School of Science and Technology, Singapore
Mathematics Department

2023 Secondary 3 ADDITIONAL MATHEMATICS
Unit 05. Linear Law

A

Name Solutions. ( ) Class S3–0

Enduring Understanding

Students will understand the following:

- Relationships in the form of non-linear functions can be transformed into linear relationships.

Essential Questions

- How can relationships be represented using a linear function?
- How can a non-linear equation be converted into a linear equation?
- How can we determine the unknown constants in an algebraic equation?

Big Ideas

- Notations: Notations are symbols and conventions of writing used to represent mathematical objects, and their operations and relationships in a concise and precise manner.
- Equivalence: Equivalence is a relationship that expresses the 'equality' of two mathematical objects that may be represented in two different forms based on a criterion. Transformation or conversion of an expression or equation from one form to another equivalent form is the basis of many manipulations for analyzing and comparing them and algorithms for finding solutions.

Knowledge & Skills

At the end of the unit, students should be able to

- Explain how non-linear relationship in linear form can be used to derive the relationship between two variables
- Transform given relationships, including $y = ax^n$ and $y = kb^x$, to linear form to determine the unknown constants from a straight line graph.
- Plot linear graph given set of experimental data (with no scale given).
- Determine unknowns in relations using experimental data by applying linear law to obtain
- straight line graphs
- Understand and identify outliers or incorrect readings

Common Symbols & Language Used

- Line of best fit, linear function, equation of a straight line, variables, constants

Resources

- Yeo, B. W. J., Choy, B, H., Teh, K. S., Wong, L. F., & Lee, S. (2020). Linear Law. In *Think! Additional Mathematics Book A* (10th ed. pp. 175-198). Singapore: Shinglee Publishers Pte Ltd.
- Yan, K. C., Chng, B. K. E., & Khor, N. H. D. (2020). Applications of Straight Line Graphs. In *Additional Maths 360 Volume A* (2nd ed. pp. 212-233). Singapore: Marshall Cavendish Education Pte Ltd.

Lesson sequence in the unit								
Student Learning Outcomes	Dimensions (Please tick the appropriate boxes)							
	FUNCTIONS **F**	INVARIANCE **I**	NOTATIONS **N**	DIAGRAMS **D**	MEASURES **M**	EQUIVA-LENCE **E**	PROPOR TIONALITY **P**	MODELS **M**
Conversion of non-linear to linear			✓			✓		
Determine unknowns by applying linear law			✓	✓		✓		
Applications of linear law			✓			✓		✓

5.1 Introduction to Linear Law

Why study linear law?

Some phenomena in the sciences or in the real world can be **modelled** using a function and its equation. We can conduct an experiment and plot the data collected on a graph. Sometimes, we do not obtain a linear function and it may be difficult to find the unknown parameters (or coefficients) of the equation.

Activity 1

Learning objective: To appreciate the use of linear law in the sciences.

Part A. To find a formula for the resistance of a resistor

In this experiment, we want to determine the resistance, R ohms, of a resistor.
A circuit is set up as shown in Figure **(a)**, where R is a resistor, V is a voltmeter to measure the potential difference, V volts, and A is an ammeter to measure the current, I amperes, flowing through the resistor. The data collected are shown in Figure **(b)**.

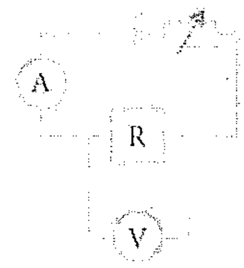
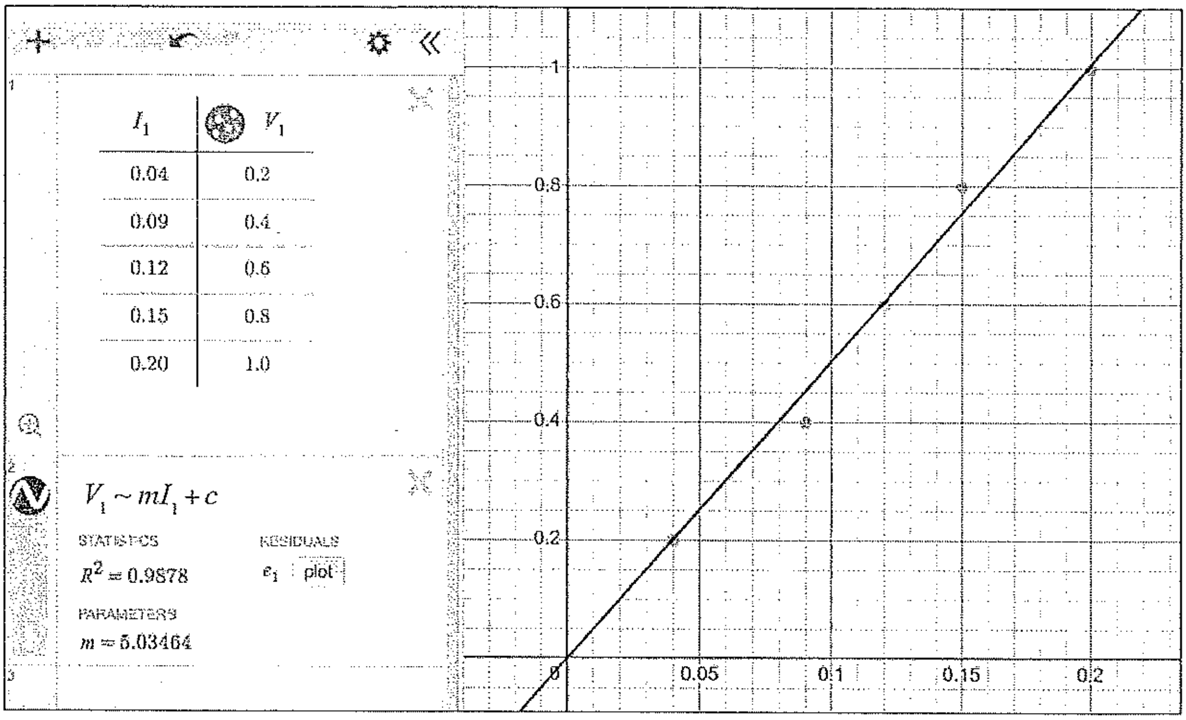


Figure **(a)**

I / ampere	V / volt
0.04	0.2
0.09	0.4
0.12	0.6
0.15	0.8
0.20	1.0

Figure **(b)**

A graph of V against I is plotted using Desmos as shown below:



We observe that the points almost lie in a straight line, so a straight line can be drawn to pass through as many points as possible. This line is called **the line of best fit**. Its equation is $V = mI + c$, where m and c are constants to be determined.

Using the straight line graph,
1. calculate the gradient, m, of the straight line,
2. find the V-intercept, c, of the straight line, and
3. hence, state the equation of the straight line.

We have just **modelled** the relationship between the potential difference, V volts, across a resistor and the current, I amperes, flowing through it by the equation $V = 5.03I$, by finding the unknowns m and c of the straight line graph. In this case, the resistance of the resistor is given by $R = \frac{V}{I} = 5.03$ ohms. Once we have obtained the equation, we can also use it to ***predict*** the value of I for any given V. This is one of the reasons why a model is useful.

Part B. To find a formula for the period of a pendulum

[Pendulum Lab: https://phet.colorado.edu/sims/html/pendulum-lab/latest/pendulum-lab_en.html]

The time taken for a simple pendulum to travel from X to Y and back to X (i.e. the period), T seconds, is measured for different lengths of string, L metres, as shown in Figure (**c**). The data collected are shown in Figure (**d**).

