1. Find work done in force field $\bar{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$

Along line $(0,0,0)$ to $(2,1,3)$

Work done $\int_C F \cdot dr = \int_{(000)}^{(2\,1\,3)} (3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k}) \cdot d\bar{r}$

$dr = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$F \cdot dr = 3x^2\,dx + (2xz - y)\,dy + z\,dz \quad -(1)$

$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz \quad -(2)$

Comparing common terms in (1)

$d\phi = d(x^3) + d\left(2xzy - \frac{y^2}{2}\right) + \frac{d(z^2)}{2}$

$\int_C F \cdot dr = \int_C d\phi = \left[x^3 + xzy - \frac{y^2}{2} + \frac{z^2}{2}\right]_{000}^{213}$

$= 8 + 6 - 4 = 10$

(2) $\int_A^B (3xy\,dx - y^2\,dy)$ x in terms of y

C: $y = 2x^2$ $A(0,0) \to B(1,2)$

what is integral if path is straight line?

Find if F is conservative

$F \cdot dr = 3xy\,dx - y^2\,dy$

$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$

$d\phi = d(3x^2 y) - d(y^3)$

$\int_C d\phi = \left(3x^2 y - y^3\right)_{0\,0}^{1\,2}$

$= 6 - 8 = -2$

Work done is independant of path

$\therefore$ w/d $= -2$ always

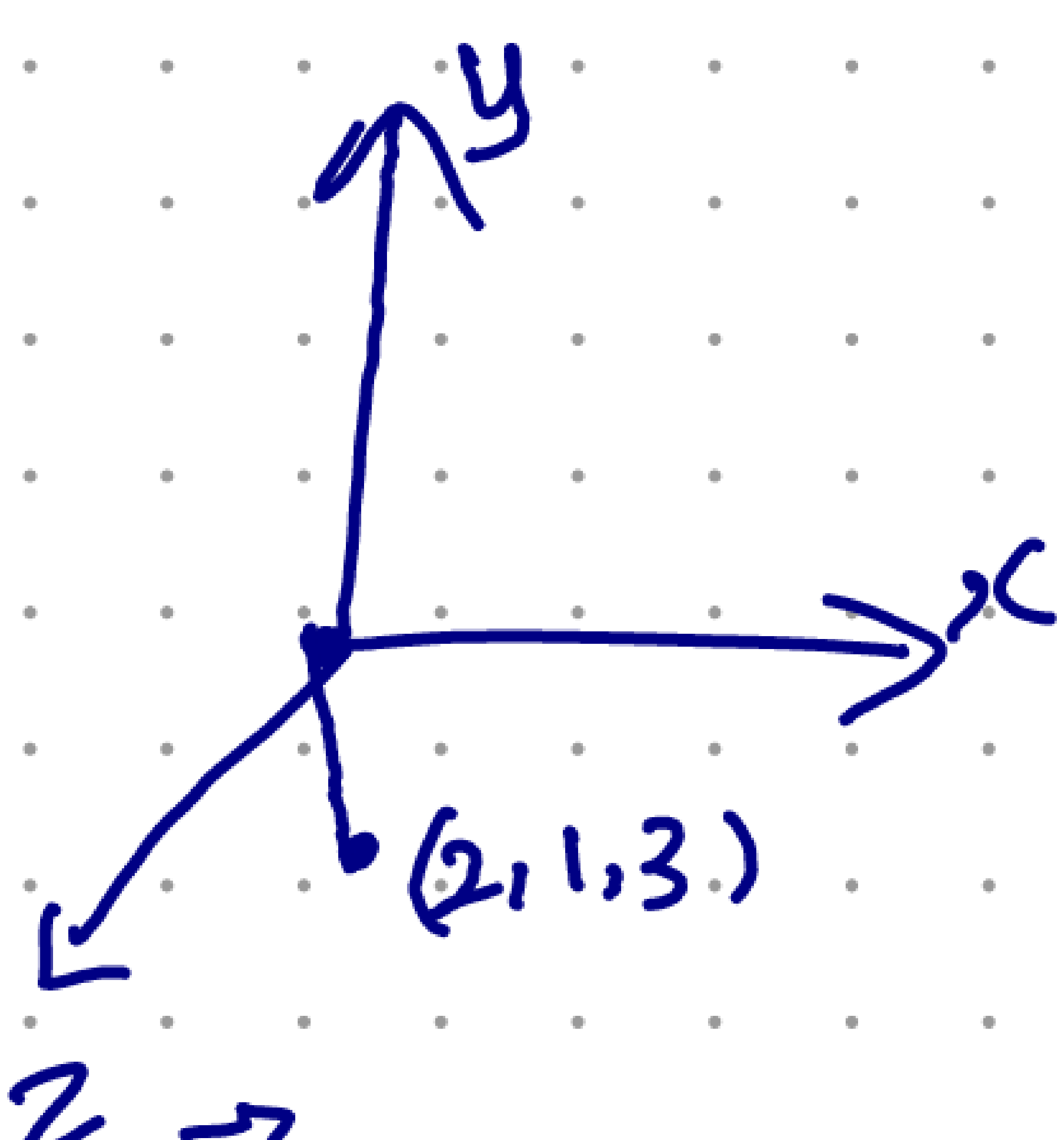
$F = 3xy\hat{i} + y^2\hat{j}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xy & y^2 & 0 \end{vmatrix} = 3x$$

