School of Science and Technology, Singapore
Mathematics Department
2023 Secondary 3 ADDITIONAL MATHEMATICS

Unit 07. POLYNOMIALS & PARTIAL FRACTIONS

Name: Solutions ( ) Class: ____________

A

POLYNOMIALS AND PARTIAL FRACTIONS

ENDURING UNDERSTANDING

At the end of the topic, students will understand that

- the Remainder Theorem relates divisors in the division algorithm to evaluate remainders and Factor Theorem is a special case of the Remainder Theorem.
- the factorisation of polynomials utilises the Factor Theorem to maintain the identity-nature of the 2 expressions in the manipulation of algebraic expressions. **(equivalence)**
- the solutions of cubic equations utilises the Factor Theorem and describes the roots of the **function** of the polynomials.
- utilising the division algorithm expresses an improper fraction as a sum of a polynomial and a proper fraction.
- the denominators of algebraic fractions determine the forms of partial fractions.

KNOWLEDGE AND SKILLS

(Source: SEAB 4049_ Additional Mathematics O-Level)

A4	Polynomials and partial fractions	• Multiplication and division of polynomials • Use of remainder and factor theorems, including factorizing polynomials and solving cubic equations • Use of: - $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ - $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ • Partial fractions with cases where the denominator is no more complicated than: - $(ax + b)(cx + d)$ - $(ax + b)(cx + d)^2$ - $(ax + b)(x^2 + c^2)$

APPLICATIONS OF POLYNOMIALS

Since polynomials are used to describe curves of various types, people use them in the real world to graph curves. For example, roller coaster designers may use polynomials to describe the curves in their rides. Combinations of polynomial functions are sometimes used in economics to do cost analyses, for example. Engineers use polynomials to graph the curves of roller coasters and bridges.

Reference: https://prezi.com/ozcnjnwvoc0_/the-use-of-polynomial-functions-in-real-life/

CONTENTS

RESOURCES

1. K.C. Yan, B.K. Chng & N.H. Khor (2020). "*Additional Maths 360 2nd Edition*". Singapore: Marshall Cavendish Education.
2. Chow, W.K. (2010). "*Discovering Additional Mathematics*". Singapore: Star Publishing Pte Ltd.
3. Lee, L.K. (2011). "*Pass with Distinction: Additional Mathematics (By Topic)*". Singapore: Shinglee Publishers Pte Ltd.
4. Sadler, A.J. and Thorning, D.W.S. (1987). "Understanding Pure Mathematics". UK: Oxford University Press.

TEACHING TO THE BIG IDEA …

Lesson sequence in the unit								
Student Learning Outcomes	Dimensions (Please tick the appropriate boxes)							
	FUNCTIONS F	INVARIANCE I	NOTATIONS N	DIAGRAMS D	MEASURES M	EQUIVALENCE E	PROPORTIONALITY P	MODELS M
Multiplication and division of polynomials								
Identities								
Remainder and factor theorem								
Factorising polynomials and solving cubic equations								
Partial Fractions								

1.1 – Polynomials & Identities

Topical Key Understandings

1. Students will be able to **define** the following key mathematical terms
   - polynomials
   - coefficients
   - identity

2. Students will also be able to **differentiate** between
   - coefficients and constants
   - equations and identity
   - equations and expressions

Basic terms

What are the similarities and differences between equation, expression and identity?

- A ***polynomial*** is a mathematical expression (or ***algebraic expression***) involving a sum of terms of the form ax^n , n is a non negative integers

- Thus, algebraic expressions consisting a collection of terms, all of the form ax^n, where a is the coefficient of x^n and $n \in Z^+$, i.e. n is a positive integer, are polynomials ('poly' means many and 'nomials' meaning numbers or terms).

- Polynomials can be arranged systematically, either in ascending order of size of powers of x, or in descending order of size of such powers.

  For example,

$$6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x + 1$$

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5.$$

> **THINK!**
> What is the main difference in expressing a polynomial in either ascending or descending power of x?

Note: $x + \frac{1}{x}$, $\sqrt{x+1}$, $x^{\frac{4}{3}}$ are not polynomials, since not all the powers of x are positive integers.

Question 1

Determine whether each of the following functions are polynomials and provide reasons to support your conclusion. Circle Y or N. power of x is $\frac{1}{2}$.

(a) $3x^2 - 4x^3 + 2x - 0.5$ Y / N	(b) $3x - \frac{1}{x}$ Y / N (power of x is -1)	(c) $8x^3 - 2x + 3\sqrt{x} + 5$ Y / N
(d) $3x^3 + 4x + 2 - 7x^{-2}$ Y / N (negative power)	(e) $5x^3 - 4x^2 - y + 8$ Y / N (multi-variate polynomial)	

Polynomials are often denoted as P(x), Q(x), f(x), g(x) and so on. We evaluate a polynomial by substituting in the given value of the variable.

Question 2

Given that $P(x) = 2x^5 + 7x^4$, find the value of $P(x)$ when

(i) When $x = 3$,

$P(3) = 2(3)^5 + 7(3)^4$

$= 1053$

(ii) When $x = -2$,

$P(-2) = 2(-2)^5 + 7(-2)^4$

$= 48$

The **highest order power** in a polynomial is known as the d egree (or **order**) of the polynomial.

Question 3

Polynomial	Degree of polynomial
$x^7 + 8$	7
$2x^5 + 7x^4$	5
$9 + x$	1

Addition and Subtraction of Polynomials

We can add and subtract polynomials by **combining like terms.**

Question 4

Consider the polynomials $P(x) = x^2 + x + 1$ and $Q(x) = 2x^2 - 3x + 2$, find

(a) $Q(x) - P(x)$

(b) $P(x) + 2Q(x)$

(a) $Q(x) - P(x)$

$= (2x^2 - 3x + 2) - (x^2 + x + 1)$

$= x^2 - 4x + 1$

(b) $P(x) + 2Q(x)$

$= (x^2 + x + 1) - 2(2x^2 - 3x + 2)$

$= -3x^2 + 7x - 3$

Multiplication of Polynomials
We can **multiply** two polynomials by using the ***distributive law***.

Question 5a
Expand (i) $(7x-3)(2x^2+4x-1)$ (ii) $(x^2+3x+2)(8x^2-5x-4)$.

(i) $(7x-3)(2x^2+4x-1)$

$= 14x^3+28x^2-7x-6x^2-12x+3$

$= 14x^3+22x^2-19x+3$

(ii) $(x^2+3x+2)(8x^2-5x-4)$

$= 8x^4-5x^3-4x^2+24x^3-15x^2-12x+16x^2-10x-8$

$= 8x^4+19x^3-3x^2-22x-8$

Question 5b
Without expanding the expression, find the coefficient of x^2 in the expansion of

(i) $(7x-3)(2x^2+4x-1)$

$= (7x)(4x)-3(2x^2)+\cdots$

$= 22x^2+\cdots$

coefficient of $x^2 = 22$

(ii) $(x^2+3x+2)(8x^2-5x-4)$.

$= -4x^2+3(-5)x^2+2(8x^2)+\cdots$

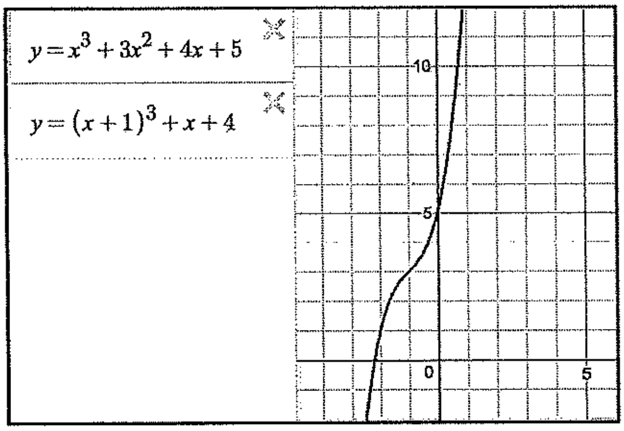
$= -3x^2+\cdots$

coefficient of $x^2 = -3$

A ***polynomial equation*** (***algebraic equation***) is an equation of the form $P(x) =$ ___0___, where $P(x)$ is a polynomial. So far, **solving** equations would yield a fixed number of solutions.

Consider the equation $x^3 + 3x^2 + 4x + 5 = (x + 1)^3 + x + 4$.

Now, we will need to interpret the algebraic solution GRAPHICALLY using **Desmos.**



What do you notice?

NUMERICALLY, we can assign values to the variable x – for instance

$$\text{LHS: } x^3 + 3x^2 + 4x + 5 = (x + 1)^3 + x + 4 \text{ : RHS}$$

When $x = -1$, LHS = 3 RHS = 3

When $x = 0$, LHS = 5 RHS = 5

When $x = 2$, LHS = 33 RHS = 33

Whatever value x takes, the equation is always valid. i.e., when the __left__ side of an equation is exactly identical to the __right__ side of the equation for all values of x, such equation is an *i*dentity and we use the symbol " $\equiv$ " to denote it.

Notation: An **identity**

$x^3 + 3x^2 + 4x + 5 = (x + 1)^3 + x + 4$ for *all values of* x,

or, $x^3 + 3x^2 + 4x + 5 \equiv (x + 1)^3 + x + 4$.

Note that $3x^2 + 4x = x + 5$ is NOT an identity. Explain.

Examples of identities:

- $x^2+4x \equiv x(x+4)$
- $(a \pm b)^2 \equiv a^2 \pm 2ab + b^2$
- $(a+b)(a-b) \equiv a^2 - b^2$

When an identity is given, you are often required to find values of the unknown constants. In such calculations, the "=" sign is used. Usually, one or both of the following methods are employed:

(1)	comparing the coefficients of like terms,
(2)	substitution of suitable values for x.

Example
Given $3x^2 + 5x - 7 = 3x^2 + ax + 2x + b - 3$ for all real values of x, find the values of a and b.

By comparing coefficients,

$a+2=5$

$a=3$.

$b-3=-7$

$b=-4$

Example 1

Find the values of A and B in the following identities.

$$7x^2 - 17x - 12 \equiv A(x-3)(x+1) + B(x-3)$$

comparing the coefficients of **like** terms	substitution of suitable values for x.

comparing the coefficients of like terms

$7x^2 - 17x - 12 \equiv A(x-3)(x+1) + B(x-3)$

RHS $= A(x-3)(x+1) + B(x-3)$

$= A(x^2 - 2x - 3) + B(x-3)$

$= Ax^2 - 2Ax + Bx - 3A - 3B$

$= Ax^2 + (B-2A)x - (3A+3B)$

comparing coefficients:

$A = 7$

$B - 2A = -17$

$B - 2(7) = -17$

$B = -3$

$\therefore A = 7, B = -3$

substitution of suitable values for x.

When $x = 0$

$7(0)^2 - 17(0) - 12 = A(-3)(1) + B(-3)$

$-3A - 3B = -12$

$A + B = 4$ ——— (1)

When $x = -1$,

$7(-1)^2 - 17(-1) - 12 = A(-4)(0) + B(-4)$

$-4B = 12$

$B = -3$ —— (2)

subs (2) into (1): $A = 7$.

$\therefore A = 7, B = -3$

Classwork 1

1. Find the values of A, B and C in the following identities.
   $4x^2+3x-7 \equiv A(x-1)(x+3)+B(x-1)+C$

2. For all values of x, $3x^3+5x^2-4x-3 \equiv (Ax+2)(x+B)(x-1)+C$. Find the values of A, B and C.

1. When $x=1$,

$4(1)^2+3(1)-7 = 0+0+C$

$C=0$

When $x=-3$

$4(-3)^2+3(-3)-7 = 0-4B+C$

$-4B=20$

$B=-5$

When $x=0$

$4(0)^2+3(0)-7 = A(-1)(3)+B(-1)+C$

$-3A-B+C=-7$

$-3A-(-5)+0=-7$

$A=4$

$\therefore A=4,\ B=-5,\ C=0$

2. $(Ax+2)(x+B)(x-1)+C$

$= Ax^3+(-A+AB+2)x^2+(-AB-2+2B)x-2B+C$

$= 3x^3+5x^2-4x-3$

comparing coefficients

of x^3:

$A=3$

of x^2:

$(-A+AB+2)=5$

$(-3+3B+2)=5$

$3B=6$

$B=2$

constant term:

$-3=-2B+C$

$-3=-2(2)+C$

$C=1$

$\therefore A=3,\ B=2,\ C=1$

Checklist for Self-Assessment on Polynomials and Partial Fractions		
I am unsure of…	I am able to…	
☐	☐	Add, subtract and multiply polynomials
☐	☐	Find unknowns in identities using substitution method or comparing coefficients of the corresponding terms and/ or constants on both sides of the identity.

