Module 1

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① Laplace transform

② Inverse Laplace transform

③ Differential equations Using L. T.

① $L\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\, dt$

② $L\left[e^{at}\right] = \frac{1}{s-a}$

③ $L\left[\cos at\right] = \frac{s}{s^2+a^2}$

④ $L\left[\sin at\right] = \frac{a}{s^2+a^2}$

⑤ $L\left[\cosh at\right] = \frac{s}{s^2-a^2}$

⑥ $L\left[\sinh at\right] = \frac{a}{s^2-a^2}$

(7) $L[t^n] = \dfrac{\lceil n+1}{s^{n+1}}$

(8) $L[f(at)] = \dfrac{1}{a}\,\phi\left(\dfrac{s}{a}\right)$

(9) $L[e^{at} f(t)] = \phi(s-a)$

(10) $g(t) \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$

$L[g(t)] = e^{-as}\, L[f(t)]$

(11) $L(t^n f(t)) = (-1)^n \dfrac{d^n}{ds^n}\,\phi(s)$

(12) $L\left[\dfrac{f(t)}{t}\right] = \int_s^{\infty} \phi(s)\,ds$

Usually answer comes in ln form

(13) $L\left[\dfrac{d^n}{dt^n} f(t)\right] = s^n \phi(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots f'''(0)$

eg.

$L\left[\dfrac{d^2}{dt^2} f(t)\right] = s^2 L(f(t)) - s f(0) - f'(0)$

(14) $L\left[\displaystyle\int_0^t f(u)\,du\right] = \dfrac{\phi(s)}{s}$

(15) $L[u(t-a)\ f(t)] = e^{-as} L[f(t+a)]$

(16) $L[\delta(t-a)\ f(t)] = e^{-as} f(a)$

(17) $L[f(t-a)] = \dfrac{1}{1-e^{-as}} \displaystyle\int_0^a e^{-st} f(t)$

If f has period a

(18) $L^{-1}(\phi(s)) = f(t)$

$$L^{-1}(\phi(s-a)) = e^{-at} f(t)$$

$s \to s-a$, Multiply by e^{-at}

(19) $L^{-1}\left\{ \frac{d^n}{ds^n} \phi(s) \right\} = (-1)^n t^n f(t)$

(20) $L^{-1}(\phi(s)) = -\frac{1}{t} L^{-1}(\phi'(s))$

used for Tan & log

$$\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

(21) $L^{-1}(\phi_1(s)) = f(u)$

$$L^{-1}(\phi_2(s)) = g(u)$$

$$L^{-1}(\phi_1(s)\,\phi_2(s)) = \int_0^t f(u)\, g(t-u)\, du$$

For solving differential equations, first take Laplace then take laplace inverse

1. Take Laplace on both sides using derivative
2. Shift common terms to denominator
3. Use Laplace Inverse

Module 2

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1. Fourier Series
2. Fourier Integral
3. Fourier Transform
4. Inverse fourier transform

(1) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$

$$a_0 = \frac{1}{L}\int_a^{a+2L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_a^{a+2L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_a^{a+2L} f(x)\sin\frac{n\pi x}{L}\,dx$$

(2) Parsevals' Identity

$$\frac{1}{L}\int_0^{a+2L} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

(3) Half range fourier Series

a_0, a_n becomes $\frac{2}{L}$ instead of $\frac{1}{L}$

Parseval becomes $\frac{2}{L}\int_0^{a+2L} f^2(x)$

(4) Complex form of fourier series

$$f(x) = \sum_{n=0}^{\infty} C_n e^{\frac{in\pi x}{L}}$$

$$C_n = \frac{1}{2L} \int_{a}^{a+2L} f(x) e^{-\frac{in\pi x}{L}} dx$$

Dont forget

Write answer in form of Sinh or Cosh if possible

For solving any fourier series question

① Write formula

② Find a_0 a_n b_n

③ If function is odd or even, accordingly check if limits are $(-L$ to $L)$

$$\int_{-L}^{L} \text{odd} = 0 \qquad \int_{-L}^{L} \text{even} = 2\int_{0}^{L} \text{even}$$

$$\int_{-L}^{L} f(x) = \int_{-L}^{0} f(x) + \int_{0}^{L} f(x)$$

$$= \int_{0}^{L} f(-x) + f(x)$$

use this property to solve any question

* If $n-1$ term is in denominator, calculate separately for $n=1$

step
4 Put value of $x = 0$ or $x = \pi$
to get the required reduction

5 Use parseval's Identity if π^2 or π^4 terms
are present

Fourier Integral

① $$f(x) = \frac{1}{\pi}\int_{\omega=0}^{\infty}\int_{s=-\infty}^{\infty} f(s)\cos s\omega \cos \omega x \; d\omega\, ds$$

$$+ \frac{1}{\pi}\int_{\omega=0}^{\infty}\int_{s=-\infty}^{\infty} f(s)\sin s\omega \sin \omega x \; d\omega\, ds$$