$3x \neq 0 \therefore$ Not Conservative

③ $\int F$

Q1 $\vec{F} = 3x^2 \hat{i} + (2xz - y)\hat{j} + z\hat{k}$



$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$$\int_C \vec{F}\cdot d\vec{r} = \int 3x^2\,dx + (2xz - y)\,dy + z\,dz$$

$\vec{F} = \nabla \phi$

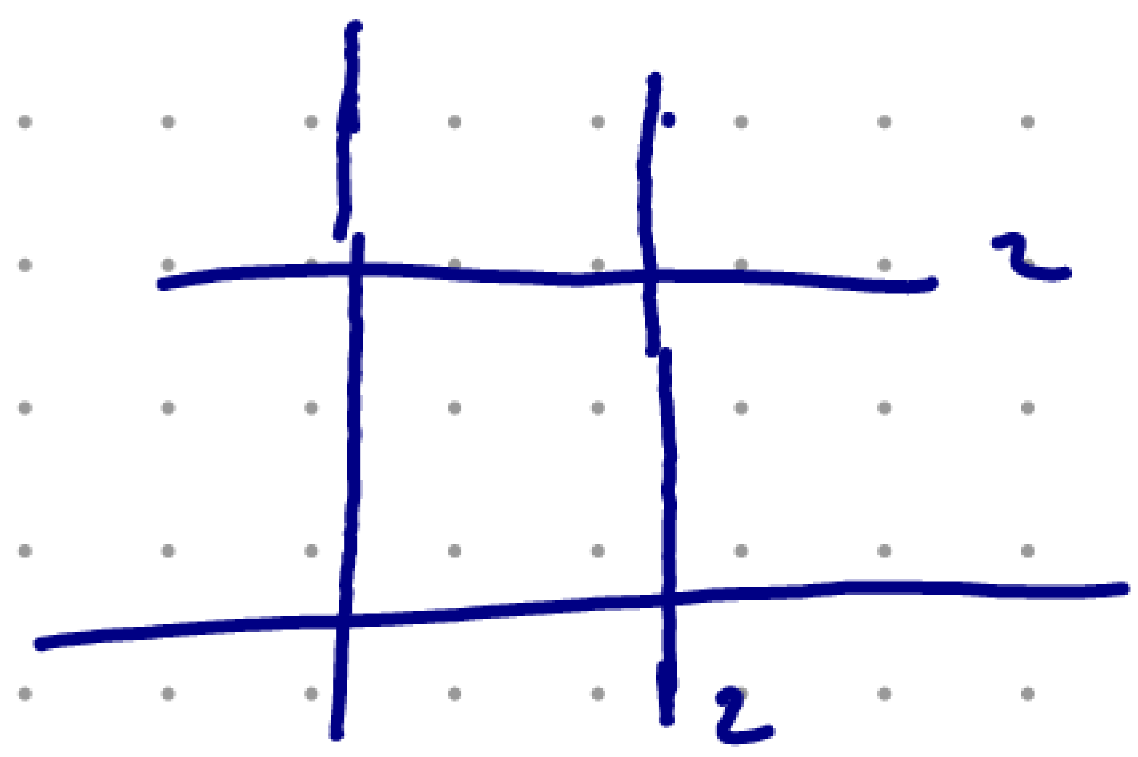
$$= \left[x^3\right] + \left[2xyz - \frac{y^2}{2}\right] + \left[\frac{z^2}{2}\right]$$

