Initial Value theorem

$$f(0) = \lim_{z \to \infty} F(z)$$

$$f(1) = \lim_{z \to \infty} z \left[F(z) - f(0) \right]$$

$$f(2) = \lim_{z \to \infty} z^2 \left[F(z) - f(0) - \frac{f(1)}{z} \right]$$

$\vdots \qquad \vdots$

$$f(n) = \lim_{z \to \infty} z^n \left[F(z) - \sum_{p=0}^{n-1} \frac{f(p)}{z^p} \right]$$

Find $f(0)$ $f(1)$ $f(2)$ for $F(z) = \dfrac{2z^2+5z+14}{(z-1)^4}$

$$f(0) = \lim_{z \to \infty} F(z)$$

$$= \lim_{z \to \infty} \frac{2z^2+5z+14}{(z-1)^4} = 0$$

$$\therefore f(0) = 0$$

$$f(1) = \lim_{z \to \infty} z\left(F(z)\right) = \frac{2z^3+5z^2+14z}{(z-1)^4} = 0$$

$$f(2) = \lim_{z \to \infty} z^2 \left[F(z) - f(0) - \frac{f(1)}{z} \right)$$

$$= \lim_{z \to \infty} z^2 \left(\frac{2z^2+5z+14}{(z-1)^4} \right)$$

$$= 2$$

$$f(3) = \lim_{z \to 0} z^3 \left(f(z) - 0 - \frac{0}{z} - \frac{2}{z^2} \right)$$

$$= \frac{2z^5 + 5z^4 + 14z^3}{(z-1)^4} - 2z^3$$

$$= 5$$

Convolution

If $Z^{-1}[F(z)] = f(n)$

$Z^{-1}[G(z)] = g(n)$

$$Z^{-1}[f(3)\,G(3)] = \sum_{n=0}^{n} f(m)\,g(n-m)$$

① $$Z^{-1}\left[\frac{z^2}{(z-a)(3-b)}\right]$$ $$Z^{-1}\left(\frac{z}{z-a}\right) = a^n$$

$$Z^{-1}\left[\frac{z}{z-a} * \frac{z}{z-b}\right] = \sum_{n=0}^{n} a^{m}\, b^{n-m}$$

GP $_{(a>b)}$

$$= b^n \sum_{n=0}^{n} \left(\frac{a}{b}\right)^m$$

$$= b^n \frac{\left(\frac{a}{b}\right)^{n-1}}{\frac{a}{b}-1} = \frac{a^{n+1}-b^{n+1}}{a-b}$$

(2) $Z^{-1}\left(\left(\frac{z}{z-a}\right)^3\right)$

first find $Z^{-1}\left(\frac{z^2}{(z-a)^2}\right)$

$$= Z^{-1}\left(\frac{z}{z-a} \cdot \frac{z}{z-a}\right)$$

$$= \sum_{m=0}^{n} a^m a^{n-m}$$

$$= \sum_{m=0}^{n} a^n$$

$$= a^n \sum_{m=0}^{n} 1$$

$$= a^n (n+1)$$

$$F^{-1}\left(\frac{z^3}{(z-a)^3}\right) = \frac{z^2}{(z-a)^2} \cdot \frac{z}{z-a}$$

$$= \sum_{m=0}^{n} a^m (m+1)\, a^{n-m}$$

$$= a^n \sum_{m=0}^{n} (m+1)$$

$$= a^n \left(\frac{n(n+1)}{2} + n\right)$$

(3) $z^{-1}\left[\dfrac{8z^2}{(2z-1)(4z-1)}\right]$

$$= z^{-1}\left[\frac{z}{(z-1/2)} \cdot \frac{z}{(z-1/4)}\right]$$

$$= \sum_{m=0}^{n} (1/2)^m \, (1/4)^{n-m}$$

Binomial expansion

Find $z^{-1}\left(\frac{4z}{z-a}\right)$ for $|z|>|a|$

$|z|<|a|$

Solution :→ $\frac{4z}{z-a} = 4z(z-a)^{-1}$

$= -\frac{4z}{a}\left(1-\frac{z}{a}\right)^{-1}$

$\left|\frac{z}{a}\right|<1$

$= -\frac{4z}{a}\left(1+\frac{z}{a}+\frac{z^2}{a^2}+\frac{z^3}{a^3}+\cdots\right)$