A Math Assignment 03A Polynomials and Identities

A Math Textbook: Marshall Cavendish Additional Math 360 (2nd Edition) Volume A

Tier A

- Textbook Exercise 4.1 (pages 73 to 75): Questions 2(b), 3(c), 4(b), 4(e), 4(f)

Tier B

- Textbook Exercise 4.1 (pages 73 to 75): Questions 8, 9, 10(b), 10(d)

Tier C

- Textbook Exercise 4.1 (pages 73 to 75): Questions 19

1.2 Division of Polynomials

Topical Key Understandings

1. Students will be able to translate performing numeric LONG DIVISION to its equivalent form for ALGEBRAIC long division.

2. Students will be able to apply SYNTHETIC DIVISION in place of long division and will be able to articulate the limitations of synthetic division method.

3. Students will be able to express algebraic division into the form

   Dividend ≡ Divisor × Quotient + Remainder

Essential Question

Why is the remainder's degree <u>at most one less than</u> that of the divisor?

Long Division Method

Recall: Long division for integers.

$$
\begin{array}{r}
146 \\
7\overline{)1025} \\
\underline{-7} \\
32 \\
\underline{-28} \\
45 \\
\underline{-42} \\
\underline{3}
\end{array}
$$

Quotient

Divisor

Dividend

Remainder

Dividend = Divisor X Quotient + Remainder

The process of long division for polynomials:

Steps for Long Division

Division of Polynomials

Let's compare the long division of numerals and polynomials:

$$\begin{array}{r} 4\,7 \\ 5\,\overline{)\,2\,3\,7} \\ \underline{2\,0} \\ 3\,7 \\ \underline{3\,5} \\ 2 \end{array} \qquad \begin{array}{r} 2x + 3 \\ x + 1\,\overline{)\,2x^2 + 5x + 5} \\ \underline{2x^2 + 2x\phantom{{}+5}} \\ 3x + 5 \\ \underline{3x + 3} \\ 2 \end{array}$$

Step 1	$\frac{23}{5} \approx 4$	$\frac{2x^2}{x} = 2x$
Step 2	Place 4 directly above 3	Place 2x directly above an x term in the dividend
Step 3	$4 \times 5 = 20$	$2x(x + 1) = 2x^2 + 2x$
Step 4	20 is placed directly below 23	$(2x^2 + 2x)$ is placed directly below $(2x^2 + 5x)$
Step 5	$23 - 20 = 3$	$(2x^2 + 5x) - (2x^2 + 2x) = 3x$
Step 6	Bring 7, the next number from the dividend, down	Bring (+ 5), the next number from the dividend, down
Step 7	$\frac{37}{5} \approx 7$ Place 7 directly above 7	$\frac{3x}{x} = 3$ Place 3 directly above the constant term in the dividend
Step 8	$7 \times 5 = 35$	$3(x + 1) = 3x + 3$
Step 9	35 is place directly below 37	$(3x + 3)$ is place directly below $(3x + 5)$
Step 10	$37 - 35 = 2$	$(3x + 5) - (3x + 3) = 2$

Example 2

Using LONG DIVISION, find the quotient and remainder of the following divisions

(a) $(6x^3 + 31x^2 + 25x - 12 \div (x + 4)$

(b) $(2x^4 + 6x^3 - 17x^2 + 4x - 2) \div (x^2 + 2x - 1)$

(c) $(2x^3 + 4x - 5) \div (x - 1)$

(a)

$$\begin{array}{r l}
 & 6x^2 + 7x - 3 \\
x+4 \,\big) & 6x^3 + 31x^2 + 25x - 12 \\
 & -(6x^3 + 24x^2) \\
\hline
 & 7x^2 + 25x \\
 & -(7x^2 + 28x) \\
\hline
 & -3x - 12 \\
 & -(-3x - 12) \\
\hline
 & 0
\end{array}$$

Quotient $= 6x^2 + 7x - 3$

Remainder $= 0$

(b)

$$\begin{array}{r l}
 & 2x^2 + 2x - 19 \\
x^2+2x-1 \,\big) & 2x^4 + 6x^3 - 17x^2 + 4x - 2 \\
 & -(2x^4 + 4x^3 - 2x^2) \\
\hline
 & 2x^3 - 15x^2 + 4x \\
 & -(2x^3 + 4x^2 - 2x) \\
\hline
 & -19x^2 + 6x - 2 \\
 & -(-19x^2 - 38x + 19) \\
\hline
 & 44x - 21
\end{array}$$

Quotient $= 2x^2 + 2x - 19$

Remainder $= 44x - 21$

(c)

$$\begin{array}{r l}
 & 2x^2 + 2x + 6 \\
x-1 \,\big) & 2x^3 + 0x^2 + 4x - 5 \\
 & -(2x^3 - 2x^2) \\
\hline
 & 2x^2 + 4x \\
 & -(2x^2 - 2x) \\
\hline
 & 6x - 5 \\
 & -(6x - 6) \\
\hline
 & 1
\end{array}$$

Quotient $= 2x^2 + 2x + 6$

Remainder $= 1$

Synthetic Division
If the DIVISOR happens to be LINEAR, we will be able to simplify the division process by simply performing SYNTHETIC DIVISION (see next page):

Steps for Synthetic Division (Source: http://www.purplemath.com/modules/synthdiv.htm)

Example: $(x^2 + 5x + 6) \div (x - 1)$.

First, write the coefficients ONLY inside an upside-down division symbol:

$$\begin{array}{|rrr} 1 & 5 & 6 \\ & & \\ \hline \end{array}$$

Make sure you leave room inside, underneath the row of coefficients, to write another row of numbers later.

Put the test zero, $x = 1$, at the left:

$$\begin{array}{r|rrr} 1 & 1 & 5 & 6 \\ & & & \\ \hline \end{array}$$

Take the first number inside, representing the leading coefficient, and carry it down, unchanged, to below the division symbol:

$$\begin{array}{r|rrr} 1 & 1 & 5 & 6 \\ & \downarrow & & \\ \hline & 1 & & \end{array}$$

Multiply this carry-down value by the test zero, and carry the result up into the next column:

$$\begin{array}{r|rrr} 1 & 1 & 5 & 6 \\ & & 1 & \\ \hline & 1 & & \end{array}$$

Add down the column:

$$\begin{array}{r|rrr} 1 & 1 & 5 & 6 \\ & & 1 & \\ \hline & 1 & 6 & \end{array}$$

Multiply the previous carry-down value by the test zero, and carry the new result up into the last column:

$$\begin{array}{r|rrr} 1 & 1 & 5 & 6 \\ & & 1 & 6 \\ \hline & 1 & 6 & \end{array}$$

Add down the column:
This last carry-down value is the remainder.

$$\begin{array}{r|rrr} 1 & 1 & 5 & 6 \\ & & 1 & 6 \\ \hline & 1 & 6 & 12 \end{array}$$

Comparing, you can see that we got the same result from the synthetic division, the same quotient (namely, $1x + 6$) and the same remainder at the end (namely, 12), as when we did the long division:

$$
\begin{array}{r|l}
 & x + 6 \\
\hline
x - 1 & x^2 + 5x + 6 \\
 & x^2 - x \\
\hline
 & \quad 6x + 6 \\
 & \quad 6x - 6 \\
\hline
 & \qquad 12
\end{array}
\qquad\qquad
\begin{array}{r|rrr}
1 & 1 & 5 & 6 \\
 & & 1 & 6 \\
\hline
 & 1 & 6 & 12
\end{array}
$$

Example 3

Using SYNTHETIC DIVISION, find the quotient and value of the remainder of the following divisions

a) $(6x^3 + 31x^2 + 25x - 12) \div (x + 4)$

b) $(x^3 - 7x^2 + 24x - 9) \div (x + 3)$

(a)

$$
\begin{array}{r|rrrr}
 & x^3 & x^2 & x & c \\
-4 & 6 & 31 & 25 & -12 \\
 & & -24 & -28 & 12 \\
\hline
 & 6 & 7 & -3 & 0
\end{array}
$$

Quotient $= 6x^2 + 7x - 3$

Remainder $= 0$

(b)

$$
\begin{array}{r|rrrr}
 & x^3 & x^2 & x & c \\
-3 & 1 & -7 & 24 & -9 \\
 & & -3 & 30 & -162 \\
\hline
 & 1 & -10 & 54 & -171
\end{array}
$$

Quotient $= x^2 - 10x + 54$

Remainder $= -171$

which of these can we use synthetic division?

Classwork 2

Find the quotient and remainder of the following divisions

a) Divide $2x^6 - 3x^4 - 27$ by $x^2 - 3$

b) Divide $6x^3 + 5x^2 - 8x - 3$ by $2x + 3$

c) Divide $2x^3 - 7x^2 - 9x + 36$ by $x - 3$

(a)

$$2x^4 \quad + 9x^2 + 27$$

$$x^2+0x-3 \overline{)\, 2x^6 + 0x^5 - 3x^4 + 0x^3 + 0x^2 + x - 27}$$

$$-(2x^6 + 0 - 12x^4)$$

$$9x^4 + 0x^3 + 0x^2$$

$$-(9x^4 + 0x^3 - 27x^2)$$

$$27x^2 + x - 27$$

$$-(27x^2 + 0x - 81)$$

$$54$$

Quotient: $2x^4 + 9x^2 + 27$

Remainder: 54

(b)

$$3x^2 - 2x - 1$$

$$2x+3 \overline{)\, 6x^3 + 5x^2 - 8x - 3}$$

$$-(6x^3 + 9x^2)$$

$$-4x^2 - 8x$$

$$-(-4x^2 - 6x)$$

$$-2x - 3$$

$$-2x - 3$$

$$0$$

Quotient $= 3x^2 - 2x - 1$

Remainder $= 0$

(c)

$$\begin{array}{r|rrrr} 3 & 2 & -7 & -9 & 36 \\ & & 6 & -3 & -36 \\ \hline & 2 & -1 & -12 & 0 \end{array}$$

Quotient: $2x^2 - x - 12$

Remainder $= 0$

THE DIVISION ALGORITHM

Dividend $\equiv$ Divisor $\times$ Quotient $\div$ Remainder

Note the symbol used is "$\equiv$" not "$=$", because it is an ***identity***.

<u>Example</u>
When $(2x^3 + 4x - 5) \div (x - 1)$, quotient $= (2x^3 + 2x + 6)$ and remainder $= 1$. Thus, we have
$2x^3 + 4x - 5 \equiv (x - 1)(2x^3 + 2x + 6) + 1$

Similarly, consider polynomial of <u>degree 4</u>:

$$f(x) = x^4 - 2x^3 + 5x^2 - 4$$

Expressing the terms in a different form,

$$x^4 - 2x^3 + 5x^2 - 4 \equiv (x^2 - 1)(x^2 - 2x + 6) - 2x + 2;$$

<u>Observe the relationship</u>
Fill in the blanks.

i. $2x^4 + 13x^3 + 4x^2 - 13x - 6 \equiv (x+2)(2x^3 + 9x^2 - 14x + 15) - 36$

Divisor $= x + 2$ Degree of divisor = 1

Quotient = $2x^3 + 9x^2 - 14x + 15$ Degree of quotient = 3

Remainder = -36 Degree of remainder = 0

Dividend = $2x^4 + 13x^3 + 4x^2 - 13x - 6$ Degree of dividend = 4

ii. $2x^4 + 13x^3 + 4x^2 - 13x - 6 \equiv (x^2 + 1)(2x^2 + 13x + 2) - 26x - 8$

Divisor $= x^2 + 1$ Degree of divisor = 2

Quotient = $2x^2 + 13x + 2$ Degree of quotient = 2

Remainder = $-26x - 8$ Degree of remainder = 1

Dividend = $2x^4 + 13x^3 + 4x^2 - 13x - 6$ Degree of dividend = 4

iii. $2x^4+13x^3+4x^2-13x-6 \equiv (x^2-1)(2x^2+13x+6)$

Divisor = x^2-1 Degree of divisor = 2

Quotient = $2x^2+13x+6$ Degree of quotient = 2

Remainder = 0 Degree of remainder = 0

Dividend = $2x^4+13x^3+4x^2-13x-6$ Degree of dividend = 4

iv. $2x^4+13x^3+4x^2-13x-6 \equiv (2x^3-3x^2-2x+3)(x+8)+30x^2-30$

Divisor = $2x^3-3x^2-2x+3$ Degree of divisor = 3

Quotient = $x+8$ Degree of quotient = 1

Remainder = $30x^2-30$ Degree of remainder = 2

Dividend = $2x^4+13x^3+4x^2-13x-6$ Degree of dividend = 4

v. $2x^4+13x^3+4x^2-13x-6 \equiv (2x^3+13x^2+4x+12)(x)-25x-6$

Divisor = $2x^3+13x^2+4x+12$ can this be simplified further? Degree of divisor = 3

Quotient = x Degree of quotient = 1

Remainder = $-25x-6$ Degree of remainder = 1

Dividend = $2x^4+13x^3+4x^2-13x-6$ Degree of dividend = 4

What is the relationship between the degrees of divisor, quotient, remainder and dividend?

