School of Science and Technology, Singapore
Mathematics Department 2023
Secondary 3 Elementary Mathematics

SCHOOL OF SCIENCE AND TECHNOLOGY, SINGAPORE

E

Name: Solutions ( ) Class: ________

E MATH UNIT 04: GEOMETRICAL PROPERTIES OF CIRCLES

a. ENDURING UNDERSTANDING

Students understand that

- diagrams are succinct, visual representations of the real world.
- diagrams are succinct, visual representations of mathematical objects that serve to communicate properties of the objects and facilitate problem solving.
- symmetrical properties of circles stem from the symmetry of the circle.
- angle properties of circles stem from 'Angle at centre is twice angle at circumference'

b. KNOWLEDGE & SKILLS

At the end of the unit, my current levels for the following knowledge and skills are

c. ESSENTIAL QUESTIONS

- How are diagrams (and their features/the information that they contain) used to solve and communicate problems related to the geometrical figures or the real-world objects that they model?
- What is a concentric circle and how is a circle symmetrical?
- What properties follow from 'Angle at center is twice angle at circumference'?

d. COMMON SYMBOLS/LANGUAGE USED

- Concentric circles, chord, minor arc, major arc, minor segment, major segment, cyclic quadrilaterals, tangent, secant

e. RESOURCES

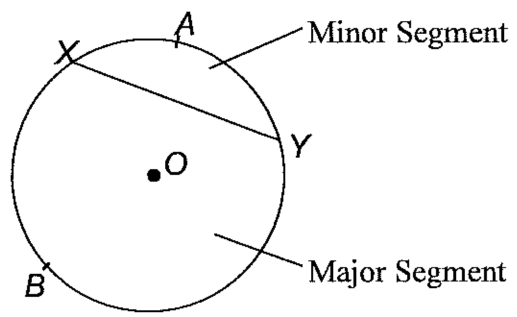
- Yeap B. H., J. Yeo, The K. S., Loh C. Y., I. Chow (2013). "New Syllabus Mathematics". 7th Ed. PP 171 – 241. Singapore: Shinglee Publishers Pte Ltd. Reference Text Chapter 6, 7
- Yeap B. H., J. Yeo, The K. S., Loh C. Y., I. Chow (2013). "New Syllabus Additional Mathematics". 9th Ed. PP 105 – 153. Singapore: Shinglee Publishers Pte Ltd. Reference Text Chapter 4
- Chow, W.K. (2010). "Discovering Additional Mathematics". Singapore: Star Publishing Pte Ltd. PP 21 – 36
- Lee, L.K. (2011). "Pass with Distinction: Additional Mathematics (By Topic)". Singapore: Shinglee Publishers Pte Ltd.
- Sadler, A.J. and Thorning, D.W.S. (1987). "Understanding Pure Mathematics". UK: Oxford University Press.
- Chow W. K. (2007). "Discovering Mathematics 3". Singapore: Star Publishing Pte Ltd.
- Ho S. T. and Khor N. H. (2007). "Additional Mathematics". Singapore: Panpac

TEACHING TO THE BIG IDEA ...

Lesson sequence in the unit								
Student Learning Outcomes	Dimensions (Please tick the appropriate boxes)							
	FUNCTIONS F	INVARIANCE I	NOTATIONS N	DIAGRAMS D	MEASURES M	EQUIVALENCE E	PROPORTIONALITY P	MODELS M
Symmetric properties of circles								
Angle properties of circles								
Properties involving cyclic quadrilaterals								
Properties involving tangents of circles								

4.1 KEY TERMINOLOGY

Here are some of the common terms used in this chapter:



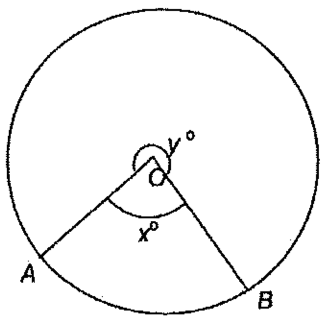
Chord – a line segment joining two points on the circumference of a circle ie XY.

Minor arc – the minor part of the circumference of the circle ie arc XAY.

Major arc – the major part of the circumference of the circle ie arc XBY.

Minor segment – the region bounded by the minor arc and a chord.

Major segment – the region bounded by the major arc and a chord.



$OA = OB$, which is the radius of the circle.

Minor arc AB **subtends** angle x at the centre of the circle.

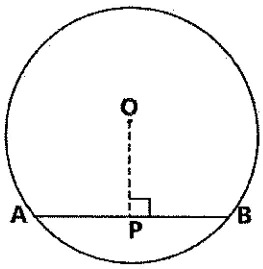
Major arc AB **subtends** angle y at the centre of the circle.

Useful Links: Definitions and Proofs (Circles, sectors and arcs) https://www.bbc.com/bitesize/guides/zqrdxfr/revision/1 (Angles in a circle) https://www.bbc.com/bitesize/guides/zcsgdxs/test (Online quiz) https://www.bbc.com/bitesize/guides/zcsgdxs/test

4.2 SYMMETRIC PROPERTIES OF CIRCLES

<u>Symmetric Property of Circle #1: A straight line from the centre of a circle that bisects a chord is perpendicular to the chord.</u>

* (**Abbreviation**: Perpendicular bisector of chords passes through centre)



Visualising this:

Draw the line from O to P, which is the centre of chord AB. If you pick up another copy of the diagram (labelled A', P', B' and O') and flip it horizontally, you will be able to place O on O', B' on A, A' on B and P' on P. This means hence these angles are right angles.

Let O be the centre of the circle. When $AP = PB$, $\angle OPB = 90°$.

Proof

(Hint: Look for congruent triangles)

Conversely, the perpendicular to a chord drawn from the centre of the circle bisects the chord.

(**Abbreviation**: (perpendicular from centre bisects the chord)

Let O be the centre of the circle. When $\angle OPB = 90°$, $AP = PB$.

Proof

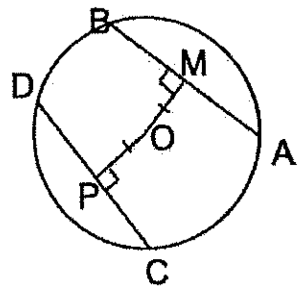
$AP = BP$ (given)
$OP = OP$ (common side)
$OA = OB$ (radius of circle)
$\therefore \triangle OAP \equiv \triangle OBP$ (SSS)
$\therefore \angle OPA = \angle OPB$
$= \frac{180°}{2}$ (adjacent angles on a straight line)
$= 90°$

$OA = OB$ (radius of circle)
$\angle OPA = \angle OPB = 90°$ (given)
$OP = OP$ (common side)
$\therefore \triangle OAP \equiv \triangle OBP$ (SAS)
$\therefore AP = PB$ (shown)

Level	Knowledge & Skills	I need help!	I can do on my own	I can explain to others
1	the perpendicular bisector of a chord passes through the centre			
1	symmetric properties of circles			

Symmetric Property of Circle #2: Equal chords are equidistant from the centre (or centres of equal circles).

(**Abbreviation**: Equal chords are equidistant from the centre)



> **Visualising this:**
>
> If you rotate the circle about O such that A coincides with D, B will also coincide with C. This means that $OM = OP$.

When chords $AB = CD$, $OM = OP$.

Proof

(Hint: Look for congruent triangles)

$AB = CD$ (given)

$\therefore AM = CP$ (perpendicular bisector of chord passes through centre)

$OA = OC$ (radius of circle)

$\angle OMA = \angle OPC = 90^\circ$ (given)

$\triangle OAM \equiv \triangle OCP$ (RHS)

$\therefore OM = OP$ (proven)

Conversely, chords which are equidistant from the centre (or centres of equal circles) are equal in length.

(**Abbreviation**: Chords equidistant from centre are equal)

Proof

$OP = OM$ (given)

$OC = OA$ (radius of circle)

$\angle OPC = \angle OMA = 90^\circ$ (given)

$\therefore \triangle OPC \equiv \triangle OMA$ (RHS)

$\therefore PC = MA$

$DC = 2 \times PC$
$BA = 2 \times MA$ } (perpendicular bisector of chords passes through centre)

$\therefore AB = CD$ (proven)

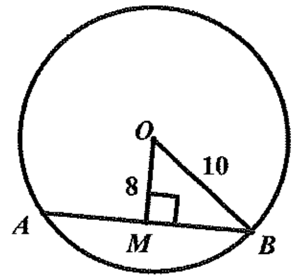
Level	Knowledge & Skills	I need help!	I can do on my own	I can explain to others
1	equal chords are equidistant from the centre			

Note: For all your solutions, you are required to state explicitly the reasons using the appropriate properties of circles.