$= -\frac{4}{a}\sum_{n=0}^{\infty}\frac{z^{n+1}}{a^n}$

$= -4\sum_{n=0}^{\infty} a^{-(n+1)} z^{n+1}$

Replace $n+1 \to -k$

$= -4\sum_{k=-1}^{-\infty} a^{k} z^{-k} = \left[-4a^k\right]_{-\infty}^{-1}$

For $|z| < |a|$

$$\frac{4z}{z-a} = \frac{4z}{z\left(1-\frac{a}{z}\right)}$$

$$= 4\left(1-\frac{a}{z}\right)^{-1}$$

$$= 4\left[1+\frac{a}{z}+\frac{a^2}{z^2}+\cdots\right]$$

$$= 4\sum_{0}^{\infty} a^n z^{-n}$$

$$= Z[a^n]$$

$$\therefore Z^{-1} = 4a^n$$

(4) $Z^{-1}\{(z-5)^{-3}\}$ for $|z| > 5$

$(|5/z| < 1)$

$$(z-5)^{-3} = z^{-3}\left(1 - \frac{5}{z}\right)^{-3}$$

$$= \frac{1}{z^3}\left(1 + \frac{3 \times 5}{z} + \frac{6 \times 5^2}{z^2} + \cdots\right)$$

$$= \frac{1}{z^3}\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} \frac{5^n}{z^n}$$

$$= \sum_{n=0}^{\infty} \frac{n\ n+1}{2}\ 5^n\ z^{-(n+3)}$$

$n+3 \to k$

$$= \sum_{k=0}^{\infty} \frac{(k-3)(k-2)}{2}\ 5^{k-3}\ z^{-k}$$

$$Z^{-1} = \left\{\frac{1}{2}\ \frac{(k-3)(k-2)}{2}\ 5^{k-3}\right\}_0^{\infty}$$

$$(1-x)^{-1} = 1+x+x^2+x^3+\cdots$$

$$(1-x)^{-2} = 1+2x+3x^2+\cdots\cdots$$

$$(1-x)^{-3} = 1+3x+6x^2+10\cdots$$

$1 \quad 3 \quad 6 \quad 10 \longrightarrow$ Triangular numbers

$$\frac{(n)(n+1)}{2}$$

Partial fraction method

$$\frac{4z^2-2z}{z^3-5z^2+8z-4}$$

$$= \frac{2z(2z-1)}{(z-2)(z-2)(z-1)}$$

$$= \frac{2z(2z-1)}{(z-1)(z-2)^2}$$

Partial fractions

$$= \frac{2z}{z-1} + \frac{2z}{z-2} + \frac{2z}{(z-2)^2}$$

$$z^{-1} = 2(1)^n + 2(2)^n + 2z^{-1}\left(\frac{z}{(z-2)^2}\right)$$

$$Z\left[a^n\right] = \frac{z}{(z-a)}$$

$$Z\left[n\, a^n\right] = \frac{a z}{(z-a)^2}$$

$$Z\left[n^2 a^n\right] = \frac{a z (z+a)}{(z-a)^3}$$

$$\therefore \quad Z^{-1}\left(\frac{3 \times 2 z}{(z-2)^2}\right) = 3 \times n 2^n$$

① Find inverse z transform using Partial fraction

$$F(z) = \frac{3^3 - 20z}{(z-2)^3(z-4)}$$

$z^3 - 6z^2 + \underline{\underline{6z}}$

-8

$-12z \quad +6z$

-14

$$(z^2 - 20) = A(z-1)^2(z-4)$$

$$+ B(3-2)(3-4)$$

$$+ C(3-4)$$

$$+ D(3-2)^3$$

at $3 = 2$, $\quad C = 8$

at $3 = 4$, $\quad D = -1/2$

at $z=0$

$$\frac{z^3-20}{(z-2)^3(z-4)} =$$

$$-\frac{20}{(8)(-4)} = \frac{A}{-2} + \frac{B}{4} + \frac{C}{8} + \frac{D}{-4}$$