$$\Rightarrow \left\{(8) + \left(12 - \frac{1}{2}\right) + \left(\frac{9}{2}\right)\right\} - \left\{0 + 0 + 0\right\}$$

$12 + 8 + 4 \Rightarrow \underline{\underline{24}}$

③ $\int_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$

C: $x=0$, $y=0$, $x=2$, $y=2$ } Boundary



$F \cdot dr = P\,dx + Q\,dy$

$P = 2x^2 - y^2$

$Q = x^2 + y^2$

$\frac{\partial P}{\partial y} = -2y \qquad \frac{\partial Q}{\partial x} = 2x$

$$\int_C F \cdot dr = \iint_C \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \int_0^2 \int_0^2 (2x + 2y)\,dx\,dy$$

$$= \int_0^2 \left[2x^2 + 2xy\right]_0^2 dy$$

$16 + 8 = \underline{\underline{24}}$

(4) $\int \frac{1}{y}\,dx + \frac{1}{x}\,dy$

$P = 1/y \qquad \frac{\partial P}{\partial y} = \frac{-1}{y^2}$

$Q = 1/x \qquad \frac{\partial Q}{\partial x} = \frac{-1}{x^2}$

$$\iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \int_1^4 \int_1^{\sqrt{x}} -\frac{1}{x^2} + \frac{1}{y^2} \, dx\, dy$$

$$= \int_1^4 \left[-\frac{1}{x^2} y - \frac{2}{y^3} \right]_1^{\sqrt{x}}$$

$$= \int_1^4 \left[-\frac{1}{x^{3/2}} - \frac{2}{x^{3/2}} \right] - \left[\frac{1}{x^2} - 2 \right]$$

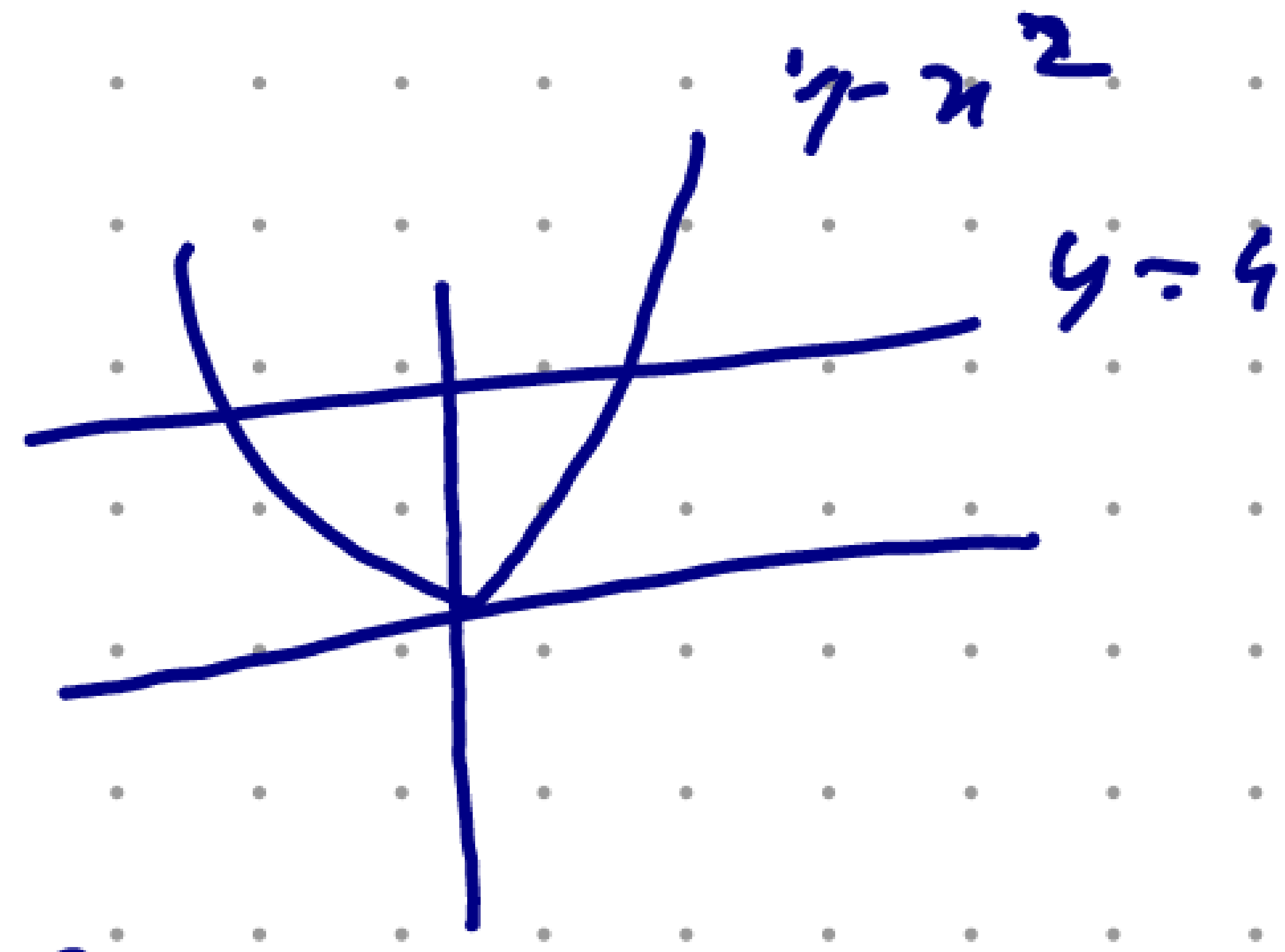
$$= \left[\frac{-3x^{5/2}}{5/2} + \frac{3}{x^3} + 2x \right]_1^4$$