Ans in terms of s

For Sine or Cosine series only, use $\frac{2}{\pi}$ instead

Use Laplace to evaluate $\int_{-\infty}^{\infty} e^{s} f(x)$

Fourier Transform & Inverse

① $F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

② $F^{-1}(F(s)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s)\, e^{-isx}\, ds$

③ $F(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos sx\, dx$

④ $F^{-1}(F(s)) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos sx\, ds$

⑤ $F(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, dx$

⑥ $F^{-1}(F(s)) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, ds$

For proving value of integral, Take fourier Transform and then Take inverse fourier transform

Module 3

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① Z Transform
② Inverse Z Transform

① $z[f(k)] = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$

② $z[a^k f(k)] = F\left(\frac{z}{a}\right)$

③ $z[f(k \pm n)] = z^{\pm n} F(z)$

④ $z[e^{\alpha k}] = \frac{z}{z - e^{\alpha}}$

⑤ $z[\sin \alpha k] = \frac{z \sin \alpha}{z^2 - 2z \cos \alpha + 1}$

⑥ $z[\cos \alpha k] = \frac{z \cos \alpha}{z^2 - 2z \cos \alpha + 1}$

⑦ $z[k^n] = -z \frac{d}{dz} z(k^{n-1})$

⑧ $z[1] = \frac{z}{z-1}$

⑨ $z(k) = \frac{z}{(z-1)^2}$

Inverse z

① Convolution

$$z^{-1}[f(z)] = f(n)$$

$$z^{-1}[g(z)] = g(n)$$

$$z^{-1}[f(z)\,g(z)] = \sum_{m=0}^{n} f(m)\,g(n-m)$$

Similar to Convolution theorem of Laplace

② $z^{-1}\left[\dfrac{z}{z-a}\right] = a^n$

③ Binomial expansion

① expand $\left(1-\frac{z}{a}\right)^{-1} = 1+\frac{z}{a}+\frac{z^2}{a^2}+\frac{z^3}{a^3}+\dots$

② Write in Series formed $\sum_{0}^{\infty} f(n)\,z^{-n}$

③ $f(n)$ is the Inverse z Transform

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$(1-x)^{-2} = 0 + 1 + 2x + 3x^3 + \cdots$$

$$(1-x)^{-3} = 0 + 0 + 3x + 6x^2 + 10 + \cdots$$

1 3 6 10

Triangular numbers

$$\frac{n(n+1)}{2}$$

(4) Partial fraction method

$$z^{-1}\left[\frac{az}{(z-a)^2}\right] = n a^n$$

$$z^{-1}\left[\frac{az(z+a)}{(z-a)^3}\right] = n^2 a^n$$

Module 4

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① Vector Algebra

② Vector differentiation

① Scalar Triple product

$$\bar{a} \cdot (\bar{b} \times \bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

If $a\,b\,c$ coplanar $\quad [a\,b\,c] = 0$

② Vector Triple product

$$(\bar{a} \times \bar{b}) \times c = (\bar{a} \cdot \bar{c})\,\bar{b} - (b \cdot c)\,\bar{a}$$

③ Scalar 4 product

$$(\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d}) = \begin{vmatrix} \bar{a} \cdot \bar{c} & \bar{b} \cdot \bar{c} \\ \bar{a} \cdot \bar{d} & \bar{b} \cdot \bar{d} \end{vmatrix}$$

Langarge's Identity

④ Vector 4 Product

$$(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) = [a\,c\,d]\,\bar{b} - [b\,c\,d]\,\bar{a}$$