Example 1

In the diagram, O is the centre of the circle. Find the length of the chord AB.

(Answer: 12 cm)



By Pythagoras' Theorem,

$MB^2 = 10^2 - 8^2$

$\therefore MB = 6$ (-6 rejected)

$AB = MB \times 2$ (perpendicular bisector of chord passes through centre)

$= 6 \times 2$

$= 12$ units

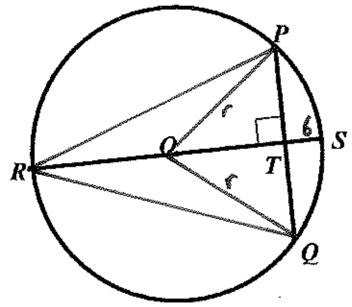
Example 2

In the diagram, the chord PQ is perpendicular to the diameter ROS.

If $PQ = 28$ cm, $TS = 6$ cm. Calculate

(a) the radius of the circle, (b) the area of Triangle PQR.

(Answer: (a) $\frac{58}{3}$ cm, (b) $\frac{1372}{3}$ cm^2)



(a) Let the radius be r cm.

$PT = \frac{28}{2}$ (perpendicular bisector of chord passes through centre)

$= 14$ cm

By Pythagoras' Theorem

$r^2 = (r-6)^2 + 14^2$

$r^2 = r^2 - 12r + 36 + 14^2$

$12r = 232$

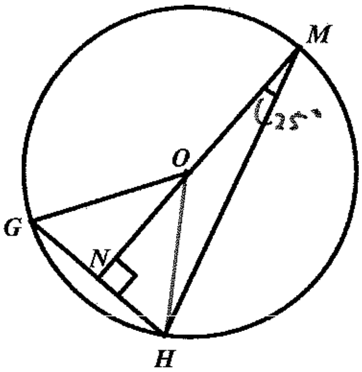
$r = \frac{58}{3}$

radius $= \frac{58}{3}$ cm

(b) Area $= \frac{1}{2} \times (\frac{58}{3} \times 2 - 6) \times 28$

$= \frac{1372}{3}$ cm^2

Example 3
In the diagram, MON is perpendicular to chord GH and O is the centre of the circle. Given that $\angle OMH = 25°$ and $GH = 12$ cm, Find
(a) $\angle OGN$,
(b) OM in exact form.
(Answer: (a) $\cancel{65°}$ $40°$*, (b)* $\frac{6}{\cos \cancel{65°} 40°}$*)*

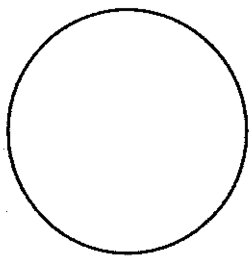


(a) $\angle OHM = 25°$ (base angles of isosceles triangle)
$\angle NHM = 90° - 25°$
$= 65°$ (sum of angles in a triangle)
$\angle OGN = \angle OHN$ (base angles of isosceles triangle)
$= 65° - 25°$
$= 40°$

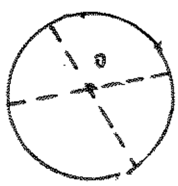
(b) $NH = \frac{GH}{2}$ (perpendicular bisector of chord passes through centre)
$= \frac{12}{2}$
$= 6$ cm
$\frac{NH}{OH} = \cos 40°$
$OH = \frac{NH}{\cos 40°}$
$\therefore OM = OH$ (radius of circle)
$= \frac{NH}{\cos 40°} = \frac{6}{\cos 40°}$

Class Discussion
Given a circle (as shown), how do you locate the centre of the circle,

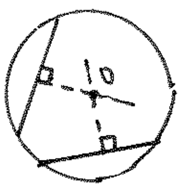
(i) with folding
(ii) without folding



(i) Fold into two twice
centre is at intersection



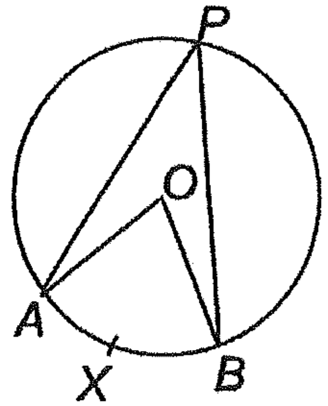
(ii) Draw 2 chords
construct ⊥ bisector of chords
centre is at the intersection of the bisectors



E Maths Textbook: Shinglee New Syllabus Mathematics 3 (7th Edition) Tier A: Exercise 11A (pg 370 - 371) Qn 1, 3 Tier B: Exercise 11A (pg 370 - 371) Qn 7, 9 Tier C: Exercise 11A (pg 370 - 371) Qn 11

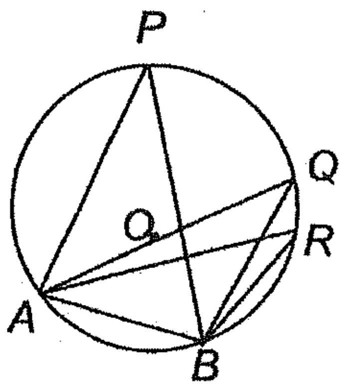
4.3 ANGLE PROPERTIES OF CIRCLES

Angle at the centre & angle at the circumference



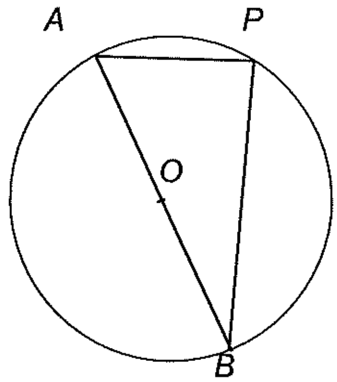
1. A, X and B are 3 points on the circumference of the circle which form an arc.
2. P is another point on the circumference.
3. Arc AXB **subtends** angle AOB at the centre of the circle, O.
4. Arc AXB **subtends** angle APB at the circumference of the circle.
5. Angle AOB is known as the **angle at the centre**.
6. Angle APB is known as the **angle at the circumference**.

Angles in the same segment



1. AB is a chord of a circle with centre O.
2. P, Q and R are 3 points on the circumference.
3. Angles APB, AQB and ARB are subtended by the same chord AB (or same arc AB).
4. They are on the same side of major segment $APQRB$.
5. Therefore, angles APB, AQB and ARB are known as **angles in the same segment.**

Angle in a semicircle

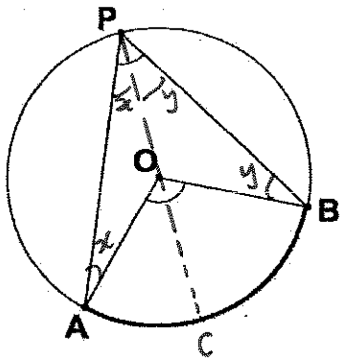


1. AB is the diameter of the circle.
2. P is a point on the circumference.
3. Segment APB is a semi-circle.
4. Angle APB is known as the **angle in a semicircle.**

Level	Knowledge & Skills	I need help!	I can do on my own	I can explain to others
0	State the symmetric properties of circles			
0	State each of the angle properties of circles			

Angle Property of Circle #1: The angle at the centre of a circle is twice any angle at the circumference subtended by the same arc.

(**Abbreviation**: Angle at centre is twice the angle at circumference)



$AOB = 2 \text{ x } APB$

Proof

(Hint: Let angle $OAP = x$ and angle $OBP = y$. Express angle AOB and angle APB in terms of x and y.)

let $\angle OAP = x$ and $\angle OBP = y$

Then $\angle OPA = x$ and $\angle OPB = y$ (base angles of isosceles triangle)

$\angle AOC = 2x$ and $\angle BOC = 2y$ (exterior angle is equal to sum of 2 opposite angles)

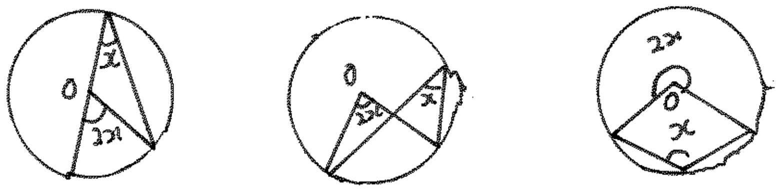
$\therefore \angle AOB = 2x + 2y$

$= 2(x+y)$

$= 2 \angle APB$ (proven)

Class Discussion

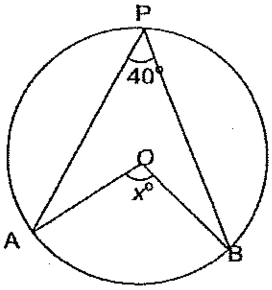
Are there other possible diagrams to illustrate this property?