$\therefore 2B = 4A + C + 20 \quad \therefore \quad 1$

at $z=1$

$-19 = -3A + 3B - 3C - D \quad 2$

Sub C & D in 1 2 & solve

$A = 1/2$

$B = 2$

$$\therefore F(3) = \frac{3}{2(3-2)} + \frac{2\,3}{(3-2)^2}$$

$$+ \frac{8\,3}{(3-2)^3} - \frac{3}{2(3-4)}$$

combine

$$F^{-1} = \frac{2^n}{2} + \cancel{2n2^n}$$

$$+ \quad - n2^n$$

$$F^{-1}\left(\frac{8\,3}{(3-2)^3} + \frac{2\,3}{(3-2)^2} \right)$$

$$= \frac{4\,3\,(3+2)}{(3-2)^3}$$

$$F^{-1}\left(\frac{4z(z+2)}{(z-2)^3} \right) = n^2 \, 2^n$$

② $F(z) = \dfrac{2(z^2 - 5z + 6.5)}{(z-2)(z-3)^2}$ $2 < |z| < 3$

$$= \frac{A}{(z-2)} + \frac{C}{(z-3)^2}$$

$$+ \frac{B}{(z-3)}$$

$\therefore 2(z^2 - 5z + 6.5) = A(z-3)^2$
$+ B(z-3)(z-2)$
$+ C(z-2)$

at $z = 2$ $A = 1$

$z = 3$ $C = 1$ } Subs value

$z = 0$ $B = 1$

$\therefore F(z) =$ $\left|\frac{2}{z}\right| < 1$ $\left|\frac{z}{3}\right| < 1$

$$\frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2}$$

$$z^{-1} = \frac{1}{z} \frac{1}{\left(1 - \frac{2}{z}\right)} + \frac{1}{3}\left(\frac{1}{\frac{z}{3} - 1}\right) + \frac{1}{9} \frac{1}{\left(\frac{z}{3} - 1\right)^2}$$

$$= \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{3}\left(1 - \frac{z}{3}\right)^{-1} + \frac{1}{9}\left(1 - \frac{z}{3}\right)^{-2}$$

$$= \frac{1}{z}\left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \cdots\right)$$

$$- \frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \cdots\right)$$

$$+ \frac{1}{9}\left(1 + 2\frac{z}{3} + 3\frac{z^2}{3^2} + \cdots\right)$$

$$= \sum_{n=0}^{\infty} 2^{n} z^{-(n+1)} + -\sum_{n=0}^{\infty} \frac{1}{3^{n+1}} z^{n} + \frac{1}{9} \sum_{0}^{\infty} (n+1) z^{n}$$

$$= + \sum_{k=1}^{\infty} 2^{k-1} z^{-k} + \dots$$

$$z^{-1} = \left(2^{n-1}\right)_{1}^{\infty} + \left[-\frac{1}{3^{n+1}} + \frac{(n+1)}{9}\right]_{-\infty}^{0}$$

$$z^{-1} \begin{cases} 2^{n-1} & n = 1 - \infty \\ -\frac{1}{3^{n+1}} + \frac{n+1}{9} & n = -\infty : 0 \end{cases}$$

$$\frac{1}{z-a} = -\frac{1}{a}\left(1-\frac{z}{a}\right)^{-1}$$

$$= -\frac{1}{a}\sum \frac{z^n}{a^n}$$

Z transform of $f(k)$

$$Z\{f(k)\} = F(z) = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}$$

two sided z transform

⊛ $Z\{af(k) + bg(k)\} = aZ\{f(k)\} + bZ\{g(k)\}$

⊛ If $Z[f(k)] = F(z)$ (z is complex number)

