Fourier Series

Consider a single valued function $f(x)$ in the interval a to $a+2l$ which satisfies Dirichelet conditions.

Dirichlet Conditions

1. f is defined in $(a, a+2l)$, $f(x+2l) = f(x)$ ie length of Interval is period of the function.

2. f is continuouse in $(a, a+2l)$ or There must be a finite number of points of discontunity

3. f has no local maxima or minima or has finite number of maxima and minima in $(a, a+2l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \underbrace{\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)}_{\text{fourier series}}$$

$$a_0 = \frac{1}{L} \int_a^{a+2L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_a^{a+2l} f(x) \cos\frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{L} \int_a^{a+2L} f(x) \sin\frac{n\pi x}{L}\, dx$$

fourier coeficiants

① Obtain fourier series for $f(x) = \left(\frac{\pi - x}{2}\right)^2$

in 0 to 2π deduce that $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

→ $a = 0,\ l = \pi$

$$\therefore a_0 = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi - x)^2}{4}\, dx$$

$$= \frac{1}{4\pi} \left[\frac{(\pi - x)^3}{-3} \right]_0^{2\pi}$$

$$= \frac{-1}{12\pi} \left\{ -\pi^3 - \pi^3 \right\} = \frac{2\pi^2}{12} = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} (\pi - x)^2 \cos nx \, dx$$

By parts

$$= \frac{1}{\pi} \left[(\pi - x)^2 \frac{\sin nx}{n} - 2(\pi - x)(-1)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right) \right]_0^{2\pi}$$

$$= \frac{-1}{2\pi n^2} \left((\pi - x) \cos nx \right]_0^{2\pi}$$

$$= \frac{-1}{2\pi n^2} (-\pi - \pi) = +\frac{1}{n^2}$$

$$\therefore a_n = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \sin nx\,(\pi - x)^2$$

$$= \frac{1}{4\pi}\Big[(\pi-x)^2\left(-\frac{\cos nx}{n}\right) - 2(\pi-x)(-1)\left(-\frac{\sin nx}{n^2}\right)$$

$$+ 2\left(\frac{\cos nx}{n^3}\right)\Big]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2}{n} + \frac{2}{n^3}\right) - \left(-\frac{\pi^2}{n} + \frac{2}{n^3}\right)\right]$$

$$= 0$$

$$\therefore f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$+ \sum_{n=1}^{\infty} \cancel{b_n}^{\,0} \sin nx$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$$

at $x=0$,

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\therefore \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots$$

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at $x=\pi$

$$0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\therefore \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

Add ① & ②

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Parseval's Identity

$$\frac{1}{l}\int_0^{a+2l} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$\therefore \frac{1}{\pi}\int_0^{2\pi} (\pi-x)^4\,dx = \frac{\pi^4}{36\times 2} + \sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right)$$

$$\therefore \frac{1}{16\pi}\left[\frac{(\pi-x)^5}{-5}\right]_0^{2\pi} = \frac{\pi^4}{72} + \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\therefore -\frac{1}{80\pi}\left[-\pi^5 - \pi^5\right] = \frac{\pi^4}{72} + \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\therefore \frac{\pi^4}{40} - \frac{\pi^4}{72} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\therefore \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

(2) find fourier series for e^{-x} in $(0, 2\pi)$

& find (1) $\sum_{2}^{\infty} \frac{(-1)^n}{n^2+1}$ (2) $\operatorname{cosech} \pi$

$$f(x) = \frac{a_n}{2} + \sum_{1}^{\infty} a_n \cos n\pi + \sum_{1}^{\infty} b_n \sin n\pi$$

$$a_0 = \frac{1}{l}\int_{0}^{a+2l} f(x)\, dx \qquad a_n = \frac{1}{l}\int_{0}^{a+2l} \cos(nx) f(x)$$