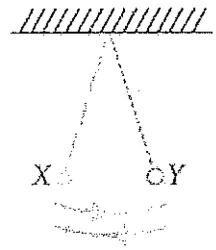
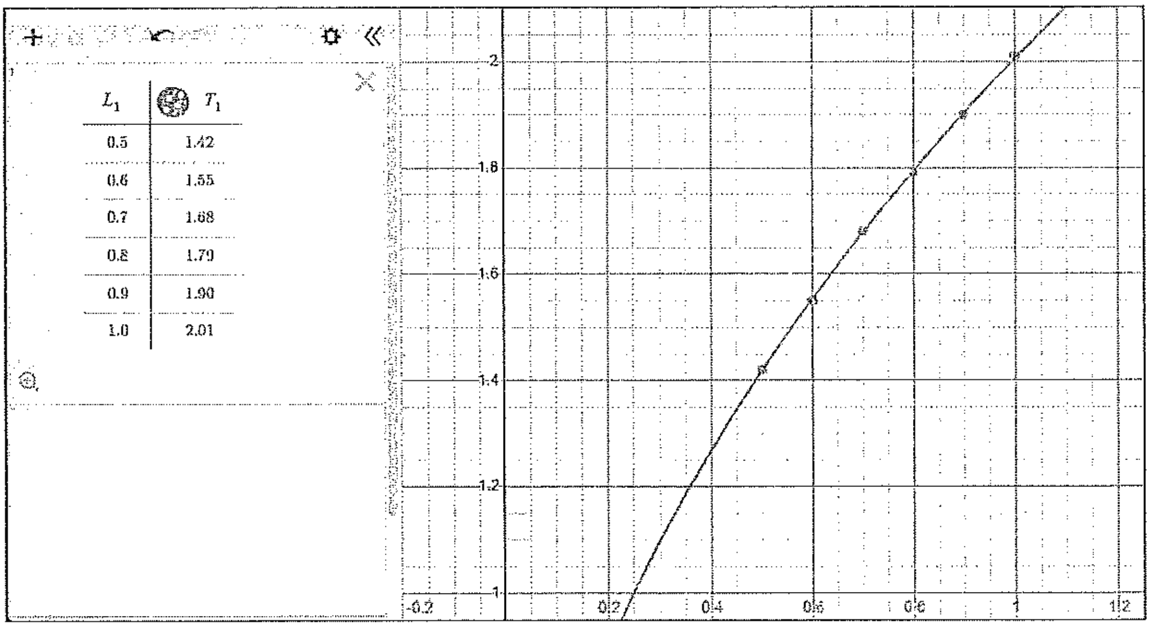


Figure (**c**)

L / metre	T / second
0.5	1.42
0.6	1.55
0.7	1.68
0.8	1.79
0.9	1.90
1.0	2.01

Figure (**d**)

A graph of T against L is plotted using Desmos as shown below:

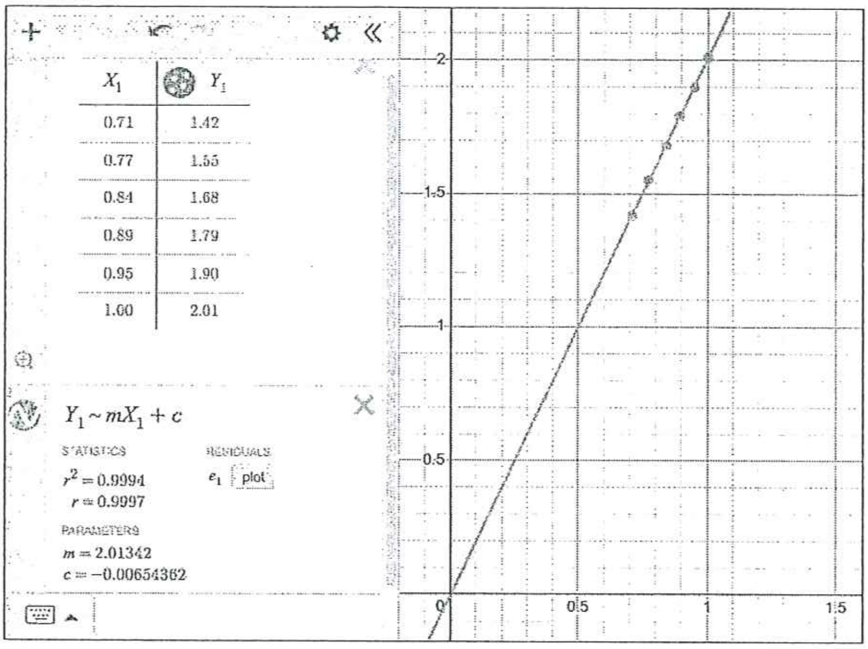


We observe that the points do not lie in a straight line, but on a curve. How can we find the equation of the graph? Can we use the graph to *predict* the value of T when $L = 2$?

If the equation of the curve is given in the form $T = k\sqrt{L}$, then let $Y = T$ and $X = \sqrt{L}$.
Create a new table of values of X and Y (in Figure (**e**)) by using the data from Figure (**c**).

$X = \sqrt{L}$	$Y = T$
0.71	1.42
0.77	1.55
0.84	1.68
0.89	1.79
0.95	1.90
1.00	2.01

The data from Figure (**e**) is plotted using Desmos as shown below:



The points now lie in a straight line of equation $Y = mX + c$, where m and c are constants to be determined.

Using the straight line graph,

1. calculate the gradient, *m*, of the straight line,
2. find the *Y*-intercept, *c*, of the straight line, and
3. hence, state the equation of the straight line.

We obtain $Y = 2X$.

Since $Y = T$ and $X = \sqrt{L}$, we obtain the equation of the function $T = 2\sqrt{L}$ which models the relationship between the period, *T*, of a simple pendulum and the length of the string, *L*.
Can we use the graph to *predict* the value of *T* when $L = 2$?

Pause & Think

What are the advantages of linear functions compared to non-linear functions?

Linear functions can be plotted more accurately.
Important information like the gradient of graph and vertical-axis intercept can be found easily.

$Y = mX + c$ is known as the **linear form** and we say that the equation $Y = mX + c$ is the **linear law** relating the variables X and Y or the variables X and Y are linearly related.

The **linear law** is a tool that allows us to transform non-straight line equations to straight line equations so that we can plot a straight line.

In research work, when two variables are believer to be related, an experiment can be carried out to obtain a set of corresponding values. This set of values is then plotted and a graph is drawn. If the points lie on a straight line, for example in **Part A of Activity 1**, it is easy to derive the relation between the two variables by $Y = mX + c$. This relation is then used to *predict* further values. However, not all experimental results obey a straight line relation, for example in **Part B**. By choosing suitable variables for X and Y, we convert a non-linear relation to a linear form and then form an equation for the non-linear relation.

In real life, we will not be given that the equation of the curve is in the form $T = k\sqrt{L}$. How then do we know what form the equation will take? How do we know it is not other forms such as $T = kL^2$ or $T = k\sqrt[3]{L}$? We need to know which form the equation of the curve will take so as to decide what variables to plot in order to obtain a straight line. In rea life, we may have to use *trial and error* to determine what variables to plot in order to obtain a straight line graph.

5.2 Converting non-linear equation into linear form

Let's learn how to choose suitable variables for Y and X so that we can convert the original non-linear equation into the linear form $Y = mX + c$, where Y and X are the new vertical and horizontal variables respectively, m is the gradient of the straight line and c is the vertical intercept.

Worked Example 1

Converting non-linear equations involving power functions into linear form

Express $y = -ax + \frac{b}{x}$, where a and b are constants, in the linear form $Y = mX + c$.

Solution:

Method 1. Multiply throughout by x

$$y = -ax + \frac{b}{x}$$

Multiply throughout by x, $\quad xy = -ax^2 + b$

Let $\quad Y = xy \quad$ and $\quad X = x^2$.

Then the equation becomes $Y = mX + c$, where $m = -a$ and $c = b$.

Method 2. Divide throughout by x

$$y = -ax + \frac{b}{x}$$

Divide throughout by x,

Let $\quad Y = \frac{y}{x} \quad$ and $\quad X = \frac{1}{x^2}$

Then the equation becomes $Y = mX + c$, where $m = b$ and $c = -a$

What is the Problem Solving Strategy for such problem?

Take note that the variables X and Y in $Y = mX + c$ must contain only the original variables x and/or y (and ***not*** the original unknown constants a and/or b). Similarly, the constants m and c must contain only the original unknown constants a and/or b (and ***not*** the original variables x and/or y).

Psst!

Method 3. Divide throughout by b

$$y = -ax + \frac{b}{x}$$

$$\frac{y}{bx} = -\frac{a}{b} + \frac{1}{x^2}$$

Divide throughout by bx,

$$\frac{1}{x^2} = \frac{1}{b}\left(\frac{y}{x}\right) + \frac{a}{b}$$

Rearranging for $\frac{1}{x^2}$ as the subject,

Let $\quad Y = \frac{1}{x^2} \quad$ and $\quad X = \frac{y}{x}$

Then the equation becomes $Y = mX + c$, where $m = \frac{1}{b}$ and $c = \frac{a}{b}$

Pause & Think

Which of the 3 methods do you prefer? Why?

Practise Now 1

Convert each of the following non-linear equations, where a and b are constants, into the linear form $Y = mX + c$. State what the variables X and Y and the constants m and c represent. There may be more than one method to do this.