(5) For proving Colinearity of 4 Points ABCD, $[A-B,\ B-C,\ C-D] = 0$

Vector differentiation

1. Gradient $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

2. Differentiation $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$

3. Directional derivative $\nabla f \cdot \hat{u}$

4. $\nabla f = f' \frac{\bar{r}}{r}$

Steps for finding Directional derivative

1. find ∇f
2. find Direction of u
3. find $\hat{u} = \frac{\bar{u}}{|u|}$
4. Substitute value of point

Types of Angles for directional derivatives

(a) A, B . $\bar{u} = B - A$

(b) $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ $\bar{u} = (a, b, c)$

(c) Tangent to curve $\frac{dt}{d\gamma}$ at B

(d) Normal to Surface g ∇g

(6) Divergence

$$\bar{f} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}$$

$$\nabla \cdot \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$\nabla \cdot \bar{f} = 0$ then f is solenoidal

(7) $\nabla \times \bar{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$

$\nabla \times \bar{f} = \bar{0}$ $\bar{f}$ is irrotational

∇f	Gradient of Scalar is Vector
$\nabla \circ f$	Divergence of Vector is Scalar
$\nabla \times f$	Curl of Vector is Vector

Example scalar function xyz^2

vector function $x\hat{i} + y\hat{j} + z^2\hat{k}$

(8) $\nabla(f\,g) = g\nabla f + f\nabla g$

(9) $\nabla\circ(\bar{f}\,g) = g\,\nabla\circ\bar{f} + \bar{f}\circ\nabla g$

(10) $\nabla\circ(\bar{f}\times\bar{g}) = \bar{g}\circ(\nabla\times\bar{f}) - \bar{f}\circ(\nabla\times\bar{g})$

(11) $\nabla\times(\bar{f}\,g) = g(\nabla\times\bar{f}) + (\nabla g)\times\bar{f}$

(12) $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\nabla\circ\bar{r} = 3$

$\nabla\times\bar{r} = \bar{0}$

Module 5

Vector Integration

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1. Line Integral
2. Greens theorem
3. Stokes theorem
4. Gauss divergence theorem

① $\oint F \circ d\bar{r} = \int_A^B d\phi$ = work done in displacing particle from A to B

$d\phi$ = Common terms of $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}$

② $\int P\,dx + Q\,dy$ write P in terms of x
Q in terms of y

③ $\int P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Anticlockwise

④ $\int F \circ dr = \iint_S \hat{N} \circ (\nabla \times \bar{F})\, ds$

$\hat{N} \rightarrow$ Unit outward vector ds

$ds = \frac{dx\,dy}{|\hat{N} \circ \hat{k}|}$

Steps for Stokes theorem

① Find $\hat{N}$ $\quad \hat{N} = \dfrac{\nabla \phi}{|\nabla \phi|}$ $\quad \phi =$ Plane

② Find ds $\quad ds = \dfrac{dx\,dy}{|\hat{N} \cdot \hat{k}|}$

③ Find Curl $\quad \nabla \times F$

④ Find Dot Product $\quad \dfrac{\hat{N} \cdot (\nabla \times F)}{|\hat{N} \cdot \hat{k}|}$ Must be constant

⑤ Find Limits → ① Draw the Curve in 3D
② Project on 2D (Put $z=0$ works many times)
③ Find Limits

⑥ Evaluate Integral usually equal to area of Surface

$$\int F \cdot dr = \iint \hat{N} \cdot (\nabla \times F)\, ds$$

⑤ $\oiint \hat{N} \cdot \bar{F} \, ds = \iiint_V \nabla \cdot \bar{F} \, dv$

① Draw 3D diagram
② Put limits of z
③ Substitute values for x & y

Cylinder $\quad z : \quad a \quad : \quad b$

$x, y \rightarrow r\cos\theta, r\sin\theta$

Cone $\quad z : \quad a \quad : \quad \sqrt{x^2+y^2}$

$x, y \rightarrow r\cos\theta, r\sin\theta$

Sphere

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

Definations

① <u>Greens theorem</u> :-

Let $P(x,y)$ & $Q(x,y)$ be continuouse functions with $\frac{\partial P}{\partial y}$ & $\frac{\partial Q}{\partial x}$ which are also continuouse on closed region R in xy plane. Let c denote positively oriented boundary of the region R. Then

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

② <u>Stokes Theorem</u> :-

If $\bar{F}$ is a continuouse vector field over S then $\int_C F \cdot d\bar{r} = \iint_S \hat{N} \cdot (\nabla \times F)\, ds$ where

$\hat{N}$ is unit outward normal vector ds