$$b_n = \frac{1}{l}\int_{0}^{a+2l} \sin(nx) f(x)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{x}\,dx = \frac{\left[e^{-x}\right]_0^{2\pi}}{\pi} = -\frac{e^{-2\pi}}{\pi} + \frac{1}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{n^2+1}\left(-\cos nx + n\sin nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi(n^2+1)}\left(1-e^{-2\pi}\right)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}\left(-\sin nx + n\cos nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\frac{1}{(n^2+1)}\left[e^{-2\pi}(n)\right] - n$$

$$= \frac{-2n}{\pi i(n^2+1)}\left[1-e^{-2i\pi}\right]$$

$$\therefore e^{-x} = \frac{1-e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty} \frac{(1-e^{-2\pi})}{\pi(n^2+1)} \cos nx$$

$$+ \sum_{n=1}^{\infty} \frac{(1-e^{-2\pi})\, n}{\pi(n^2+1)} \sin nx$$

at $x = \pi$

$$e^{-\pi} = 1-e^{-2\pi}\left(\frac{1}{2\pi} + \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi(n^2+1)} + 0\right)$$

$$\frac{e^{-\pi}}{1-e^{-2\pi}} - \cancel{\frac{1}{2\pi}} = \cancel{\frac{-1}{2\pi}} + \sum_{n=2}^{\infty} \frac{(-1)}{\frac{(n^2+1)}{\pi}}$$

$$\therefore A_n = \frac{e^{-\pi}\,\pi}{1-e^{-2\pi}}$$

$$\operatorname{cosech} x = \frac{2}{e^{x} - e^{-x}}$$

$$\operatorname{cosech} \pi = \frac{2}{e^{\pi} - e^{-\pi}} = \frac{2e^{-\pi}}{1 - e^{-2\pi}}$$

Substitute value.

$$\operatorname{cosech} \pi = \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^{n}}{n^{2}+1}$$

③ find fourier series for $f(x) = \cos px$

$p \notin Z$ in $(0, 2\pi)$

Deduce that

① $\pi \operatorname{cosec} p\pi = \frac{1}{p} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{p+n} + \frac{1}{p-n} \right]$

② $\pi \cot 2p\pi = \frac{1}{2p} + p \sum_{n=1}^{\infty} \frac{1}{p^2 - n^2}$

$$a_0 = \frac{1}{L} \int_a^{a+2L} f(x)\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \cos px\, dx$$

$$= \frac{1}{\pi} \left[\frac{\sin px}{p} \right]_0^{2\pi} = \frac{1}{\pi p} \sin(2\pi p x)$$

$$a_n = \frac{1}{L}\int_a^{a+2L} \cos nx\, f(x)\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \cos px \cos nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\cos(p+n)x + \cos(p-n)x\right)$$

$$= \frac{1}{2\pi}\left[\frac{\sin(p+n)x}{(p+n)} + \frac{\sin(p-n)x}{(p-n)}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\sin(p+n)2\pi}{(p+n)} + \frac{\sin(p-n)x}{(p-n)}\right]$$

$$= \frac{1}{2\pi} \sin 2p\pi\left(\frac{1}{(p+n)} + \frac{1}{(p-n)}\right)$$

$$b_n = \frac{1}{l}\int_a^{a+2l} \sin nx \, f(x)\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sin nx \cos(px)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\sin(n+p)x - \sin(n-p)x\right) dx$$

$$= \frac{1}{2\pi}\left(\frac{\cos (n+p)x}{n+p} - \frac{\cos(n-p)}{n-p}\right)\Bigg)_0^{2\pi}$$

$$= \frac{1}{2\pi}\left(\frac{\cos(n+p)2\pi}{n+p} - \frac{\cos(n-p)2\pi}{n-p}\right)$$

$$- \frac{1}{2\pi}\left(\frac{-1}{n+p} + \frac{1}{n-p}\right)$$

$$\therefore \frac{1}{2\pi}\cos 2p\pi \left(\frac{1}{n+p} + \frac{1}{n-p}\right) - \frac{1}{2\pi}\left(\frac{1}{n-p} - \frac{1}{n+p}\right)$$

