rewrite a quadratic expression into vertex form.
☐	☐	graph a quadratic functions, identifying key features such as the intercepts, maximum and/or minimum values and symmetry of the graph.
☐	☐	find the maximum or minimum value of a quadratic function using the method of completing the square.

1.4 – Applications of Quadratic Functions

Quadratic Functions can be used as a mathematical model for many problems in real world context.

In solving Mathematics problems, we will need to analyse the given information then select appropriate methods to solve the problem. The Polya's 4-step Problem Solving Technique guides us to think critically and solve Mathematics problems logically.

Polya's 4-step Problem Solving Technique

Stage 1: Understand the problem

a. What is the problem? What am I trying to figure out?
b. What is the given information? What else do I know from the given information?
c. What information/knowledge/skills do I need to solve the problem?
d. Draw a properly labelled diagram (if diagram is not given).

Stage 2: Devise a plan

e. What problems like this have I solved before?
f. Do I know a related problem? Do I know a theorem that could be useful?
g. What rule and/or theorem connect the given information and the unknown?
h. What is an appropriate strategy to solve the problem?

Stage 3: Carry out the plan (solve the problem)

i. Implement the plan of the solution. Perform any necessary actions or computations.
j. Check each step. Can I see clearly that the step is correct? Can I prove that it is correct?
k. Persist with the plan devised in Stage 2. If it continues not to work, discard it and choose another plan.

Stage 4: Looking Back

l. Examine the solution obtained. Can I check the result?
m. Why did my solution work/ not work?
n. How can I derive the solution differently?
o. How might the solution or the method be used to solve some other problems?

Source: How To Solve It, by George Polya, 2nd ed., Princeton University Press, 1957

Equivalent Forms of a Quadratic Function

Activity 1A

Using a graphing software, observe the graphs of the following functions:

(i) $y = 2x^2 - 2x - 12$ (ii) $y = 2(x - 0.5)^2 - 12.5$ (iii) $y = 2\,(x + 2)(x - 3)$

Then fill in the blanks below:

See	Think	Wonder
What do you observe about the graphs of these three functions?	What is the relationship between the functions that result in your observation?	What is the purpose for expressing a quadratic function in different forms?
the 3 functions are **equivalent** to each other. It is just presented in different **forms**.	(i) this form gives the (a) nature of parabola & (b) y-intercept (ii) this form gives the (a) nature of parabola & (b) turning point (iii) this form gives (a) nature of parabola & (b) x-intercepts	to enable us **make a decision** on what date/information that we would like to obtain easily.

Summary

A quadratic function generally takes the form:

$$f(x) = ax^2 + bx + c \text{ where } a, b, c \in \mathcal{R} \text{ and } a \neq 0 \text{ ---------------- (1)}$$

Through the process of COMPLETING THE SQUARE, we can express this quadratic function in the form:

$$y = a(x - h)^2 + k \text{ where } a, h, k \in \mathcal{R} \text{ and } a \neq 0 \text{ ------------------------- (2)}$$

Through the process of FACTORISATION, we can express this quadratic function in the form:

$$y = a(x - \alpha)(x - \beta) \text{ where } a, \alpha, \beta \in \mathcal{R} \text{ and } a \neq 0 \text{ ------------------------- (3)}$$

A quadratic function may be represented in the above 3 forms.

How does each form of the quadratic function useful in finding the key features of the graphs of quadratic functions?

General Form : ax^2+bx+c (useful to find y-intercept)

complete the square form : $y = a(x-h)^2+k$ (useful to find turning point)

Factorisation form: $a(x-\alpha)(x-\beta)$ (useful to find x-intercepts.

Activity 1B

The profit that a company makes selling an item (in thousands of dollars) depends on the price of the item (in dollars). Let be the price of the item and three equivalent expressions for the profit are:

(i) General form: $y = -x^2 + 12x - 27$

(ii) Factorised form: $y = -(x-3)(x-9)$

(iii) Completed square form: $y = -(x-6)^2 + 9$

Match each case to the most useful form of quadratic function. Give reasons and/or show suitable working to justify your answer.

Case	Form of Quadratic function	Justification
(a) the prices that give a profit of zero dollars?	factorised form.	$y = a(x-\alpha)(x-\beta)$, then $\$\alpha$ and $\$\beta$ are the prices that give a profit of zero dollars.
(b) the profit when the price is zero?	general form	$y = ax^2 + bx + c$, $\$c$ is the profit when the price is zero.
(c) the price that gives the maximum profit?	completed square form	$y = a(x-h)^2 + k$. The price that gives the maximum profit is $\$h$.

Example 6 [A Math Textbook: Marshall Cavendish Additional Math 360 (2nd Edition) Volume A page 16 Try 11]

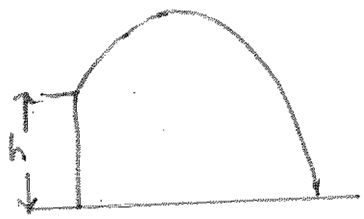
Try 11

Projectile Motion. A projectile was launched from a catapult to smash a defence structure on a fort. Its height, h m, above the ground is given by $h = -\frac{1}{2500}x^2 + \frac{2}{25}x + 3$, where x m is the horizontal distance from the catapult.

(i) Find the height of the projectile when it just left the catapult.
(ii) Find the greatest height of the projectile after it was launched from the catapult.
(iii) If the defence structure is 150 m horizontally from the catapult and 5 m above the ground, justify if the projectile will smash the structure.

Answers: (i) 3 m (ii) 7 m (iii) No

- What is the problem? What information is provided?
- What strategy to use for solving the problem?
- Work it out - ensure process is clear and presentation succinct.
- Check the solution - for clarity and relevance



(i) When $x=0$, $h = -\frac{1}{2500}(0)^2 + \frac{2}{25}(0)+3$
$= 3$

Height $= 3$ m

(ii) $h = -\frac{1}{2500}x^2 + \frac{2}{25}x + 3$

$= -\frac{1}{2500}(x^2 - 200x) + 3$

$= -\frac{1}{2500}[(x-100)^2 - 100^2] + 3$

$= -\frac{1}{2500}(x-100)^2 + \frac{10000}{2500} + 3$

$= -\frac{1}{2500}(x-100)^2 + 7$

At greatest height, $(x-100)^2 = 0$
Greatest height $= 7$ m.

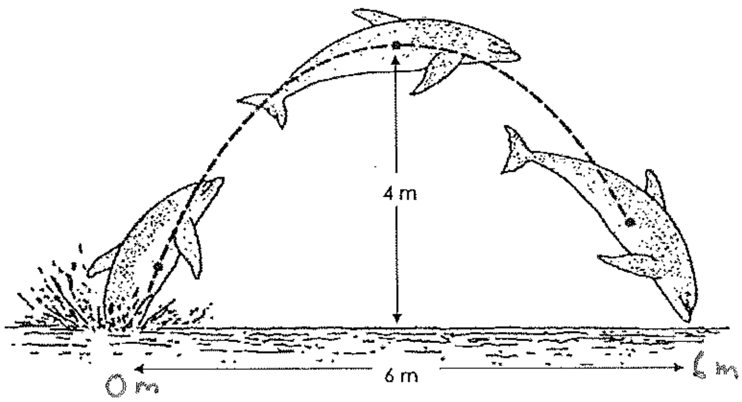
When $x = 150$
$h = -\frac{1}{2500}(150)^2 + \frac{2}{25}(150) + 3$
$= 6\text{m} > 5\text{m}$

The projectile will not smash the structure.

Activity 2

During her recent holiday to Norway, Mrs Tay sighted some dolphins during a cruise trip. One of the dolphins leapt out of the water. The path looks like a parabola. Mrs Tay estimated that the maximum height reached by the dolphin was 4 metres above sea level while the horizontal distance covered was 6 metres.

Find an equation to model the path of the dolphin. State the assumption(s) you made.



- What is the problem? what information is provided?
- What strategy to use for solving the problem?
- Work it out - ensure process is clear and presentation succinct.
- Check the solution - for clarity and relevance

Assume the dolphin travels in a parabola, and the horizontal distance at the point the dolphin leaps from the water is 0 m

$y = a(x-3)^2 + 4$, a is a constant.

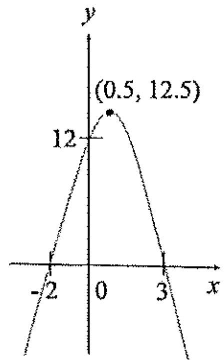
when $x = 0$, $9a + 4 = 0$

$a = -\frac{4}{9}$

$\therefore y = -\frac{4}{9}(x-3)^2 + 4$

Example 7
For the following sketches, find the equation of the graph in the required format stated in the question.