Level	Knowledge & Skills	I need help!	I can do on my own	I can explain to others
2	angles in opposite segments are supplementary			
2	angles in the same segment are equal			
2	angle at the centre is twice the angle at the circumference			

Example 4
In each of the following circles, *OA* and *OB* are radii of the circle centre *O*. Find the value of *x*.

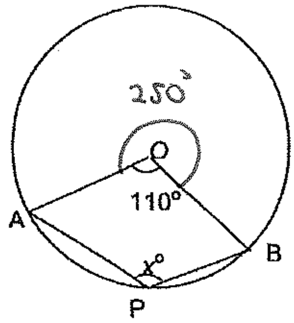
(a)



$x = 40 \times 2$ (angle at centre is twice the angle at circumference)

$= 80$

(b)



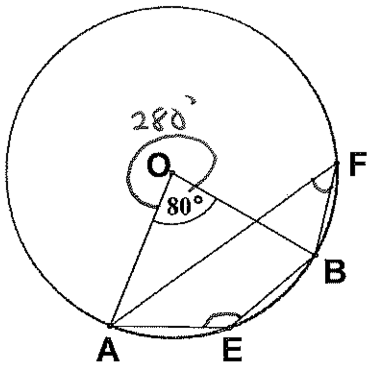
Reflex $\angle AOB = 360° - 110°$ (angles at a point)

$= 250°$

$x = \frac{250}{2}$ (angle at centre is twice the angle at circumference)

$= 125$

Example 5
In the diagram, $\angle AOB = 80°$. Find
(a) $\angle AFB$,
(b) $\angle AEB$.



(a) $\angle AFB = \frac{80°}{2}$ (angle at centre is twice the angle at circumference)

$= 40°$

(b) Reflex $\angle AOB = 360° - 80°$ (angles at a point)

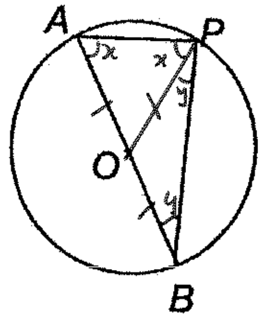
$= 280°$

$\angle AEB = \frac{280°}{2}$ (angle at centre is twice the angle at circumference)

$= 140°$

<u>Angle Property of Circle #2: The angle in a semicircle is a right angle.</u>

(**Abbreviation**: Angle in semicircle is a right angle.)



$APB = 90°$

Proof

(Hint: Consider angle at the centre and circumference subtended by arc *AB*)

$\angle AOB = 180°$

$\angle APB = \frac{180°}{2}$ (angle at centre is twice the angle at circumference)

$= 90°$

Alternate proof:

Let $\angle OAP = x$

$\therefore$ $\angle OPA = x$ (base angles of isosceles triangle)

Let $\angle OBP = y$

$\therefore$ $\angle OPB = y$ (base angles of isosceles triangle)

$2x + 2y = 180°$ (sum of angles in a triangle)

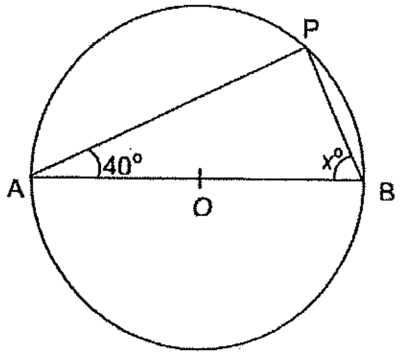
$\therefore$ $x + y = 90°$

$\therefore$ $\angle APB = 90°$ (shown)

Example 6

Find the value of x as shown in the following diagrams.

(a)

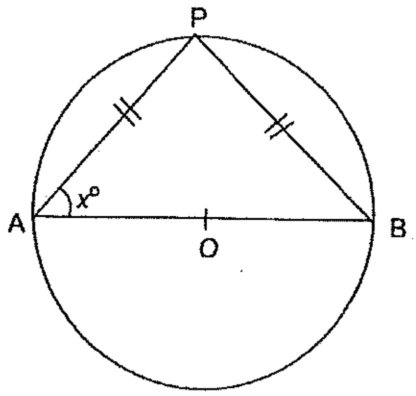


$\angle APB = 90^\circ$ (angle in semicircle is 90°)

$x^\circ = 180^\circ - 90^\circ - 40^\circ$ (sum of angles in a triangle)

$= 50^\circ$ ∦

(b)



$\angle APB = 90^\circ$ (angle in semicircle is 90°)

$x = \dfrac{180^\circ - 90^\circ}{2}$ (base angles of isosceles triangle)

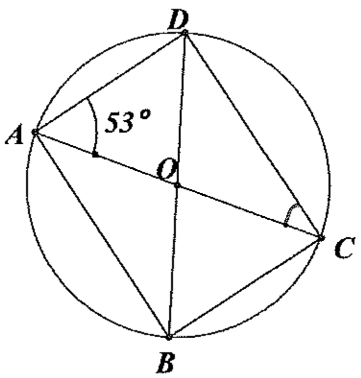
$= 45^\circ$ ∦

Example 7

In the diagram, O is the centre of the circle and angle $OAD = 53°$.
Given that AOC and BOD are diameters of the circle, find

(a) $\angle ACD$,

(b) $\angle BOC$.



(a) $\angle ADC = 90^\circ$ (angle in semi-circle is 90°)

$\angle ACD = 180^\circ - 90^\circ - 53^\circ$ (angle sum of triangle)

$= 47^\circ$ ∦

(b) $\angle ADO = 53^\circ$ (base angles of isosceles triangle)

$\angle AOB = 53^\circ + 53^\circ$ (exterior angle of triangle is equal to sum of two interior opposite angles)

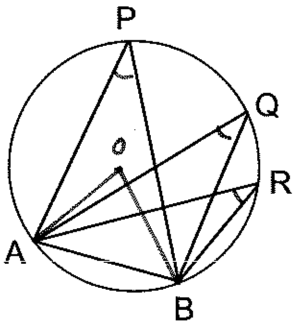
$= 106^\circ$

$\angle BOC = 180^\circ - 106^\circ$ (angles in a straight line)

$= 74^\circ$ ∦

Angle Property of Circle #3: Angles in the same segment of a circle are equal.

(**Abbreviation**: Angles in the same segment are equal)



$\angle APB = \angle AQB = \angle ARB$

Proof
(Hint: How are $\angle APB$, $\angle AQB$ and $\angle ARB$ related to $\angle AOB$?)

$\angle AOB = 2 \times \angle APB$
$\angle AOB = 2 \times \angle AQB$
$\angle AOB = 2 \times \angle ARB$
(Angle at centre is equal to twice angle at circumference)

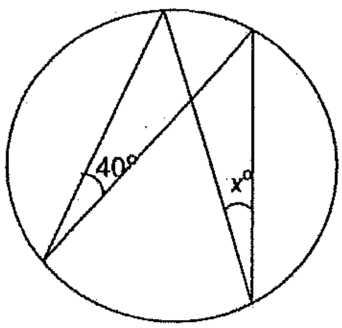
$2 \times \angle APB = 2 \times \angle AQB = 2 \times \angle ARB$

$\therefore \angle APB = \angle AQB = \angle ARB$ //

Level	Knowledge & Skills	I need help!	I can do on my own	I can explain to others
1	angle in a semicircle is a right angle			
1	angles in the same segment of a circle are equal			

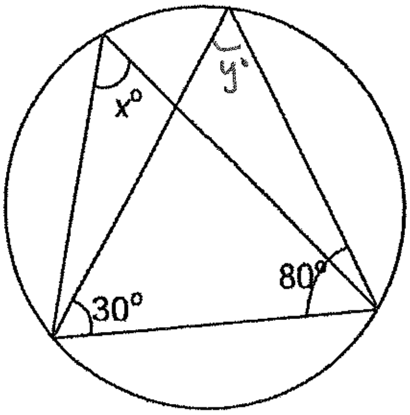
Example 8
Find the value of x as shown in the following diagrams.