- The degree (highest index of x) of the remainder is always less than the degree of the divisor. Otherwise, division is still possible.
- The degree of the divisor and the degree of the quotient add up to the degree of the dividend.

Question 6

(a) $x^4 + 2x^3 - 2x^2 - 2x + 4 \equiv (x+2)Q(x)$:

degree of remainder = ___0___ and degree of quotient, $Q(x)$ = ___1___.

(b) $x^4 + 2x^3 - 2x^2 - 2x + 4 \equiv (x^2 + x + 3)P(x) + x + 22$:

degree of remainder = ___1___ and degree of quotient, $P(x)$ = ___2___.

(c) $x^4 + 2x^3 - 2x^2 - 2x + 4 \equiv (x^2 - 1)A(x) + 3$:

degree of remainder = ___0___ and degree of quotient, $A(x)$ = ___2___.

(d) $x^4 + 2x^3 - 2x^2 - 2x + 4 \equiv (x^3 - x^2 + x - 1)B(x) - 4x + 7$:

degree of remainder = ___1___ and degree of quotient, $B(x)$ = ___3___.

Checklist for Self-Assessment on Polynomials and Partial Fractions		
I am unsure of…	**I am able to…**	
☐	☐	Divide one polynomial by another using the division algorithm.
☐	☐	Find the remainder and quotient of a polynomial using synthetic division.

A Math Assignment 03B Division of Polynomials

A Math Textbook: Marshall Cavendish Additional Math 360 (2nd Edition) Volume A

Tier A

- Textbook Exercise 4.1 (pages 73 to 75): Questions 5, 6

Tier B

- Textbook Exercise 4.1 (pages 73 to 75): Questions 12, 14, 15, 18

Tier C

- Textbook Exercise 4.1 (pages 73 to 75): Questions 20, 21

1.3 REMAINDER THEOREM

Topical Key Understandings

- If the DIVISOR is not a factor of the DIVIDEND, a non-zero REMAINDER will be left. i.e. if $f(a) \neq 0 \Rightarrow (x - a)$ is NOT a factor of the polynomial
- If the divisor is LINEAR, remainder will be a CONSTANT.
- The degree of the remainder will always be one degree less that that of the divisor.
- If you equate the divisor to zero and determine the value of x for that specific divisor and substitute it into the dividend, you will obtain the value of the remainder.

Essential Question

If $(x - a)$ produces a remainder of A and $(x - b)$ produces a remainder of B, does the remainder of $(x - a)(x - b)$ produces a remainder of AB? Why?

Short answer: No. Since $(x-a)(x-b)$ is quadratic, it will produce a linear remainder (i.e. $px+q$)

Investigation

Dividend $\equiv$ Divisor $\times$ Quotient $\div$ Remainder

Given that $f(x) = x^3 - 2x^2 + 2x - 1$, find by long division/ synthetic division,

(a) the remainder when $f(x)$ is divided by

(i) $x - 1$,

$$\begin{array}{r|rrrr} 1 & 1 & -2 & 2 & -1 \\ & & 1 & -1 & 1 \\ \hline & 1 & -1 & 1 & 0 \end{array}$$

Remainder $= 0$

(ii) $x - 2$.

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 2 & -1 \\ & & 2 & 0 & 4 \\ \hline & 1 & 0 & 2 & 3 \end{array}$$

Remainder $= 3$

(b) Evaluate

(i) $f(1) = (1)^3 - 2(1)^2 + 2(1) - 1$
$= 0$

(ii) $f(2) = (2)^3 - 2(2)^2 + 2(2) - 1$
$= 3$

Findings

Comparing the results obtained, we notice that when $f(x) = x^3 - 2x^2 + 2x - 1$ is divided by $x - 1$ and $x - 2$ respectively, the values of the remainders are the ___same___ as those obtained by calculating the values of ___f(1) and f(2)___ respectively.

Calculating the value of a function is **faster** than getting the remainder by long division. Thus, when $f(x) = x^3 - 2x^2 + 2x - 1$ is divided by $x - 1$, the remainder = f(1) = ___0___, and when $f(x) = x^3 - 2x^2 + 2x - 1$ is divided by $x - 2$, the remainder = f(2) = ___3___.

Therefore, when a polynomial $f(x)$ is divided by $(x - c)$, the remainder is $f(c)$.

**

In general, the **Remainder Theorem** states that when a polynomial $f(x)$ is divided by a linear divisor, the remainder is $f\left(\frac{b}{a}\right)$.

Fill in the following table. Given a function f(x),

Divisor	**Remainder**
$x-3$	f(3)
$x+3$	$f(-3)$
$2x-1$	$f(\frac{1}{2})$
$3x+2$	$f(-\frac{2}{3})$

Using the division algorithm, show the proof of the Remainder Theorem when a polynomial $f(x)$ is divided by a linear divisor $(ax-b)$.

Let $f(x) \equiv (ax-b)\,Q(x) + R$

when $x = \frac{b}{a}$, $f(\frac{b}{a}) = [a(\frac{b}{a}) - b]\,Q(\frac{b}{a}) + R$

$= 0 + R$

$= R$

$\therefore f(\frac{b}{a}) = R$ #

Example 3

(a) Find the remainder when $x^3 + 2x - 5$ is divided by $x + 3$.

(b) Find the value of k if $5x^3 - 4x^2 + (k+1)x - 5k$ has a remainder of -2 when divided by $x + 2$.

(a) Remainder

$= (-3)^3 + 2(-3) - 5$

$= -38$ #

(b) $5(-2)^3 - 4(-2)^2 + (k+1)(-2) - 5k = -2$

$-40 - 16 - 2k - 2 - 5k = -2$

$7k = -56$

$k = -8$ #

Classwork 3

(a) The expression $8x^3 - 4x^2 + 2px - q$ leaves a remainder of -19 when divided by $x + 1$ and a remainder of 2 when divided by $2x - 1$. Find the values of p and of q.

(b) When the expression $3x^3 + px^2 + qx + 8$ is divided by $(x^2 - 3x + 2)$, the remainder is $5x + 6$. Find the values of p and q.

(a) $8(-1)^3 - 4(-1)^2 + 2p(-1) - q = -19$

$-8 - 4 - 2p - q = -19$

$2p + q = 7$ ——— (1)

$8(\frac{1}{2})^3 - 4(\frac{1}{2})^2 + 2p(\frac{1}{2}) - q = 2$

$1 - 1 + p - q = 2$

$p - q = 2$ ——— (2)

(1) + (2): $3p = 9$

$p = 3$ #

$q = 1$ #

(b) $3x^3 + px^2 + qx + 8 \equiv (x^2 - 3x + 2)Q(x) + (5x + 6)$

$3x^2 + px^2 + qx + 8 \equiv (x - 2)(x - 1)Q(x) + (5x + 6)$

When $x = 2$

$3(2)^3 + p(2)^2 + q(2) + 8 = 0 + 5(2) + 6$

$4p + 2q + 32 = 16$

$4p + 2q = -16$

$2p + q = -8$ ——— (1)

When $x = 1$

$3(1)^3 + p(1)^2 + q(1) + 8 = 0 + 5(1) + 6$

$p + q + 11 = 11$

$p + q = 0$ ——— (2)

(1) − (2): $p = -8$ #

$q = 8$ #

RECALL: THE DIVISION ALGORITHM

Dividend $\equiv$ Divisor $\times$ Quotient $\div$ Remainder

Note the symbol used is "≡" not "=", because it is an ***identity***.

- The degree (highest index of x) of the remainder is always less than the degree of the divisor. Otherwise, division is still possible.

- The degree of the divisor and the degree of the quotient add up to the degree of the dividend.

When a polynomial f(x) is divided by a **linear divisor**, the degree of the divisor = 1.

Then the possible degree(s) of the remainder is/are 0. Hence, when a polynomial f(x) is divided by a linear divisor, the remainder will always be a constant.

For example, if $f(x) \equiv (2x + 3)Q(x) + R$, then R must be a constant.

When a polynomial f(x) is divided by a **quadratic divisor**, the degree of the divisor = 2.

Then the possible degree(s) of the remainder is/are 0 or 1. Hence, when a polynomial f(x) is divided by a quadratic divisor, the remainder can be a constant or a linear polynomial.

For example, if $f(x) \equiv (3x^2 - 4)Q(x) + R(x)$, then R($x$) can be in the form $ax+b$.

When a polynomial f(x) is divided by a **cubic divisor**, the degree of the divisor = 3.

Then the possible degree(s) of the remainder is/are 0 or 1 or 2.

Hence, when a polynomial f(x) is divided by a cubic divisor, the remainder can be a constant or a linear polynomial or a quadratic polynomial

For example, if $f(x) \equiv (x^3 + 1)Q(x) + R(x)$, then $R(x)$ can be in the form ax^2+bx+c

Example 4

For Questions I and II, recall what we learnt in the subtopic of Identities.

I. Given that $x^5 + ax^3 + 2x^2 - b = (x + 1)(x - 1)Q(x) + x + 3$ for all real values of x, where $Q(x)$ is a polynomial,

(a) state the degree of $Q(x)$.

(b) calculate the values of a and b.

[(a) 3 (b) $a = 0$, $b = -1$]

(a) degree of $Q(x) = 5-2$
$= 3$

(b) when $x = -1$

$(-1)^5 + a(-1)^3 + 2(-1)^2 - b = 0 + (-1) + 3$

$-1 - a + 2 - b = 2$

$a + b = -1$ ——— (1)

when $x = 1$

$1^5 + a(1)^3 + 2(1)^2 - b = 1 + 3$

$a - b = 1$ ——— (2)

(1)+(2) : $2a = 0$, $a = 0$ #

$b = -1$ #

II. It is given that for all real values of x, $x^{10} - px^3 + q = (x^2 - 1)Q(x) + 4x + 3$ where $Q(x)$ is a polynomial.

(a) State the degree of the polynomial $Q(x)$.

(b) Find the value of p and of q.

[(a) 8 (b) $p = -4$, $q = 2$]

(a) degree $= 10-2$
$= 8$

(b) $x^{10} - px^3 + q \equiv (x+1)(x-1)Q(x) + 4x + 3$

when $x = -1$

$(-1)^{10} - p(-1)^3 + q = 4(-1) + 3$

$1 + p + q = -1$

$p + q = -2$ ——— (1)

when $x = 1$

$(1)^{10} - p(1)^3 + q = 4(1) + 3$

$1 - p + q = 7$

$-p + q = 6$ ——— (2)

(1) + (2): $2q = 4$

$q = 2$

$p = -4$ #

III. Given that $f(x) = 3x^5 - 11x^3 + 30x^2 + 39 = (x-1)(x+3)Q(x) + ax + b$ for all values of x and that $Q(x)$ is a polynomial,

a) find the values of a and of b.

b) find the remainder when $f(x) + 2$ is divided by $x^2 + 2x - 3$.

[(a) $a = 46$, $b = 15$ (b) $46x + 17$]

(a) When $x = 1$

$3(1)^5 - 11(1)^3 + 30(1)^2 + 39 = a(1) + b$

$a + b = 61$ —— (1)

When $x = -3$

$3(-3)^5 - 11(-3)^3 + 30(-3)^2 + 39 = -3a + b$

$-3a + b = -123$ —— (2)

(1) − (2): $4a = 184$

$a = 46$ #

$b = 15$ #

(b) Since $(x-1)(x+3) = x^2 + 2x - 3$.

$f(x) + 2 = (x-1)(x+3) + Q(x) + 46x + (15+2)$

Remainder $= 46x + 17$ #

IV When the expression $3\cancel{x}^{5}$ $+ px^2 + qx + 8$ is divided by $(x^2 - 3x + 2)$, the remainder is $5x + 6$. Find the values of p and q.