⑤ $\int_C (x^2 - y)\,dx + (2y^2 + x)\,dy$

$P = x^2 - y \qquad \dfrac{\partial P}{\partial y} = -1$

$Q = 2y^2 + x \qquad \dfrac{\partial Q}{\partial y} = 4y$

$= \iint_C (4y + 1)$



$= \int_0^2 \int_{x^2}^{4} (4y+1)\,dy\,dx$

$y : x^2 : 4$

$x : 0 : 2$

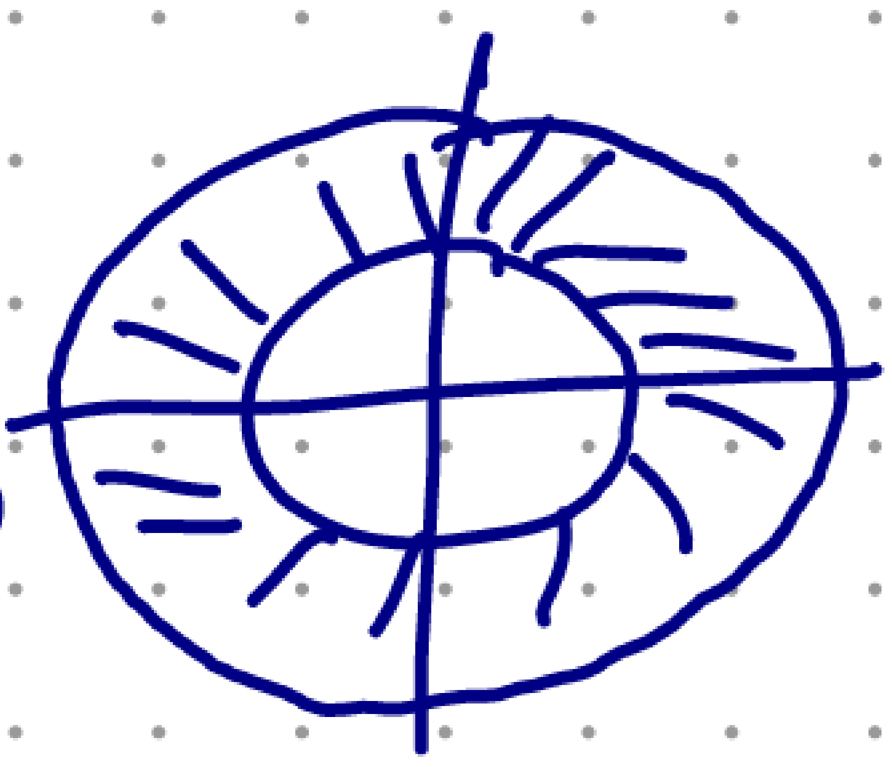
⑥ $\int (x^2+y^2)\hat{i} + (x^2-y^2)\hat{j}\,)\, d\bar{r}$