(a)



$x = 40°$ (angles in same segment are equal)

(b)

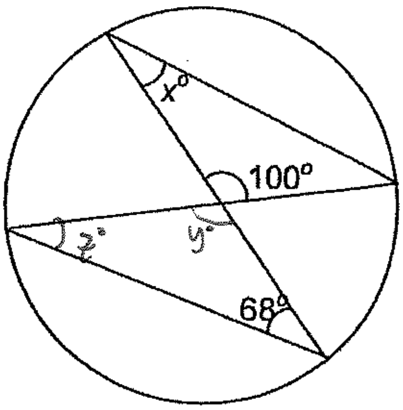


$y° = 180° - 30° - 80°$

$= 70°$ (angle sum of triangle)

$x = 70$ (angles in same segment are equal)

(c)



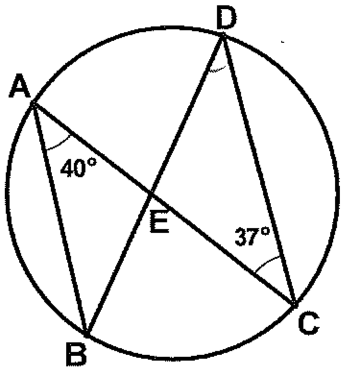
$y° = 100°$ (vertically opposite angles)

$z° = 180° - 68° - 100°$ (angle sum of triangle)

$= 12°$

Example 9

In the diagram, *AC* and *BD* intersect at *E*, ∠*BAC* = 40° and ∠*ACD* = 37°.
Find
(a) ∠*BDC*,
(b) ∠*BEC*.



(a) $\angle BDC = 40°$ (angles in same segment are equal)

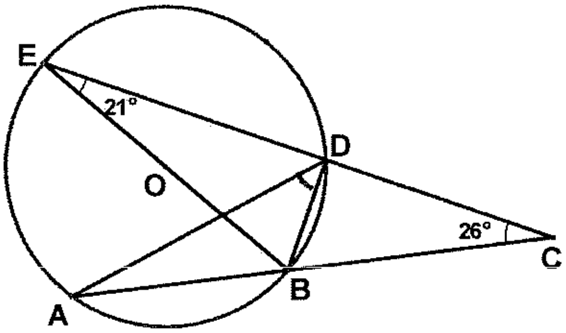
(b) $\angle BEC = 40° + 37°$

$= 77°$ (exterior angle of triangle)

Example 10

In the diagram, BOE is the diameter of the circle with centre O. ABC and EDC are straight lines. Given that $\angle BED = 21°$ and $\angle ACE = 26°$, find $\angle ADB$.

(Answer: 43°)



$\angle EDB = 90°$ (angle in semi-circle is 90°)

$\angle EBC = 180° - 21° - 26°$

$= 133°$ (angle sum of triangle)

$\angle EBA = 180° - 133°$

$= 47°$ (angles in a straight line)

$\angle EDA = 47°$ (angles in same segment are equal)

$\therefore \angle ADB = 90° - 47°$

$= 43°$

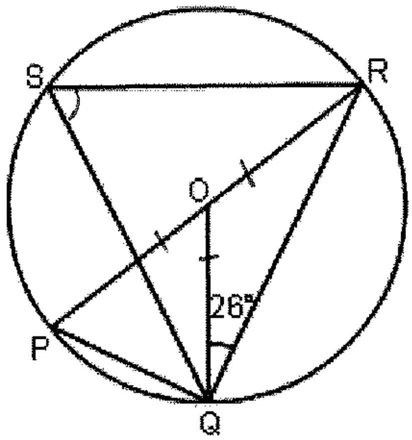
Example 11

In the diagram, $\angle POR$ is a diameter of the circle with centre O and angle $OQR = 26°$.

Find

(a) $\angle PQO$,

(b) $\angle QSR$.



(a) $\angle PQR = 90°$ (angle in semicircle is 90°)

$\angle ORQ = 26°$ (base angles of isosceles triangle)

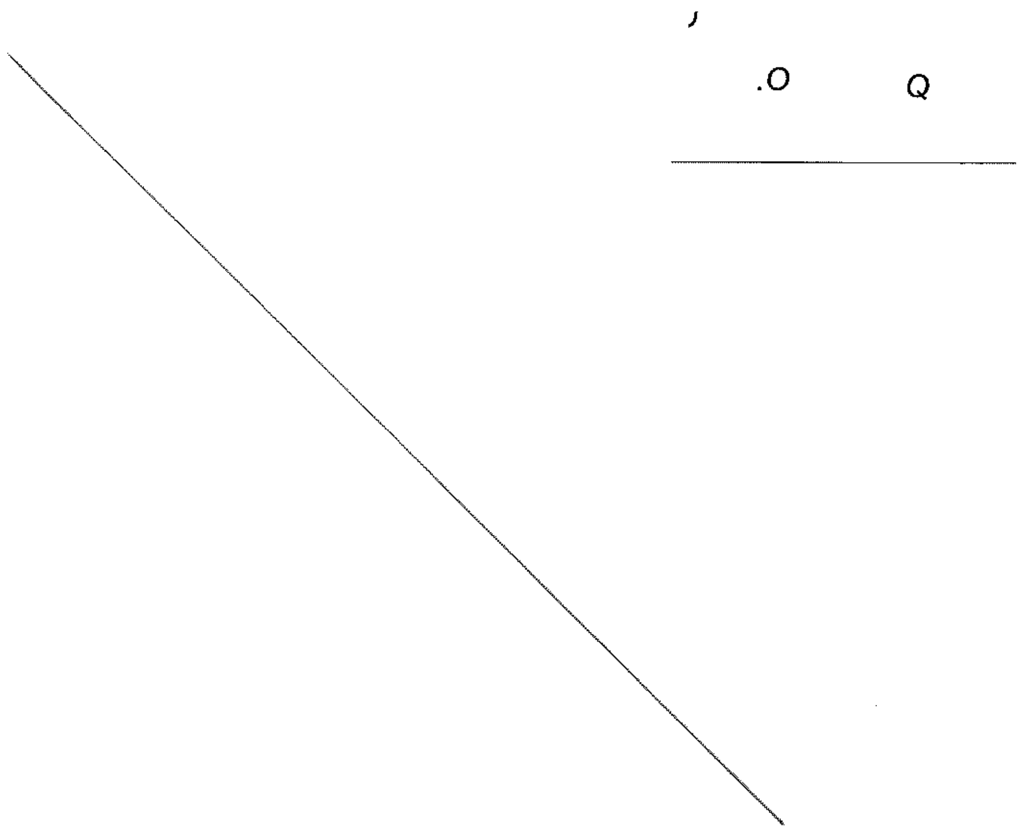
$\angle PQO = 90° - 26°$

$= 64°$

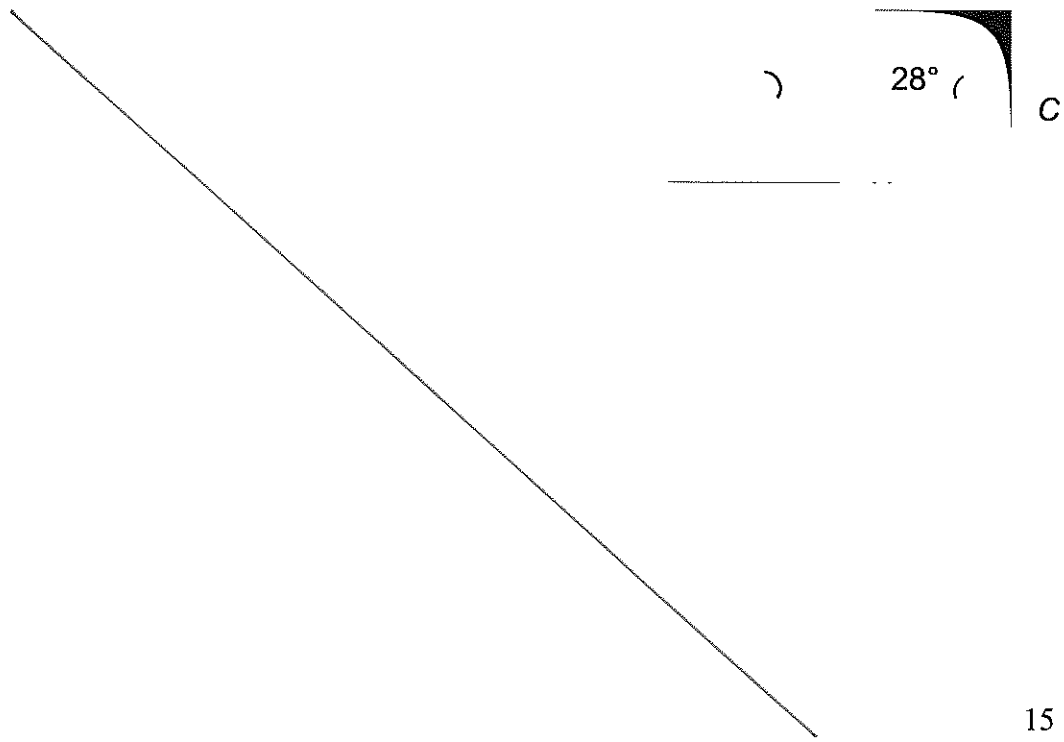
(b) $\angle OPQ = \angle PQO = 64°$ (base angles of isosceles triangle)

$\angle QSR = 64°$ (angles in same segment are equal)

Example 12
In the diagram, O is the centre of the circle and AB is the diameter of the circle.
Given that $\angle BAP = 24°$ and $\angle BPA = 35°$, find $\angle BQX$.
(Answer: 31°)



Example 13
In the diagram, $\angle ABC = 43°$ and $\angle ACB = 28°$. Find $\angle OBA$ and $\angle OCA$.
(Answer: $\angle OBA$ = 62°, $\angle OCA$ = 47°)



Example 14

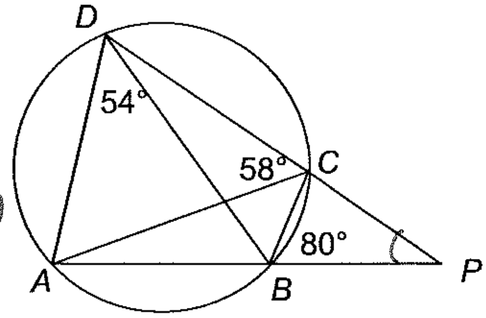
In the diagram, $\angle ADB = 54°$, $\angle ACD = 58°$ and $\angle CBP = 80°$. Find $\angle APD$.