$$Z[a^k f(k)] = f\left(\frac{z}{a}\right)$$

⊛ Shifting property

$Z[f(k)] = F(z)$ then

$$Z[f(k \pm n)] = z^{\pm n} F(z)$$

① $Z[e^{\alpha k}]$ $k > 0$

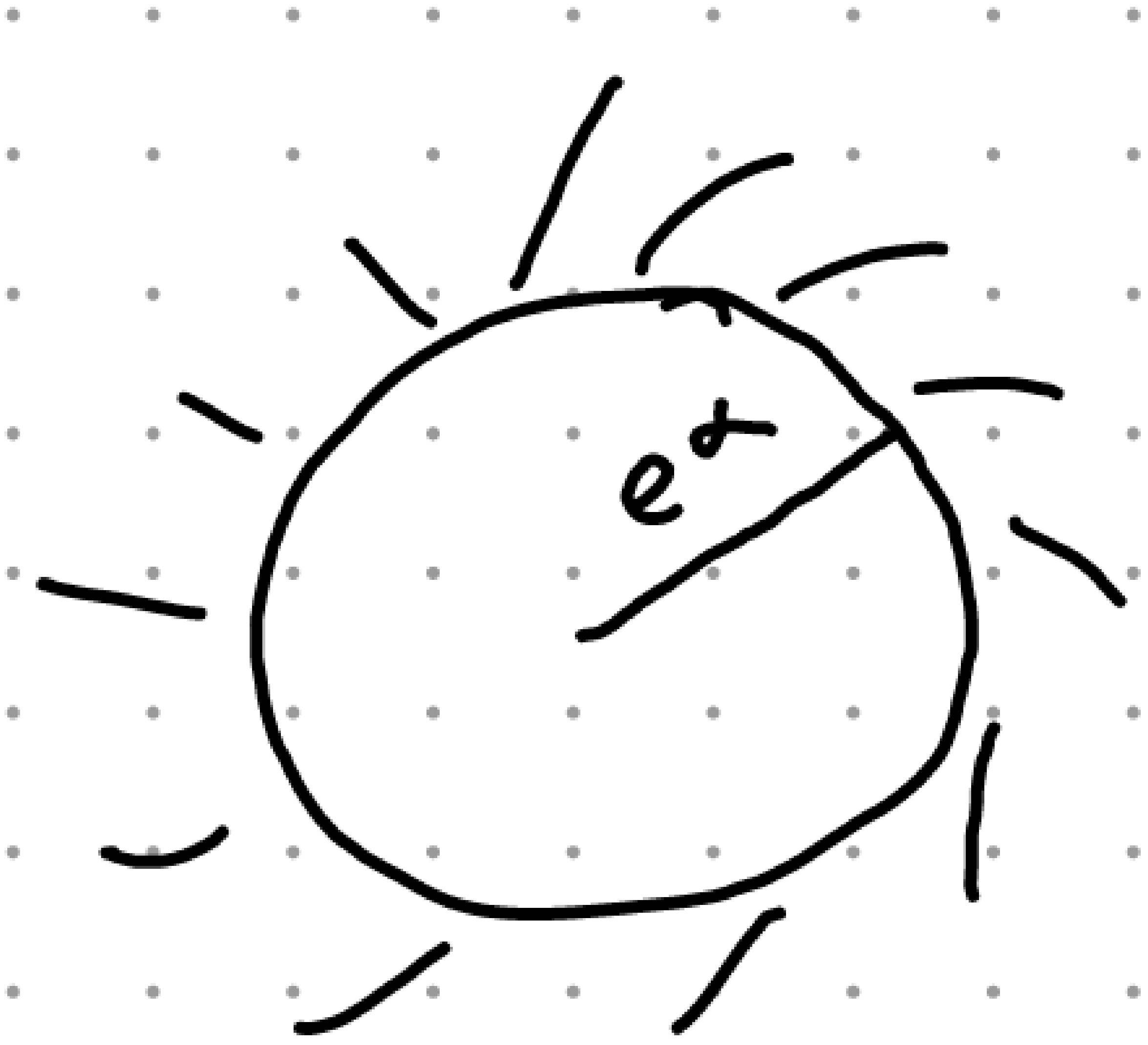
$$Z(e^{\alpha k}) = \sum_{k=0}^{\infty} e^{\alpha k} z^{-k}$$

$$= 1 + \frac{e^{\alpha}}{z} + \frac{e^{2\alpha}}{z^2} + \dots \infty$$

$$= \frac{1}{1 - \frac{e^{\alpha}}{z}} = \frac{z}{z - e^{\alpha}}$$

$$\forall \left|\frac{e^{\alpha}}{z}\right| < 1 \quad \text{ie} \quad e^{\alpha} < |z|$$