(a) $y = a\,(x - p)\,(x - q)$



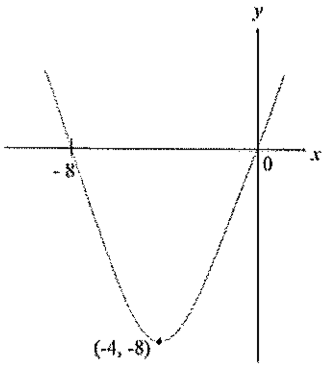
$y = a(x+2)(x-3)$

when $x = 0$, $y = 12$: $12 = a(0+2)(0-3)$

$a = \frac{12}{-6} = -2$

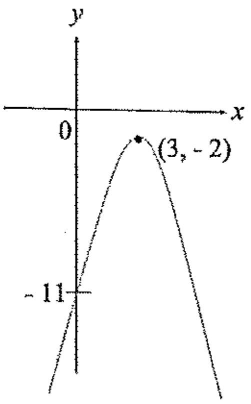
$\therefore y = -2(x+2)(x-3)$

(b) $y = a\,(x - h)^2 + k$



$y = a(x+4)^2 - 8$

(c) $y = ax^2 + bx + c$



$y = a(x-3)^2 - 2$

$y = a(x^2 - 6x + 9) - 2$

$y = ax^2 - 6ax + 9a - 2$

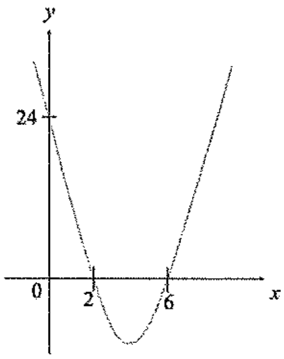
$9a - 2 = -11$

$9a = -9$

$a = -1$

$\therefore y = -x^2 + 6x - 11$

(d) $y = ax^2 + bx + c$



$y = a(x-2)(x-6)$

$y = a(x^2 - 8x + 12)$

$y = ax^2 - 8ax + 12a$

$12a = 24$

$a = 2$

$y = 2x^2 - 16x + 24$

Example 8 [A Math Textbook: Marshall Cavendish Additional Math 360 (2nd Edition) Volume A page 16 Try 12]

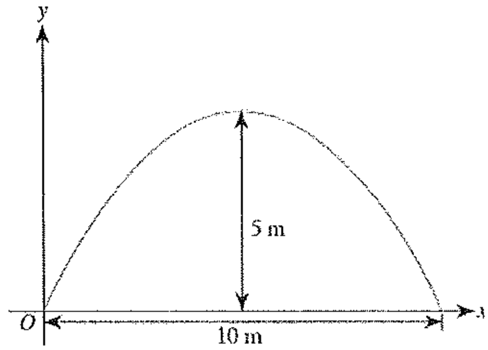
Try 12

Architectural Design. The opening of a tunnel can be modelled by a quadratic function with its graph shown. In this model, x m is the horizontal distance from one end of the tunnel and y m is the height of the tunnel. The tunnel is 10 m wide at its base and 5 m high in the middle.

(i) Write a quadratic function in the form $y = a(x - p)(x - q)$ to represent the opening of the tunnel.

(ii) A point on the opening of the tunnel is 2 m horizontally from one end. What is the height of the tunnel at this point?

(iii) Another point on the opening of the tunnel is 4.2 m vertically from the base. What is the width of the tunnel at this point?



Answers: (i) $y = -\frac{1}{5}x(x - 10)$ (ii) $3\frac{1}{5}$ m (iii) 4 m

- What is the problem? What information is provided?
- What strategy to use for solving the problem?
- Work it out - ensure process is clear and presentation succinct.
- Check the solution - for clarity and relevance

(i) $y = a(x-0)(x-10)$

when $x = 5$, $y = 5$

$5 = a(5-0)(5-10)$

$a = \frac{5}{-25} = -\frac{1}{5}$

$\therefore y = -\frac{1}{5}x(x-10)$

(ii) When $x = 2$, $y = -\frac{1}{5}(2)(2-10)$

$= \frac{16}{5}$

$= 3\frac{1}{5}$

Height of tunnel $= 3\frac{1}{5}$ m

(iii) $4.2 = -\frac{1}{5}x(x-10)$

$-21 = x(x-10)$

$x^2 - 10x + 21 = 0$

$(x-3)(x-7) = 0$

$x = 3$ or $x = 7$

width $= 7 - 3 = 4$ m.

Classification of Justification Questions

DIVINE Framework (Chua, 2017)

Nature of Justification Tasks	**Purpose of Justification Tasks**	**Expected Element in the Justification**
Making Decision	Explain Whether … Explain which …	Make a decision about the mathematical claim with evidence to support or refute the claim
Inference	Explain what …	Infer the meaning of the mathematical result, with the key words in the task addressed
Validation	Explain why …	Give a reason or evidence to support or refute the mathematical claim
Elaboration	Explain how …	Give a clear description of the method or strategy used to obtain the mathematical result

Note: Questions may also be phrased as "Prove…", "Show…"

A suggested structure to answer justification questions

1. **Evidence:** < the math working>

2. **Reason (interpretation of evidence):** <what can we interpret from the math working to get the conclusion>

3. **Conclusion:** <what can we conclude/ decide about the mathematical claim from the reason and evidence?>

...ple 9 [A Math Textbook: Marshall Cavendish Additional Math 360 (2nd Edition) Volume A page 16 Try 12]

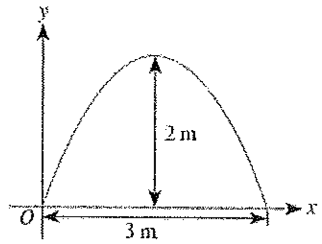
Try 13

Architectural Design. An arched underpass has the shape of a parabola as shown. In the diagram, x m is the horizontal distance from one end of the arch and y m is the height of the arch. A river passing under the arch is 3 m wide, and the maximum height of the arch is 2 m.

(i) Write a quadratic function in the form $y = a(x-h)^2 + k$ to represent the arch.

(ii) Find the height of the arch when its width is 1 m.

(iii) Decide whether it is possible for a boat that is 1 m wide and 1.8 m tall to navigate through the underpass. Explain your answer.



Answers: (i) $y = -\frac{8}{9}\left(x - \frac{3}{2}\right)^2 + 2$ (ii) $1\frac{7}{9}$ m (iii) No

(i) $y = a\left(x - \frac{3}{2}\right)^2 + 2$

when $x = 0$, $y = 0$

$0 = a\left(0 - \frac{3}{2}\right)^2 + 2$

$\frac{9}{4}a = -2$

$a = -\frac{8}{9}$

$y = -\frac{8}{9}\left(x - \frac{3}{2}\right)^2 + 2$

(ii) when $x = 1$

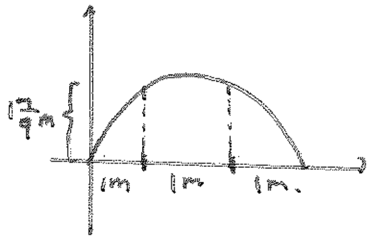
$y = -\frac{8}{9}\left(1 - \frac{3}{2}\right)^2 + 2$

$y = -\frac{8}{9}\left(\frac{1}{4}\right) + 2$

$y = 1\frac{7}{9}$

Height $= 1\frac{7}{9}$ m

(iii)



At 1m, the height of the underpass is 1.77m < 1.8m.
The boat cannot navigate the underpass if its cross section is a rectangle.

Checklist for Self-Assessment on Quadratic Functions, Equations and Inequalities		
I am unsure of...	I am able to...	
☐	☐	use and solve quadratic functions as models.

1.5 – Solving of Linear and Non-linear Simultaneous Equations

Revision: Solving of Simultaneous Linear Equations (Review from Elementary Mathematics)

Solve the following pairs of simultaneous equations

$$2x + y = 9,$$
$$2x - 3y = 29.$$

Substitution Method	Elimination Method
$2x + y = 9$	$2x + y = 9$ ---------- (1)
$y = 9 - 2x$ ---------- (1)	$2x - 3y = 29$ ---------- (2)
$2x - 3y = 29$ ---------- (2)	(1) – (2):
	$2x + y - (2x - 3y) = 9 - 29$
	$4y = -20$
Sub (1) into (2):	$y = -5$
$2x - 3(9 - 2x) = 29$	
$2x - 27 + 6x = 29$	Sub $y = -5$ into (1):
$8x = 56$	$2x + (-5) = 9$
$x = 7$	$2x = 14$
	$x = 7$
Sub x=7 into (1):	$\therefore x = 7, y = -5$
$y = 9 - 2(7)$	
$y = -5$	
$\therefore x = 7, y = -5$	

<u>Think:</u> Can you describe, in general, the substitution method of solving linear simultaneous equations.

1. Express **one variable as a subject**
2. Substitute one variable in terms of another **to find the value of the first variable**
3. Solve **for the other variable.**
4. Substitute back **both values to check for accuracy.**

Revision Question

1. Solve the following pairs of simultaneous equations by substitution method.

$$5x - 2y = 16$$

$$x + 3y = -7$$

Do you know how the calculator can be used to check your solutions?

$5x - 2y = 16$ —(1)

$x + 3y = -7$ —(2)

(2): $x = -7 - 3y$ —(3)

subs (3) into (1):

$5(-7-3y) - 2y = 16$

$-35 - 15y - 2y = 16$

$17y = -35 - 16 = -51$

$y = -3$ #

$x + 3(-3) = -7$

$x - 9 = -7$

$x = 2$ #

General Strategy for Solving Linear and Non-linear Simultaneous Equations

Exactly the same as that described in the previous page!

Example 10
Solve the simultaneous equations

$$y = 2x + 1,$$
$$y = x^2 + 2x - 3.$$

$y = 2x + 1$ —— (1)

$y = x^2 + 2x - 3$ —— (2)

subs (1) into (2):

$2x + 1 = x^2 + 2x - 3$

$x^2 - 4 = 0$

$(x+2)(x-2) = 0$

$x = -2$ or $x = 2$

When $x = -2$, $y = 2(-2) + 1 = -3$ #

when $x = 2$, $y = 2(2) + 1 = 5$ #

Example 11

Find the coordinates of the points of intersection of the line $y - x + 3 = 0$ and $y = x^2 - 4x + 1$.