$$\cos px = \frac{\sin 2p\pi}{2p\pi} + \sum_{n=1}^{\infty} p\, \frac{p \sin 2p\pi}{\pi(p^2-n^2)} \cos nx$$

$$+ \sum_{n=1}^{\infty} -\frac{p \cos(2p\pi)}{\pi}\, \frac{p}{(p^2-n^2)} \sin nx$$

$$+ \sum_{n=1}^{\infty} \frac{-1}{2\pi}\, \frac{\sin nx}{p^2-n^2}$$

at $x = \pi$

$$\cos p\pi = \frac{\sin 2p\pi}{2p\pi} + \sum_{n=1}^{\infty} (-1)^n \frac{\sin 2p\pi}{2\pi(p^2-n^2)}\, p$$

$$\frac{1}{2\sin p\pi} = \frac{1}{p\pi} + \sum_{n=1}^{\infty} (-1)^n \frac{p}{\pi(p^2-n^2)}$$

$$\therefore \pi \operatorname{cosec} p\pi = \frac{1}{p} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{p+n} - \frac{1}{p-n} \right)$$

$$x = 2\pi$$

$$\cos 2\pi p = \frac{\sin 2p\pi}{2p\pi} + \sum_{n=1}^{\infty} \frac{\sin 2p\pi \; p}{\pi(p^2-n^2)}$$

$$\therefore \pi \cot 2p\pi = \frac{1}{2p} + \sum_{n=1}^{\infty} \frac{1}{p^2-n^2}.$$

In interval $(-\pi, \pi)$

find fourier series for $f(x)$ $\begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 \le x \le \pi \end{cases}$

Deduce that $\frac{1}{1\cdot 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \dots = \frac{1}{2}$

$$\frac{1}{1\times 3} - \frac{1}{3\times 5} + \dots = \frac{1}{4}(\pi - 2)$$

$L \to \pi$

$a \to -\pi$, $a + 2L = \pi$

$$a_0 = \frac{1}{L}\int_a^{a+2L} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_0^{\pi} \sin x\,dx$$

$$= \frac{-1}{\pi}\left[\cos x\right]_0^{\pi} = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} \cos x\,(0)\, dx + \frac{1}{\pi}\int_{0}^{\pi} \cos nx \cdot \sin x$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \sin(n+1)x - \sin(n-1)x$$

$$= -\frac{1}{\pi}\left[\frac{\cos n+1x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} = \frac{1}{2\pi}\left(\frac{(-1)^{n-1}-1}{n-1} + \frac{(-1)^{n+1}-1}{n+1}\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} \sin nx\,(0)\, dx + \frac{1}{\pi}\int_{0}^{\pi} \sin nx \sin x$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \cos(n-1)x - \cos(n+1)x$$

$$n \neq 1 \quad = \frac{1}{\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{(n+1)} \right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[0 \right] = 0$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2 x \, dx = 1/2$$

$$f(x) = \frac{1}{\pi} + \sum_{n=1}^{\infty} \frac{2\left((-1)^{n+1} - 1\right)\cos nx}{(n^2-1)\,2\pi} + \frac{1}{2}\sin x$$

at $x = 0 \quad f(x) = 0$

$$0 = \frac{1}{\pi} + \sum_{n=1}^{\infty} \frac{\cancel{2}\left((-1)^{n+1} - 1\right)}{(n^2-1)\cancel{2}\pi}$$

$$-1 = \sum_{n=1}^{\infty} \frac{\left((-1)^{n+1} - 1\right)}{(n+1)(n-1)}$$

$$\therefore -1 = a_1 + \frac{-2}{1 \times 3} + 0 + \frac{-2}{3 \times 5} + \cdots$$