Region of convergance



② $Z[\sin \alpha k]_{k \geq 0} = \sum_{k=0}^{\infty} \sin \alpha k \, z^{-k}$

$= 0 + \frac{\sin\alpha}{z} + \frac{\sin 2\alpha}{z} + \cdots$

$= \sum_{k=0}^{\infty} \frac{e^{i\alpha k} - e^{-i\alpha k}}{2i} z^{-k}$

$= \frac{1}{2i} \sum_{k=0}^{\infty} e^{i\alpha k} z^{-k} - e^{-i\alpha k} e^{-k}$

for $\left|\frac{e^{i\alpha}}{z}\right| < 1$

$= \frac{1}{2i}\left(\frac{z}{z - e^{i\alpha}} - \frac{z}{z - e^{-i\alpha}}\right)$ $\quad \left|\frac{e^{-i\alpha}}{z}\right| < 1$

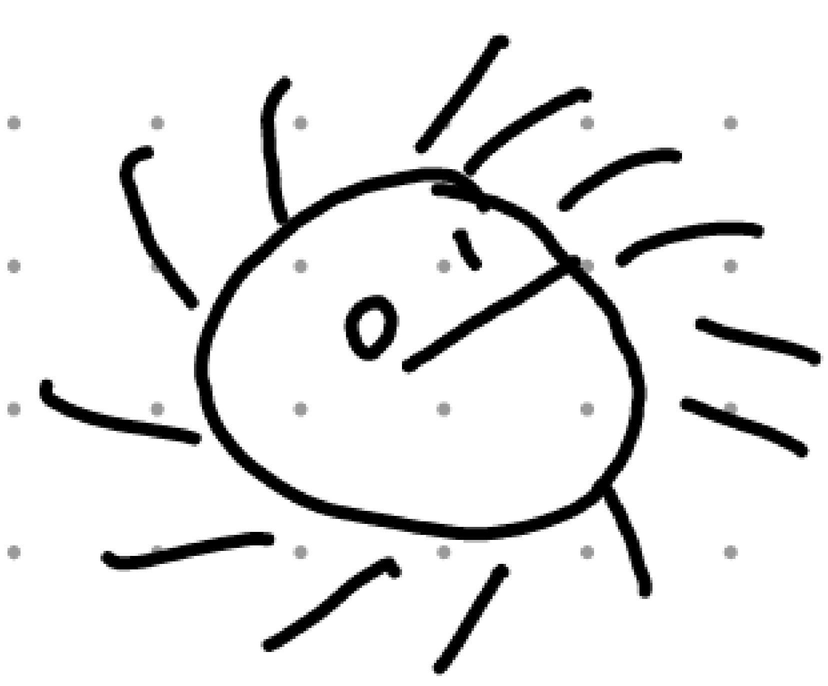
$= \frac{1}{2i} z \left(\frac{e^{i\alpha} - e^{-i\alpha}}{z^2 - z(e^{i\alpha} + e^{-i\alpha}) + 1}\right)$

But $|e^{i\alpha}| = 1$

$= \frac{z \sin\alpha k}{z^2 - 2\cos\alpha z + 1}$

$$Z[\sin \alpha n] = \frac{z \sin \alpha}{z^2 - 2z\cos\alpha + 1}$$

For $|z| > 1$



Region of Convergence

Similarly $Z(\cos \alpha n) = \frac{z \cos \alpha}{z^2 - 2z\cos\alpha + 1}$

③ $z[e^k \sin\alpha k] = \dfrac{\frac{3}{e}\sin\alpha}{\left(\frac{3}{e}\right)^2 - 2\left(\frac{3}{e}\right)\cos\alpha + 1}$

P.T.

(Apply change of scale)

④ $z[a^{|k|}] = \sum_{-\infty}^{\infty} z^{k} a^{|k|}$

$= \sum_{0}^{\infty} z^{-k} a^{k} + \sum_{-\infty}^{-1} a^{-k} z^{-k}$

$= \dfrac{1}{1-\frac{z^{1}}{a}} + \dfrac{az}{1-az}$

For $|z| < \frac{1}{a}$, $|z| > a$

$\therefore a < |z| < 1/a$

$\therefore 0 < a < 1$

⑤ P.T. $Z[k^n] = -z \dfrac{d}{dz} Z(k^{n-1})$

Proof :→ $Z(k^{n-1}) = \sum_{k=0}^{\infty} k^{n-1} z^{-k}$

$$-z \cdot \frac{d}{dz} Z(k^{n-1}) = -k \sum_{0}^{\infty} z^{-(k+1)} k^{n-1} \times -z$$