[$p = \cancel{-8}$, $q = \cancel{8}$]
-44 44

$3x^3 + px^2 + qx + 8 \equiv (x^2 - 3x + 2)\, Q(x) + (5x + 6)$

$3x^5 + px^2 + qx + 8 \equiv (x-1)(x-2)\, Q(x) + (5x + 6)$

When $x = 1$

$3(1)^5 + p(1)^2 + q(1) + 8 = 5(1) + 6$

$3 + p + q + 8 = 11$

$p + q = 0$ —— (1)

When $x = 2$

$3(2)^5 + p(2)^2 + q(2) + 8 = 5(2) + 6$

$4p + 2q + 104 = 16$

$4p + 2q = -88$

$2p + q = -44$ —— (2)

(2) − (1): $p = -44$ #

$q = 44$ #

Classwork 4 ↗ repeated question (see pg 23 classwork 3)

a) The expression $8x^3 - 4x^2 + 2px - q$ leaves a remainder of -19 when divided by $x + 1$ and a remainder of 2 when divided by $2x - 1$. Find the values of p and of q.

b) Given the identity $x^5 - x^4 - 2x^3 + 2x^2 - 3 = (x + 1)(x - 2)Q(x) + ax + b$ where $Q(x)$ is a polynomial,
   (i) state the degree of $Q(x)$,
   (ii) find the value of a and b.
   (iii) Hence state the remainder when $x^5 - x^4 - 2x^3 + 2x^2 - 3$ is divided by $x^2 - x - 2$.

[(i) 3 (ii) a = 2, $b = 1$ (iii) $2x + 1$]

(a) Let $f(x) = 8x^3 - 4x^2 + 2px - q$.

$f(-1) = 8(-1)^3 - 4(-1)^2 + 2p(-1) - q = -19$

$-8 - 4 - 2p - q = -19$

$2p + q = 7$ —— (1)

$f(\frac{1}{2}) = 8(\frac{1}{2})^3 - 4(\frac{1}{2}) + 2p(\frac{1}{2}) - q$

$1 - 1 + p - q = 2$

$p - q = 2$ —— (2)

(1) + (2) : $3p = 9$

$p = 3$

$q = 1$

(b) (i) degree of $Q(x) = 5 - 2 = 3$

(ii) When $x = -1$

$(-1)^5 - (-1)^4 - 2(-1)^3 + 2(-1)^2 - 3 = -a + b$

$-a + b = -1$ —— (1)

When $x = 2$,

$2^5 - 2^4 - 2(2)^3 + 2(2)^2 - 3 = 2a + b$

$2a + b = 5$ —— (2)

(2) − (1): $3a = 6$

$a = 2$

$b = 1$

(iii) $(x+1)(x-2) = (x^2 - x - 2)$

Remainder = $2x + 1$

1.4 The Factor Theorem

Topical Key Understandings
If the DIVISOR is a factor of the DIVIDEND, a ZERO remainder will be left. If $(x - b)$ is a factor of $f(x)$, $f(b) = 0$. If $(ax - b)$ is a factor of $f(x)$, $f\left(\frac{b}{a}\right) = 0$.

Essential Question

1. How do you think that the factor theorem can help us solve non-linear and non-quadratic equations?

2. Do you think that there is a general formula to solving cubic and higher-order polynomial equations?

The ***Factor Theorem*** states that

**

If a polynomial $f(x)$ is divided by $(x - a)$, and that if $f(a) = 0$, then $(x - a)$, is a factor of $f(x)$.

Conversely,

if $(x - a)$ is a factor of $f(x)$, then $f(a) = 0$ and $f(x)$ is divisible by $(x - a)$.

i.e. $\boxed{(x - a) \text{ is a factor of } f(x) \leftrightarrow f(a) = 0}$

In general,

$$\boxed{(ax - b) \text{ is a factor of } f(x) \leftrightarrow f\left(\frac{b}{a}\right) = 0}$$

Note: When $(ax - b)$ is a factor of $f(x)$, $f(x)$ is **exactly divisible** by $(ax - b)$.

Example 5

a) Verify that $x + 2$ is a factor of $f(x) = 2x^3 - 3x^2 - 11x + 6$ and factorise $f(x)$ completely.

b) Given that $x^2 - x - 2$ and $x^3 + kx^2 - 10x + 6$ have a common factor, find the possible values of k.

c) Given that $x + 2$ is a common factor of $ax^2 + (a + k)x + 6$ and $(k - a)x^2 + 4x - a$, find the values of a and k.

(a) $f(-2) = 2(-2)^3 - 3(-2)^2 - 11(-2) + 6$

$= -16 - 12 + 22 + 6$

$= 0$

$\therefore (x+2)$ is a factor of $f(x)$

$2x^3 - 3x^2 - 11x + 6 = (x+2)(\underline{\ ?\ })$

$= (x+2)(2x^2 - 7x + 3)$

$= (x+2)(2x-1)(x-3)$

Method 1

$$\begin{array}{r|rrrr} -2 & 2 & -3 & -11 & 6 \\ & & -4 & 14 & -6 \\ \hline & 2 & -7 & 3 & 0 \end{array}$$

$\therefore 2x^3 - 3x^2 - 11x + 6 = (x+2)(2x^2 - 7x + 3)$

Method 2

Long Division

Method 3

Let $2x^3 - 3x^2 - 11x + 6 = (x+2)(ax^2 + bx + c)$

comparing coefficient of:

x^3: $a = 2$

constant: $2c = 6 \Rightarrow c = 3$

x^2: $b + 2a = -3$

$b = -7$

$\therefore 2x^3 - 3x^2 - 11x + 6 = (x+2)(2x^2 - 7x + 3)$

(b) $x^2 - x - 2 = (x-2)(x+1)$

when $x = 2$:

$2^3 + k(2)^2 - 10(2) + 6 = 0$

$4k = 6$

$k = \frac{3}{2}$

when $x = -1$

$(-1)^3 + k(-1)^2 - 10(-1) + 6 = 0$

$k = -15$

$\therefore k = \frac{3}{2}$ or -15

(c) when $x = -2$

$a(-2)^2 + (a+k)(-2) + 6 = 0$

$4a - 2a - 2k + 6 = 0$

$2a - 2k = -6$

$a - k = -3$

$a = k - 3$ ——— (1)

$(k-a)(-2)^2 + 4(-2) - a = 0$

$4k - 4a - 8 - a = 0$

$4k - 5a = 8$ ——— (2)

subs (1) into (2):

$4k - 5(k-3) = 8$

$-k + 15 = 8$

$k = 7$, $a = 4$

Classwork 5

a) Given that $x+1$ and $x-3$ are factors of the expression $x^4+px^3+5x^2+5x+q$, find the values of p and q. Hence, find the other two factors of the expression.

b) If x^2-2x-3 is a factor of the expression $x^4+px^3+qx-81$, find the values of p and q. Hence, factorise the expression completely.

(a) when $x=-1$

$(-1)^4+p(-1)^3+5(-1)^2+5(-1)+q=0$

$-p+q=-1$ —— (1)

when $x=3$,

$3^4+p(3)^3+5(3)^2+5(3)+q=0$

$27p+q=-141$ —— (2)

(2) − (1): $28p=-140$

$p=-5$

$q=-6$ ✱

$x^4-5x^3+5x^2+5x-6$

$=(x+1)(x-3)(ax^2+bx+c)$

comparing coefficients:

x^4: $a=1$

constant: $-3c=-6$

$c=2$

x: $-3c+c-3b=5$

$-4-3b=5$

$b=3$

$\therefore x^4-5x^3+5x^2+5x-6$

$=(x+1)(x-3)(x^2+3x+2)$

$=(x+1)(x-3)(x+1)(x+2)$

$=(x+1)^2(x+2)(x-3)$

Different methods shown.

(b) $x^2-2x-3=(x-3)(x+1)$

when $x=3$,

$3^4+(3^3)p+3q-81=0$

$27p+3q=0$

$9p+q=0$ —— (1)

when $x=-1$

$(-1)^4+(-1)^3p+(-1)q-81=0$

$-p-q=80$ —— (2)

(1) + (2): $8p=80$

$p=10$

$q=-90$ ✱

$$
\begin{array}{r l}
 & x^2+12x+27 \\
x^2-2x-3 \,\big) & x^4+10x^3+0x^2-90x-81 \\
 & -(x^4-2x^3-3x^2) \\
\hline
 & 12x^3+3x^2-90x \\
 & -(12x^3-24x^2-36x) \\
\hline
 & 27x^2-54x-81 \\
 & -(27x^2-54x-81) \\
\hline
 & 0
\end{array}
$$

$\therefore x^4+10x^3-90x-81=(x^2-2x-3)(x^2+12x+27)$

$=(x+1)(x-3)(x+3)(x+9)$ ✱

Checklist for Self-Assessment on Polynomials and Partial Fractions		
I am unsure of…	**I am able to…**	
☐	☐	Use the Remainder Theorem to find the remainder when an expression is divided by a linear factor
☐	☐	Use the Factor Theorem to find the remainder when an expression is divided by a linear factor

The other 2 factors are $(x+1)$ and $(x+2)$ ✱

Exercise: Spot the Mistake!

Question
The expression $ax^3 + ax^2 + bx - 12$ has $x - 3$ as a factor and gives a remainder of -4 when divided by $x + 1$. Find the values of a and b.

Incorrect Solution	**Identify** and **describe** mistake	**Correct** Solution
$f(3)=a(3)^3+a(3)^2+b(3)-12$	Define f(x) first	Let $f(x) = ax^3+ax^2+bx-12$
$=36a+3b-12$ $=12a+b-4$	Expressions cannot be divided by 3 throughout	$f(3)=36a+3b-12=0$ $36a+3b=12$
$12a+b=4$ -------- (1)	Expression to equation is unclear	$12a+b=4$ —— (1)
$f(-1)=a(-1)^3+a(-1)^2+b(-1)-12$ $=2a-b-12$	manipulation error	$f(-1)=a(-1)^3+a(-1)^2+b(-1)-12$ $=-b-12$
$2a-b-12=-4$ $2a-b=-16$ ----------- (2)		$-b-12=-4$ $b=-8$
(1)+(2): $12a+b+2a-b=4+(-16)$ $14a=-12$ $a=-\frac{6}{7}$		Subs $b=-8$ into (1) $a=1$ $\therefore a=1,\ b=-8$
Sub $a=-\frac{6}{7}$ into (1): $12\left(-\frac{6}{7}\right)+b=4$		
$b=14.3$ (3sf)	Answer is exact at $14\frac{2}{7}$. should not round off to 3 s.f.	

A Math Assignment 03C The Remainder Theorem and The Factor Theorem

A Math Textbook: Marshall Cavendish Additional Math 360 (2nd Edition) Volume A

Tier A

- Textbook Exercise 4.2 (pages 78 to 79): Questions 1, 3, 4, 5

Tier B

- Textbook Exercise 4.2 (pages 78 to 79): Questions 9, 10, 8, 14, 16, 18, 20, 21

Tier C

- Textbook Exercise 4.2 (pages 78 to 79): Questions 24

Past Years GCE 'O' Level Questions from Ten Year Series (TYS)

1. N10/I/1

The function f is defined by $f(x) = x^4 - x^3 + kx - 4$, where k is a constant.

(i) Given that $x - 2$ is a factor of $f(x)$, find the value of k. [2]

(ii) Using the value of k found in part **(i)**, find the remainder when $f(x)$ is divided by $x + 2$. [2]

2. N09/I/1

The expression $2x^3 + ax^2 + bx + 3$, where a and b are constants, has a factor of $x - 1$ and leaves a remainder of 15 when divided by $x + 2$. Find the value of a and of b. [4]

1.5 FACTORISATION OF CUBIC EXPRESSIONS

Topical Key Understandings

- That in order to factorise cubic equations, there is a need to first find a linear factor.
- The quotient of dividing the cubic expression with a linear factor will be quadratic, which is easily factorisable.

Essential Questions

- Can a cubic equation have no real roots? Why?
- Can this process work for the factorisation of higher-order polynomials?
- Is there a general formula for the solving of cubic equation?

Factorising Cubic Expressions using Special Identities

Recall the following algebraic identities:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$a^2 - b^2 = (a-b)(a+b)$$

2 new algebraic identities (2 new golden rules):

Sum of Cubes $\quad a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Difference of Cubes $\quad a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Example 6

Use the above algebraic identities to **factorise** each of the following.

(a) $x^3 + 27$ (b) $8x^3 - 27y^3$ (c) $216 - (3x-4)^3$

(a) $x^3 + 27$

$= (x+3)(x^2 - 3x + 9)$

(b) $8x^3 - 27y^3$

$= (2x)^3 - (3y)^3$

$= (2x-3y)(4x^2 + 6xy + 9y^2)$ ※

(c) $216 - (3x-4)^3$

$= 6^3 - (3x-4)^3$

$= [6 - (3x-4)][6^2 + 6(3x-4) + (3x-4)^2]$

$= (10-3x)(36 + 18x - 24 + 9x^2 - 24x + 16)$

$= (10-3x)(9x^2 - 6x + 28)$ ※

Factorising Cubic Expressions

Investigation:
Consider the general form of a cubic expression is $ax^3 + bx^2 + cx + d$:

The polynomial is of degree $\underline{\quad 3 \quad}$ (the highest power is $\underline{\quad 3 \quad}$).