$a_1 = \quad (0/0 \text{ form})$

$$= \frac{1}{\pi} \int_0^{\pi} \sin x \cos x \, dx$$

$$= \frac{1}{2\pi} \left(- \frac{\cos 2x}{2} \right)_0^{\pi}$$

$$= -\frac{1}{4\pi} \left(\cos 2\pi - \cos 0 \right) = 0$$

$$\therefore \quad -1 = -2 \left[\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots \right]$$

$$\therefore \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \frac{1}{2}$$

at $x = \pi/2$

$$\sin\frac{\pi}{2} = \frac{1}{\pi} + \frac{1}{\pi} \sum_{1}^{\infty} \frac{\left((-1)^{n+1} - 1\right)}{(n^2-1)} \cos n\frac{\pi}{2} + \frac{1}{2}\sin\frac{\pi}{2}$$

$$= \frac{1}{\pi} + \frac{1}{\pi} \left(-\frac{2\cos\pi}{1\cdot 3} - \frac{2\cos 3\pi}{3\cdot 5} - \cdots \right)$$

$$\therefore \; i = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\left[\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \cdots\right]$$

$$\therefore \left(\frac{i}{2} - \frac{1}{\pi}\right) = -\frac{2}{\pi}\left(\qquad \right)$$

$$-\frac{\pi}{2}\left(\frac{1}{2} - \frac{1}{\pi}\right) = (\quad)$$

$$\frac{1}{4}(\pi - 2) = (\quad)$$

⑤ Obtain fourier expansion of $f(x) = |\cos x|$ in the interval ($-\pi$ to π) Hence obtain fourier series for $|\sin x|$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x| = \frac{2}{\pi}\int_{0}^{\pi} |\cos x|$$

$$= \frac{2}{\pi}\left[\int_{0}^{\pi/2} \cos x - \int_{\pi/2}^{\pi} \cos x\right]$$

$$= \frac{2}{\pi}\left[(\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}\right] = \frac{2}{\pi}\left[1-0-0-1\right]$$

$$= \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos x| \cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} |\cos x| \cos x$$

$$= \frac{1}{\pi}\int_{0}^{\pi/2} 2\cos x \cos nx - \int_{\pi/2}^{\pi} 2\cos x \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} (\cos(n+1)x + \cos(n-1)x \,] \, dx \right.$$

$$\left. - \int_{\pi/2}^{\pi} \cos(n+1)x + \cos(n-1)x \, dx \right]$$

$$= \frac{1}{\pi}\left[\left[\frac{\sin(n+1)x}{n+1} - \frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2} \right.$$

$$\left. - \left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_{\pi/2}^{\pi} \right)$$

$$= \frac{2}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} \right]$$

for odd n, $\quad \sin(n+1)\pi/2 = 0$

$n = 2k+1 \quad \sin(n-1)\pi/2 = 0$

for even $n \quad \sin(n+1)\pi/2 = (-1)^k$

$n = 2k \quad \sin(n-1)\pi/2 = (-1)^{k+1} = -(-1)^k$

$$\therefore a_n = \frac{-4(-1)^k}{\pi(n^2-1)} \quad \text{for even } n = 2k$$

Alternative

$$\sin(n \pm 1)\frac{\pi}{2} = \sin n\frac{\pi}{2}\cos\frac{\pi}{2} \pm \cos n\frac{\pi}{2}\sin\frac{\pi}{2}$$

$$= \underline{+}\cos n\frac{\pi}{2}$$

$$\cos n\frac{\pi}{2} \begin{cases} \cos k\pi = (-1)^k & n = 2k \\ 0 & n = 2k+1 \end{cases}$$

Since $n > 1$ ($n-1$ in denominator)

find a_1

$$a_1 = \frac{2}{\pi}\int_{-\pi}^{\pi} \cos^2 x \, dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} \sin nx \cos x + \frac{1}{\pi}\int_{0}^{\pi} \sin nx \cos x$$

$$= -\frac{1}{\pi}\int_{0}^{\pi} \sin nx \cos x + \frac{1}{\pi}\int_{0}^{\pi} \sin nx \cos x$$

$$= 0$$

for all even functions $b_n = 0$

$$\therefore |\cos x| \doteq \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{4k^2-1}\cos 2kx$$

$$x \to x + \pi/2$$

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{4k^2-1}\cos(2kx + k\pi)$$

$$\cos(2kx + k\pi) = \cos 2kx\,(-1)^k + 0$$

$$\therefore |\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2kx}{(4k^2-1)}$$