Equation	Linear form	Y	X	m	c
(a) $y = 3x^2 + x$	$\frac{y}{x} = 3x + 1$	$\frac{y}{x}$	x	3	1
(b) $y = \sqrt{x} + \frac{2}{\sqrt{x}}$	$y\sqrt{x} = x + 2$	$y\sqrt{x}$	x	1	2
(c) $y^2 = 4x - 2$	$y^2 = 4x - 2$	y^2	x	4	-2
(d) $y = \frac{x+1}{x}$	$xy = x + 1$. OR $y = \frac{1}{x} + 1$	xy y	x $\frac{1}{x}$	1 1	1 1
(e) $\frac{1}{y} = \frac{\pi x}{x-1}$	$y = \frac{x-1}{\pi x}$ $xy = \frac{1}{\pi}(x) - \frac{1}{\pi}$	xy	x	$\frac{1}{\pi}$	$-\frac{1}{\pi}$
(f) $y = x^3 + ax + b$	$y - x^3 = ax + b$	$y - x^3$	x	a	b

Worked Example 2

Converting non-linear equations involving exponential functions into linear form

For each of the following, explain how each equation can be expressed in a form suitable for plotting a straight line graph to determine the values of the unknown constants a and b.

(a) $y = ae^{bx}$

(b) $y = ax^{b}$

(c) $y = ab^{x}$

Solution:

(a)

$$y = ae^{bx}$$

Take $\ln$ on both sides,

$$\begin{aligned} \ln y &= \ln ae^{bx} \\ &= \ln a + \ln e^{bx} && \textit{Apply laws of logarithms} \\ &= \ln a + bx && \textit{since } \ln e = 1 \\ &= bx + \ln a \end{aligned}$$

Thus, **a straight line graph is obtained when** $\ln y$ **is plotted against** x, where **the gradient is** b and **the vertical intercept is** $\ln a$.

(b)

$$y = ax^{b}$$

Take $\lg$ on both sides,

$$\begin{aligned} \lg y &= \lg ax^{b} \\ &= \lg a + \lg x^{b} && \textit{Apply laws of logarithms} \\ &= \lg a + b\lg x \\ &= b\lg x + \lg a \end{aligned}$$

Thus, **a straight line graph is obtained when** $\lg y$ **is plotted against** $\lg x$, where **the gradient is** b and **the vertical intercept is** $\lg a$.

(c)

$$y = ab^{x}$$

Take $\lg$ on both sides,

$$\begin{aligned} \lg y &= \lg ab^{x} \\ &= \lg a + \lg b^{x} && \textit{Apply laws of logarithms} \\ &= \lg a + x\lg b \\ &= (\lg b)x + \lg a \end{aligned}$$

Thus, **a straight line graph is obtained when** $\lg y$ **is plotted against** x, where **the gradient is** $\lg b$ **and the vertical intercept is** $\lg a$.

What is the Problem Solving Strategy for such problem?

- The mathematical constant e has a fixed value, unlike the unknown constants a and b.
- Use ln if base e appears in the equation; use lg if base 10 appears in the equation; otherwise, we can use either ln or lg.

Practise Now 2

Convert each of the following non-linear equations, where a and b are constants, into the linear form $Y = mX + c$. State what the variables X and Y and the constants m and c represent. There may be more than one method to do this.

Equation	Linear form	Y	X	m	c
(a) $y = a(10^{bx})$	$\lg y = \lg a + \lg 10^{bx}$ $\lg y = bx + \lg a$	$\lg y$	x	b	$\lg a$
(b) $y = Ae^{-kx}$	$\ln y = \ln A - kx \ln e$ $\ln y = -kx + \ln A$	$\ln y$	x	$-k$	$\ln A$
(c) $y + 10 = Ak^x$	$\lg(y+10) = \lg A + x \lg k$ $\lg(y+10) = x \lg k + \lg A$	$\lg(y+10)$	x	$\lg k$	$\lg A$
(d) $\frac{a}{y} = b^x$	$\lg a - \lg y = x \lg b$ $\lg y = -x \lg b + \lg a$	$\lg y$	x	$-\lg b$	$\lg a$
(e) $y^a x^b = 10$	$\lg y^a + \lg x^b = \lg 10$ $a \lg y + b \lg x = 1$ $\lg y = -\frac{b}{a} \lg x + \frac{1}{a}$	$\lg y$	$\lg x$	$-\frac{b}{a}$	$\frac{1}{a}$
(f) $y = x^3 + a^{x-b}$	$y - x^3 = a^{x-b}$ $\lg(y-x^3) = (x-b)\lg a$ $\lg(y-x^3) = x \lg a - b \lg a$	$\lg(y-x^3)$	x	$\lg a$	$-b \lg a$

More Practice

Convert each of the following non-linear equations, where a and b are constants, into the linear form $Y = mX + c$, where either the vertical variable Y or the horizontal variable X is given. State the values of m and c, where m is the gradient and c is the vertical intercept.

Equation	Linear form	Y	X	m	c
(a) $y = ax^2 + b\sqrt{x}$	$\frac{y}{\sqrt{x}} = ax\sqrt{x} + b$	$\frac{y}{\sqrt{x}}$	$x\sqrt{x}$	a	b
(b) $ax^3 + by^2 = 1$	$by^2 = -ax^3 + 1$ $y^2 = -\left(\frac{a}{b}\right)x^3 + \frac{1}{b}$	y^2	x^3	$-\left(\frac{a}{b}\right)$	$\frac{1}{b}$
(c) $y = \frac{a}{x+b}$	$y(x+b) = a$ $xy + by = a$ $xy = a - by$ $x = \frac{a}{y} - b$	x	$\frac{1}{y}$	a	$-b$
(d) $xy = \frac{a}{x} + bx$	$y = \frac{a}{x^2} + b$	y	$\frac{1}{x^2}$	a	b
(e) $y = \sqrt{ax+b}$	$y^2 = ax + b$	y^2	x	a	b
(f) $\ln y = a(x^2 + b)$	$\ln y = ax^2 + ab$	$\ln y$	x^2	a	ab

Equation	Linear form	Y	X	m	c
(g) $y = e - ax^b$	$y - e = -ax^b$ $\frac{y-e}{x^{b-1}} = -ax$	$\frac{y-e}{x^{b-1}}$	x	$-a$	0
(h) $y = \frac{(x+b)^2}{a}$, where $x > \cancel{0}$ $-b$	$ay = (x+b)^2$ $x+b = \sqrt{ay}$ $x = -b + \sqrt{ay}$ $= -b + \sqrt{a}\sqrt{y}$	x	$\sqrt{y}$	$\sqrt{a}$	$-b$

Concept Quiz 1 | Assignment 1

Checklist for Self-Assessment on Linear Law				
Learning Objectives	Knowledge & Skills	I need help. I need to see an example.	I can do this on my own.	I can do this on my own as well as explain my solution to my peer or teacher.
Transforming non-linear relations to linear form so as to determine the unknown constants from a straight line graph	**Algebra**: Distinction between constants and variables	☐	☐	☐
	Coordinate Geometry: Linear equation in gradient-intercept form $Y = mX + c$	☐	☐	☐
	Logarithmic Functions: Laws of logarithms	☐	☐	☐

5.3	**Converting linear form into non-linear equation**

Recap:

1. Gradient of line segment joining (x_1, y_1) and $(x_2, y_2) = \dfrac{y_2 - y_1}{x_2 - x_1}$
2. Equation of a straight line is given by
   (i) $y = mx + c$ where m is the gradient and c is the y-intercept
   (ii) $y - y_1 = m(x - x_1)$ where m is the gradient and (x_1, y_1) is a point on the line

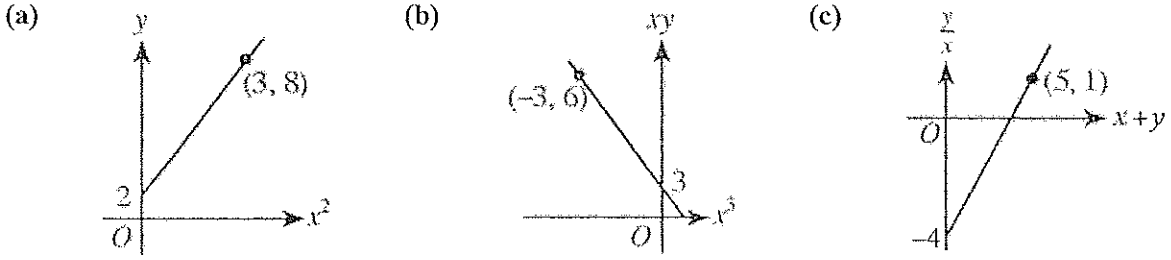
Steps to converting linear form to non-linear equation

Step 1: Find the gradient, m
Step 2: Form the equation in linear format
- $Y = mX + c$ if vertical intercept is given
- $Y - Y_1 = m(X - X_1)$ if vertical intercept is NOT given

Step 3: Express y in terms of x.