$y - x + 3 = 0$

$y = x - 3$ —— (1)

$y = x^2 - 4x + 1$ —— (2)

(1) = (2): $x - 3 = x^2 - 4x + 1$

$x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

$x = 4$ or $x = 1$

when $x = 4$, $y = 4 - 3 = 1$

when $x = 1$, $y = 1 - 3 = -2$

The coordinates are $(4, 1)$ and $(1, -2)$ #

Example 12

Refer to A Math Textbook: Marshall Cavendish Additional Math 360 (2nd Edition) Volume A

Use the space below to write your solutions. For (ii), draw in the textbook.

Tier B

- Textbook Exercise 2.1 (pages 31 to 32): Question 6

Checklist for Self-Assessment on Quadratic Functions, Equations and Inequalities		
I am unsure of…	I am able to…	
☐	☐	solve simultaneous equations with at least one linear equation by substitution.

1.6 – Solving Quadratic Inequalities

Statements such as $13x-7>1-6x$, $2x^2+13x<7$ are examples of inequalities in one variable. They represent relationships between two quantities listed on both sides of the inequality sign. They illustrate the comparison of how one quantity is more than or less than another quantity.

When we **solve** inequalities, we find values of the variable (eg. x) that would make the relationship or statement true.

Revision: Solving Simple Linear Inequalities (Review from Elementary Mathematics)

Recall: How would you solve $13x - 7 > 1 - 6x$? What about $-3 < \frac{2x-1}{3} \leq 10$?

Rules involving inequalities

Fill in the boxes below with $\geq$, $>$, $<$ or $\leq$.

(a) If $a < b$ and c > 0, then $a + c \boxed{<} b + c$.	(b) If $a < b$ and c > 0, then $a - c \boxed{<} b - c$.
(c) If $a < b$ and c > 0, then $ac \boxed{<} bc$.	(d) * If $a < b$ and c < 0, then $ac \boxed{>} bc$.
(e) If $a < b$ and c > 0, then $\frac{a}{c} \boxed{<} \frac{b}{c}$.	(f) * If $a < b$ and c < 0, then $\frac{a}{c} \boxed{>} \frac{b}{c}$.

** When an inequality is multiplied or divided by a negative number, the inequality sign changes.

1. Solve the following inequalities.

(a) $2x < 4$

$x < \frac{4}{2}$

$x < 2$

(b) $2x < -4$

$x < -\frac{4}{2}$

$x < -2$

(c) $-2x < 4$

$x > \frac{4}{-2}$

$x > -2$

(d) $-2x < -4$

$x > \frac{-4}{-2}$

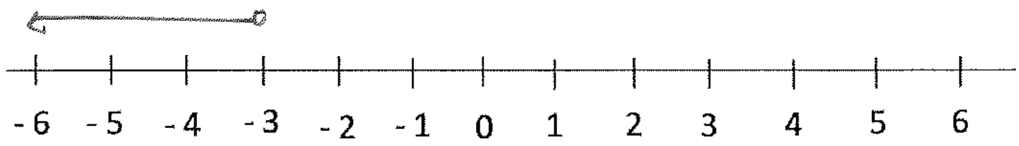
$x > 2$

Representing inequalities on a number line

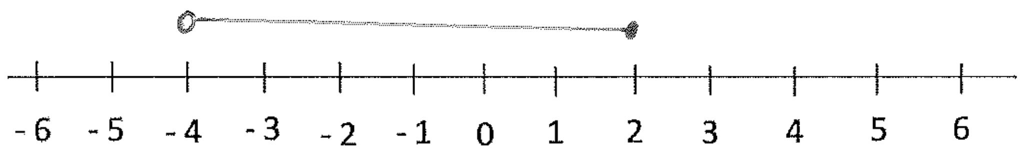
- Hollow circle: <, >
- Shaded circle: ≤, ≥

2 Represent the following range of x on the number line.

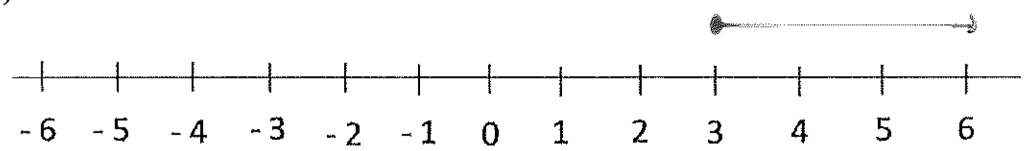
(i) $x < -3$



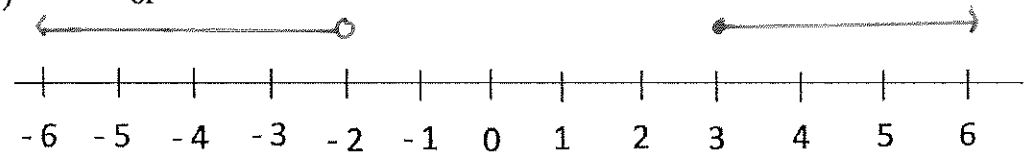
(ii) $-4 < x \leq 2$



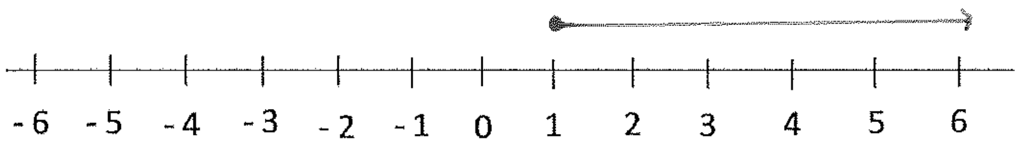
(iii) $x > 3$



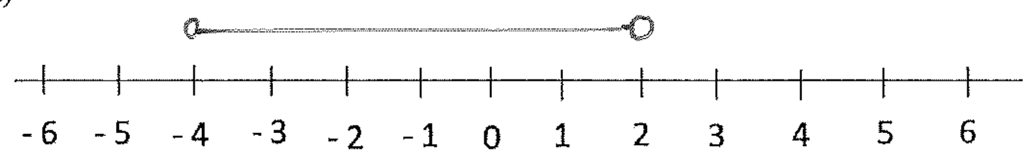
(iv) $x < -2$ or $x > 3$



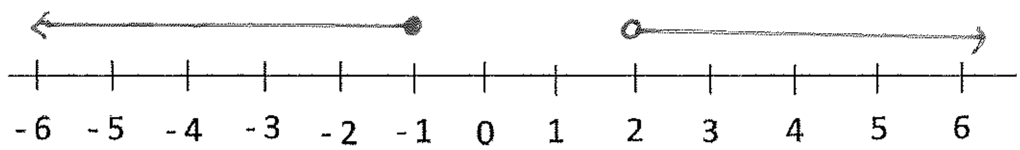
(v) $x \geq 1$



(vi) $-4 < x < 2$

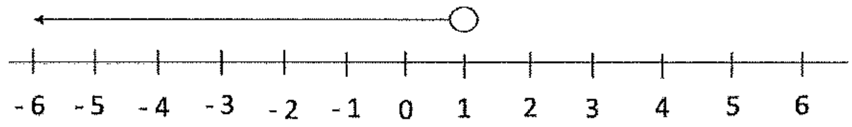


(vii) $x \leq -1$ or $x > 2$



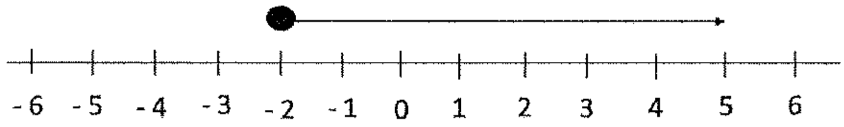
3. Write the range of x in the box in the following shadings on the number line.

(i)



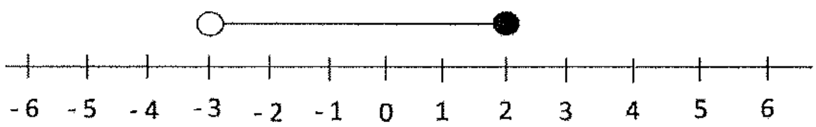
$x < 1$

(ii)



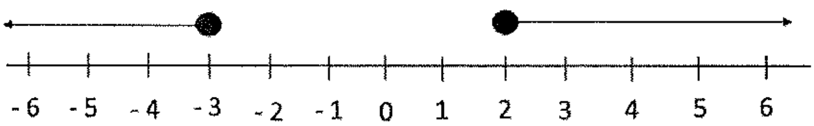
$x \geq -2$

(iii)



$-3 < x \leq 2$

(iv)



$x \leq -3$ or $x \geq 2$

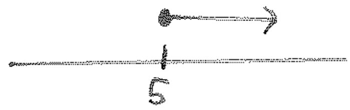
Difference between equation and inequality

Equation	**Inequality**
Solve $2x + 3 = 11$. $2x + 3 = 11$ $2x = 8$ $x = 4$	Solve $2x + 3 < 11$. $2x + 3 < 11$ $2x < 8$ $x < 4$

The difference: Equation has <u>a finite number</u> of solutions whereas inequality has a <u>range</u> of solutions.