⑥ $f(x) = x + \pi/2 \qquad -\pi < x < 0$

$\qquad = \frac{\pi}{2} - x \qquad 0 < x < \pi$

Deduce ① $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

② $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) = \frac{1}{\pi}\int_{-\pi}^{0}\left(x + \frac{\pi}{2}\right)dx$$

$$+ \frac{1}{\pi}\int_{0}^{\pi}\left(\frac{\pi}{2} - x\right)dx$$

$$\cdot -\frac{1}{\pi}\int_{0}^{-\pi}\left(x + \frac{\pi}{2}\right)dx = \frac{1}{\pi}\int_{0}^{\pi}\left(\frac{\pi}{2} - x\right.$$

$$\therefore a_0 = \frac{2}{\pi}\int_{0}^{\pi}(\pi - x)\,dx$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2}x - \frac{x^2}{2} \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2} \right) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \left(\frac{\pi}{2} - x\right) \cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\pi}{2}\cos nx - x\cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\pi}{2}\cos nx \quad - \frac{2}{\pi}\left[x\,\frac{\sin nx}{n} - \int \frac{\sin nx}{n}\right]$$

$$= \frac{2}{\pi}\frac{\pi}{2}\left[\frac{\sin nx}{n}\right]_0^{\pi} - \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= 0 + -\frac{2}{\pi} \frac{\cos n\pi}{n^2} + \frac{2}{\pi n^2}$$

$$= \frac{2}{\pi (n^2)} (1 - \cos n\pi)$$

$$= \frac{2}{\pi n^2} (1 - (-1)^n)$$

$$n = 2k-1 \longrightarrow \frac{2 \cdot 2}{\pi (2k+1)^2}$$

$$n = 2k \rightarrow 0$$

$b_n = 0$ for even function.

$$f(x) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{4}{\pi (2k-1)^2} \cos(2k-1)$$

at $x = 0$ $\quad f(0) = \frac{\pi}{2}$ (point of discontinuous function)

$$\frac{\pi}{2} = \frac{4}{\pi}\left(\frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Parseval's Identity

$$\frac{1}{l}\int_a^{a+2l} (f(x))^2 = \frac{a_0}{2} + \sum_{i=1}^{\infty} (a_i^2 + b_i^2)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2 = \frac{2}{\pi}\int_0^{\pi} (f(x))^2 = \frac{2}{\pi}\int_0^{\pi}\left(x - \frac{\pi}{2}\right)^2 dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 - \pi x + \frac{\pi^2}{4} = \frac{2}{\pi}\left[\frac{x^3}{3} - \pi\frac{x^2}{2} + \frac{\pi^2}{4}x\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{3} - \frac{\pi^3}{2} + \frac{\pi^3}{4}\right] = 2\pi^2\left[\frac{1}{3} - \frac{1}{2} + \frac{1}{4}\right]$$

$$= 2\pi^2 \left[\frac{4}{12} - \frac{6}{12} + \frac{3}{12} \right]$$

$$= \frac{\pi^2}{6}$$

$$\therefore \frac{\pi^2}{6} = 0 + \sum_{k=1}^{\infty} \left(\frac{4}{\pi (2k+1)^2} \right)^2$$

$$\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{16}{(2k+1)^4}$$

$$\therefore \frac{\pi^2}{96} = \frac{1}{1} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$$