$$P = x^2+y^2 \qquad Q = x^2-y^2$$

$$\frac{\partial P}{\partial y} = 2y \qquad \frac{\partial Q}{\partial x} = 2x$$

$$= \iint_C (2x-2y)\, dy\, dx$$

$$= \int_0^{2\pi}\int_2^4 (2r\sin\theta - 2r\cos\theta)\, r\, dr\, d\theta$$



$r \rightarrow 2 : 4$

$\theta : 0 \rightarrow 2\pi$

$$= \int_0^{2\pi} \left[\frac{r^3}{3}(2\sin\theta - 2\cos\theta)\right]_2^4$$

$$= 0$$

② $\int F \cdot d\bar{r}$ $\quad F = -xy(x\hat{i} - y\hat{j})$

$\bar{r} = a(1+\cos\theta)$

$P = -x^2 y \quad \frac{\partial P}{\partial y} = -x^2$

$Q = +xy^2 \quad \frac{\partial Q}{\partial x} = y^2$

$dx\,dy = r\,dr\,d\theta$

$= \iint y^2 + x^2 \; dx\,dy$

$= \int_0^{2\pi}\int_0^{a(1+\cos\theta)} r^3(\sin^2\theta + \cos^2\theta)\, dr\, d\theta$

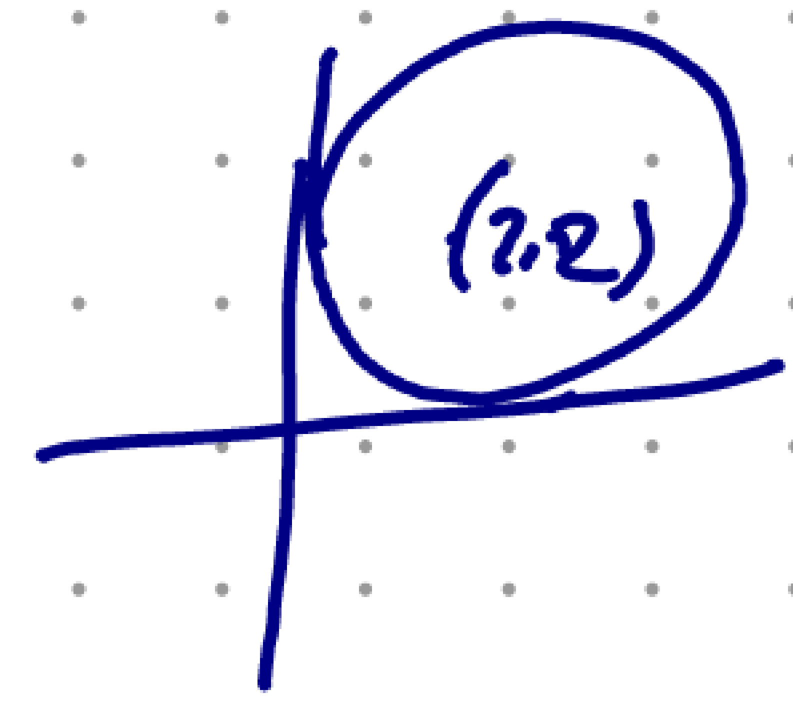
$= \int_0^{2\pi} \frac{a^4}{4}(1+\cos\theta)^4 \, d\theta$