(Answer: 32°)

$\angle ACB = 54°$ (angles in same segment are equal)

$\angle APD = 58° + 54° - 80°$ (exterior angle of triangle is equal to sum of 2 opposite angles)

$= 32°$



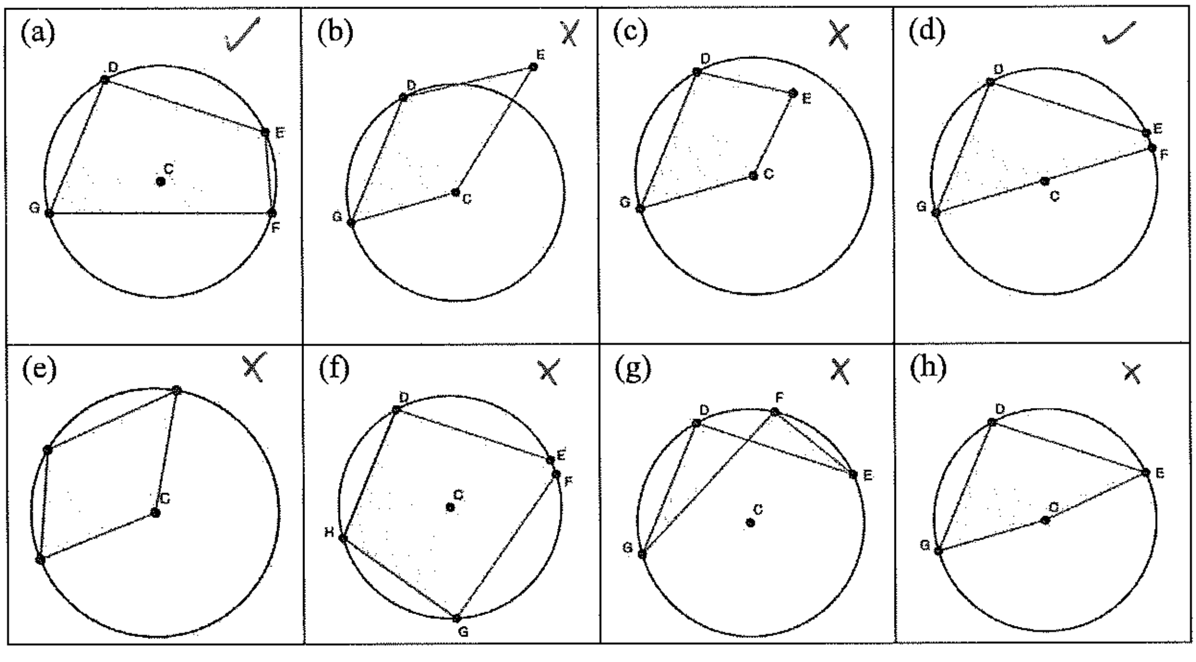
Level	Knowledge & Skills	I need help!	I can do on my own	I can explain to others
1	Apply symmetric and/or angle properties of circles to calculate unknown lengths and angles			
2	Justify mathematical claims with symmetric properties and/or angle properties of circles			

4.4 PROPERTIES INVOLVING CYCLIC QUADRILATERALS

A **cyclic quadrilateral** is a quadrilateral with its four vertices on the circumference of a circle. Given that *DEFG* is a cyclic quadrilateral, the points *D, E, F* and *G* are **concyclic**.

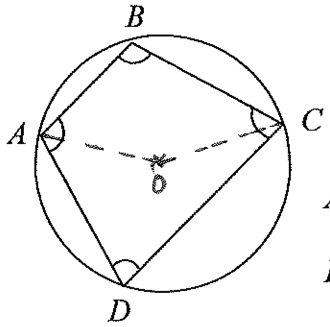
Class Discussion

Which of the following shapes are cyclic quadrilaterals?



Cyclic Quadrilaterals Property #1: In a cyclic quadrilateral, the angles in opposite segments are supplementary i.e. their sum is 180°.

(**Abbreviation**: Angles in opposite segments are supplementary)



$ABC + CDA = 180°$

$DAB + BCD = 180°$

Proof

(Hint: Express $\angle ABC$ and $\angle CDA$ in terms of $\angle COA$ and reflex $\angle COA$)

Reflex $\angle AOC = 2 \times \angle ABC$ (angle at circumference is equal to twice angle at centre)

$\angle AOC = 2 \times \angle ADC$ (angle at circumference is equal to twice angle at centre

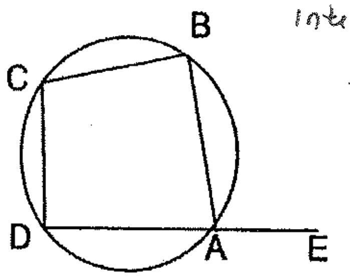
$\angle AOC +$ Reflex $\angle AOC = 360°$ (angle at point)

$\therefore 2 \times \angle ABC + 2 \times \angle CDA = 360°$

$\angle ABC + \angle CDA = 180°$ ※

(Not in syllabus) Cyclic Quadrilaterals Property #2: The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

(**Abbreviation**: Exterior angle of cyclic quadrilateral is equal to interior opposite angle.)



$\angle BAE = \angle DCB$

Proof

(Hint: Use Cyclic Quadrilaterals Property #1)

$\angle DCB + \angle BAD = 180°$ (angles in opposite segments are supplementary)

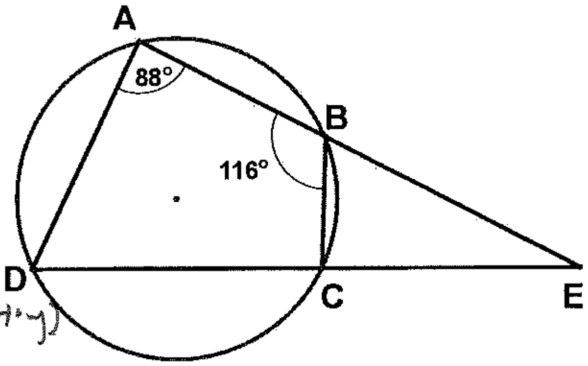
$\angle BAE + \angle BAD = 180°$ (angles in a straight line)

$\therefore \angle BAE = \angle DCB$ ※

Example 15

In the diagram, AB and DC produced meet at E.
Given that $\angle DAB = 88°$ and $\angle ABC = 116°$.
Find,

(a) $\angle ADC$,
(b) $\angle BCE$,
(c) $\angle BEC$.