Example *A*
Solve the inequality $2x - 19 \leq 7x + 6$. Represent the solution on a number line.

$$2x - 19 \leq 7x + 6$$
$$-19 - 6 \leq 7x - 2x$$
$$5x \geq -25$$
$$x \geq 5$$



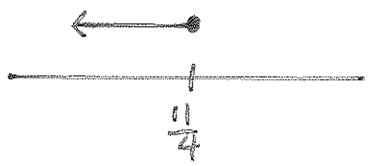
When there are fractions in the inequalities, multiply every term with the **lowest common multiple** of the denominators.

Example *B*
Solve the inequality $\frac{2x+3}{4} \geq \frac{5x-1}{6}$. Represent the solution on a number line.

$$\frac{2x+3}{4} \geq \frac{5x-1}{6}$$

LCM of 4 and 6 = 12

$$12 \times \frac{2x+3}{4} \geq 12 \times \frac{5x-1}{6}$$
$$3(2x+3) \geq 2(5x-1)$$
$$6x + 9 \geq 10x - 2$$
$$4x \leq 11$$
$$x \leq \frac{11}{4}$$



Simultaneous linear inequalities

Example *C*
Solve the inequalities $5x + 13 \geq 2x - 5$ and $3x - 7 > 9x + 11$.

Represent solution on a number line.

$5x+13 \geqslant 2x-5$ and $3x-7 > 9x+11$

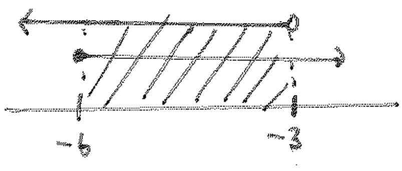
$3x \geqslant -18$

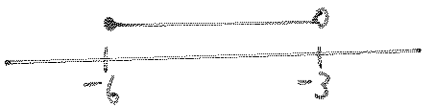
$x \geqslant -6$

$-18 > 6x$

$6x < -18$

$x < -3$





Example *D*
Solve the inequalities $3(x-2) < 5(x+1) < 3 - x$. Represent solution on a number line.

$3(x-2) < 5(x+1) < 3-x$

$3(x-2) < 5(x+1)$ and $5(x+1) < 3-x$

$3x-6 < 5x+5$

$-11 < 2x$

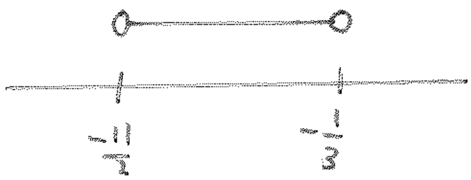
$2x > -11$

$x > -\frac{11}{2}$

$5x+5 < 3-x$

$6x < -2$

$x < -\frac{1}{3}$



Revision Exercise

Solve the following linear inequalities and illustrate your answer on a number line.

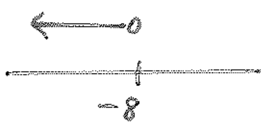
(a) $-2x < -24 - 5x$

$-2x < -24 - 5x$

$5x - 2x < -24$

$3x < -24$

$x < -8$

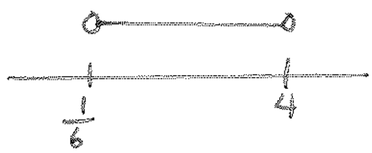


(b) $-\frac{1}{2} < 3x - 1 < 11$

$-\frac{1}{2} < 3x - 1 < 11$

$\frac{1}{2} < 3x < 12$

$\frac{1}{6} < x < 4$



(c) $x + 5 > 2 - 3(x - 4)$

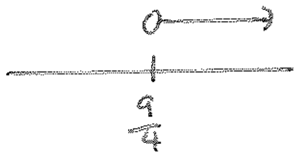
$x + 5 > 2 - 3(x - 4)$

$x + 5 > 2 - 3x + 12$

$x + 5 > 14 - 3x$

$4x > 9$

$x > \frac{9}{4}$



(d) $-3 \leq \frac{4-3x}{2} \leq 5$

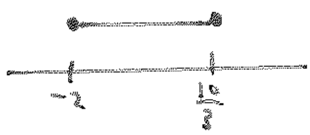
$-6 \leq 4 - 3x \leq 10$

$-10 \leq -3x \leq 6$

Divide throughout by -3

$\frac{10}{3} \geq x \geq -2$

$-2 \leq x \leq \frac{10}{3}$



Quadratic Inequalities

Quadratic inequalities are inequalities involving quadratic expressions. Examples of quadratic inequalities: $2x^2 - 3x + 1 \leq 0$, $(x - 3)(x + 1) > 0$

Examine the working.

$$x^2 - 2x - 3 > 0$$
$$(x + 1)(x - 3) > 0$$
$$x + 1 > 0 \quad \text{or} \quad x - 3 > 0$$
$$x > -1 \quad \text{or} \quad x > 3$$
$$\therefore x > 3$$

Which step is incorrect? Why is it incorrect?

Can you provide the correct solution to the problem?

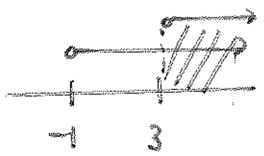
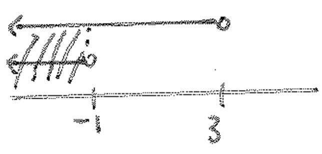
$(x+1)(x-3) > 0$ does not mean $(x+1) > 0$ or $(x-3) > 0$

if $(x+1) < 0$ and $(x-3) < 0$,

$(x+1)(x-3) > 0$.

$x^2 - 2x - 3 > 0$

$(x+1)(x-3) > 0$

Either	OR
$(x+1) > 0$ and $(x-3) > 0$	$(x+1) < 0$ and $(x-3) < 0$
$x > -1$ and $x > 3$	$x < -1$ and $x < 3$
	
$x > 3$ or	$x < -1$

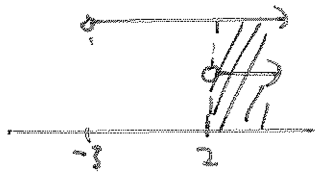
Method 1: Using the **algebraic method** to solve quadratic inequalities:

Example 13 (Algebraic Method)

Solve $(x + 3)(x - 2) > 0$.

Case 1: $(x - 2) > 0$ and $(x + 3) > 0$

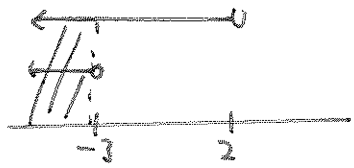
$x > 2$ and $x > -3$



$x > 2$

Case 2: $(x - 2) < 0$ and $(x + 3) < 0$

$x < 2$ and $x < -3$



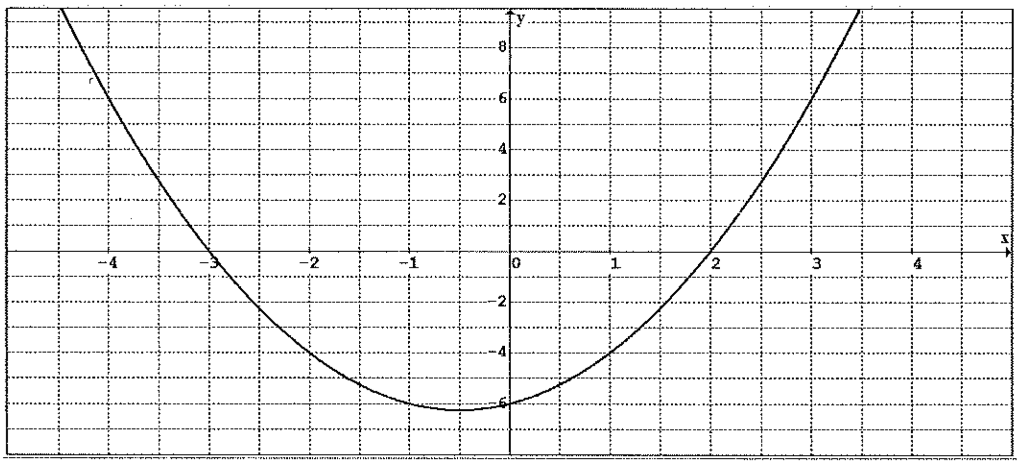
$x < -3$

$\therefore$ $x > 2$ or $x < -3$

<u>Method 2:</u> Using the **graphical method** to solve quadratic inequalities:

Example 14 (Graphical Method)

The graph of $y = (x + 3)(x - 2)$ is plotted below. Use the graph to answer the following questions.



(a)(i) Write down the values of x for which $y = 0$.

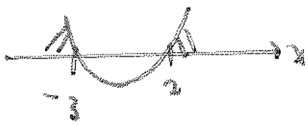
$x = -3$ or $x = 2$

(a)(ii) Write down the values of x for which $(x + 3)(x - 2) = 0$.

$x = -3$ or $x = 2$

(b)(i) Shade the part of the graph when $y > 0$.
Write down the range of values of x for which $y > 0$.

$x < -3$ or $x > 2$

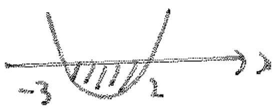


(b)(ii) Write down the range of values of x for which $(x + 3)(x - 2) > 0$.

$x < -3$ or $x > 2$

(c)(i) Shade the part of the graph when $y < 0$.
Write down the range of values of x for which $y < 0$.