$$= \sum_{0}^{\infty} k^{n} \cdot z^{-k}$$

$$= Z(k^n)$$

⑥ $Z(k) = -z \dfrac{d}{dz} Z(k^0) = -z \dfrac{d}{dz} \dfrac{z}{z-1}$

$$= -z\left(\frac{z-1}{(z-1)^2} - \frac{z}{(z-1)^2}\right)$$

$$= \frac{z}{(z-1)^2}$$

(7) $$Z(k^2) = -z\frac{d}{dz}\frac{z}{(z-1)^2}$$

$$= -z\left(\frac{1}{(z-1)^2} + -\frac{2(z-1)z}{(z-1)^4}\right)$$

$$= -z\left(\frac{1-2z(z-1)}{(z-1)^2}\right)$$

(8) $$Z(k^2a^k) = -\frac{z}{a}\left(\frac{1-2\frac{z(z-a)}{a^2}}{\left(\frac{z}{a}-1\right)^2}\right)$$

(9) $$Z(a^k \sin k\theta) = \frac{\frac{z}{a}\sin\theta}{\left(\frac{z}{a}\right)^2 - 2\frac{z}{a}\cos\theta + 1}$$

(10) $$Z(k^2e^{\alpha k}) = Z\left(k^2\left(e^{\alpha}\right)^k\right) = -\frac{z}{e^{\alpha}}\,\frac{\left(1-2\frac{z(z-e^{\alpha})}{a^{\alpha}}\right)}{\left(\frac{z}{e^{\alpha}}-1\right)^2}$$

(11) $\sin(3k+5) = \sin 3k \cos 5 + \cos 3k \sin 5$

$$= \cos 5 \frac{z \sin 3}{z^2 - 2z\cos 3 + 1} + \sin 5 \frac{z \cos 3}{z^2 - 2z \cos 3 + 1}$$

$$= \frac{z \sin(3+5)}{z^2 - 2z\cos 3 + 1}$$

(12) $$Z\left(\frac{1}{k!}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} z^{-k}$$

$$= 1 + \frac{1}{z} + \frac{1}{2!\, z^2} + \cdots$$

$$= e^{1/z}$$

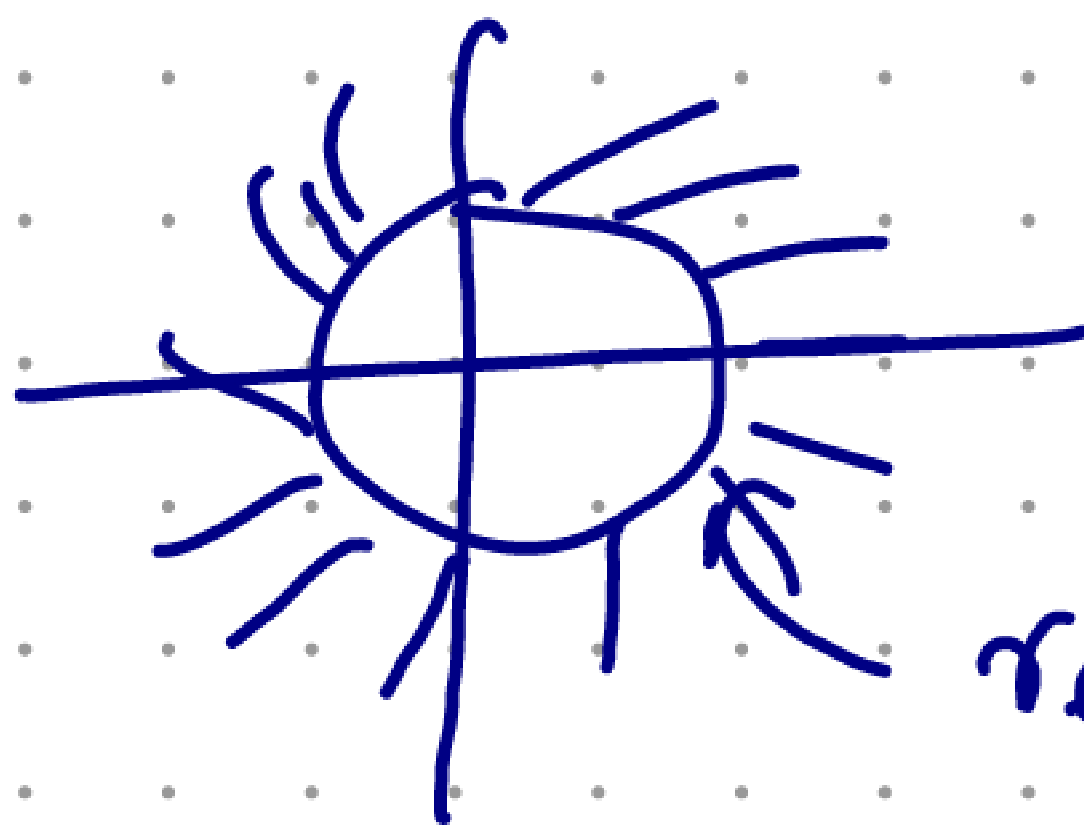
(13) $$Z(1/k) = \sum_{k=1}^{\infty} \frac{1}{k} z^{-k} = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^2} \cdots$$