- If $f(x) = 2x^3 + 11x^2 - 7x - 6$, find $f(1)$. $f(1) = 2(1)^3 + 11(1) - 7(1) - 6 = 0$
- What is the remainder when $2x^3 + 11x^2 - 7x - 6$ is divided by $x - 1$? Remainder $= 0$
- Hence, write $f(x)$ as $(x - 1)(ax^2 + bx + c) + d$. $f(x) = (x-1)(2x^2 + 13x + 6)$
- Find the factors of $f(x)$.

We see that $2x^3 + 11x^2 - 7x - 6 = \underline{(x-1)(2x+1)(x+6)}$ ie. there are $\underline{\quad 3 \quad}$ linear factors.

It is also possible that a cubic expression has $\underline{\quad 1 \quad}$ linear factors.
e.g. $x^3 + 3x^2 + 3x + 2 = (x + 2)(x^2 + x + 1)$. The quadratic factor $(x^2 + x + 1)$ cannot be further $\underline{\text{factorised}}$.

Test $x - 1$, $x + 1$, $x + 2$, for factors of $f(x) = 3x^3 - 2x^2 - 5x + 3$.

Think of a systematic way of factorising a cubic expression?

1. Find a linear factor of $f(x)$
2. Divide $f(x)$ by the linear factor using synthetic/long division to find the other quadratic factor.
3. Factorise the quadratic factor if possible.

1.6 SOLVING CUBIC EQUATIONS

In solving equations, the factorised expression is equated to zero, and the solutions obtained accordingly.

Example 7

Factorise $2x^3 + 3x^2 - 5x - 6$ completely.

let $f(x) = 2x^3 + 3x^2 - 5x - 6$

When $x = -1$,

$f(-1) = 2(-1)^3 + 3(-1)^2 - 5(-1) - 6$

$= 0$

$(x+1)$ is a factor of $f(x)$

$2x^3 + 3x^2 - 5x - 6$

$= (x+1)(2x^2 + x - 6)$

$= (x+1)(2x-3)(x+2)$ ※

Solve the equation $2x^3 + 3x^2 - 5x - 6 = 0$.

$2x^3 + 3x^2 - 5x - 6 = 0$

$(x+1)(2x-3)(x+2) = 0$

$x = -1$ or $x = \frac{3}{2}$ or $x = -2$ ※

Example 8

Factorise $x^3 - 10x^2 + 28x - 16$ completely.

Let $f(x) = x^3 - 10x^2 + 28x - 16$

Let $x = 4$

$f(4) = 4^3 - 10(4)^2 + 28(4) - 16$

$= 0$

$(x-4)$ is a factor of $f(x)$

$\therefore x^3 - 10x^2 + 28x - 16$

$= (x-4)(x^2 - 6x + 4)$

Solve the equation $x^3 - 10x^2 + 28x - 16 = 0$.

$x^3 - 10x^2 + 28x - 16 = 0$

$(x-4)(x^2 - 6x + 4) = 0$

$x = 4$ or $x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(4)}}{2}$

$x = 4$ or $x = \dfrac{6 \pm \sqrt{20}}{2}$

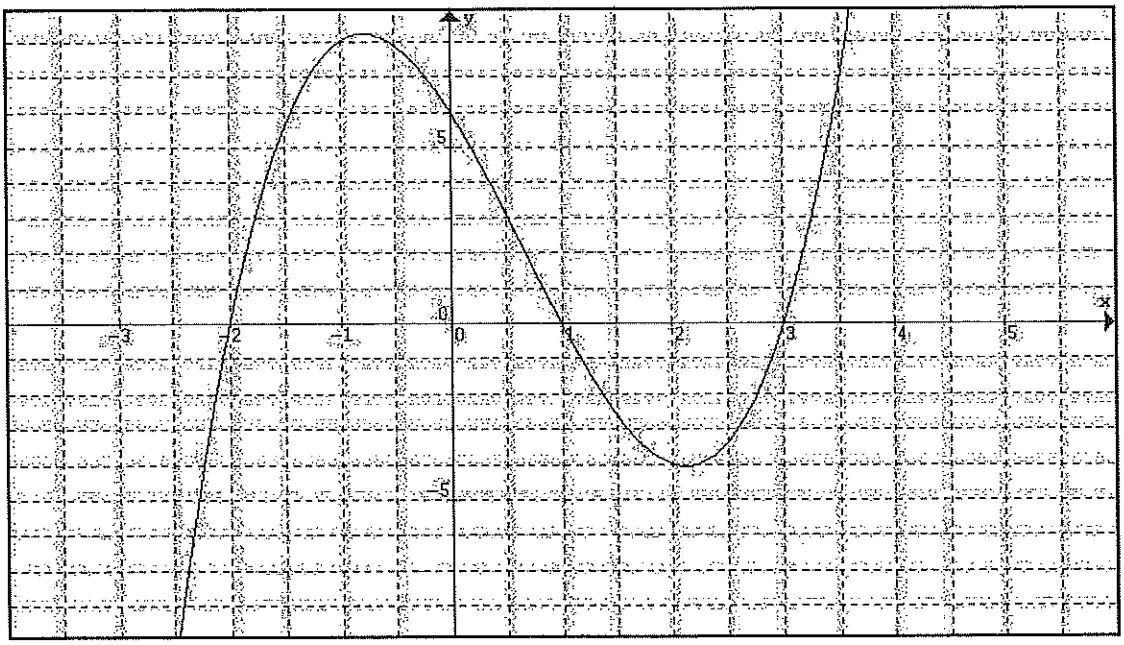
$x = 4$ or $x = 3 + \sqrt{5}$ or $x = 3 - \sqrt{5}$

$x = 4$ or $x = 5.24$ (3 sf) or $x = 0.764$ (3 sf)

Graphs of cubic functions

The graph of the cubic equation $y = (x-1)(x+2)(x-3)$ will have x-intercepts of 1, 3 and -2 and a y-intercept of $(0-1)(0+2)(0-3) = 6$.

The graph will look as follows: Analyse the graph and compare it with the solutions to $(x-1)(x+2)(x-3) = 0$. What do you notice?



The solutions to $(x-1)(x+2)(x-3) = 0$ are $x=1$, $x=-2$, $x=3$.

These are the x-intercepts on the graph.

Example 9

N08/II/5

(a) The term containing the highest power of x in the polynomial $f(x)$ is $2x^4$. Two of the roots of the equation $f(x) = 0$ are -1 and 2. Given that $x^2 - 3x + 1$ is a quadratic factor of $f(x)$, find (i) an expression for $f(x)$ in descending powers of x,

(ii) the number of real roots of the equation $f(x) = 0$, justifying your answer,

(iii) the remainder when $f(x)$ is divided by $2x - 1$.

[(i) $f(x) = 2x^4 - 8x^3 + 4x^2 + 10x - 4$ (ii) 4 (iii) $1\frac{1}{8}$]

(b) (i) Solve the cubic equation $2w^3 - w^2 - 8w + 4 = 0$.

(ii) Hence, or otherwise, solve the equation $2(y + 3)^3 = (y + 3)^2 + 8y + 20$.

[(i) w = -2, 0.5, 2 (ii) $y = -5$ or -1]
or $y = -\frac{5}{2}$

(a) (i) $f(x) = 2(x+1)(x-2)(x^2-3x+1)$

$= (2x^2-2x-4)(x^2-3x+1)$

$= 2x^4-6x^3+2x^2-2x^3+6x^2-2x-4x^2+12x-4$

$= 2x^4-8x^3+4x^2+10x-4$ ∦

(ii) Two of the roots are -1 and 2.

For quadratic factor x^2-3x+1

Discriminant $= (-3)^2-4(1)(1)$

$= 5 > 0$

so $x^2-3x+1=0$ has 2 other roots.

so $f(x)=0$ has 4 real roots.

(iii) $f(\frac{1}{2}) = 2(\frac{1}{2})^4 - 8(\frac{1}{2})^3 + 4(\frac{1}{2})^2 + 10(\frac{1}{2}) - 4$

$= \frac{1}{8} - 1 + 1 + 5 - 4$

$= \frac{9}{8}$

Remainder $= \frac{9}{8}$ ∦

(b) (i) let $g(w) = 2w^3 - w^2 - 8w + 4$

$g(2) = 2(2)^3 - 2^2 - 8(2) + 4 = 0$

$(w-2)$ is a factor of $g(w)$.

$g(w) = (w-2)(2w^2+3w-2)$

$= (w-2)(2w-1)(w+2) = 0$

$w=2$ or $w=\frac{1}{2}$ or $w=-2$ ∦

(ii)

$2(y+3)^3 = (y+3)^2 + 8y + 20$

$2(y+3)^3 - (y+3)^2 - 8(y+3) + 4 = 0$

let $w = y+3$

$2w^3 - w^2 - 8w + 4 = 0$

$\therefore w=2$ or $w=\frac{1}{2}$ or $w=-2$

$y+3=2$ or $y+3=\frac{1}{2}$ or $y+3=-2$

$y=-1$ or $y=-\frac{5}{2}$ or $y=-5$ ∦

(c) (i) Solve the equation $2x^3 + 5x^2 - x - 6 = 0$.
(ii) Using your result from (a), solve the equation $16y^3 + 20y^2 - 2y - 6 = 0$.
[(i) 1 or – 1.5 or – 2 (ii) 0.5 or – 0.75 or – 1]

(d) (i) Solve the equation $m^3 + 3m^2 - 6m - 8 = 0$.
(ii) Hence, solve $\frac{1}{x^3} + \frac{3}{x^2} - \frac{6}{x} - 8 = 0$.
[(i) $m = -1, -4, 2$ (ii) $x = -1, -0.25, 0.5$]

(c) (i) $2x^3+5x^2-x-6=0$

$2(1)^3+5(1)^2-1-6=0$

$(x-1)$ is a factor of $2x^3+5x^2-x-6$

$2x^3+5x^2-x-6$

$=(x-1)(2x^2+7x+6)$

$=(x-1)(2x+3)(x+2)$

$=0$

$x=1$ or $x=-\frac{3}{2}$ or $x=-2$

(ii) $16y^3+20y^2-2y-6=0$

$2(2y)^3+5(2y)^2-(2y)-6=0$

let $x=2y$

$2x^3+5x^2-x-6=0$

$x=1$ or $x=-\frac{3}{2}$ or $x=-2$

$2y=1$ or $2y=-\frac{3}{2}$ or $2y=-2$

$y=\frac{1}{2}$ or $y=-\frac{3}{4}$ or $y=-1$

(d) (i) $m^3+3m^2-6m-8=0$

$(-1)^3+3(-1)^2-6(-1)-8=0$

$\therefore (m+1)$ is a factor of m^3+3m^2-6m-8

$m^3+3m^2-6m-8=(m+1)(m^2+2m-8)$

$=(m+1)(m+4)(m-2)=0$

$m=-1$ or $m=-4$ or $m=2$.

(ii) Replace m with $\frac{1}{x}$.

$(\frac{1}{x})^3+3(\frac{1}{x})^2-6(\frac{1}{x})-8=0$

$\frac{1}{x^3}+\frac{3}{x^2}-\frac{6}{x}-8=0$

$\frac{1}{x}=-1$ or $\frac{1}{x}=-4$ or $\frac{1}{x}=2$

$\therefore x=-1$ or $x=-\frac{1}{4}$ or $x=\frac{1}{2}$

(e) Solve the equation $2x^4 - 19x^3 + 61x^2 - 74x + 24 = 0$.

Let $f(x) = 2x^4 - 19x^3 + 61x^2 - 74x + 24$

$f(2) = 0 \Rightarrow (x-2)$ is a factor of $f(x)$

$f(3) = 0 \Rightarrow (x-3)$ is a factor of $f(x)$

$f(4) = 0 \Rightarrow (x-4)$ is a factor of $f(x)$

$\therefore \quad 2x^4 - 19x^3 + 61x^2 - 7x + 24 = (x-2)(x-3)(x-4)(2x-1)$

$= 0$

$x = 2,\ x = 3,\ x = 4$ and $x = \frac{1}{2}$

Classwork 9

(a) A cubic polynomial f(x) is such that the coefficient of x^3 is 2 and the roots of the equation f(x) = 0 are 0.5, 2 and k. Given that f(x) has a remainder of 4 when divided by $x - 1$, find

(i) the value of k.