Worked Example 3
Converting linear form into non-linear equation given gradient & intercept

Each of the following diagrams shows part of a straight line graph. The vertical intercept and a point on each line are given. For each graph, form an equation expressing y in terms of x.



Solution:

(a) Gradient of straight line $= \dfrac{8-2}{3-0}$
$= 2$

Since the vertical intercept is 2, $\underline{\underline{y = 2x^2 + 2}}$ *Apply linear form $Y = mX + c$*

(b) Gradient of straight line $= \dfrac{6-3}{-3-0}$
$= -1$

Since the vertical intercept is 3, $xy = -1x^3 + 3$ *Apply linear form $Y = mX + c$*

$\underline{\underline{y = -x^2 + \dfrac{3}{x}}}$

(c) Gradient of straight line $= \dfrac{1-(-4)}{5-0}$
$= 1$

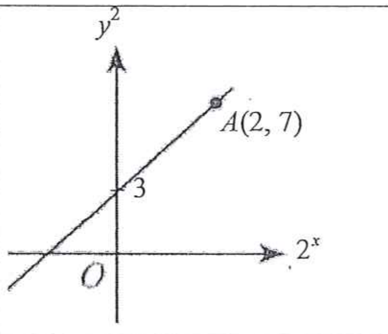
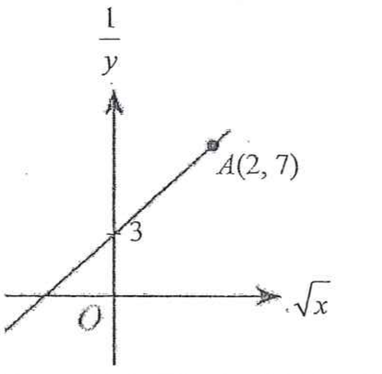
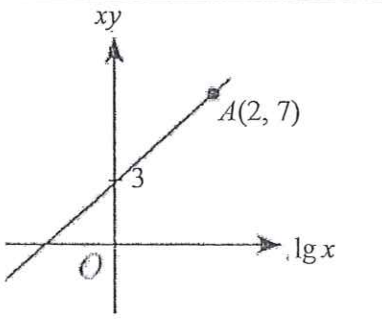
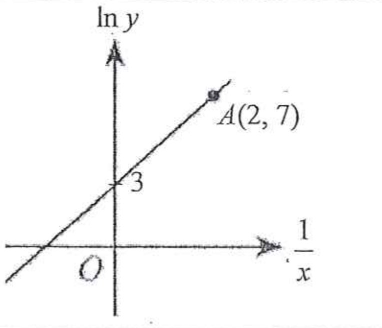
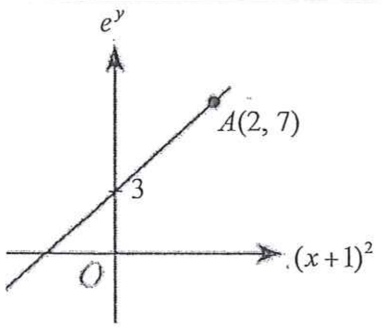
Since the vertical intercept is −4, $\dfrac{y}{x} = 1(x + y) - 4$ *Apply linear form $Y = mX + c$*

$y = x^2 + xy - 4x$

$\underline{\underline{y = \dfrac{x^2 - 4}{1 - x}}}$

Practise Now 3

Each of the following diagrams shows part of a straight line graph. The vertical intercept and a point on each line are given. For each graph, form an equation expressing y in terms of x.

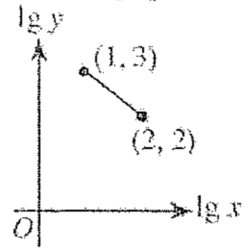
(a)		Gradient $= \frac{7-3}{2-0} = 2$ y^2-intercept $= 3$ $y^2 = 2(2^x) + 3$ $y = \pm\sqrt{2^{x+1}+3}$
(b)		Gradient $= \frac{7-3}{2-0} = 2$ $\frac{1}{y}$-intercept $= 3$ $\frac{1}{y} = 2\sqrt{x} + 3$ $y = \frac{1}{2\sqrt{x}+3}$
(c)		Gradient $= \frac{7-3}{2-0} = 2$ xy-intercept $= 3$ $xy = 2\lg x + 3$ $y = \frac{2\lg x + 3}{x}$
(d)		Gradient $= \frac{7-3}{2-0} = 2$ $\ln y$-intercept $= 3$ $\ln y = 2(\frac{1}{x}) + 3$ $y = e^{\frac{2}{x}+3}$
(e)		Gradient $= \frac{7-3}{2-0} = 2$ e^y-intercept $= 3$ $e^y = 2(x+1)^2 + 3$ $y = \ln[2(x+1)^2 + 3]$

Worked Example 4

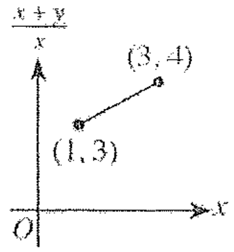
Converting linear form into non-linear equation given two points

Each of the following diagrams shows part of a straight line graph. Two points on each line are given. For each graph, form an equation expressing y in terms of x.

(a)



(b)



Solution:

(a) Gradient of straight line $= \dfrac{3-2}{1-2}$

$= -1$

Since (1, 3) lies on the line, $\lg y - 3 = -1(\lg x - 1)$ *Apply linear form $Y - Y_1 = m(X - X_1)$*

$$\lg y + \lg x = 4$$

$$\lg xy = 4$$

$$xy = 10^4$$

$$\underline{\underline{y = \frac{10000}{x}}}$$

(b) Gradient of straight line $= \dfrac{4-3}{3-1}$

$= \dfrac{1}{2}$

Since (1, 3) lies on the line, $\dfrac{x+y}{x} - 3 = \dfrac{1}{2}(x-1)$ *Apply linear form $Y - Y_1 = m(X - X_1)$*

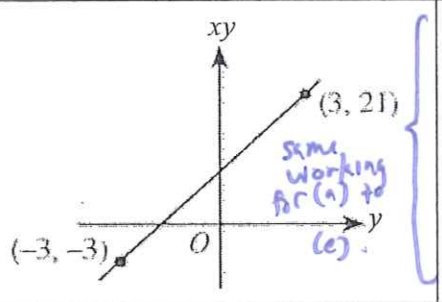
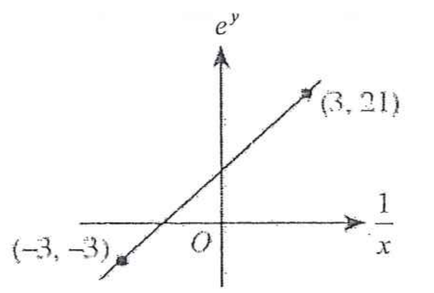
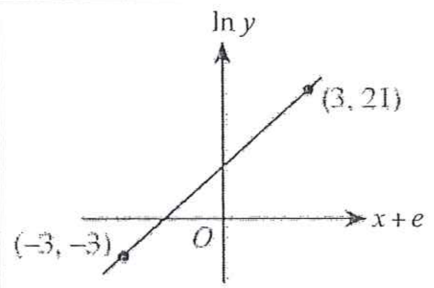
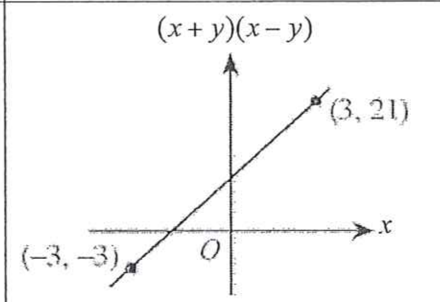
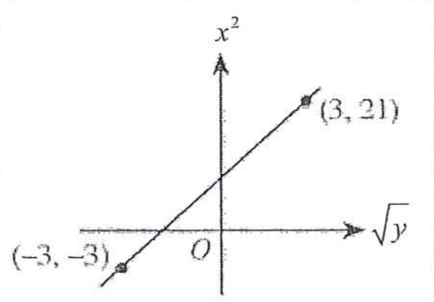
$$2x + 2y - 6x = x^2 - x$$

$$2y = x^2 + 3x$$

$$\underline{\underline{y = \frac{x^2 + 3x}{2}}}$$

Practise Now 4

Each of the following diagrams shows part of a straight line graph. Two points on each line are given. For each graph, form an equation expressing y in terms of x.