$-3 < x < 2$

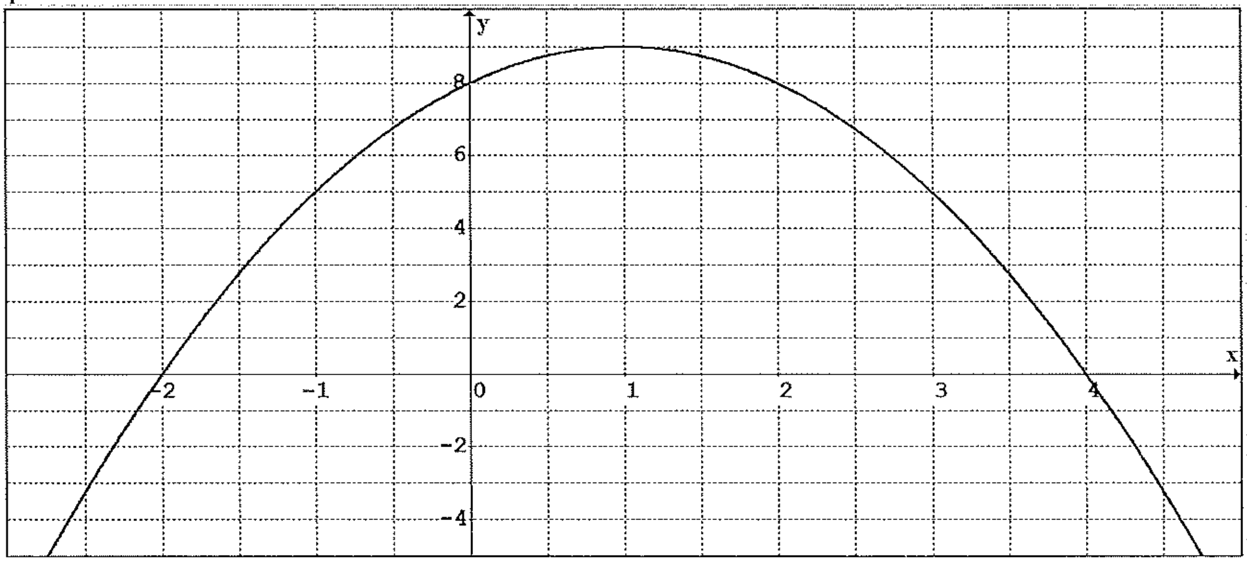


(c)(ii) Write down the range of values of x for which $(x + 3)(x - 2) < 0$.

$-3 < x < 2$

Example 15

The graph of $y = -(x+2)(x-4)$ is plotted below. Use the graph to answer the following questions.



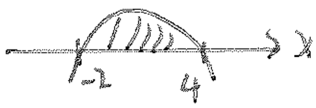
(a)(i) Write down the values of x for which $y = 0$.

$x = -2$ or $x = 4$

(a)(ii) Write down the values of x for which $-(x+2)(x-4) = 0$.

$x = -2$ or $x = 4$

(b)(i) Shade the part of the graph when $y > 0$.
Write down the range of values of x for which $y > 0$.

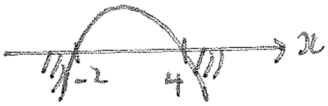


$-2 < x < 4$

(b)(ii) Write down the range of values of x for which $-(x+2)(x-4) > 0$.

$-2 < x < 4$

(c)(i) Shade the part of the graph when $y < 0$.
Write down the range of values of x for which $y < 0$.



$x < -2$ or $x > 4$

(c)(ii) Write down the range of values of x for which $-(x+2)(x-4) < 0$.

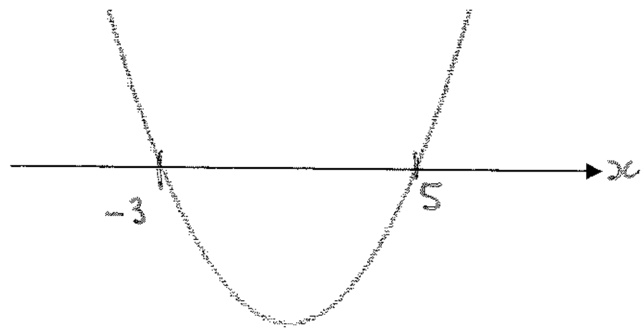
$x < -2$ or $x > 4$

Example 16

(a) Write down the values of x for which $(x + 3)(x - 5) = 0$.

$x = -3$ or $x = 5$

(b) Indicate the x-intercepts of the graph of $y = (x + 3)(x - 5)$ in the sketch below.



(c) Write down the range of values of x which satisfy the following inequalities.

(i) $(x + 3)(x - 5) > 0$

$x < -3$ or $x > 5$

(ii) $(x + 3)(x - 5) < 0$

$-3 < x < 5$

(iii) $(x + 3)(x - 5) \geq 0$

$x \leq -3$ or $x \geq 5$

(iv) $(x + 3)(x - 5) \leq 0$

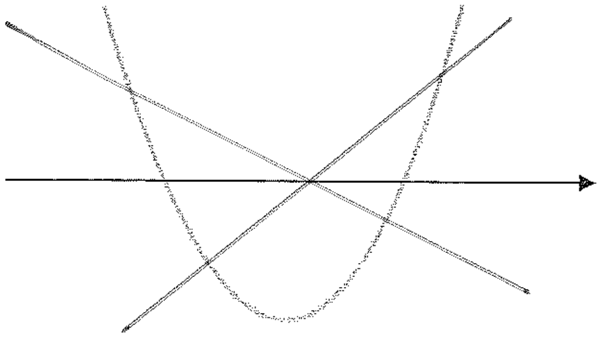
$-3 \leq x \leq 5$

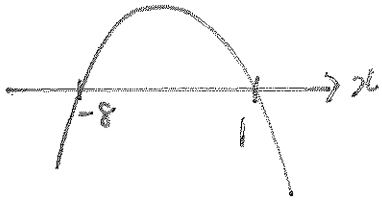
Example 17

(a) Write down the values of x for which $-(x-1)(x+8)=0$.

$x = 1$ or $x = -8$

(b) Indicate the x-intercepts of the graph of $y = -(x-1)(x+8)$ in the sketch below.





(c) Write down the range of values of x which satisfy the following inequalities.

(i) $-(x-1)(x+8) > 0$

$-8 < x < 1$

(ii) $-(x-1)(x+8) < 0$

$x < -8$ or $x > 1$

(iii) $-(x-1)(x+8) \geq 0$

$-8 \leq x \leq 1$

(iv) $-(x-1)(x+8) \leq 0$

$x \leq -8$ or $x \geq 1$

Example 18

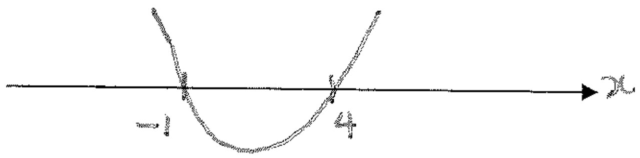
(a) Write down the values of x for which $x^2 - 3x - 4 = 0$.

$x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0$

$x = 4$ or $x = -1$

(b) Sketch $y = x^2 - 3x - 4$, indicating clearly the x-intercepts.



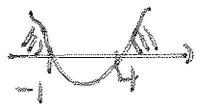
(c) Solve the following inequalities.

(i) $x^2 - 3x - 4 > 0$,

$x^2 - 3x - 4 > 0$

$(x-4)(x+1) > 0$

$x < -1$ or $x > 4$

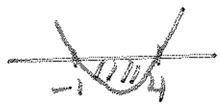


(ii) $x^2 - 3x - 4 < 0$,

$x^2 - 3x - 4 < 0$

$(x-4)(x+1) < 0$

$-1 < x < 4$



(iii) $x^2 - 3x - 4 \geq 0$,

$x^2 - 3x - 4 \geq 0$

$(x-4)(x+1) \geq 0$

$x \leq -1$ or $x \geq 4$

Example 19

Solve the following inequalities.

(a) $x^2+3x+2>0$

(b) $2-x-x^2<0$

(c) $x(x-8)<0$

(d) $4x^2-x>0$

(e) $(x+3)(x+2)<42$

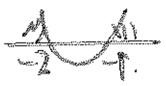
(f) $4x(x-1)>3$

(g) $x^2<4$

(h) $4x^2\geq 25$

(a) $x^2+3x+2>0$

$(x+1)(x+2)>0$

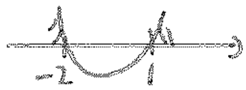


$x<-2$ or $x>-1$

(b) $2-x-x^2<0$

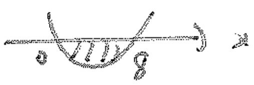
$x^2+x-2>0$

$(x+2)(x-1)>0$



$x<-2$ or $x>1$

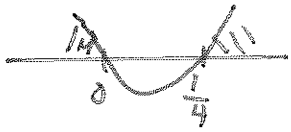
(c) $x(x-8)<0$



$0<x<8$

(d) $4x^2-x>0$

$x(4x-1)>0$



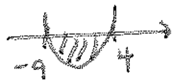
$0<x<\frac{1}{4}$

(e) $(x+3)(x+2)<42$

$x^2+5x+6-42<0$

$x^2+5x-36<0$

$(x+9)(x-4)<0$



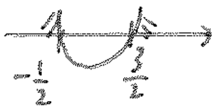
$-9<x<4$ #

(f) $4x(x-1)>3$

$4x^2-4x>3$

$4x^2-4x-3>0$

$(2x-3)(2x+1)>0$



$x<-\frac{1}{2}$ or $x>\frac{3}{2}$ #

(g) $x^2<4$

$x^2-4<0$

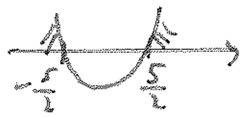
$(x+2)(x-2)<0$

$-2<x<2$ #

(h) $4x^2>25$

$4x^2-25>0$

$(2x-5)(2x+5)>0$



$x<-\frac{5}{2}$ or $x>\frac{5}{2}$ #

Example 20

Solve the following inequalities.