(a) $\angle ADC = 180° - 116°$ (angles in opposite segments are supplementary)
$= 64°$

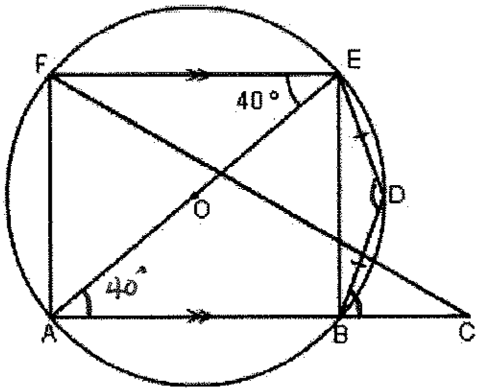
(b) $\angle BCD = 180° - 88°$ (angles in opposite segments are supplementary)
$= 92°$

$\angle BCE = 180° - 92°$ (adjacent angles on a straight line)
$= 88°$

(c) $\angle BEC = 116° - 88°$
$= 28°$ (exterior angle of triangle)

Example 16

In the figure, O is the center of the circle, ABC and AOE are straight lines and EF is parallel to AB. Find $\angle DBC$.
(Answer: 70°)



$\angle EAB = \angle AEF = 40°$ (alternating angles, FE is parallel to AB)

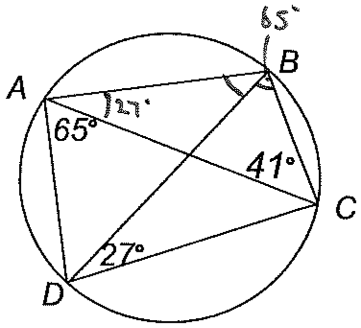
$\angle EDB = 180° - 40°$
$= 140°$ (angles in opposite segments are equal)

$\angle DEB = \angle DBE = \frac{180° - 140°}{2}$ (base angles of isosceles triangle)
$= 20°$

$\angle EBA = 90°$ (angle in semicircle is a right angle)

$\therefore \angle DBC = 180° - 90° - 20°$ (adjacent angles on a straight line)
$= 70°$

Example 17
In the diagram, $\angle DAC = 65^\circ$, $\angle ACB = 41^\circ$ and $\angle BDC = 27^\circ$.
Find $\angle ABD$.
(Answer: 47°)

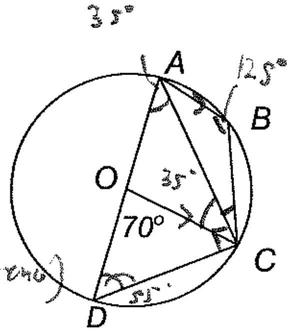


$\angle DBC = \angle CAD = 65^\circ$ (angles in same segment are equal)

$\angle CDB = \angle CAB = 27^\circ$ (angles in same segment are equal)

$\therefore \angle ABD = 180^\circ - 65^\circ - 27^\circ - 41^\circ$ (sum of angles in triangle)
$= 47^\circ$ #

Example 18
A circle, centre O, passes through the points A, B, C and D. AD is a diameter of the circle, $\angle COD = 70^\circ$ and AB is parallel to OC. Find
(a) $\angle OAC$, (b) $\angle ODC$, (c) $\angle ABC$, (d) $\angle ACB$.



(Answer: (a) 35°, (b) 55°, (c) 125°, (d) 20°)

(a) $\angle OAC = \frac{1}{2} \times \angle DOC$ (angle at centre is twice angle at circumference)
$= \frac{1}{2} \times 70^\circ$
$= 35^\circ$ #

(b) $\angle ACD = 90^\circ$ (angle in semicircle is right angle)
$\therefore \angle ODC = 180^\circ - 90^\circ - 35^\circ$ (angle sum of triangle)
$= 55^\circ$ #

OR $\angle ODC = \dfrac{180^\circ - 70^\circ}{2}$ (base angles of isosceles triangle)
$= 55^\circ$ #

(c) $\angle ABC = 180^\circ - 55^\circ$ (angles in opposite segments are supplementary)
$= 125^\circ$ #

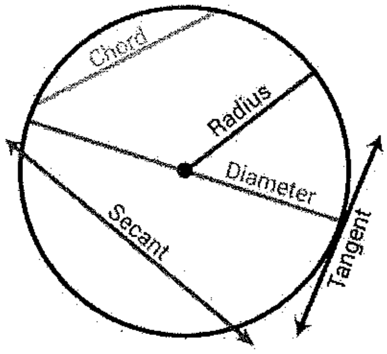
(d) $\angle OCA = \angle OAC = 35^\circ$ (base angles of isosceles triangle)
$\angle BAC = \angle OCA = 35^\circ$ (alternate angles, AB is parallel to OC)
$\angle ACB = 180^\circ - 125^\circ - 35^\circ$ (angle sum of triangle)
$= 20^\circ$ #

<u>E Maths Textbook: Shinglee New Syllabus Mathematics 3 (7th Edition)</u>
Tier A: Exercise 11C (pg 397 - 401) Qn 1f, 1h, 2d, 5, 8
Tier B: Exercise 11C (pg 397 - 401) Qn 9, 11, 13, 18, 20, 21
Tier C: Exercise 11C (pg 397 - 401) Qn 23, 25

4.5 PROPERTIES INVOLVING TANGENTS OF CIRCLES

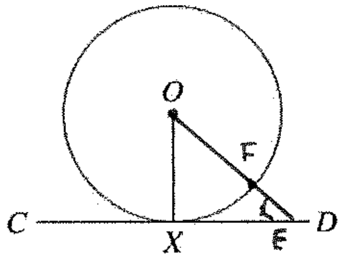
A straight line that touches a circle at one and only one point of contact is known as a **tangent**.

A straight line that cuts a circle at two distinct points is known as a **secant**.



Property of Tangent of Circle #1: The tangent is perpendicular to the radius drawn to the point of contact.

(**Abbreviation**: Angle between tangent and radius of circle is right angle.)



Given that CD is a tangent to the circle at X, $OXC = 90°$

Visualising this:

Draw a chord AB in the circle perpendicular to OX. As you move AB closer towards X while keeping it perpendicular to OX, AB will get shorter and shorter. When you reach X, the chord AB vanishes and you get the tangent line instead.

Proof

Suppose CD is not perpendicular to OX (ie. $\angle OXC \neq 90°$)

Then there exists a point E on CD such that $\angle OED = 90°$

Then $\angle OXE$ is acute and $OX > OE$

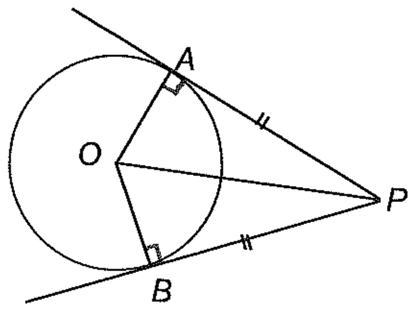
Let F be on the circumference such that OFE is a straight line

$OX = OF$, so $OE > OX$, a contradiction.

$\therefore$ CD is perpendicular to OX (proven)

Property of Tangent of Circle #2: Tangents from an external point to a circle have equal length.

(Abbreviation: Tangents from external point are equal in length)



Given that AP and BP are tangents to the circle at A and B, $AP = BP$.

Visualising this:

Reflect the diagram about OP. A should reflect onto B and vice versa. All the lengths should similarly reflect onto each other.

Proof

(Hint: Look for congruent triangles)

$OA = OB$ (radius of circle)

$OP = OP$ (common side)

$\angle OAP = \angle OBP = 90^\circ$ (angle between tangent and radius of circle is right angle)

$\therefore \triangle OAP \cong \triangle OBP$ (RHS)

$\therefore AP = BP$ (proven)

Level	Knowledge & Skills	I need help!	I can do on my own	I can explain to others
1	tangents from an external point are equal in length			
1	the line joining an external point to the centre of the circle bisects the angle between the tangents			
1	the line joining an external point to the centre of the circle bisects the angle between the tangents			

Class Discussion

What other properties of $\angle AOP$, $\angle BOP$ and $\angle APB$ do you observe?

(1) $\angle AOP = \angle BOP$ (line joining an external point to the centre of the circle bisects the angle between the tangents) ✗

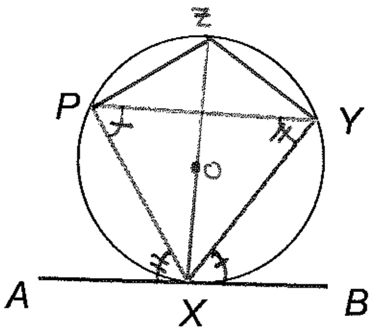
(2) $\angle APO = \angle BPO$

(**Abbreviation**: line joining an external point to centre of circle bisects angle between tangents)

<u>(AM syllabus) Property of Tangent of Circle #3: The angle between the tangent and the chord through the point of contact is equal to the angle in the alternate segment.</u>

(**Abbreviation**: tangent-chord theorem / alternate segment theorem)

This property is also known as **Alternate Segment Theorem** (or Tangent-Chord Theorem).



$\angle YXB = \angle YPX$

$\angle PXA = \angle PYX$

Proof

(Hint: Draw the diameter XZ and observe how $\angle YXZ$ and $\angle YXB$, $\angle YXZ$ and $\angle YZX$, and $\angle YZX$ and $\angle YPX$ are related.)

Let Z be a point on circumference of circle such that XZ is diameter.