(ii) the expression of $f(x)$.

(iii) the remainder when $f(x)$ is divided by $x + 1$.

[(i) 5 (ii) $2x^3 - 15x^2 + 27x - 10$ (iii) -54]

(a) (i) $f(x) = (2x-1)(x-2)(x-k)$

$f(1) = (1)(-1)(1-k) = 4$

$k - 1 = 4$

$k = 5$

(ii) $f(x) = (2x-1)(x-2)(x-5)$

$= 2x^3 - 10x^2 - 4x^2 + 20x - x^2 + 5x + 2x - 10$

$= 2x^3 - 15x^2 + 27x - 10$

(iii) Remainder $= f(-1)$

$= (2(-1)-1)(-1-2)(-1-5)$

$= (-3)(-3)(-6)$

$= -54$

(b) (i) Factorise the expression $4x^3 - 9x^2 - 10x + 3$ completely.
(ii) Hence solve the equation $4(y + 1)^3 - 9(y + 1)^2 = 10y + 7$.
[(i) $(x + 1)(4x - 1)(x - 3)$ (ii) $y = -2$ or -0.75 or 2]

(c) A polynomial $f(x)$ is given by $f(x) = ax^3 - 11x^2 + bx + 20$, where a and b are constants. $x^2 - 3x - 10$ is a factor of $f(x)$.
(i) Find the values of a and b.
(ii) Solve the equation $f(x) = 0$
(iii) Hence, solve the equation $24u^3 - 44u^2 - 48u + 20 = 0$.
[(i) $a = 3$, $b = -24$ (ii) $-2, 5, \frac{2}{3}$ (iii) $-1, \frac{5}{2}, \frac{1}{3}$]

b(i) Let $f(x) = 4x^3 - 9x^2 - 10x + 3$

$f(-1) = 4(-1)^3 - 9(-1)^2 - 10(-1) + 3 = 0$

$\therefore$ $(x+1)$ is a factor of $f(x)$

$f(x) = 4x^3 - 9x^2 - 10x + 3$

$= (x+1)(4x^2 - 13x + 3)$

$= (x+1)(4x-1)(x-3)$

(ii) $4(y+1)^3 - 9(y+1)^2 - 10y - 7 = 0$

$4(y+1)^3 - 9(y+1)^2 - 10(y+1) + 3 = 0$

Let $y + 1 = x$.

$x = -1$, $x = \frac{1}{4}$ or $x = 3$

$y + 1 = -1$, $y + 1 = \frac{1}{4}$, $y + 1 = 3$

$y = -2$, $y = -\frac{3}{4}$, $y = 2$ #

c (i) $(x^2 - 3x - 10) = (x-5)(x+2)$

$f(-2) = a(-2)^3 - 11(-2)^2 + b(-2) + 20 = 0$

$-8a - 44 - 2b + 20 = 0$

$8a + 2b = -24$

$4a + b = -12$ —— (1)

$f(5) = a(5)^3 - 11(5)^2 + b(5) + 20 = 0$

$125a - 275 + 5b + 20 = 0$

$125a + 5b = 255$

$25a + b = 51$ —— (2)

(2) − (1) : $21a = 63$

$a = 3$ #

$b = -24$ #

(ii) $f(x) = 3x^3 - 11x^2 - 24x + 20$

$= (x+2)(x-5)(3x-2) = 0$

$x = -2$, $x = 5$ or $x = \frac{2}{3}$ #

Checklist for Self-Assessment on Polynomials and Partial Fractions		
I am unsure of…	**I am able to…**	
☐	☐	Use $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
☐	☐	Factorise cubic expressions or polynomials of higher degree using the Factor theorem.
☐	☐	Solve cubic equations.
☐	☐	Solve problems involving cubic equations.

(iii) $24u^3 - 44u^2 - 48u + 20 = 0$

$3(2u)^3 - 11(2u)^2 - 24(2u) + 20 = 0$

Let $2u = x$

$3x^3 - 11x^2 - 24x + 20 = 0$

$x = -2$, $x = 5$, $x = \frac{2}{3}$

$u = -1$, $u = \frac{5}{2}$, $u = \frac{1}{3}$ #

A Math Assignment 03D Cubic Polynomials and Equations

A Math Textbook:Marshall Cavendish Additional Math 360 (2nd Edition) Volume A

Tier A

- Textbook Exercise 4.3 (pages 84 to 85): Questions 1, 2, 4(b), 4(d)

Tier B

- Textbook Exercise 4.3 (pages 84 to 85):
  Questions 6(a), 6(d), 7, 8(b), 10, 13, 14, 17, 18, 20, 21

B3. Given that $f(x) = 2x^3 - 9x^2 + ax + b$, where a and b are constants, is exactly divisible by $(x + 1)$ and leaves a remainder of 3 when divided by $(x - 2)$.

(i) Show that the values of a and b are 4 and 15 respectively.

(ii) Factorise f (x) completely.

(iii) Hence, solve the equation $2(k - 7)^3 - 9(k - 7)^2 + 4k - 13 = 0$.

[(ii) $(x+1)(2x-5)(x-3)$ (iii) k=6, 9.5 or 10]

B4.

(a) Solve the equation $2x^3 - 9x^2 + 7x + 6 = 0$.

(b) Hence, solve the equation $27y^3 - 81y^2 + 42y + 24 = 0$.

(c) Find the remainder when $2x^3 - 9x^2 + 7x + 6$ is divided by $2x - 3$.

[(a) $x = 2, 3, -0.5$ (b) $y = \frac{4}{3}, 2, -\frac{1}{3}$ (c)3]

B5.

(a) Factorize the expression $4x^3 - 5x^2 - 23x + 6$ completely.

(b) Hence, or otherwise, solve $\frac{1}{2}x^3 - \frac{5}{4}x^2 - \frac{23}{2}x + 6 = 0$.

[(a) $(x + 2)(x - 3)(4x - 1)$ (b) -4, 6 or 0.5]

B6.

It is given that $f(x) = 3x^3 - 5x^2 - 11x - 3$.

(i) Find the remainder when f(x) is divided by $x+3$.

(ii) Show that $x - 3$ is a factor of $f(x)$.

(iii) Hence, solve the equation f$(x) = 0$.

(iv) By using an appropriate substitution, solve the equation $-3y^3 - 11y^2 - 5y + 3 = 0$.

[(i) -96 (iii) x=3,-1/3 ,-1 (iv) x=1/3, -3, -1]

Tier C

- Textbook Exercise 4.3 (pages 84 to 85): Question 22, 23

Past Years GCE 'O' Level Questions from Ten Year Series (TYS)

1. N05/II/9

The function $f(x) = x^3 - 6x^2 + ax + b$, where a and b are constants, is exactly divisible by $x - 3$ and leaves a remainder of -55 when divided by $x + 2$.

(i) Find the value of a and of b. [4]

(ii) Solve the equation $f(x) = 0$. [4]

2. N13/II/3

The function $f(x) = x^3 + ax + b$, where a and b are constants, is exactly divisible by $x + 3$. Given that $f(x)$ leaves a remainder of 56 when divided by $x - 4$,

(i) find the value of a and of b. [4]

(ii) determine, showing all necessary working, the number of real roots of the equation $f(x) = 0$. [4]

1.7 Partial Fractions

Proper & Improper Fractions

Definitions: A rational fraction is a function of the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials in x, where $Q(x) \neq 0$.

If the degree of $P(x) <$ the degree of $Q(x)$, then $\frac{P(x)}{Q(x)}$ is a proper fraction.

If the degree of $P(x) \geq$ the degree of $Q(x)$, then $\frac{P(x)}{Q(x)}$ is an improper fraction.

Examples

No.	Proper Algebraic Fractions	Improper Algebraic Fractions
(a)	$\frac{1}{5-x}$	$-\frac{x^3+8}{4x^2-x+6}$
(b)	$\frac{8-x}{x^2-2x+3}$	$\frac{x^4-x^3+2x}{9-5x}$
(c)	$-\frac{4-2x+3x^4}{2x^5+7}$	$\frac{x}{x+2}$

Are the following fractions proper or improper? Fill in the table.

Fraction	Proper or Improper?
i. $\frac{3x^2+2x}{x+1}$	Improper
ii. $\frac{(x+1)(x-6)}{x^2+3x+6}$	Improper
iii. $\frac{x^2+3x+6}{(x+1)(x^2+4)}$	Proper
iv. $\frac{x^5+1}{x^3(x+2)}$	Improper

I. Division Algorithm for Polynomials

Recall long division of numbers: E.g. Divide 7 by 3.

$$\begin{array}{r} 2 \leftarrow \text{quotient} \\ \text{divisor} \rightarrow 3\overline{)\ 7} \leftarrow \text{dividend} \\ \underline{-6} \\ 1 \leftarrow \text{remainder} \end{array}$$

We can express 7 in terms of the divisor, quotient and remainder as follows:

$$\begin{array}{ccccccccc} 7 & = & 3 & \times & 2 & + & 1 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{dividend} & = & \text{divisor} & \times & \text{quotient} & + & \text{remainder} \end{array}$$

This is known as the Division Algorithm for Positive Integers.

The **Division Algorithm for Polynomials** states that:

$$\begin{array}{ccccccc} P(x) & = & D(x) & \times & Q(x) & + & R(x) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{dividend} & = & \text{divisor} & \times & \text{quotient} & + & \text{remainder} \end{array}$$

where **degree of R(x) < degree of D(x).**

Recall long division of polynomials:
E.g. Divide x^3+2x-7 by $x-2$.

$$\begin{array}{r} x^2+2x+6 \leftarrow \text{quotient} \\ \text{divisor} \rightarrow x-2\overline{)\ x^3+0x^2+2x-7} \leftarrow \text{dividend} \\ \underline{-(x^3-2x^2)} \\ 2x^2+2x \\ \underline{-(2x^2-4x)} \\ 6x-7 \\ \underline{-(6x-12)} \\ 5 \leftarrow \text{remainder} \end{array}$$

By the Division Algorithm for Polynomials, we can express $x^3 + 2x - 7$ as:

$$\begin{array}{ccccccc} x^3+2x-7 & = & (x-2) & & (x^2+2x+6) & + & 5 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{dividend} & = & \text{divisor} & \times & \text{quotient} & + & \text{remainder} \end{array}$$

II. Improper Fractions

An improper fraction can be expressed as a sum of a positive integer and a proper fraction.

e.g. $\frac{7}{3} = 2\frac{1}{3}$

$= 2 + \frac{1}{3}$

$= Quotient + \frac{Remainder}{Divisor}$

$$\begin{array}{r l} & \ 2 \leftarrow \text{quotient} \\ \text{divisor} \rightarrow 3 & \overline{)\ 7} \leftarrow \text{dividend} \\ & \ \ \underline{-6} \\ & \ \ \ \ 1 \leftarrow \text{remainder} \end{array}$$

Generally, another way to write the Division Algorithm is:

$$\boxed{\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}}$$

We use this to express an improper fraction as a **Sum of a Polynomial and a Proper Algebraic Fraction.**

Question

Express the following improper fractions as the sum of a polynomial and a proper fraction.

(a) $\dfrac{8x+3}{2x-1}$ (b) $\dfrac{15x^3+3}{3x^2-1}$ (c) $\dfrac{2x^3-4x^2-5x+3}{x^2-2x+3}$

(a)

$$\begin{array}{r l} & \ 4 \\ 2x-1 & \overline{)\ 8x+3} \\ & \underline{-(8x-4)} \\ & \ \ \ 7 \end{array}$$

$8x+3 = 4 + \frac{7}{2x-1}$

(b)

$$\begin{array}{r l} & \ 5x \\ 3x^2-1 & \overline{)\ 15x^3+0x^2+0x+3} \\ & \underline{-(15x^3 \qquad\quad -5x\ )} \\ & \qquad\qquad\quad 5x+3 \end{array}$$

$\frac{15x^3+3}{3x^2-1} = 5x + \frac{5x+3}{3x^2-1}$

(c)

$$\begin{array}{r l} & \ 2x \\ x^2-2x+3 & \overline{)\ 2x^3-4x^2-5x+3} \\ & \underline{-(2x^3-4x^2+6x\ )} \\ & \qquad\qquad -11x+3 \end{array}$$

$2x^3-4x^2-5x+3 = x^2-2x+3 + \frac{-11x+3}{x^2-2x+3}$

In algebra, we learnt to add or subtract two proper fractions. For example:

$$\frac{3}{x+2}+\frac{2}{x-1}=\frac{3(x-1)+2(x+2)}{(x+2)(x-1)}$$
$$=\frac{5x+1}{(x+2)(x-1)}$$

If we **reverse the process** and express $\frac{5x+1}{(x+2)(x-1)}$ as $\frac{3}{x+2}+\frac{2}{x-1}$, the process is called decomposing $\frac{5x+1}{(x+2)(x-1)}$ into its partial fractions.