⑧ $F = (4x-2y)\hat{i} + (2x-4y\hat{j})$

$P = 4x-2y \quad \frac{\partial P}{\partial y} = -2$

$Q = (2x-4y) \quad \frac{\partial Q}{\partial x} = 2$

$\therefore \iint 2+2 = \iint_C 4 = 4\iint_C dx\,dy$



$(x-2)^2+(y-2)^2=4$

$\iint dx\,dy = \text{Area} = \pi\, 2^2 = 4\pi$

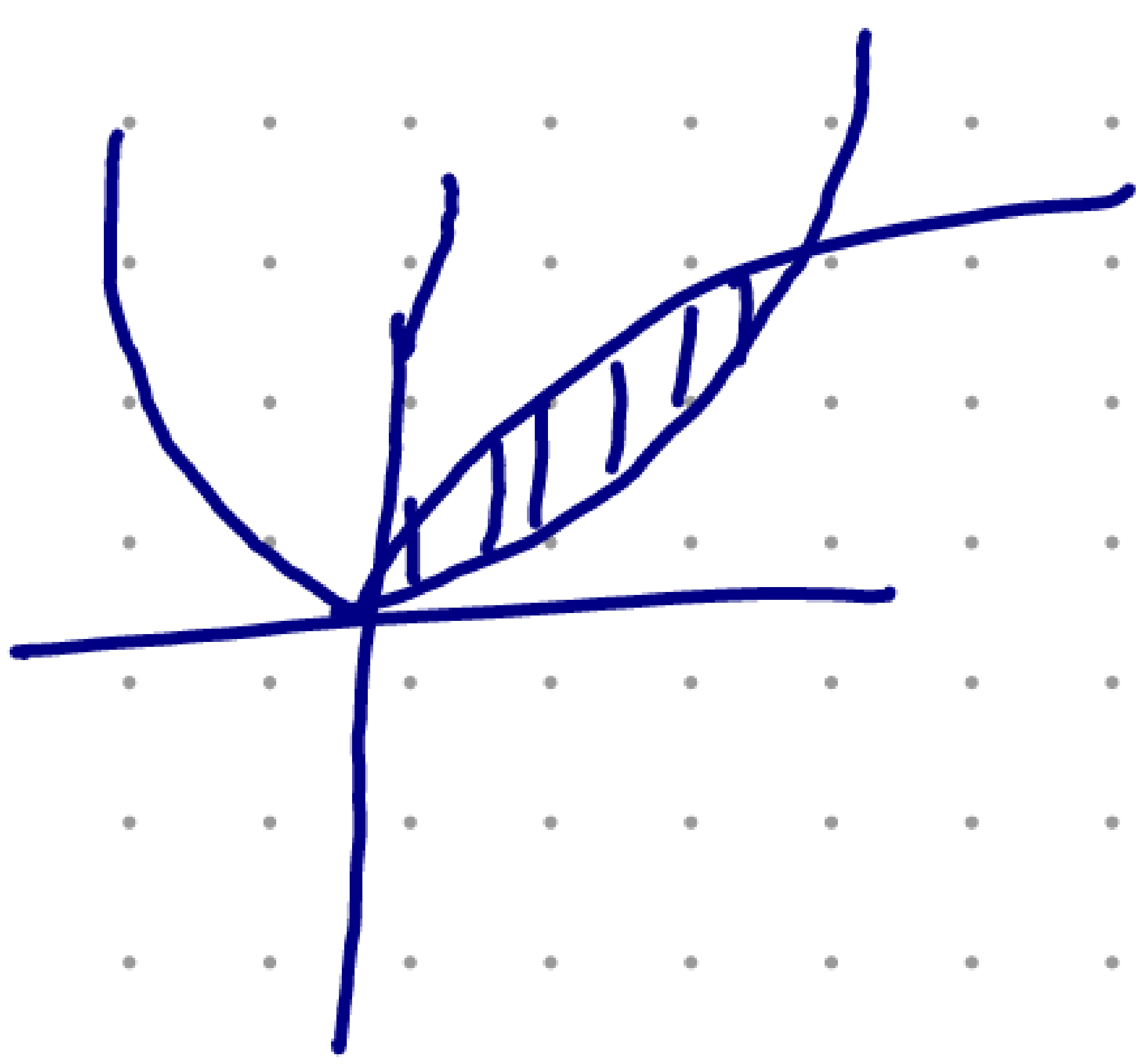
$\therefore \text{An} = 16\pi$

⑨ $\int (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

$P = 3x^2 - 8y^2$ | $Q = 4y - 6xy$

$\frac{\partial P}{\partial y} = -16y$ | $\frac{\partial Q}{\partial x} = -6y$