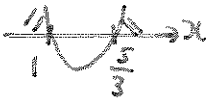
(a) $x^2 > \frac{8x-5}{3}$ (b) $\frac{2x}{x+3} > \frac{4}{2x+1}$

(c) $x(8-x) \leq 15$ (d) $3x + 4 < x^2$

(a) $x^2 > \frac{8x-5}{3}$

$3x^2 > 8x - 5$

$3x^2 - 8x + 5 > 0$



$(3x-5)(x-1) > 0$

$x < 1$ or $x > \frac{5}{3}$ ※

(b) $\frac{2x}{x+3} > \frac{4}{2x+1}$

Is this easy to solve? why not?

(c) $x(8-x) \leq 15$

$8x - x^2 \leq 15$

$x^2 - 8x + 15 \geq 0$

$(x-3)(x-5) \geq 0$

$x \leq 3$ or $x \geq 5$ ※

(d) $3x+4 < x^2$

$x^2 - 3x - 4 > 0$

$(x-4)(x+1) > 0$

$x < -1$ or $x > 4$

Checklist for Self-Assessment on Quadratic Functions, Equations and Inequalities		
I am unsure of…	I am able to…	
☐	☐	Solve quadratic inequalities and represent the solution set on a number line.

1.7 – Discriminant and the Nature of Roots

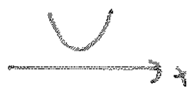
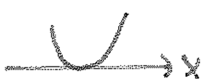
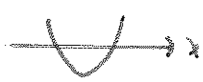
- When we solve a quadratic equation, we are finding the values of x for which the statement $ax^2 + bx + c = 0$ is true.
- The solutions of a quadratic equation $ax^2 + bx + c = 0$ can be found using the g<u>eneral</u> formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The expression $b^2 - 4ac$ is known as the d<u>iscriminant</u>.
- The solutions of a quadratic equation $ax^2 + bx + c = 0$ are called the r<u>oots</u> of the quadratic equation, which are the <u>x-intercepts</u> of the corresponding quadratic graph.

Investigating the Discriminant and the Nature of Roots

Through this investigation, you will discover the relationship between the value of the discriminant, $b^2 - 4ac$, and the nature of roots of a quadratic equation. You may use the graphing software for this investigation.

Quadratic Equation $ax^2+bx+c=0$	a	b	c	**Value of the discriminant** b^2-4ac	**Graphical Illustration**	**Nature of Roots**
(a) $2x^2 - 7x - 9 = 0$	2	-7	-9	$(-7)^2-4(2)(-9)$ $= 121$		real and distinct
(b) $2x^2 - 7x + 9 = 0$	2	-7	9	$(-7)^2-4(2)(9)$ $= -23$		no real roots
(c) $4x^2 - 20x + 25 = 0$	4	-20	25	$(-20)^2-4(4)(25)$ $= 0$		real and equal roots
(d) $\frac{x^2}{16} - 49 = 0$	$\frac{1}{16}$	0	-49	$0^2-4(\frac{1}{16})(-49)$ $= \frac{49}{4}$		real and distinct roots

Conclusion:

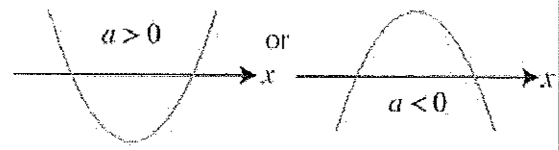
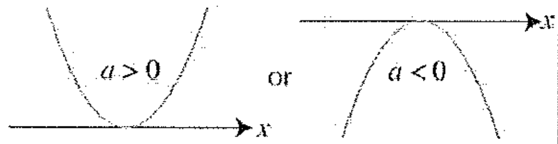
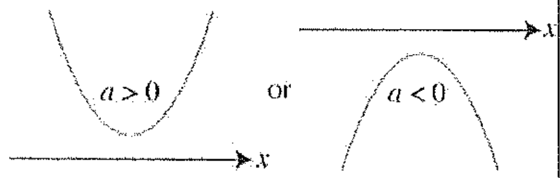
(a) If the discriminant $b^2 - 4ac > 0$, the equation has real and distinct roots.

(b) If the discriminant $b^2 - 4ac < 0$, the equation has no real roots.

(c) If the discriminant $b^2 - 4ac = 0$, the equation has real and equal roots.

(d) If the discriminant $b^2 - 4ac \geq 0$, the equation has real roots.

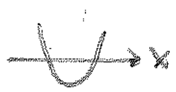
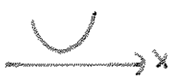
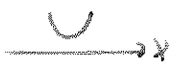
Summary:
For a quadratic equation $ax^2 + bx + c = 0$ and its corresponding curve $y = ax^2 + bx + c$,

discriminant $b^2 - 4ac$	**Nature of Roots**	**Shapes of Curve $y = ax^2 + bx + c$**	**Characteristics of Curve**
> 0	2 real and distinct roots		$y = ax^2 + bx + c$ cuts the x-axis at 2 distinct points
$= 0$	2 real and equal roots		$y = ax^2 + bx + c$ touches the x-axis at 1 point
< 0	no real roots		$y = ax^2 + bx + c$ lies entirely above (for $a > 0$) or entirely below (for $a < 0$) the x-axis

<u>**Notes**</u>

- Real roots $\Leftrightarrow$ discriminant $b^2 - 4ac \geq 0$
- Equal roots = repeated roots = coincident roots => discriminant $b^2 - 4ac = 0$

Example 21
For each of the following quadratic equations, determine the discriminant and hence determine the nature of the roots.

Quadratic Equation $ax^2 + bx + c = 0$	**Sign of the discriminant** $b^2 - 4ac$	**Nature of Roots**	**Graphical Illustration**
(a) $2x^2 + 5x - 8 = 0$	+	real and distinct	
(b) $5x^2 - 3x + 6 = 0$	−	no real roots	
(c) $(2x-1)^2 = 0$ $4x^2 - 4x + 1 = 0$	0	real and equal roots	
(d) $(3x-1)^2 + 6 = 0$ $9x^2 - 6x + 7 = 0$	−	no real roots	

General Strategy for Solving Questions involving Nature of Roots:

1. Compute the discriminant of the quadratic equation.
2. Use the information provided in the question to obtain an equation/inequality involving the discriminant.
3. Solve the equation/inequality.

<u>Example *A*</u>

Find the range of values of p for which the equation $px^2 - x - 4 = 0$ has no real roots.

Discriminant

$$= (-1)^2 - 4(p)(-4)$$
$$= 1 + 16p$$

Discriminant < 0

$$1 + 16p < 0$$
$$16p < -1$$
$$p < -\frac{1}{16}$$

<u>Example *B*</u>

Find the range of values of k for which the equation $x^2 + kx + x - k^2 + 1 = 0$ has real roots.

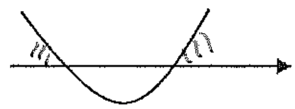
$$x^2 + (k+1)x - k^2 + 1 = 0$$

Discriminant

$$= (k+1)^2 - 4(1)(-k^2 + 1)$$
$$= k^2 + 2k + 1 + 4k^2 - 4$$
$$= 5k^2 + 2k - 3$$

Discriminant ≥ 0

$$5k^2 + 2k - 3 \geq 0$$
$$(5k-3)(k+1) \geq 0$$



$$k \leq -1 \text{ or } k \geq \frac{3}{5}$$

Example *C*

Show that the equation $(2-3k)x^2+\frac{3}{4}k=-x$ has real roots for all real values of *x*.

$$\left(2-3k\right)x^2+\frac{3}{4}k=-x$$

$$\left(2-3k\right)x^2+x+\frac{3}{4}k=0$$

Discriminant

$$=\left(-1\right)^2-4\left(2-3k\right)\left(\frac{3}{4}k\right)$$

$$=1-6k+9k^2$$

$$=9\left[k^2-\frac{6}{9}x+\frac{1}{9}\right]$$

$$=9\left[\left(k-\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)^2+\frac{1}{9}\right]$$

$$=9\left(k-\frac{1}{3}\right)^2$$

Since $9\left(k-\frac{1}{3}\right)^2\geq 0$, discriminant ≥ 0. Hence, equation has real roots for all values of *x*.

Compare Example B and Example C. What are the similarities and differences?

Example 22

Find the possible values of *k* such that the equation $25x^2+kx+4=0$ has two real and equal roots.

$25x^2+kx+4=0$

Discriminant $=k^2-4(25)(4)$

$=0$

$k^2-400=0$

$(k+20)(k-20)=0$

$k=-20$ or $k=20$ #

Example 23

Find the range of values of m such that the equation $4x^2 - 3x - mx + 1 = 0$ has two real and distinct roots.