The fractions $\frac{3}{x+2}$ and $\frac{2}{x-1}$ are known as the **partial fractions** of $\frac{5x+1}{(x+2)(x-1)}$.

A rational function which may be expressed as a sum of separate fractions is said to be resolved into its partial fractions.

<u>If the rational function is a proper fraction</u>:

The method of obtaining the partial fractions depends on the type of factors in the denominator:

- For every **LINEAR factor** $(ax + b)$ of the denominator, there will be a corresponding partial fraction $\frac{A}{(ax+b)}$.

- For every **REPEATED LINEAR factor** $(ax + b)^2$ of the denominator, there will be two corresponding partial fractions

$$\frac{B}{(ax+b)}+\frac{C}{(ax+b)^2}.$$

- For every **QUADRATIC factor that cannot be factorised** $(ex^2 + fx + g)$ of the denominator, there will be a corresponding partial fraction $\frac{Dx+E}{(ex^2+fx+g)}$.

Summary

Type	Denominator contains	Algebraic Fraction	Partial Fractions	Example
1	Distinct linear factors	$\dfrac{px+q}{(ax+b)(cx+d)}$	$\dfrac{A}{ax+b}+\dfrac{B}{cx+d}$	$\dfrac{7x+4}{(x+3)(x-2)}$ $=\dfrac{A}{x+3}+\dfrac{B}{x-2}$
2	Repeated linear factors	$\dfrac{px+q}{(ax+b)^2}$	$\dfrac{B}{(ax+b)}+\dfrac{C}{(ax+b)^2}$	$\dfrac{7x+4}{(2x+1)(x-2)^2}$ $=\dfrac{A}{2x+1}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$
3	Quadratic factor that cannot be factorised (ex^2+fx+g)	$\dfrac{px+q}{(ex^2+fx+g)^2}$	$\dfrac{Dx+E}{(ex^2+fx+g)}$	$\dfrac{4x+7}{(x+1)(x^2+1)}$ $=\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+1}$

Steps for Finding Partial Fractions

Step 1: Factorise the numerator and denominator to eliminate any common factors.
e.g.

$$\frac{x^2-4x-5}{(x^2+4x+3)(x-1)}=\frac{(x+1)(x-5)}{(x+3)(x+1)(x-1)}=\frac{x-5}{(x+3)(x-1)}$$

Step 2: Check whether the fraction is a proper fraction. → degree of numerator < degree of denominator
If it is an improper fraction, divide and express it as a sum of polynomial and a proper algebraic fraction. Then resolve the proper fraction into its partial fractions.

Step 3: Factorise the denominator completely into its factors.

Step 4: Form an identity between the original fraction and the sum of its partial fractions (the form is dependent on the type of factors in the denominators). Write all the fractions using a common denominator.

Step 5: Write all the fractions using a common denominator.

Step 6: Solve for unknown constants: Compare the two numerators by
- Comparing 'convenient coefficients' on each side of the identity

  and/or
- Substitute values of x which reduce individual factors to zero.

Step 7: CHECK your answer for ACCURACY and PRESENTATION.

Example 10: VARIOUS CASES IN PARTIAL FRACTION.

Case 1: Proper fractions with distinct linear factors in the denominator

Example A: Express $\frac{10x-11}{(x+1)(2x-5)}$ in partial fractions.

[Observations: proper fraction, denominator completely factorised, 2 linear factors]

Solution:

$$\frac{10x-11}{(x+1)(2x-5)} = \frac{A}{x+1} + \frac{B}{2x-5}$$

$$= \frac{A(2x-5)+B(x+1)}{(x+1)(2x-5)}$$

$$10x-11 = A(2x-5) + B(x+1)$$

When $x = \frac{5}{2}$,

$$10\left(\frac{5}{2}\right) - 11 = B\left(\frac{5}{2}+1\right)$$

$$\frac{7}{2}B = 14$$

$$B = 4$$

When $x = -1$,

$$10(-1) - 11 = A(-2-5)$$

$$-21 = -7A$$

$$A = 3$$

$$\therefore \frac{10x-11}{(x+1)(2x-5)} = \frac{3}{x+1} + \frac{4}{2x-5}$$

Check answer:

Subs $x = 0$, $\quad LHS = \frac{10(0)-11}{(0+1)(2\times 0-5)} = \frac{-11}{-5} = \frac{11}{5}$

$RHS = \frac{3}{1} + \frac{4}{-5} = \frac{11}{5}$

<u>Example B</u>: Express $\frac{15x+13}{3x^2-2x-1}$ in partial fractions.

[Observations: proper fraction, denominator **NOT** factorised]

Solution:

$$\frac{15x+13}{3x^2-2x-1} = \frac{15x+13}{(3x+1)(x-1)}$$

$$= \frac{A}{3x+1} + \frac{B}{x-1}$$

$$= \frac{A(x-1)+B(3x+1)}{(3x+1)(x-1)}$$

$15x+13 = A(x-1)+B(3x+1)$

when $x=1$,

$15(1)+13 = B(3+1)$

$4B = 28$

$B = 7$

when $x = -\frac{1}{3}$

$15(-\frac{1}{3})+13 = A(-\frac{1}{3}-1)$

$8 = -\frac{4}{3}A$

$A = -6$

$$\therefore \frac{15x+13}{(3x+1)(x-1)} = -\frac{6}{3x+1} + \frac{7}{x-1}$$ #

Before we consider Type 2, do the following questions.

Express the following as a single fraction in its simplest form.

(i) $\frac{x}{(x-1)^2}+\frac{1}{(x-1)^2}$

$= \frac{x+1}{(x-1)^2}$

(ii) $\frac{1}{x-1}+\frac{2}{(x-1)^2}$

$= \frac{(x-1)}{(x-1)^2}+\frac{2}{(x-1)^2}$

$= \frac{x+1}{(x-1)^2}$ ※

Hence, for proper algebraic fraction containing repeated linear factors in the denominator such as $\frac{px+q}{(ax+b)^2}$, the form of the partial fractions is $\frac{B}{(ax+b)}+\frac{C}{(ax+b)^2}$.

Case 2: Proper algebraic fraction containing repeated linear factors in the denominator

<u>Example C</u>: Express $\frac{-20x-2}{(x+1)(x-2)^2}$ in partial fractions.

[Observations: proper fraction, denominator fully factorised, 1 linear factor and 1 repeated linear factor]

Solution:

$$\frac{-20x-2}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$= \frac{A(x-2)^2 + B(x+1)(x-2) + C(x+1)}{(x+1)(x-2)^2}$$

$$-20x-2 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$$

When $x = 2$,

$$-20(2)-2 = C(2+1)$$

$$3C = -42$$

$$C = -14$$

When $x = -1$

$$-20(-1)-2 = A(-1-2)^2$$

$$9A = 18$$

$$A = 2$$

When $x = 0$,

$$-2 = 2(0-2)^2 + B(0+1)(0-2) - 14(0+1)$$

$$-2 = 8 - 2B - 14$$

$$2B = -4$$

$$B = -2$$

$$\frac{-20x-2}{(x+1)(x-2)^2} = \frac{2}{x+1} - \frac{2}{x-2} - \frac{14}{(x-2)^2} \quad \#$$

Before we consider Type 3, do the following questions.

Express the following as a single fraction in its simplest form.

(i) $\frac{4x+1}{x^2+4} - \frac{3}{x-1}$ $= \frac{(x-1)(4x+1)-3(x^2+4)}{(x^2+4)(x-1)}$

$= \frac{4x^2+x-4x-1-3x^2-12}{(x^2+4)(x-1)}$

$= \frac{x^2-3x-13}{(x^2+4)(x-1)}$ #

(ii) $\frac{8}{x^2+4} - \frac{3}{x-1}$ $= \frac{8(x-1)-3(x^2+4)}{(x^2+4)(x-1)}$

$= \frac{8x-8-3x^2-12}{(x^2+4)(x-1)}$

$= \frac{-3x^2+8x-20}{(x^2+4)(x-1)}$ #

Hence, for proper algebraic fraction containing a quadratic factor which cannot be factorised in the denominator such as $\frac{px+q}{(ex^2+fx+g)^2}$, the form of the partial fractions is $\frac{Dx+E}{(ex^2+fx+g)}$.

Case 3: Proper algebraic fraction containing a quadratic factor which cannot be factorised in the denominator

<u>Example D</u>: Express $\frac{1}{4x(x^2+4)}$ in partial fractions.

[Observations: proper fraction, denominator fully factorised, 1 linear factor and 1 quadratic factor which cannot be factorised]

Solution:

$$\frac{1}{4x(x^2+4)} = \frac{A}{4x} + \frac{Bx+C}{x^2+4}$$

$$= \frac{A(x^2+4) + 4x(Bx+C)}{4x(x^2+4)}$$

$A(x^2+4) + 4x(Bx+C) = 1$

when $x=0$,

$4A = 1$

$A = \frac{1}{4}$

$\frac{1}{4}x^2 + 1 + 4Bx^2 + 4Cx = 1$

$\frac{1}{4}x^2 + 4Bx^2 + 4Cx = 0$

comparing coefficients: $4C = 0$
$C = 0$

$\frac{1}{4} + 4B = 0$

$B = -\frac{1}{16}$

$$\therefore \frac{1}{4x(x^2+4)} = \frac{\frac{1}{4}}{4x} + \frac{-\frac{1}{16}x}{x^2+4}$$

$$= \frac{1}{16x} - \frac{x}{16(x^2+4)}$$

Partial Fractions involving Improper Fractions

Example E: Express $\frac{3x^3-5}{x^2-1}$ in partial fractions.

[Observations: **improper fraction**, denominator **NOT** fully factorised, 2 distinct linear factors]

Solution:

$$\begin{array}{r|l} & 3x \\ x^2-1 & 3x^3+0x^2+0x-5 \\ & -(3x^3 \qquad -3x) \\ \hline & \qquad\qquad 3x-5 \end{array}$$

$$\frac{3x^3-5}{x^2-1} = 3x + \frac{3x-5}{x^2-1}$$

$$\frac{3x-5}{x^2-1} = \frac{3x-5}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$= \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

$A(x-1)+B(x+1) = 3x-5$

When $x=1$,

$2B = -2$

$B = -1$

When $x=-1$,

$-2A = 3(-1)-5$

$-2A = -8$

$A = 4$

$$\therefore \frac{3x^3-5}{x^2-1} = 3x + \frac{4}{x+1} - \frac{1}{x-1}$$

remember this!

Additional Exercise on Partial Fractions

Express the following in partial fractions.

(a) $\dfrac{1+2x}{(1+x)(1-2x^2)}$

(b) $\dfrac{1}{(1-2x)(1-3x)}$

(c) $\dfrac{5x+4}{(x-1)(x^2-2x-8)}$

(d) $\dfrac{3x+1}{(x+1)^2}$

(e) $\dfrac{x^3}{(x+1)(x+2)}$

(f) $\dfrac{10-17x+14x^2}{(2+x)(1-2x)^2}$

(g) $\dfrac{2x-5}{(x+1)(x-2)(2x+3)}$

(h) $\dfrac{x^2+10x+6}{x^2+2x-8}$

(i) $\dfrac{1}{(x+1)(x+2)}$

(j) $\dfrac{5x}{(2x+1)(x^2+1)}$

(k) $\dfrac{7-3x-x^2}{(1-x^2)(2+x)}$

Checklist for Self-Assessment on Polynomials and Partial Fractions		
I am unsure of...	**I am able to...**	
☐	☐	Identify whether an algebraic fraction is a proper fraction or an improper fraction
☐	☐	Express a proper algebraic fraction in partial fractions
☐	☐	Express an improper algebraic fraction in partial fractions

Additional Exercise on Partial Fractions (see attached)

Express the following in partial fractions.