⑧ $f(x) = 2x - x^2$ for $0 \leq x \leq 3$

$$a_0 = \frac{1}{L} \int_0^{2L} f(x)\,dx \qquad 2l = 3$$

$$l = 3/2$$

$$= \frac{2}{3} \int_0^3 (2x - x^2)\,dx$$

$$= \frac{2}{3} \left[x^2 - \frac{x^3}{3} \right]_0^3 = \frac{2}{3} [9 - 9) = 0$$

$$a_n = \frac{2}{3} \int_0^3 \cos\left(\frac{n\pi x}{l}\right) (2x - x^3)$$

$$= \frac{2}{3} \int_0^3 (2x - x^3) \cos \frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[(2x-x^2)\frac{\sin\frac{2n\pi x}{3}}{\frac{2n\pi x}{3}}\right.$$

$$+ \frac{(2-2x)\cos\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^2}$$

$$\left. + \frac{-2\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi x}{3}\right)^3}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{-4-2}{2n^2\pi^2}\right]$$

$$= -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} \sin\left(\frac{n\pi x}{l}\right) f(x)\, dx$$

$$= \frac{2}{3}\int_0^{3} (2x - x^2) \sin\left(\frac{2n\pi x}{3}\right) dx$$

$$= \frac{2}{3}\left[\frac{-(2x-x^2)\cos\left(\frac{2n\pi x}{3}\right)}{\frac{2n\pi x}{3}} - \frac{(2-2x)\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi x}{3}\right)^2} - \frac{(-2)\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi x}{3}\right)^3} \right]_0^{\pi}$$

$$= \frac{2}{3}\left[+ \frac{3 \times 3}{2n\pi} \right]$$

$$= \frac{3}{n\pi}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{1}^{\infty}\left(a_n \cos \frac{n\pi}{l}\right.$$

$$\left. + b_n \sin \frac{n\pi}{l}\right)$$

$$= -\frac{9}{\pi^2}\sum_{1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{3} + \frac{3}{\pi}\sum_{1}^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{3}$$

In Interval (- L to L)

① $$f(x) = \begin{cases} 0 & -2 < x < 1 \\ 1+x & -1 < x < 0 \\ 1-x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

$f(x)$ is even $f(-x) = f(x)$

$b_n = 0$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^1 (1-x)\,dx$$

$$= x - \frac{x^2}{2} = 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}$$

$$= \int_0^1 (1-x)\cos\frac{n\pi x}{2}\,dx$$

$$= \left[\frac{(1-x)\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - \frac{\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right]_0^1$$

$$= \frac{\cos\frac{n\pi}{2}}{\left(\frac{n\pi}{2}\right)^2} - \frac{1}{\left(\frac{n\pi}{2}\right)^2}$$

$$= \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)$$

~~at $n = 2k$, $a_n = -\frac{4}{n^2\pi^2}\left((-1)^{2k} - 1\right)$~~

~~dont do This~~

~~$n = 2k+1 \quad = \frac{+4}{n^2\pi^2}$~~

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos n\frac{\pi x}{2}$$

$$= \frac{1}{4} + \frac{4}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{n^2} \cos \frac{n\pi x}{2}\left(1 - \cos\frac{n\pi}{2}\right)$$

① Find Half range cosine series for

$f(x) = x$ in $(0, 2)$ & deduce

$$\frac{1}{1} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$$

$$a_0 = \frac{2}{2}\int_0^l f(x)\,dx$$

$$= \frac{2}{2}\int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2 = 2$$

$$a_n = \frac{2}{2}\int_0^2 x \cos\frac{n\pi}{2} x$$

$$= \left(\frac{x \sin\frac{n\pi}{2}}{\frac{n\pi}{2}} - \left(-\frac{\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right)_0^2$$

$$= \frac{4}{n^2\pi^2}\left[\cos n\pi - 1\right]$$

$$a_n = \frac{4}{n^2\pi^2}\left((-1)^n - 1\right) \;\therefore \begin{cases} 0 & n = 2k \\ 1 & n = 2k-1 \end{cases}$$

$$\therefore f(x) = x = 1 - \frac{8}{\pi^2}\sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi^2}$$

$$\cos (2k-1)\frac{\pi}{2}x$$

Parseval

$$\frac{2}{L}\int_0^L f(x)^2 = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\left[\frac{x^3}{3}\right] = 2 + \frac{64}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}$$

$$\therefore \left(\frac{8}{3}-2\right)\frac{\pi^4}{6^4} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}$$

$$\therefore \frac{\pi^4}{96} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}$$