(a)	 same working for (a) to (e).	Gradient $= \frac{21-(-3)}{3-(-3)}$ $= 4$ Let $Y = 4X + C$ $21 = 4(3) + C$ $C = 9$ $\therefore Y + 4X + 9$ $xy = 4y + 9$
(b)		$e^y = 4(\frac{1}{x}) + 9$ $y \ln e = \ln(\frac{4}{x} + 9)$ $y = \ln(\frac{4}{x} + 9)$
(c)		$\ln y = 4(x+e) + 9$ $y = e^{4x+4e+9}$
(d)		$(x+y)(x-y) = 4x + 9$ $x^2 - y^2 = 4x + 9$ $y^2 = x^2 - 4x - 9$ $y = \pm\sqrt{x^2 - 4x - 9}$
(e)		$x^2 = 4\sqrt{y} + 9$ $4\sqrt{y} = x^2 - 9$ $\sqrt{y} = \frac{x^2}{4} - \frac{9}{4}$ $y = (\frac{x^2}{4} - \frac{9}{4})^2$

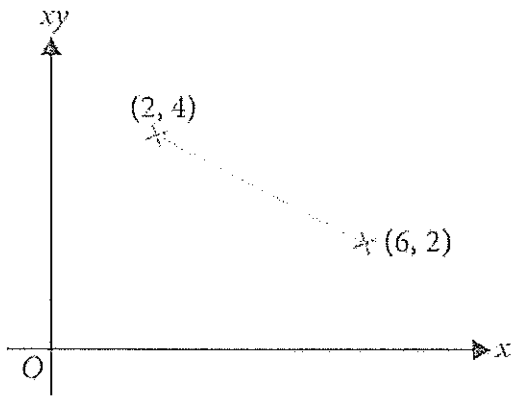
5.4 Finding unknown constants

Worked Example 5

Finding unknown constants The variables x and y are connected by the equation $y=\frac{a}{x}-b$, where a and b are constants. When a graph of xy against x is drawn, the resulting line passes through the points (2, 4) and (6, 2) as shown in the diagram. Find

(i) the value of a and of b,

(ii) the value of y when $y=\frac{6}{x}$.



Solution:

(i) Gradient of straight line $=\frac{4-2}{2-6}$

$$=-\frac{1}{2}$$

Since (2, 4) lies on the straight line, $xy-4=-\frac{1}{2}(x-2)$

$$xy=-\frac{1}{2}x+5 \qquad \text{......... (1)}$$

Method 1. Transform original non-linear equation into linear form $Y=mX+c$

Given $y=\frac{a}{x}-b$

$xy=-bx+a$ where $-b$ is gradient and a is vertical intercept

Then $-b=-\frac{1}{2} \Rightarrow \underline{\underline{b=\frac{1}{2}}}$

and $\underline{\underline{a=5}}$

Method 2. Transform linear form to original non-linear equation

$$xy=-\frac{1}{2}x+5$$

$$y=\frac{5}{x}-\frac{1}{2}$$

Comparing with the given equation $y=\frac{a}{x}-b$,

$\underline{\underline{a=5}}$ and $\underline{\underline{b=-\frac{1}{2}}}$

(b) When $y=\frac{6}{x}$, $xy=6$

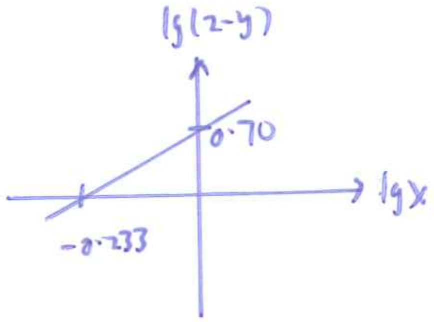
Sub. $xy=6$ into (1), $6=-\frac{1}{2}x+5$

$$x=-2$$

Sub. $x=-2$ into $y=\frac{6}{x}$, $\underline{\underline{y=-3}}$

Practise Now 5

The variables x and y are connected by the equation $y = 2 - px^q$ where p and q are unknown constants. A straight line graph is obtained by plotting $\lg(2-y)$ against $\lg x$.
The straight line cuts the horizontal and vertical axes at -0.233 and 0.70 respectively.
Find the value of p and of q.



$y = 2 - px^q$

$2 - y = px^q$

$\lg(2-y) = \lg p + q \lg x$

$\lg(2-y) = q \lg x + \lg p$

$\lg p = 0{\cdot}70$

$p = 10^{0{\cdot}70} = 5{\cdot}01$ (3 sf)

$q = \dfrac{0{\cdot}70 - 0}{0 - (-0{\cdot}233)}$

$= 3{\cdot}00$ (3 sf)

Concept Quiz 2 | Assignment 2

Checklist for Self-Assessment on Linear Law				
Learning Objectives	Knowledge & Skills	I need help. I need to see an example.	I can do this on my own.	I can do this on my own as well as explain my solution to my peer or teacher.
Converting linear form to non-linear equations given (1) gradient and vertical intercept, or (2) two points on the straight line	**Algebra**: Changing subject of a formula	☐	☐	☐
	Coordinate Geometry: Linear equation in (1) gradient-intercept form $Y = mX + c$, & (2) gradient-point form $Y - Y_1 = m(X - X_1)$	☐	☐	☐
	Logarithmic Functions: Laws of logarithms	☐	☐	☐

5.5 Applications of linear law

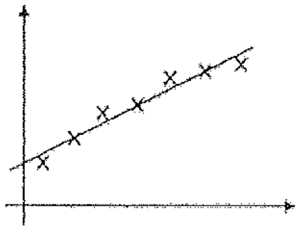
In this section, we will apply what we have learnt to convert a non-linear equation to linear form, plot a best-fit line from the data, and determine unknown constants from the straight line graph.

Steps to plotting a straight line graph

Step 1: Convert the non-linear function with variables x and y (or other variables) to a linear form $Y = mX + c$.
Step 2: Construct a table of values for X and Y.
Step 3: Choose a suitable scale & label the axes.
Vertical scale need not start from zero but horizontal scale MUST start from zero.
Step 4: Plot the points and join them with a best-fit **STRAIGHT LINE**.
Extend the line to cut the vertical axis.
Step 5: Using your graph, determine the gradient m and vertical intercept c.

What is the Problem Solving Strategy for such problem?

1. Deciding the scale
   - Choose a 'good' scale that is easy to read.
     Example: 2 cm to 1 unit, 1 cm to 0.5 unit, 4 cm to 1 unit
   - Ensure that all the points can be plotted with the axes.
   - Ensure that your graph crosses more than half of the graph paper.
   - Your line **MUST CUT THE VERTICAL AXIS**. Plan to include the value of vertical intercept on your vertical axis (instead of just the values from the table).
2. Draw the best-fit line
   - The points that are not on the line are close to the line (except for "known" error).
   - Approximate equal number of points on each side of the line (if not all points lie on the line).
     Example:



3. Calculating the gradient of line
   - Choose two points that are
     - far apart
     - easy to read

Worked Example 6 [GCEO2008/AM4038/P1/Q15]

The table shows values of variables x and y, which are related by the equation $y = ab^x$.

x	2	4	5	7	8
y	13	50	99	380	760

(i) Plot $\lg y$ against x and draw a straight line graph. [3]