Let $\angle YXB = a$

$\therefore \angle ZXY = 90° - a$ (angle between tangent and radius of circle is right angle)

Since $\angle ZYX = 90°$ (angle in semi-circle is right angle)

$\angle XZY = 180° - (90° - a) - 90° = a$ (angle sum in triangle)

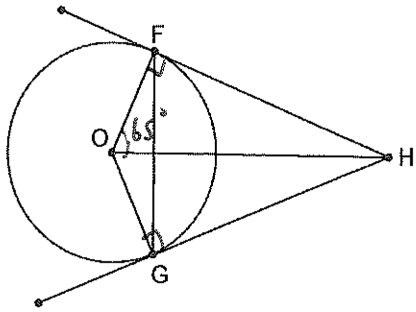
$\angle YPX = a$ (angles in same segment are equal)

$\therefore \angle YXB = \angle YPX$ (shown)

Example 19

It is given that HF and HG are tangents to the circle, centre O at F and G respectively. If $\angle FOH = 65^{\circ}$, find

(a) $\angle FHG$, (b) $\angle FGH$.



(a) $\angle OFH = 90^{\circ}$ (angle between tangent and radius of circle is 90°)

$\angle FOH = 180^{\circ} - 90^{\circ} - 65^{\circ} = 25^{\circ}$ (angle sum of triangle)

$\angle FHG = 25^{\circ} \times 2$ (line joining external point to centre of circle bisects angle between tangents)
$= 50^{\circ}$

(b) $FH = GH$ (tangents of external point are equal in length)

$\therefore \triangle FHG$ is isosceles.

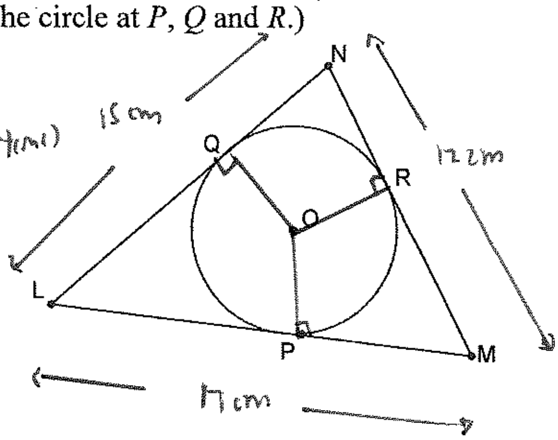
$\angle FGH = \frac{180^{\circ} - 50^{\circ}}{2} = 65^{\circ}$ (base angles of isosceles triangle) ✱

Example 20

Circle PQR is inscribed in ΔLMN. $LN = 15$ cm, $LM = 17$ cm and $NM = 12$ cm. (This means that the sides of ΔLMN are tangent to the circle at P, Q and R.)

Find the lengths of

(a) NR, (b) LP.



(a) Let $NR = x$ cm. Then $NQ = x$ cm (tangents from external point are equal in length)

$QL = 15 - x$ cm

$PL = 15 - x$ cm (tangents from external point are equal in length)

$MR = 12 - x$ cm

$MP = 12 - x$ cm (tangents from external point are equal in length)

$(15 - x) + (12 - x) = 17$

$27 - 2x = 17$

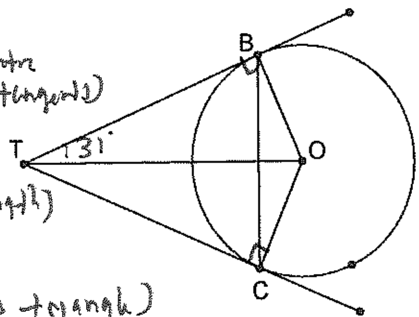
$x = 5$, $NR = 5$ cm ✱

(b) $LP = 15 - 5$
$= 10$ cm ✱

Example 21

In the diagram, TB and TC are tangents to the circle, centre O at B and C respectively. It is also given that $\angle OTB = 31^{\circ}$. Find

(a) $\angle TCB$, (b) $\angle COB$.



(a) $\angle OTC = \angle OTB = 31^{\circ}$ (line joining external point to centre of circle bisects angle between tangents)

$\angle BTC = 31^{\circ} \times 2 = 62^{\circ}$

$BT = CT$ (tangents from external point are equal in length)

$\therefore \triangle BTC$ is an isosceles triangle

$\angle TCB = \frac{180^{\circ} - 62^{\circ}}{2} = 59^{\circ}$ (base angles of isosceles triangle) ✱

(b) $\angle OCT = 90^{\circ}$ (angle between tangent and radius of circle is 90°)

$\angle COT = 180^{\circ} - 31^{\circ} - 90^{\circ}$ (angle sum of triangle)
$= 59^{\circ}$

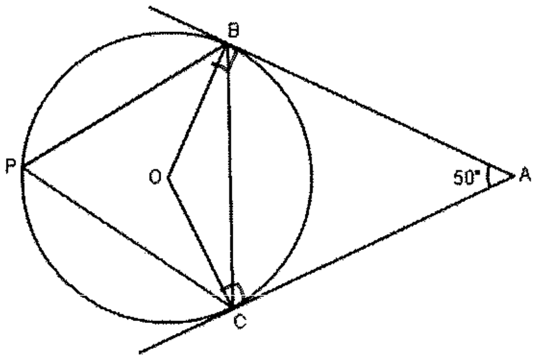
$\therefore \angle COB = 59^{\circ} \times 2$ (line joining external point to centre of circle bisects angle between tangents)
$= 118^{\circ}$ ✱

Example 22

In the diagram below, O is the centre of the circle. AB and AC are tangents to the circle and $\angle BAC = 50°$.

(a) Find $\angle BOC$.

(b) Given that $\angle OCP = 1.5\angle OBP$, find $\angle OCP$.



(a) $\angle OBA = \angle OCA = 90°$ (angle between tangent and radius of circle is right angle)

$\angle BOC = 360° - 90° - 90° - 50°$ (angle sum of quadrilateral)

$= 130°$

(b) $\angle BPC = \frac{130°}{2} = 65°$ (angle at centre is twice angle at circumference)

Reflex $\angle OCP = 360° - 130° = 230°$ (angles at a point)

$\angle OCP + \angle OBP = 65° + 230° = 360°$ (sum of angles in quadrilateral)

$\angle OCP + \frac{2}{3}\angle OCP = 65°$

$\frac{5}{3}\angle OCP = 65°$

$\angle OCP = 39°$

Example 23

In the diagram below, $PQRS$ are points on the circumference of the circle with centre at O. QS is the diameter of the circle. TSU is a tangent to the circle at S. $\angle PST = 80°$ and $\angle RSU = 70°$.

Find

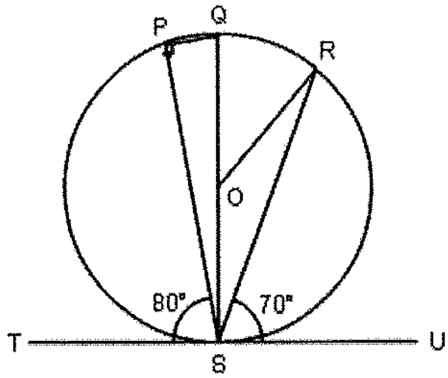
(a) $\angle QPS$,

(b) $\angle QSU$,

(c) $\angle PQS$,

(d) $\angle SOR$.

(Ans: (a) 90° (b) 90° (c) 80° (d) 140°)



(a) $\angle QPS = 90°$ (angle in semi-circle is right angle)

(b) $\angle QSU = 90°$ (angle between tangent and radius of circle is right angle)

(c) $\angle QST = 90°$ (angle between tangent and radius of circle is right angle)

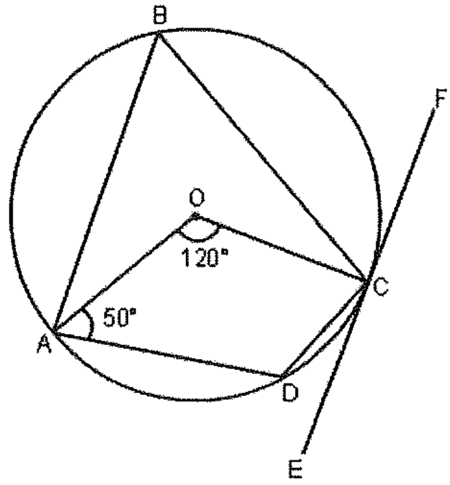
$\angle PQS = 80°$ (alternate segment theorem)

(d) $\angle QSR = 90° - 70° = 20°$

$\angle SOR = 180° - 20° - 20°$ (base angles of isosceles triangle)

$= 140°$

Example 24



A, B, C and D are points on the circumference of a circle with centre O. ECF is a tangent to the circle. Given that $\angle AOC = 120°$, and $\angle OAD = 50°$, calculate

(a) $\angle ABC$,

(b) $\angle ADC$,

(c) $\angle DCE$.