(2) Find Cosine series for period 2π to represent $\sin x$ in $(0,\pi)$ Deduce

(1) $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots=\frac{1}{2}$

(2) $\frac{1}{1^2\cdot 3^2}+\frac{1}{3^2\cdot 5^2}+\cdots=\frac{\pi^2-8}{16}$

(3) $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$

$f(x) = \sin x$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} \sin x \, dx$$

$$= \frac{2}{\pi} - \left[\cos x\right]_0^{\pi} = -\frac{2}{\pi} \{-1 - +1\}$$

$$= +\frac{4}{\pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin(n+1)x + \sin(1-n)x$$

$$= \frac{1}{\pi} \left[\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)}{n-1} \right]_0^{\pi}$$

$n \neq 1$

$$= \frac{1}{\pi}\left[-\frac{1}{(n+1)} + \frac{1}{(n-1)} + \frac{(-1)^{n+1}}{(n+1)} - \frac{(-1)^{n-1}}{(n-1)}\right]$$

$$\therefore \frac{-(-1)^n - 1}{(n+1)} + \frac{(-1)^n + 1}{(n-1)}$$

$$= \frac{-1}{\pi}\left((-1)^n + 1\right)\left(\frac{1}{(n-1)} - \frac{1}{n+1}\right) = \frac{1}{\pi}\frac{2}{n^2-1}\left((-1)^n + 1\right)$$

$n = 2k+1 = 0$

$n = 2k$

$$= \frac{-2}{\pi}\left[\frac{2}{(2k+1)^2 - 1)}\right]$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \cos x \sin x$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x = -\frac{1}{\pi}\frac{1}{2}\left[\cos 2x\right]_0^{\pi}$$

$$= 0$$

at $x = \pi/2$,

$$1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)(2k+1)}$$

$$\frac{\pi-2}{4} = -\frac{1}{2}\sum_{k=1}(-1)^k \frac{1}{(2k-1)} - \frac{1}{2k+1}$$

$$= +\frac{1}{2}\left(1 - \frac{1}{3} - \frac{1}{3} + \frac{1}{5} + \frac{1}{5} + \cdots\right)$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{5} + \cdots\cdots$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \cdots$$

③ Prove that In $0 < x < \pi$

$$\frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi}\left[\frac{\sin x}{a^2+1} - \frac{2\sin 2x}{a^2+4} + \frac{3\sin 3x}{a^2+9} + \cdots \right]$$

$$f(x) = e^{ax} - e^{-ax}$$

find Half range Sine series

$$b_n = \frac{2}{\pi} \int_0^{\pi} \left(e^{ax} - e^{-ax}\right) \sin nx \, dx$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b\cos bx\right]$$

$$\therefore \; = \frac{2}{\pi} \int_0^{\pi} e^{ax} \sin bx \, dx - \int_0^{\pi} e^{-ax} \sin nx \, dx$$

$$= \frac{2}{\pi}\left[\frac{e^{ax}}{a^2+n^2}\left(a \sin nx - n\cos nx\right)_0^{\pi}\right.$$

$$\left. - \frac{e^{-ax}}{a^2+n^2}\left(-a \sin nx - n \cos nx\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2} - n(-1)^n - \frac{1}{a^2+n^2}(-n)\right.$$

$$\left. - \frac{e^{-a\pi}}{a^2+n^2}\left[-n)(-1)^n\right) + \frac{1}{a^2+n^2}(-n))\right]$$

$$= \frac{2}{\pi} \frac{n(-1)^n}{(a^2+n^2)} \left(e^{a\pi} - e^{-a\pi}\right)$$

$$\frac{2\left(e^{ax} - e^{-a\pi}\right)}{\pi} = \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{\left(e^{a\pi} - e^{-a\pi}\right)(n)}{(n^2+a^2)}$$