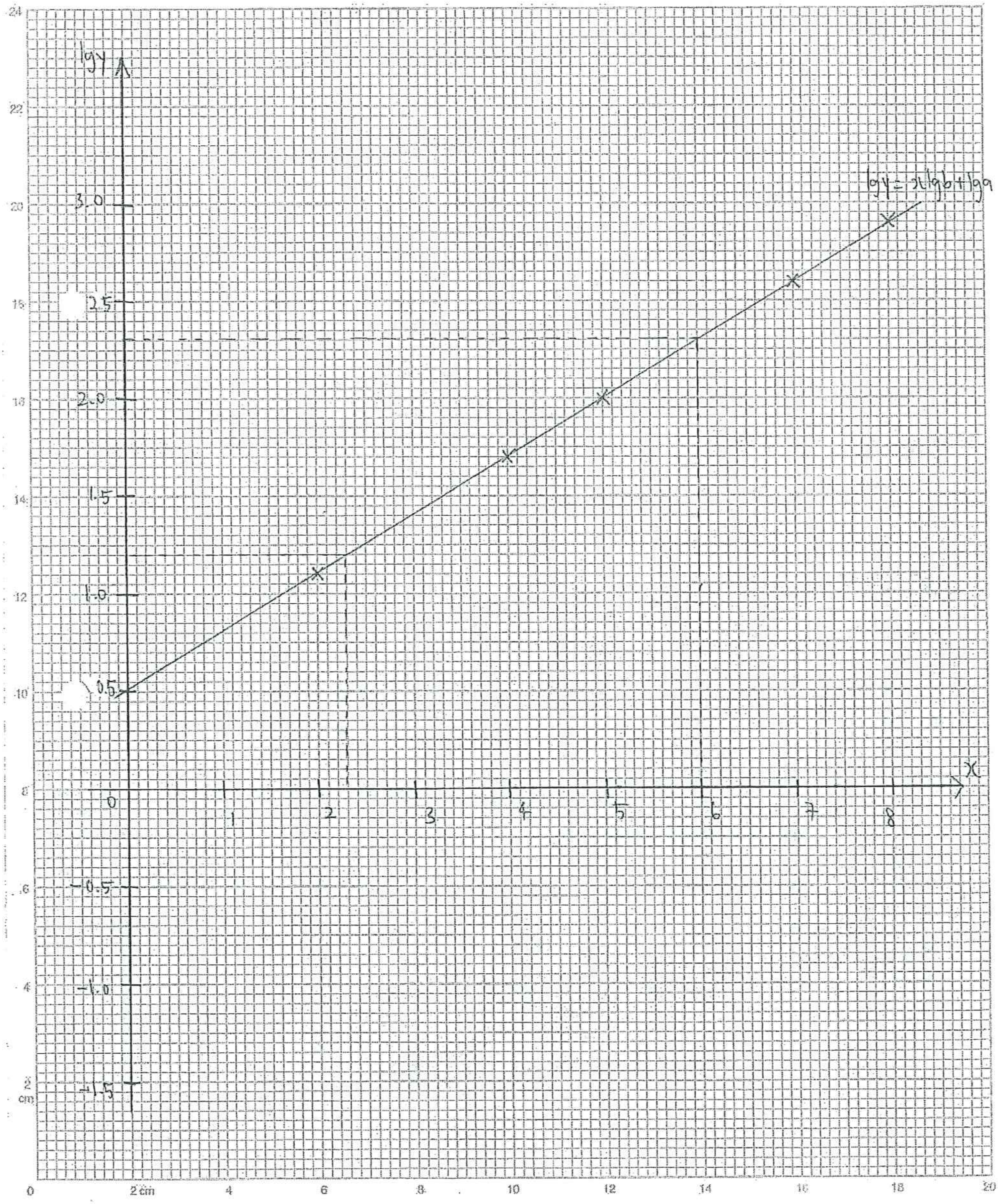
Use your graph to estimate, to 1 decimal places,

(ii) the value of a and of b, [4]

(iii) the value of x when $y = 200$. [2]

Name .... Index No ....

Subject .... worked Example 6 ....Class .... Date..



Worked Example 6

i)

x	2	4	5	7	8
$\lg y$	1.11	1.70	2.00	2.58	2.88

ii)

$y = ab^x$

$\lg y = \lg a + x \lg b$

$\lg y = (\lg b)x + \lg a$

From graph, vertical intercept $= 0.5$

$\therefore \lg a = 0.5$

$a = 10^{0.5}$

$a = 3.2$ (1dp)

$m = \frac{2.6 - 0.5}{7 - 0}$ } Most of graph?

$= 0.3$

$\therefore \lg b = 0.3$

$b = 10^{0.3}$

$b = 2.0$ (1dp)

iii) When $y = 200$,

$\lg y = \boxed{2.30 \text{ (2dp)}}$ Why not 5sf?

From graph, $\boxed{x = 1.2?}$

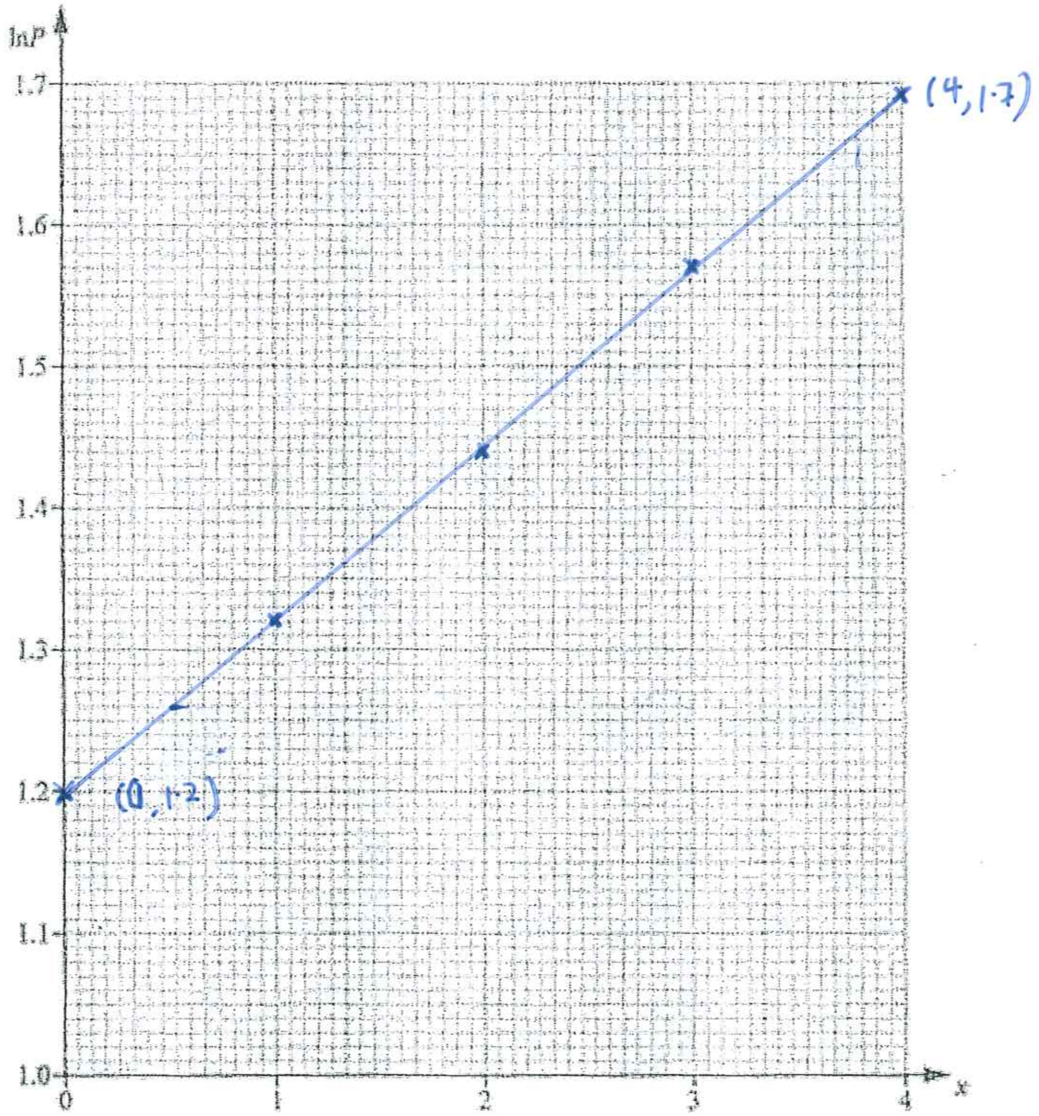
$x = 6$.

Worked Example 6 [GCEO2020/AM4047/P2/Q6]

The table shows, to 3 significant figures, the population, P, in millions, of a country on January 1st at intervals of five years from 1995 to 2015. The variable x is measured in units of 5 years.

Year	1995	2000	2005	2010	2015
x	0	1	2	3	4
P	3.32	3.75	4.24	4.80	5.43
$\ln P$	1.20	1.32	1.44	1.57	1.69

(i) On the grid below plot $\ln P$ against x and draw a straight line graph. [2]



(ii) Find the gradient of your straight line and hence express P in the form Ae^{kx}, where A and k are constants. [4]

(iii) If this model for the population remains valid, find the first year of the interval in which the population first exceeds 8 million. [3]

(ii) Gradient $= \dfrac{1.69 - 1.2}{4 - 0}$

$= 0.1225$

$\ln P = 0.1225x + 1.2$

$P = e^{0.1225x + 1.2}$

$P = e^{1.2}e^{0.1225x} = 3.32e^{0.123x}$ #

(iii) $P > 8000000$

$\ln P > 15.895$

$0.1225x + 1.2 > 15.895$

$0.1225x > 14.695$

$x > 119.96$

$x = 120$ #

Worked Example 7

The table shows experimental values of two variables x and y.

x	20	30	40	50	60
y	300	410	530	670	810

It is known that x and y are related by an equation of the form $y = \dfrac{a\sqrt{x}}{\sqrt{x}+b}$, where a and b are constants. Using the vertical axis for y and the horizontal axis for $\dfrac{y}{\sqrt{x}}$, draw a straight line graph of y against $\dfrac{y}{\sqrt{x}}$ for the given data.