(Ans: (a) 60° (b) 120° (c) 20°)

(a) $\angle ABC = \frac{120°}{2} = 60°$ (angle at centre is twice angle at circumference)

(b) Reflex $\angle AOC = 360° - 120°$ (angles at a point)

$= 240°$

$\angle ADC = \frac{240°}{2} = 120°$ (angle at centre is twice angle at circumference)

(c) $\angle OCD = 360° - 120° - 50° - 120°$ (angle sum of quadrilateral)

$= 70°$

$\angle OCE = 90°$ (angle between tangent and radius of circle is right angle)

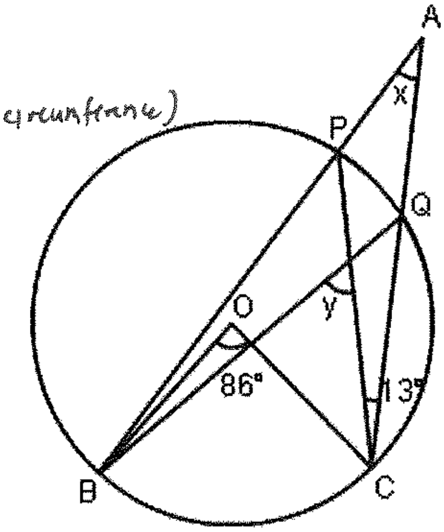
$\angle DCE = 90° - 70°$

$= 20°$

Example 25

Given that O is the centre of the circle, $\angle BOC = 86°$ and $\angle PCA = 13°$, find the value of x and y in the following diagram.

(Ans: x = 30°, y = 56°)



$\angle BPC = \frac{86°}{2} = 43°$ (angle at centre is twice angle at circumference)

$x = 43° - 13°$ (exterior angle of triangle)

$= 30°$

$\angle BQC = 43°$ (angles in same segment are equal)

$y = 43° + 13°$ (exterior angle of triangle)

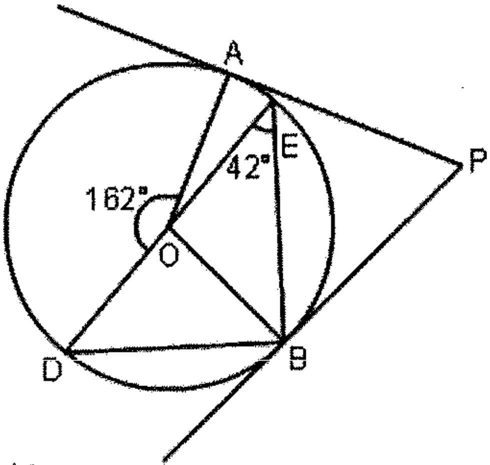
$= 56°$

Example 26

O is the centre of the circle and DOE is a straight line. AP and BP are both tangents to the circle at points A and B respectively. Given $\angle DCE = 42°$ and $\angle AOD = 162°$, find

(a) $\angle DOB$,
(b) $\angle BDE$,
(c) $\angle EBP$,
(d) $\angle APB$.

(Ans: (a) 84° (b) 48° (c) 48° (d) 66°)



(a) $\angle DOB = 42° \times 2 = 84°$ (angle at centre is twice angle at circumference)

(b) $\angle DBE = 90°$ (angle in semicircle is right angle)
$\angle BDE = 180° - 90° - 42° = 48°$ (angle sum of triangle)

(c) $\angle EBP = 48°$ (alternate segment theorem)

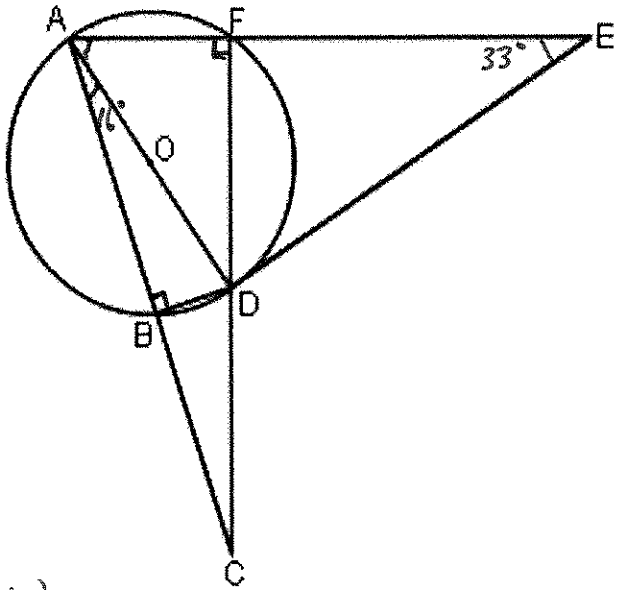
(d) $\angle AOE = 180° - 162° = 18°$ (angles on straight line)
$\angle BOE = 180° - 84° = 96°$ (angles on straight line)
$\angle AOB = 18° + 96° = 114°$
$\angle OAP = \angle OBP = 90°$ (angle between tangent and radius of circle is right angle)
$\therefore \angle APB = 360° - 90° - 90° - 114°$ (angle sum of quadrilateral)
$= 66°$

Example 27

In the diagram, AD is the diameter of the circle centre O. BD and AF is produced to meet at E. ABC and FDC are straight lines. Given that $\angle FED = 33°$ and $\angle DAB = 16°$, calculate

(a) $\angle AFC$,
(b) $\angle ADB$,
(c) $\angle FDE$,
(d) $\angle DAF$.

(Ans: (a) 90° (b) 74° (c) 57° (d) 41°)



(a) $\angle AFC = 90°$ (angle in semicircle is right angle)

(b) $\angle ABD = 90°$ (angle in semicircle is right angle)
$\therefore \angle ADB = 180° - 16° - 90°$ (angle sum of triangle)
$= 74°$

(c) $\angle FDE = 90° - 33°$ (exterior angle of triangle)
$= 57°$

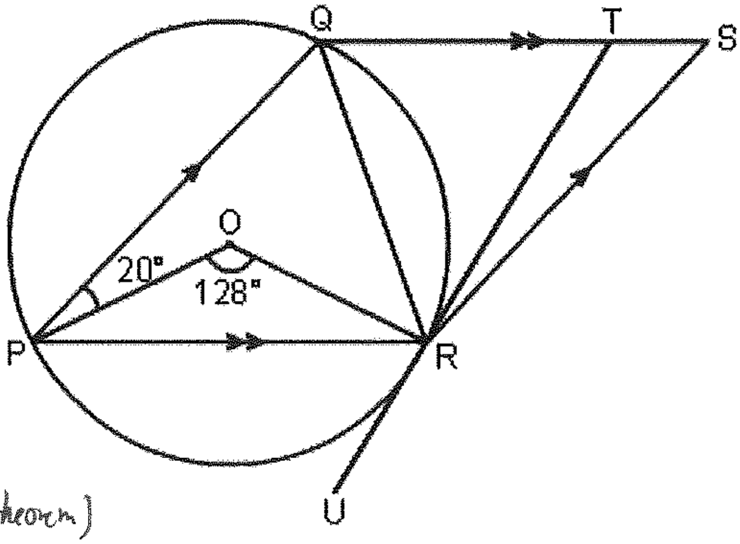
(d) $\angle DAF = 180° - 90° - 33° - 16°$ (angle sum of triangle)
$= 41°$

Example 28

P, *Q* and *R* are three points on a circle with centre *O* and *PQSR* is a parallelogram. *URT* is a tangent to the circle at *R*. $\angle POR = 128°$ and $\angle OPQ = 20°$. Calculate

(a) $\angle PQR$,
(b) $\angle OPR$,
(c) $\angle QRT$,
(d) $\angle TRS$.

(Ans: (a) 64° (b) 26° (c) 46° (d) 18°)



(a) $\angle PQR = \frac{128°}{2}$ (angle at centre is twice angle at circumference)
$= 64°$

(b) $\angle OPR = \frac{180° - 128°}{2}$ (base angles of isosceles triangle)
$= 26°$

(c) $\angle QRT = 20° + 26°$ (alternate segment theorem)
$= 46°$

(d) $\angle ORT = 90°$ (angle between tangent and radius of circle is 90°)
$\angle TRS = 180° - 20° - 26° - 90° - 26°$ (interior opposite angles, PQ is parallel to RS)
$= 18°$

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Tier A: Exercise 11A (pg 199) Qn 3, 6, 12
Tier B: Exercise 11A (pg 201) Qn 15, 17, 21
Tier C: Exercise 11A (pg 202) Qn 28