$$\log(1+x) = 1 - \frac{x}{2} + \frac{x^3}{3} - \cdots\cdots$$

$$\log(1-x) = -\left(1 + \frac{x}{2} + \frac{x^3}{3} + \cdots\right)$$

$$\therefore \frac{1}{z} + \frac{1}{2z} + \cdots = -\log\left(1 - \frac{1}{z}\right) = \log\left(\frac{z}{z-1}\right)$$

$$|z| > 1$$



region of Convergence

(14) $$\frac{a^k}{k!} = e^{a/z} \quad \text{(first shifting)}$$

(15) $Z\left[\frac{1}{k(k+1)}\right] = Z\left[\frac{1}{k} + \frac{1}{k+1}\right]$

$$Z\frac{1}{(k+1)} = 1 - 1\frac{1}{2z} + \frac{1}{3z^2} - \cdots$$

$$= 3\left(\frac{1}{3} + \frac{1}{2z^2} - \cdots\right.$$

$\therefore Z\left[\frac{1}{k}\right] + Z\left[\frac{1}{k+1}\right] = (1-3)\, Z\left[1/k\right]$

$$= (1-3)\log\frac{z}{z-1}$$

(16) $Z\left(k^2 e^{\alpha k}\right] = \left(-z\frac{d}{dz}\right)^2 Z\left(e^{\alpha k}\right)$

☆ Imp

Put z in diff

$$= \left(-z\frac{d}{dz}\right)^2 \frac{z}{z-e^{\alpha}}$$

$$= \left(-z\frac{d}{dz}\right)^{-3} \times \frac{e^{\alpha}}{(3-e^{\alpha})^2}$$

$$= -3\frac{d}{dz}\;\frac{-3e^{\alpha}}{(3-e^{\alpha})^2}$$

$$= -3\,e^{\alpha}\,\frac{(2+e^{a})}{(z-e^{a})^3}$$

Note Don't do $z^2\dfrac{d^2}{dz^2}f()$

it is $-3\dfrac{d}{dz}\left(-3\dfrac{d}{dz}f()\right)$

(12) $Z[k \sin \alpha] = +3 \dfrac{d}{dz}\left[Z(\sin \alpha)\right]$

$$= -3 \frac{d}{dz} \frac{z \sin \alpha}{(z^2 - 2z\cos\alpha + 1)}$$

$$z[e^{\alpha k}] = \frac{3}{3-e^{\alpha}}$$

$$z[\sin \alpha k) = \frac{3 \sin \alpha}{3^2 - 2 3 \cos \alpha + 1}$$

$$z(\cos \alpha k) = \frac{3 \cos \alpha}{3^2 - 2 3 \cos \alpha + 1}$$

$$z[1] = \frac{3}{3-1}$$

$$z(k^n) = -3 \frac{d}{dz} z(k^{n-1})$$

$$z(a^k f(k)) = F\left(\frac{k}{a}\right)$$

$$z(1/k) = \log\left(\frac{3}{3-1}\right)$$

$$z(f(k+\alpha)) = z^{\alpha} z(f(k))$$

$$z(k f(k)) = -3 \frac{d}{dz} z(f(k))$$