$$(-1)^{n+1} \sin nx$$

$$\therefore \frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi} \left(\frac{\sin x}{a^2+1} - 2\frac{\sin 2x}{a^2+4} + 3 \ldots \right.$$

$$\therefore \sin x = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_n \cos\left(2k+1 \frac{\pi x}{\pi}\right)$$

$$\sin x = \frac{4}{2\pi} + -\frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k+1)(2k-1)} \cos\left((2k+1)\frac{\pi x}{\pi}\right)$$

at $x = 0$

$$\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \cdots \infty$$

Parseval Identity

$$\frac{2}{l}\int_0^l f(x)^2 = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_n^2$$

$$\frac{2}{l}\int_0^l \sin^2 x = \frac{2}{\pi}\int_0^\pi \frac{1}{2} - \frac{\cos 2\theta}{2} = \frac{2}{\pi}\left[\frac{\pi}{2} - \frac{[\sin 2x]_0^\pi}{4}\right.$$

$$= 1$$

$$1 = \frac{16}{2 \times 4 \pi^2} + \frac{16}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2 (2k+1)^2}$$

$$\frac{\pi^2 - 8}{16} = \frac{1}{1 \cdot 3^2} + \frac{1}{3^2 5^2} + \frac{1}{5^2 7^2} + \dots \infty$$

Half range cosine series for $f(x)$ in $(0, l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$a_0 = \frac{2}{l} \int_0^L f(x)\, dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}$$

Half range Sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$= \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

① $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_a^{a+2L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_a^{a+2L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_a^{a+2L} f(x)\sin\frac{n\pi x}{L}\,dx$$

② $$\frac{1}{L}\int_a^{a+2L} f(x)^2\,dx = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Parseval Identity

* for series in interval $-\pi$ to π check for odd or even function

$$f(x)\begin{cases} \to -\pi & -\pi < x < 0 \\ \to x & 0 < x < \pi \end{cases}$$

$$f(-x)\begin{cases} \to -x & -\pi < x < 0 \\ \to -\pi & \pi > x > 0 \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\pi + \int_{0}^{\pi} x$$

$$\pi\, a_0 = -\pi\,\left[x\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi}$$

$$\pi a_0 = -\pi\left[0 - -\pi\right) + \frac{\pi^2}{2}$$

$$a_0 = -\pi + \frac{\pi}{2} = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx$$

$$= \int_{-\pi}^{0} -\pi \cos nx + \int_{0}^{\pi} x \cos nx$$

$$= -\pi\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[x\,\frac{\sin nx}{n}\right.$$

$$\left. + \frac{\cos nx}{n^2}\right]_{0}^{\pi}$$

$$= -\pi\left[0 + \cancel{\frac{\sin n\pi}{n}}\right] +$$

$$\left[\pi\,\cancel{\frac{\sin n\pi}{n}} + \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$\frac{(\cos n\pi - 1)}{n^2}$$

$$\cos n\pi = (-1)^n$$

$$\therefore a_n = \frac{(-1)^n - 1}{\pi n^2}$$

$$b_n = \int_{-\pi}^{\pi} f(x) \sin(nx)$$

$$= \int_{-\pi}^{0} -\pi \sin nx + \int_{0}^{\pi} x \sin nx$$

$$= +\pi \left[\frac{\cos nx}{n}\right]_{-\pi}^{0} + \left[\frac{x \cos nx}{-n} + \frac{\sin nx}{n^2}\right]_{0}^{\pi}$$

$$\pi b_n = \pi \left[\frac{1}{n} - \frac{\cos n\pi}{n} \right]$$

$$- \pi \frac{\cos n\pi}{n} + \cancel{\frac{\sin n\pi}{n^2}}^{\,0}$$

$$= \frac{1}{n} - \frac{2\cos n\pi}{n}$$

$$\frac{1 - 2(-1)^n}{n}$$

$n \to 2k \pm 1$ $\qquad$ $\frac{3}{n}$