$4x^2 - 3x - mx + 1 = 0$

$4x^2 - (3+m)x + 1 = 0$

Discriminant > 0

$(3+m)^2 - 4(4)(1) > 0$

$m^2 + 6m + 9 - 16 > 0$

$m^2 + 6m - 7 > 0$

$(m+7)(m-1) > 0$

$m < -7$ or $m > 1$ #

Example 24

Find the range of values of n such that the equation $3x(x + 8) = n + 5$ has no real roots.

$3x(x+8) = n + 5$

$3x^2 + 24x - n - 5 = 0$

Equation has no real roots

$\therefore$ Discriminant < 0

$24^2 - 4(3)(-n-5) < 0$

$576 + 12n + 60 < 0$

$12n < -636$

$n < -53$ #

Example 25

Find the range of values of p such that the equation $2x^2 - 6x + p = 7$ has real roots.

$2x^2 - 6x + p - 7 = 0$

Equation has real roots

Discriminant ≥ 0

$(-6)^2 - 4(2)(p-7) \geq 0$

$36 - 8p + 56 \geq 0$

$8p \leq 92$

$p \leq \frac{23}{2}$ #

Example 26

Find the values of m if the roots of the quadratic equation $x^2 + 2mx + m + 2 = 0$ are equal.

$x^2 + 2mx + m + 2 = 0$

Roots are equal $\Rightarrow$ Discriminant $= 0$

$(2m)^2 - 4(1)(m+2) = 0$

$4m^2 - 4m - 8 = 0$

$m^2 - m - 2 = 0$

$(m-2)(m+1) = 0$

$m = 2$ or $m = -1$

Example 27

Given that the curve $y = -x^2 + (k-6)x - 1$ lies entirely below the x-axis for all real values of x, find the range of values of k.

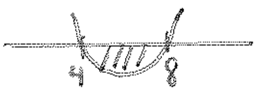
Discriminant < 0

$(k-6)^2 - 4(-1)(-1) < 0$

$k^2 - 12k + 36 - 4 < 0$

$k^2 - 12k + 32 < 0$

$(k-8)(k-4) < 0$



$4 < k < 8$

Example 28

Find the range of values of a such that $3x^2 - 9x + a$ is always positive for all real values of x.

$3x^2 - 9x + a$ is always positive

Discriminant < 0

$(-9)^2 - 4(3)(a) < 0$

$81 - 12a < 0$

$12a > 81$

$a > \frac{27}{4}$

Example 29 [N08/I/10b]

Find the smallest value of the integer b for which $-5x^2+bx-2$ is negative for all values of x.

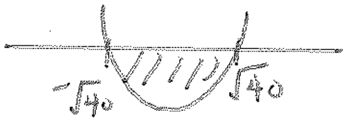
$-5x^2+bx-2$ is negative

discriminant <0

$b^2-4(-5)(-2)<0$

$b^2-40<0$

$(b+\sqrt{40})(b-\sqrt{40})<0$



Smallest value of integer $b=-6$

Example 30 [Modified from 2019 SST S3 AM CT]

Show that, for $k>2$, the equation $(1-2k)x^2+(4-k)x-\frac{k}{4}=0$ has no real roots.

$$\begin{aligned}\text{Discriminant} &= (4-k)^2-4(1-2k)\left(-\frac{k}{4}\right)\\ &= k^2-8k+16+k-2k^2\\ &= -k^2-7k+16\end{aligned}$$

when $k>2$.

$-k^2<-4$

$-k^2-7k<-4-14$

$-k^2-7k+16<-4-14+16<0$

Discriminant <0

$\therefore$ Equation has no real roots.

Checklist for Self-Assessment on Quadratic Functions, Equations and Inequalities		
I am unsure of…	I am able to…	
☐	☐	apply the conditions for a quadratic equation to have • two real roots • two equal roots • no real roots

1.8 – Conditions for quadratic functions to be always positive or always negative

For all real values of x,

For $y = ax^2 + bx + c$ to be always positive ($ax^2 + bx + c > 0$),

the conditions are

- Discriminant $<$ 0
- $a > 0$

In this case,
the graph of $y = ax^2 + bx + c$ lies entirely above the x-axis.

For $y = ax^2 + bx + c$ to be always negative ($ax^2 + bx + c < 0$),

the conditions are

- Discriminant $<$ 0
- $a < 0$

In this case,
the graph of $y = ax^2 + bx + c$ lies entirely below the x-axis.

Example 31

Explain why $-x^2 + 4x - 7$ is always negative for all real values of x.

· coefficient of $x^2 < 0$

Discriminant $= 4^2 - 4(-1)(-7) = -12 < 0$

$\therefore -x^2+4x-7$ is always negative for all real x.

Example 32 [N15/I/4]

(i) Given that $ax^2 + 6x + c$ is always negative, what conditions must apply to the constants a and c?

(ii) Given an example of values of a and c which satisfy the conditions found in part (i).

(i) $a < 0$ #

Discriminant < 0

$6^2 - 4ac < 0$

$36 - 4ac < 0$

$4ac > 36$

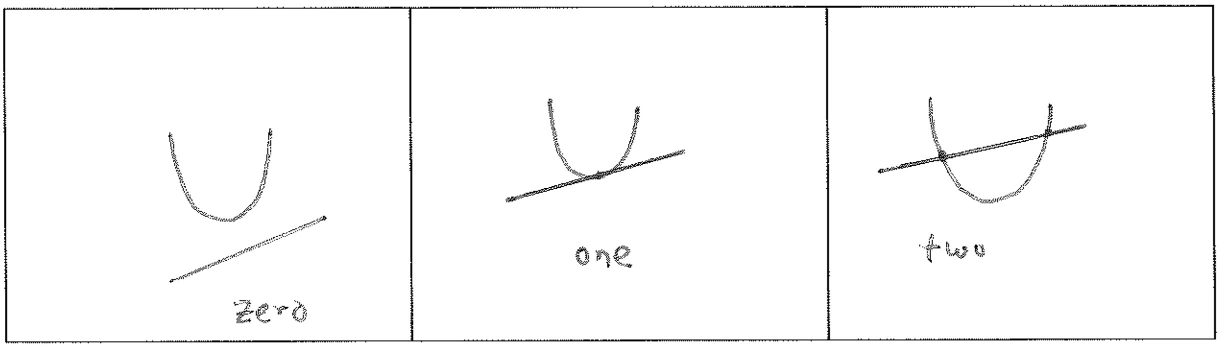
$ac > 9$ #

(ii) $a = -1$, $c = -10$

Checklist for Self-Assessment on Quadratic Functions, Equations and Inequalities		
I am unsure of…	I am able to…	
☐	☐	apply the conditions for $ax^2 + bx + c$ to be always positive or always negative.

1.9 – Intersection between a line and a curve

When a line intersects a quadratic curve, how many possible points of intersection can there be? Illustrate the possibilities in the boxes below.



In this section, we make use of the important fact that the number of intersections of 2 curves corresponds to the number of solutions of simultaneous equations. Once we eliminate one variable and reduce the simultaneous equation to a quadratic equation, we can then make use of the fact that the number of solutions to a quadratic equation is determined by the discriminant of the quadratic equation.

General Strategy for Solving Questions involving Intersections of Curves:

1. By substitution/elimination, reduce the set of simultaneous equations into a quadratic equation.
2. Compute the discriminant of the quadratic equation.
3. Use the information provided in the question to obtain an equation/inequality involving the discriminant.
4. Solve the equation/inequality.

Example E

Find the range of values of k for which the line $y = 3x + k$ will not intersect the curve $y^2 = 3x - k$.

Sub. $y = 3x + k$ into $y^2 = 3x - k$,

$$(3x+k)^2 = 3x - k$$
$$9x^2 + 6kx + k^2 - 3x + k = 0$$
$$9x^2 + (6k-3)x + (k^2+k) = 0$$

Since line does not intersect the curve,

Discriminant < 0

$$(6k-3)^2 - 4(9)(k^2+k) < 0$$
$$36k^2 - 36k + 9 - 36k^2 - 36k < 0$$
$$-72k < -9$$
$$k > \frac{-9}{-72}$$
$$k > \frac{1}{8}$$

Example 33

Find the possible values of k for which the line $y = kx - 5$ is tangent to the curve $2y = x^2 - 1$.

$y = kx - 5$ —(1)

$2y = x^2 - 1$ —(2)

subs (1) into (2):

$2(kx - 5) = x^2 - 1$

$2kx - 10 = x^2 - 1$

$x^2 - 2kx + 9 = 0$

Line is tangent to curve

Discriminant $= 0$

$(-2k)^2 - 4(1)(9) = 0$

$4k^2 - 36 = 0$

$k^2 = 9$

$k = 3$ or -3 #

Example 34

Find the range of values of k for which the line $x + 3y = k - 1$ meets the curve $y^2 = 2x + 5$.

$x + 3y = k - 1$

$x = -3y + k - 1$ —(1)

$y^2 = 2x + 5$ —(2)

subs (1) into (2):

$y^2 = 2(-3y + k - 1) + 5$

$y^2 = -6y + 2k + 3$

$y^2 + 6y - 2k - 3 = 0$

line meets curve, discriminant $\geqslant 0$

$(6)^2 - 4(1)(-2k-3) \geqslant 0$

$36 + 8k + 12 \geqslant 0$

$48 + 8k \geqslant 0$

$8k \geqslant -48$

$k \geqslant -6$ #

Example 35

Given that the line $x + y = m$ does not intersect the curve $x^2 + y^2 = 8$, find the range of values of m.