(i) Use your graph to estimate

(a) the value of a and of b,

(b) the value of x when $y = 80\sqrt{x}$.

(ii) By drawing a suitable straight line, find the value of x and y which satisfy the simultaneous equations

$$y = \frac{a\sqrt{x}}{\sqrt{x}+b},$$
$$y\sqrt{x} = 900\sqrt{x} - 5y.$$

(iii) If a straight line graph of $\dfrac{\sqrt{x}}{y}$ is plotted against $\sqrt{x}$ instead, find the value of the gradient of this straight line.

Worked Example 7

i) $y = \frac{a\sqrt{x}}{\sqrt{x}+b}$

$y\sqrt{x} + by = a\sqrt{x}$

$y\sqrt{x} = -by + a\sqrt{x}$

$y = -b\frac{y}{\sqrt{x}} + a$

$\frac{y}{\sqrt{x}}$	67.08	74.86	83.80	94.75	104.57
y	300	410	530	670	810

From graph, vertical intercept $= -600$

$\therefore a = -600$

$m = \frac{880-(-600)}{110-0}$

$= 13.5$ (3sf).

$\therefore -b = 13.5$

$b = -13.5$

ii) When $y = 80\sqrt{x}$, $\frac{y}{\sqrt{x}} = 80$

From graph, when $\frac{y}{\sqrt{x}} = 80$, $y = 480$

Check
$y = \frac{1480}{110}(80) - 600$
$= 476.36$ (5sf)
$\boxed{\text{OK!}}$

iii) $y\sqrt{x} = 900\sqrt{x} - 5y$

$y = 900 - 5\frac{y}{\sqrt{x}}$

When $\frac{y}{\sqrt{x}} = 0$, $y = 900$ $(0, 900)$

When $\frac{y}{\sqrt{x}} = 180$, $y = 0$ $(180, 0)$

From graph, $y = 490$, $\frac{y}{\sqrt{x}} = 81 \Rightarrow \frac{490}{\sqrt{x}} = 81$

$\sqrt{x} = \frac{490}{81}$

$x = 36.6$ (3sf).

Ans: $x = 36.6$, $y = 490$

Check
① LHS $= 490$
RHS $= \frac{-600\left(\frac{490}{81}\right)}{\left(\frac{490}{81}\right) - \left(\frac{1480}{110}\right)}$
$= 490.15$ (5sf) $\boxed{\text{OK!}}$
② LHS $= 490 \times \frac{490}{81}$
$= 2964.2$ (5sf)
RHS $= 900\left(\frac{490}{81}\right) - 5(490)$
$= 2994.4$ $\boxed{\text{OK!}}$

iv) $y = \frac{-600\sqrt{x}}{\left(\sqrt{x} - \frac{1480}{110}\right)}$

$y\sqrt{x} - \frac{1480}{110}y = -600\sqrt{x}$

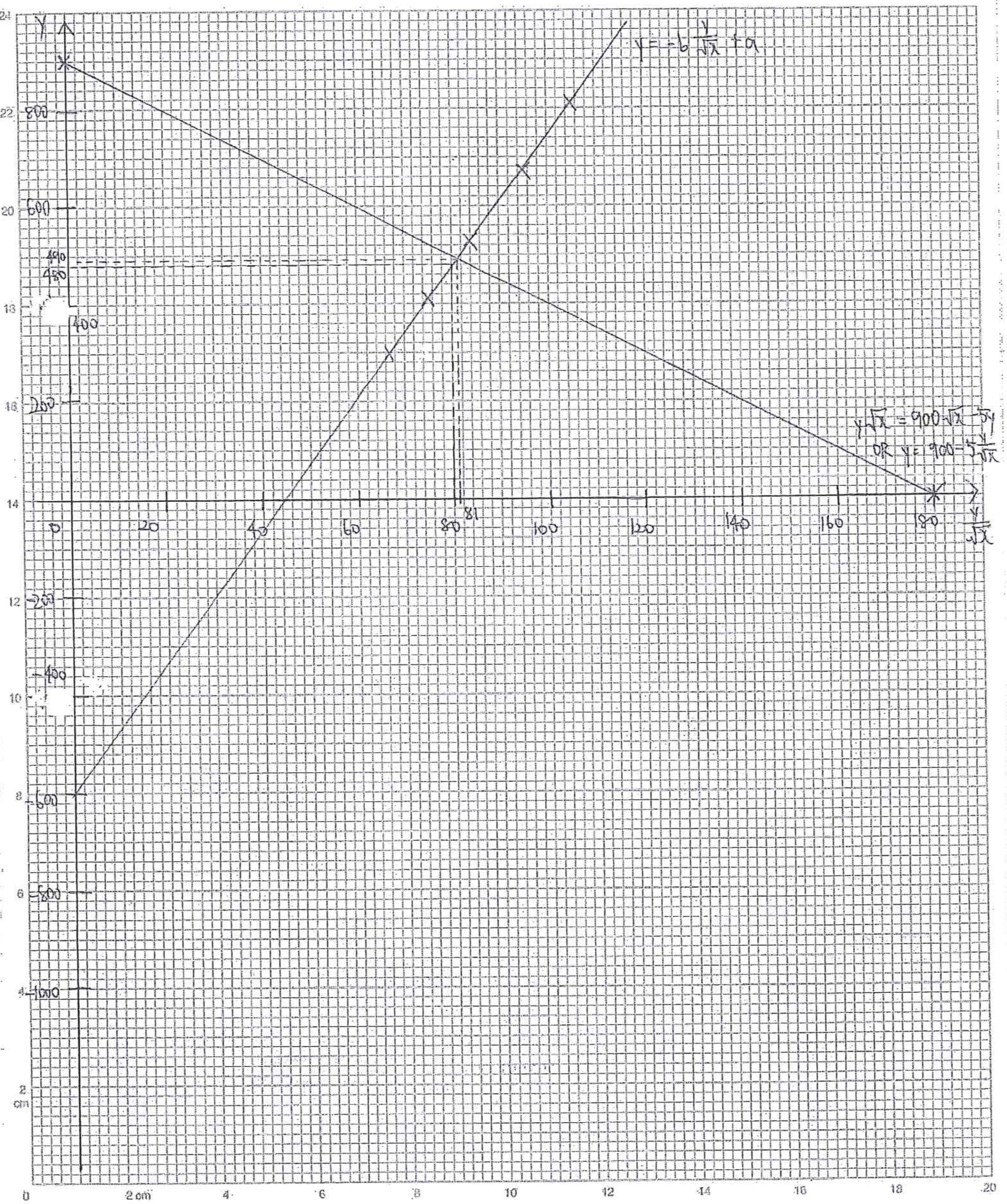
$-600\frac{\sqrt{x}}{y} = \sqrt{x} - \frac{1480}{110}$

$\frac{\sqrt{x}}{y} = -\frac{1}{600}\sqrt{x} + \frac{37}{1650}$

$\therefore m = -\frac{1}{600}$

Name .... Index No ....

Subject .... worked example 7 .... Class .... Date..................



Practise Now [TYS Questions]

[GCEO2007/AM4018/P1/Q12-OR] modified

The table shows experimental values of two variables x and y.

x	0.5	1.0	1.5	2.0
y	15.9	19.1	23.4	30.2

It is known that x and y are related by the equation $y = 10 + Ab^x$, where A and b are constants.

(i) Plot $\lg(y-10)$ against x and draw a straight line graph. [3]

(ii) Use your graph to estimate the value of A and of b. [4]

(iii) By drawing a suitable line on your graph, solve the equation $Ab^x = 10^{2x}$. [3]

[GCEO2002/AM4018/P1/Q12]

A rectangle of area y m^2 has sides of length x m^2 and $(Ax+B)$ m, where A and B are constants, and x and y are variables. Values of x and y are given in the table below.

x	0.5	1.0	1.5	2.0
y	15.9	19.1	23.4	30.2

(i) Use the data above in order to draw, on graph paper, the straight line of $\frac{y}{x}$ against x.

(ii) Use your graph to estimate the value of A and of B.