(a) $\dfrac{1+2x}{(1+x)(1-2x^2)}$

(b) $\dfrac{1}{(1-2x)(1-3x)}$

(c) $\dfrac{5x+4}{(x-1)(x^2-2x-8)}$

(d) $\dfrac{3x+1}{(x+1)^2}$

(e) $\dfrac{x^3}{(x+1)(x+2)}$

(f) $\dfrac{10-17x+14x^2}{(2+x)(1-2x)^2}$

(g) $\dfrac{2x-5}{(x+1)(x-2)(2x+3)}$

(h) $\dfrac{x^2+10x+6}{x^2+2x-8}$

(i) $\dfrac{1}{(x+1)(x+2)}$

(j) $\dfrac{5x}{(2x+1)(x^2+1)}$

(k) $\dfrac{7-3x-x^2}{(1-x^2)(2+x)}$

PARTIAL FRACTIONS (Notes pg. 64)

a) $$\frac{1+2x}{(1+x)(1-2x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1-2x^2}$$

$$= \frac{A(1-2x^2) + (Bx+C)(1+x)}{(1+x)(1-2x^2)}$$

$$1+2x \equiv A(1-2x^2) + (Bx+C)(1+x)$$

Let $x=-1$:

$$1+2(-1) = A(1-2(-1)^2)$$
$$-1 = A(-1)$$
$$A = 1$$

Let $x=0$:

$$1 = 1(1-0) + (C)(1)$$
$$1 = 1 + C$$
$$C = 0$$

Let $x=1$:

$$1+2(1) = 1(1-2) + (B)(2)$$
$$3 = -1 + 2B$$
$$B = 2$$

$$\cancel{\frac{1+2x}{(1+x)(1-2x^2)} = \frac{1}{1+x} + \frac{2x}{1-2x^2}}$$

$$\frac{1+2x}{(1+x)(1-2x^2)} = \frac{1}{1+x} + \frac{2x}{1-2x^2}$$

b) $$\frac{1}{(1-2x)(1-3x)} = \frac{A}{1-2x} + \frac{B}{1-3x}$$

$$= \frac{A(1-3x) + B(1-2x)}{(1-2x)(1-3x)}$$

$$1 \equiv A(1-3x) + B(1-2x)$$

Let $x = \frac{1}{3}$:

$$1 = B(1-2(\tfrac{1}{3}))$$
$$B = 3$$

Let $x = \frac{1}{2}$

$$1 = A(1-3(\tfrac{1}{2}))$$
$$A = -2$$

$$\frac{1}{(1-2x)(1-3x)} = -\frac{2}{1-2x} + \frac{3}{1-3x}$$

c) $$\frac{5x+4}{(x-1)(x^2-2x-8)}$$

$$= \frac{5x+4}{(x-1)(x-4)(x+2)}$$

$$= \frac{A}{x-1} + \frac{B}{x-4} + \frac{C}{x+2}$$

$$= \frac{A(x-4)(x+2) + B(x-1)(x+2) + C(x-1)(x-4)}{(x-1)(x-4)(x+2)}$$

$$5x+4 \equiv A(x-4)(x+2) + B(x-1)(x+2) + C(x-1)(x-4)$$

Let $x=1$:

$$5+4 = A(1-4)(1+2)$$
$$A = -1$$

Let $x=4$:

$$5(4)+4 = B(4-1)(4+2)$$
$$B = \frac{4}{3}$$

Let $x=-2$:

$$5(-2)+4 = C(-2-1)(-2-4)$$
$$C = -\frac{1}{3}$$

$$\frac{5x+4}{(x-1)(x^2-2x-8)} = -\frac{1}{x-1} + \frac{4}{3(x-4)} - \frac{1}{3(x+2)}$$

PARTIAL FRACTIONS (notes pg 64)

d) $\dfrac{3x+1}{(x+1)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2}$

$= \dfrac{A(x+1)+B}{(x+1)^2}$

$3x+1 \equiv A(x+1)+B$

Let $x=-1$:

$3(-1)+1 = B$

$B = -2$

Let $x=0$:

$1 = A-2$

$A = 3$

$\dfrac{3x+1}{(x+1)^2} = \dfrac{3}{x+1} - \dfrac{2}{(x+1)^2}$

e) $\dfrac{x^3}{(x+1)(x+2)} = \dfrac{x^3}{x^2+3x+2}$

$$\begin{array}{r l}
 & x - 3 \\
x^2+3x+2 \;\big) & x^3 \\
 & -(x^3+3x^2+2x) \\ \hline
 & -3x^2-2x \\
 & -(-3x^2-9x-6) \\ \hline
 & 7x+6
\end{array}$$

$\dfrac{x^3}{(x+1)(x+2)} = x-3 + \dfrac{7x+6}{(x+1)(x+2)}$

$\dfrac{7x+6}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$

$= \dfrac{A(x+2)+B(x+1)}{(x+1)(x+2)}$

$7x+6 \equiv A(x+2)+B(x+1)$

Let $x=-1$:

$7(-1)+6 = A(-1+2)$

$A=-1$

Let $x=-2$:

$7(-2)+6 = B(-2+1)$

$B = 8$

$\dfrac{x^3}{(x+1)(x+2)} = x-3 - \dfrac{1}{x+1} + \dfrac{8}{x+2}$

f) $\dfrac{10-17x+14x^2}{(2+x)(1-2x)^2}$

$= \dfrac{A}{2+x} + \dfrac{B}{1-2x} + \dfrac{C}{(1-2x)^2}$

$= \dfrac{A(1-2x)^2+B(1-2x)(2+x)+C(2+x)}{(2+x)(1-2x)^2}$

$10-17x+14x^2 \equiv A(1-2x)^2+B(1-2x)(2+x)+C(2+x)$

Let $x=-2$:

$10-17(-2)+14(-2)^2 = A(1-2(-2))^2$

$A = 4$

Let $x=\frac{1}{2}$:

$10-17(\frac{1}{2})+14(\frac{1}{2})^2 = C(2+\frac{1}{2})$

$C = 2$

Let $x=0$:

$10 = 4 + B(1)(2) + 2(2)$

$B = 1$

$\dfrac{10-17x+14x^2}{(2+x)(1-2x)^2} = \dfrac{4}{2+x} + \dfrac{1}{1-2x} + \dfrac{2}{(1-2x)^2}$

g) $\dfrac{2x-5}{(x+1)(x-2)(2x+3)} = \dfrac{A}{x+1} + \dfrac{B}{x-2} + \dfrac{C}{2x+3}$

$= \dfrac{A(x-2)(2x+3)+B(x+1)(2x+3)+C(x+1)(x-2)}{(x+1)(x-2)(2x+3)}$

$2x-5 \equiv A(x-2)(2x+3)+B(x+1)(2x+3)+C(x+1)(x-2)$

Let $x=-1$:

$2(-1)-5 = A(-1-2)(-2+3)$

$A = \frac{7}{3}$

Let $x=2$:

$2(2)-5 = B(2+1)(2(2)+3)$

$B = -\frac{1}{21}$

Let $x=-\frac{3}{2}$:

$2(-\frac{3}{2})-5 = C(-\frac{3}{2}+1)(-\frac{3}{2}-2)$

$-8 = C(\frac{7}{4})$

$C = -\frac{32}{7}$

$\dfrac{2x-5}{(x+1)(x-2)(2x+3)} = \dfrac{7}{3(x+1)} - \dfrac{1}{21(x-2)} - \dfrac{32}{7(2x+3)}$

PARTIAL FRACTIONS (notes pg 64)

h)

$$x^2+2x-8 \overline{\big)\, x^2+10x+6}\quad 1$$
$$-(x^2+2x-8)$$
$$8x+14$$

$$\frac{x^2+10x+6}{x^2+2x-8} = 1 + \frac{8x+14}{x^2+2x-8}$$

$$\frac{8x+14}{x^2+2x-8} = \frac{8x+14}{(x+4)(x-2)}$$

$$= \frac{A}{x+4} + \frac{B}{x-2}$$

$$= \frac{A(x-2)+B(x+4)}{(x+4)(x-2)}$$

$$8x+14 \equiv A(x-2)+B(x+4)$$

Let $x=2$:

$$8(2)+14 = B(2+4)$$
$$B=5$$

Let $x=-4$:

$$8(-4)+14 = A(-4-2)$$
$$A=3$$

$$\frac{x^2+10x+6}{x^2+2x-8} = 1 + \frac{3}{x+4} + \frac{5}{x-2}$$

i)

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$= \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

$$1 \equiv A(x+2)+B(x+1)$$

Let $x=-1$:

$$1 = A(-1+2)$$
$$A=1$$

Let $x=-2$:

$$1 = B(-2+1)$$
$$B=-1$$

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

j)

$$\frac{5x}{(2x+1)(x^2+1)} = \frac{A}{2x+1} + \frac{Bx+C}{x^2+1}$$

$$= \frac{A(x^2+1)+(Bx+C)(2x+1)}{(2x+1)(x^2+1)}$$

$$5x \equiv A(x^2+1)+(Bx+C)(2x+1)$$

Let $x=-\frac{1}{2}$:

$$5(-\tfrac{1}{2}) = A\left((-\tfrac{1}{2})^2+1\right)$$
$$A=-2$$

Let $x=0$:

$$0=(-2)(1)+(C)(1)$$
$$C=2$$

Let $x=1$:

$$5=(-2)(2)+(B+2)(3)$$
$$B=1$$

$$\frac{5x}{(2x+1)(x^2+1)} = -\frac{2}{2x+1} + \frac{x+2}{x^2+1}$$

PARTIAL FRACTIONS (notes pg 64)

k) $\dfrac{7-3x-x^2}{(1-x^2)(2+x)}$

$= \dfrac{7-3x-x^2}{(1+x)(1-x)(2+x)}$

$= \dfrac{A}{1+x} + \dfrac{B}{1-x} + \dfrac{C}{2+x}$

$= \dfrac{A(1-x)(2+x) + B(1+x)(2+x) + C(1+x)(1-x)}{(1+x)(1-x)(2+x)}$

$7-3x-x^2 \equiv A(1-x)(2+x) + B(1+x)(2+x) + C(1+x)(1-x)$

Let $x=-1$:

$7-3(-1)-(-1)^2 = A(1+1)(2-1)$

$A = \dfrac{9}{2}$

Let $x=1$:

$7-3-1 = B(2)(3)$

$B = \dfrac{1}{2}$

Let $x=-2$:

$7-3(-2)-(-2)^2 = C(1-2)(1+2)$

$C = -3$

$$\frac{7-3x-x^2}{(1-x^2)(2+x)} = \frac{9}{2(1+x)} + \frac{1}{2(1-x)} - \frac{3}{2+x}$$

A Math Assignment 03E Partial Fractions

A Math Textbook:Marshall Cavendish Additional Math 360 (2nd Edition) Volume A

Tier A

- Textbook Exercise 4.4 (pages 89 to 91):
  Questions 1(a), 1(d), 2(b), 2(c), 3(a), 3(d), 4

Tier B

- Textbook Exercise 4.4 (pages 89 to 91):
  Questions 5, 6(b), 6(c), 9, 11, 12, 13, 14, 15

Tier C

- Textbook Exercise 4.4 (pages 89 to 91): Question 16

Past Years GCE 'O' Level Questions from Ten Year Series (TYS)

1. N09/II/2(i)

Express $\frac{7}{2x^2-x-6}$ in partial fractions. [3]

(N09/II/2(i))

2. N15/II/8

(i) Find the remainder when $2x^3-3x^2-5$ is divided by $2x+1$. [2]

(ii) Factorise completely the cubic polynomial $2x^3-3x^2+1$. [4]

(iii) Express $\frac{4-5x-8x^2}{2x^3-3x^2+1}$ as the sum of 3 partial fractions. [4]

A Math Assignment 03E Partial Fractions

A Math Textbook: Marshall Cavendish Additional Math 360 (2nd Edition) Volume A

Tier A

- Textbook Exercise 4.4 (pages 89 to 91):
  Questions 1(a), 1(d), 2(b), 2(c), 3(a), 3(d), 4

Tier B

- Textbook Exercise 4.4 (pages 89 to 91):
  Questions 5, 6(b), 6(c), 9, 11, 12, 13, 14, 15

Tier C

- Textbook Exercise 4.4 (pages 89 to 91): Question 16

Past Years GCE 'O' Level Questions from Ten Year Series (TYS)

1. N09/II/2(i)

Express $\frac{7}{2x^2-x-6}$ in partial fractions. [3]

2. N15/II/8

(i) Find the remainder when $2x^3 - 3x^2 - 5$ is divided by $2x + 1$. [2]

(ii) Factorise completely the cubic polynomial $2x^3 - 3x^2 + 1$. [4]

(iii) Express $\frac{4-5x-8x^2}{2x^3-3x^2+1}$ as the sum of 3 partial fractions. [4]

Mathematics Homework Reflection Question	
Your response to the question(s) should be detailed. Please write in complete sentences and be ready to share your response in class.	
1.	What were the main mathematical concepts or ideas that you learned today or that we discussed in class today?
2.	Describe a mistake or misconception that you or a classmate had in class today. What did you learn from this mistake or misconception?
3.	What questions do you still have about ....? If you don't have a question, write a similar problem and solve it instead.