$x^2 + y^2 = 8$ —(1)

$x + y = m$

$y = -x + m$ —(2)

subs (2) into (1):

$x^2 + (m - x)^2 = 8$

$x^2 + m^2 - 2mx + x^2 = 8$

$2x^2 - 2mx + m^2 - 8 = 0$

Line does not meet curve. Discriminant < 0

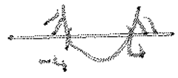
$(-2m)^2 - 4(2)(m^2 - 8) < 0$

$4m^2 - 8m^2 + 64 < 0$

$4m^2 - 64 > 0$

$m^2 - 16 > 0$

$(m - 4)(m + 4) > 0$



$m < -4$ or $m > 4$

Example 36

Find the range of values of k for which $x^2 + 6x - 5$ is always greater than $8x + k$.

$x^2 + 6x - 5 = 8x + k$

$x^2 - 2x - 5 - k = 0$

Discriminant < 0

$(-2)^2 - 4(1)(-5-k) < 0$

$4 + 20 + 4k < 0$

$4k < -24$

$k < -6$ #

Example 37

Show that the line $y + px = p$ will intersect the curve $y = (p + 1)x^2 + px - 1$ at two distinct points for all real values of p, where $p \neq -1$.

$y + px = p$

$y = -px + p$ ——— (1)

$y = (p+1)x^2 + px - 1$ ——— (2)

Subs (1) into (2):

$-px + p = (p+1)x^2 + px - 1$

$(p+1)x^2 + 2px - 1 - p = 0$

Discriminant $= (2p)^2 - 4(p+1)(-1-p)$

$= 4p^2 + 4(p+1)^2 > 0$ for all real p, $p \neq -1$

$\therefore$ Line will intersect the curve at 2 distinct points

STEM in Polynomials

1. The height, h(t), of an object is given by

$$h(t) = 200t - 4.9t^2$$

Determine t for h(t) = 0

$-4.9t^2 + 200t = 0$

$4.9t^2 - 200t = 0$

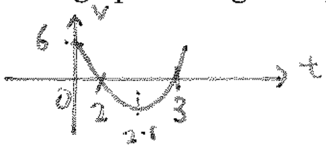
$t(4.9t - 200) = 0$

$t = 0$ or $t = \frac{200}{4.9} = 40.8$ (3 sf)

2. The voltage, V, of a circuit is given by

$V = t^2 - 5t + 6, \ t \geq 0$

Sketch the graph of V against t, indicated the minimum value of V.



$V = (t-2)(t-3)$

when $t = 2.5$, $V = (0.5)(-0.5) = -0.25$

minimum value of $V = -0.25$

3. The displacement, $x(t)$, of a particle is given by $x(t) = t^2 - t - 2$.

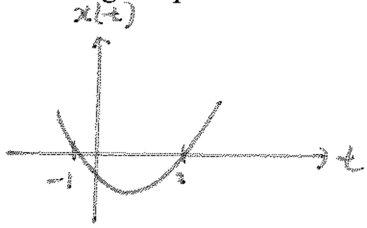
Sketch the graph $x(t)$ against t, marking the points where the graph crosses the axes.

$x(t) = t^2 - t - 2$

$= (t-2)(t+1)$

when $x(t) = 0$

$t = 2$ or $t = -1$



Computational Thinking in Polynomials

1 Given a quadratic equation in the form $ax^2 + bx + c = 0$, write a program to determine the solutions if they exist. Otherwise, indicate that the equation has no real solutions.

Function: QuadRoots

Input: A quadratic equation in the form $ax^2 + bx + c = 0$

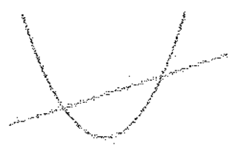
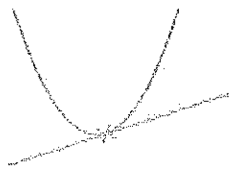
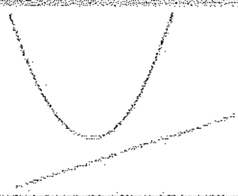
Output: Solution(s) if they exist. Otherwise, output "No real solutions".

Summary: Intersection between line and curve
For a quadratic curve $y = f(x)$ and a straight line $y = mx + c$, solving the two equations simultaneously will give us a quadratic equation of the form $ax^2 + bx + c = 0$.

Once we eliminate one variable and reduce the simultaneous equation to a quadratic equation, we can then make use of the fact that the number of solutions to a quadratic equation is determined by the discriminant of the quadratic equation.

Discriminant

$b^2 - 4ac$	Nature of Solutions	Characteristics of Line and Curve
> 0	2 real and distinct roots	Line cuts the curve at 2 distinct points
$= 0$	2 real and equal roots	Line is a tangent to the curve
< 0	no real roots	Line does not intersect the curve

Case 1	Case 2	Case 3
		
The line cuts the curve at 2 real and distinct points.	The line touches the curve at 1 real point, i.e. the line is a tangent to the curve.	The line does not intersect the curve.

Line meets curve

- line cuts curve at 2 real and distinct point or line touches curve at 1 point
- discriminant $b^2 - 4ac \geq 0$

- line touches the curve at 1 real point (ie line is a tangent to the curve)
- discriminant $b^2 - 4ac = 0$

- The line does not intersect the curve
- discriminant $b^2 - 4ac < 0$

Checklist for Self-Assessment on Quadratic Functions, Equations and Inequalities		
I am unsure of…	**I am able to…**	
☐	☐	apply the conditions for a given line to intersect a given curve be a tangent to a given curve not intersect a given curve

A Math Assignment 01A – Quadratic Functions as Models

A Math Textbook: Marshall Cavendish Additional Math 360 (2nd Edition) Volume A

Tier A
- Textbook Exercise 1.3 (pages 20 to 22): Question 3

Tier B
- Textbook Exercise 1.3 (pages 20 to 22): Question 4, 5, 6, 9

Tier C
- Textbook Exercise 1.3 (pages 20 to 22): Question 10

A Math Assignment 01B – Simultaneous Equations

A Math Textbook: Marshall Cavendish Additional Math 360 (2nd Edition) Volume A

Tier A
- Textbook Exercise 2.1 (pages 31 to 32): Questions 2(b), 3(c)

Tier B
- Textbook Exercise 2.1 (pages 31 to 32): Questions 4, 5(b), 7, 10, 11

Tier C
- Textbook Exercise 2.1 (pages 31 to 32): Questions 12, 14, 15

A Math Assignment 01C – Quadratic Inequalities

A Math Textbook: Marshall Cavendish Additional Math 360 (2nd Edition) Volume A

Tier A
- Textbook Exercise 2.3 (pages 44 to 46): Questions 1a, c, e, 2a, c, d, e

Tier B
- Textbook Exercise 2.3 (pages 44 to 46): Questions 4, 6, 9, 13

Mathematics Homework Reflection Question	
Your response to the question(s) should be detailed. Please write in complete sentences and be ready to share your response in class.	
1.	What were the main mathematical concepts or ideas that you learned today or that we discussed in class today.
2.	Describe a mistake or misconception that you or a classmate had in class today. What did you learn from this mistake or misconception?
3.	What questions do you still have about ….? If you don't have a question, write a similar problem and solve it instead.

A Math Assignment 01D: Discriminant and Nature of Roots

A Math Textbook: Marshall Cavendish Additional Math 360 (2nd Edition) Volume A

Tier A

- Textbook Exercise 1.2 (pages 13 to 14): Questions 3, 5
- Textbook Exercise 2.2 (pages 38 to 39): Questions 2a, b, d
- Textbook Exercise 2.3 (pages 44 to 46): Questions 5

Tier B

- Textbook Exercise 1.2 (pages 13 to 14): Questions 7ii, 10, 11, 13
- Textbook Exercise 2.2 (pages 38 to 39): Questions 5b, e, 10, 13
- Textbook Exercise 2.3 (pages 44 to 46): Questions 9, 11

Tier C

- Textbook Exercise 1.2 (pages 13 to 14): Questions 15, 16
- Textbook Exercise 2.2 (pages 38 to 39): Questions 14
- Textbook Exercise 2.3 (pages 44 to 46): Questions 14

A Math Assignment 01E: Intersection between line and curve

A Math Textbook: Marshall Cavendish Additional Math 360 (2nd Edition) Volume A

Tier B

- Textbook Exercise 2.2 (pages 38 to 39): Questions 4, 8a, b, 11

Q1. Find the range of values of k for which
(a) $3x^2 - 3x > x + k$ for all real values of x,
(b) $kx^2 + 1 > 2kx - k$ for all real values of x.

Q2. Show that the solutions of the equation $x^2 + kx > 3 - k$ are real for all real values of k.

Q3. Find the range of the exact values of c for which the line $y = 2x + c$ does not intersect the curve $2xy + 6 = 0$.

Mathematics Homework Reflection Question	
Your response to the question(s) should be detailed. Please write in complete sentences and be ready to share your response in class.	
1.	What were the main mathematical concepts or ideas that you learned today or that we discussed in class today.
2.	Describe a mistake or misconception that you or a classmate had in class today. What did you learn from this mistake or misconception?
3.	What questions do you still have about ….? If you don't have a question, write a similar problem and solve it instead.