(iii) On the same diagram, draw the straight line representing the equation $y = x^2$ and explain the significance of the value of x given by the point of intersection of the two lines.

(iv) State the value approached by the ratio of the two sides of the rectangle as x becomes increasing large.

Concept Quiz 3 | Assignment 3

Checklist for Self-Assessment on Linear Law

Learning Objectives	Knowledge & Skills	I need help. I need to see an example.	I can do this on my own.	I can do this on my own as well as explain my solution to my peer or teacher.
Transforming non-linear relations to linear form so as to determine the unknown constants from a straight line graph Using interpolation and extrapolation to obtain data values using graph reading	**Algebra**: Algebraic manipulation	☐	☐	☐
	Coordinate Geometry: Linear equation in gradient-intercept form $Y = mX + c$, calculating gradient	☐	☐	☐
	Graphing: Choice of scale, plotting points, reading off data values	☐	☐	☐

5.6 Real-life applications of linear law

Radioactive Decay

The equation $A = A_0 e^{-\lambda t}$ describes radioactive decay, where A_0 denotes the initial activity, λ is the decay constant and t is the time.

A particular radioactive source has a decay constant of λ per second. After exactly 5 minutes, its activity had fallen to 30 000 disintegrations per second. After another 5 minutes, its activity had fallen to 20 000 disintegrations per second. Calculate its initial activity (A_0) and its decay constant (λ).

Simple Pendulum

The periodic time T (time for each complete swing) of a simple pendulum is given by the formula $T = 2\pi\sqrt{\frac{L}{g}}$ where L is the length of pendulum, g is the acceleration due to gravity. Measurement can be made of the period T for various values of length L. The table shows some experimental data.

T (s)	1.1	1.2	1.6	1.8
L (m)	0.3	0.4	0.6	0.8

Determine the value of g.

Alternating Current

The formula $I = \dfrac{V}{\sqrt{R^2 + (2\pi f L)^2}}$ applies to the alternating current I that flows in a circuit containing inductance (L) and resistance (R) in series. V is the alternating (sinusoidal) voltage causing the current and f is the frequency.

In a typical experiment f is varied and corresponding value of I are recorded.

What graph should be plotted for a straight line to be obtained and how could measurements from the graph allow L and R to be obtained?

TYS Q2

i) $Y = 10 + Ab^x$

$y - 10 = Ab^x$

$\lg(y-10) = \lg A + x \lg b$

x	0.5	1.0	1.5	2.0
$\lg(y-10)$	0.77085	0.95904	1.1271	1.3053

From graph, vertical intercept = 0.6

$\therefore \lg A = 0.6$

$A = 3.98$ (3sf)

$m = \dfrac{1.275 - 0.6}{1.925 - 0}$

$= 0.35065$ (5sf)

$\therefore \lg b = 0.35065$

$b = 2.24$ (3sf)

ii) $Ab^x = 10^{2x}$

$Ab^x + 10 = 10^{2x} + 10$

Plot $y = 10^{2x} + 10$

When $x = 0$, $y = 11 \Rightarrow \lg(y-10) = \lg(11-10)$

$= 0$

When $x = 1$, $y = 110 \Rightarrow \lg(y-10) = \lg(110-10)$

$= 2$

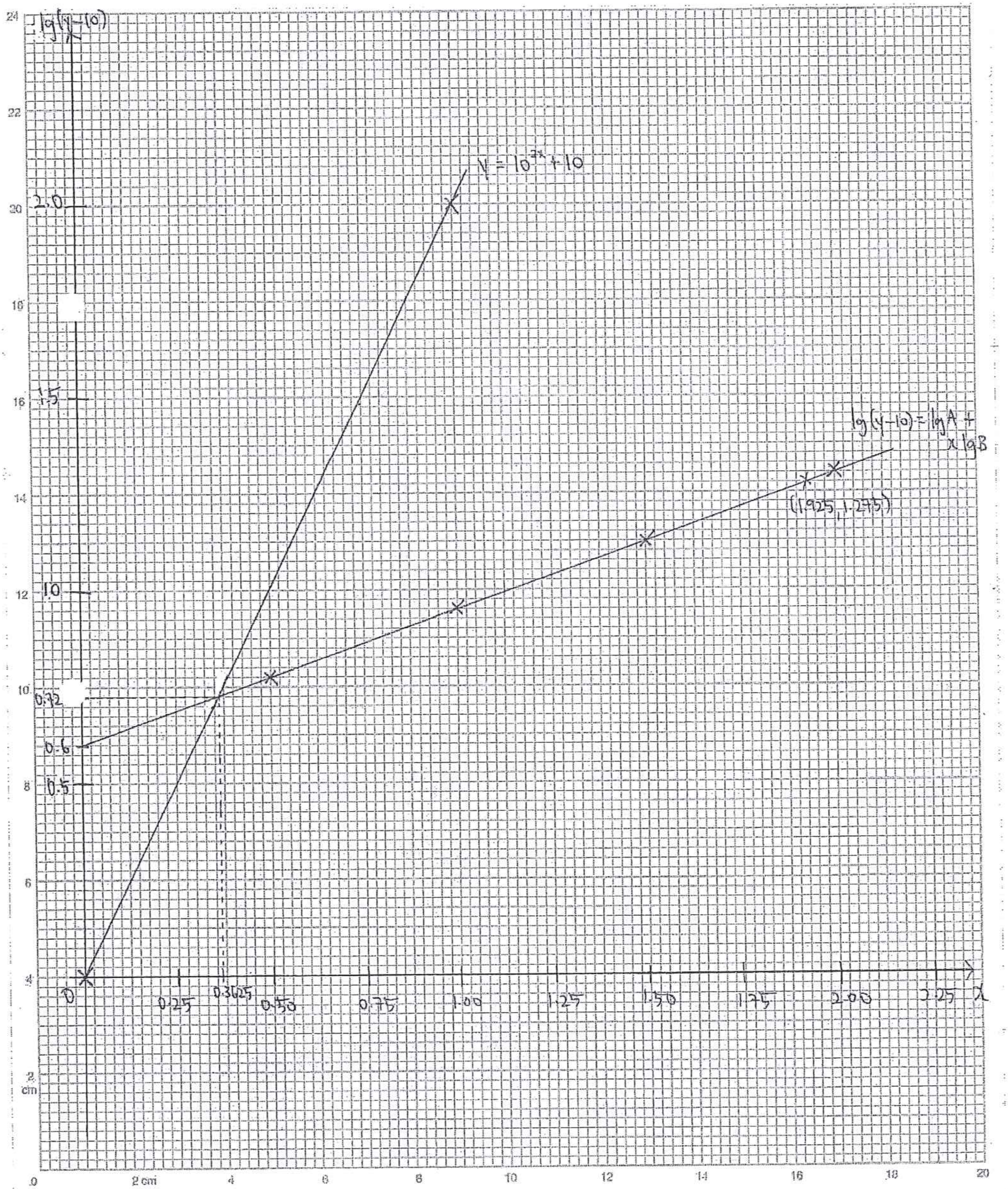
From graph, $x = 0.3625$, $\lg(y-10) = 0.725 \Rightarrow y - 10 = -0.13966$ (5sf)

$Y = 9.86$ (3sf)

Ans: $x = 0.3625$, $y = 9.86$

Name .... Index No ....

Subject TYS ..., Q2 .... Class .... Date ....



REVIEW & REFLECT

- What were the mathematical concepts or ideas that you have learnt?
- Reflect on a mistake that you have made or a misconception that you used to have. What do you learn from this mistake or misconception?
- What questions or uncertainties do you still have about… ? If you don't have a question or uncertainty, write a similar problem to a skill you wish to practise more and solve it.

Assignment 1: Converting non-linear equation into linear form

Textbook: Additional Maths 360 Volume A (2nd Edition), Marshall Cavendish
Exercise 9.1, pages 218 – 219

- **Tier A**: Questions 1, 2
- **Tier B**: Questions 8, 9

Assignment 2: Converting linear form into non-linear equation

Textbook: Additional Maths 360 Volume A (2nd Edition), Marshall Cavendish
Exercise 9.1, pages 219 – 220

- **Tier A**: Question 6
- **Tier B**: Questions 11, 12, 15
- **Tier C**: Questions 16, 17

Assignment 3: Applications of linear law

Textbook: Additional Maths 360 Volume A (2nd Edition), Marshall Cavendish
Exercise 9.2, pages 227 – 229

- **Tier A**: Questions 3, 4
- **Tier B**: Questions 5, 11, 12
- **Tier C**: Questions 13, 15