$= \iint -6y + 16y = \iint_C 10y\,dy\,dx$



$y : x^2 \to \sqrt{x}$

$x = 0$ to 1

$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\,dy\,dx$

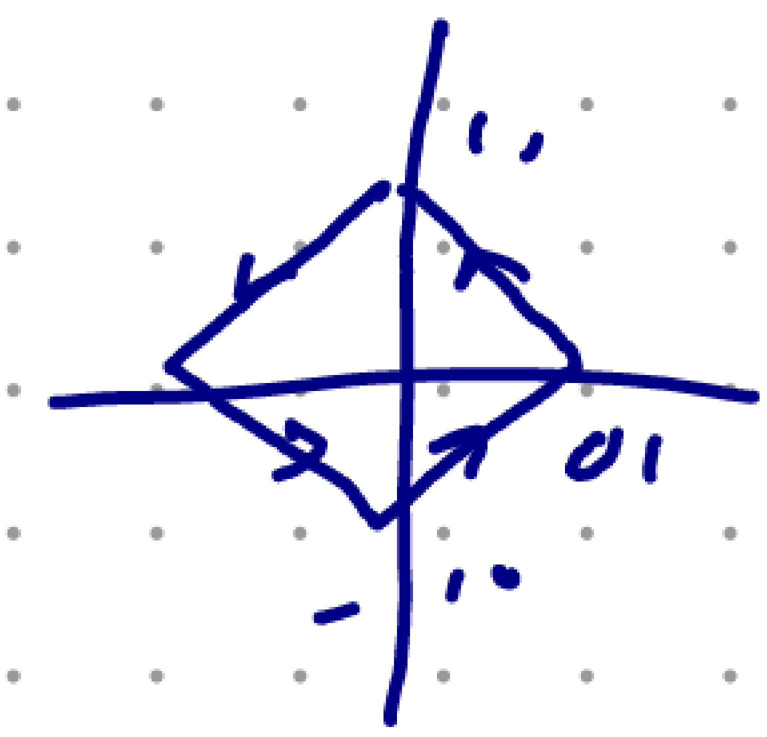
$= 10 \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$

$= 10 \int_0^1 \left[\frac{x}{2} - \frac{x^4}{2}\right] dx = 10\left[\frac{x^2}{4} - \frac{x^5}{10}\right]_0^1$

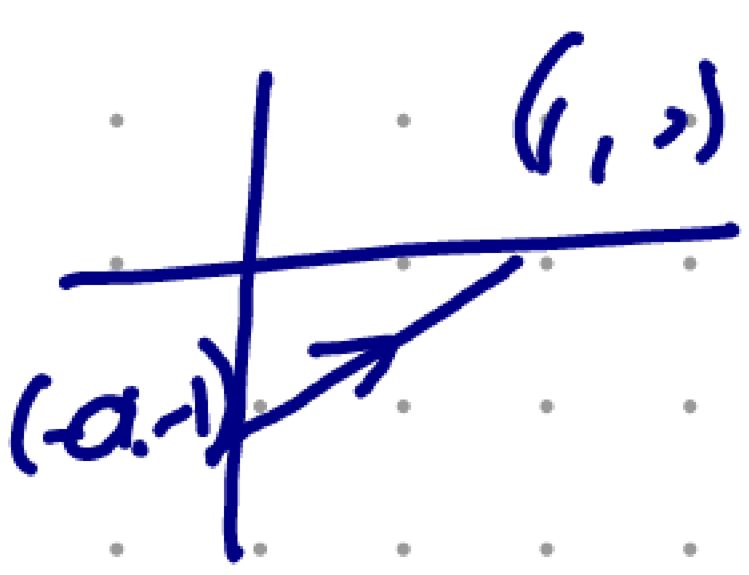
$= 3/2$

(10) $\int_C xy\,dx + xy^2\,dy$

$= \iint_C y^2 - x$



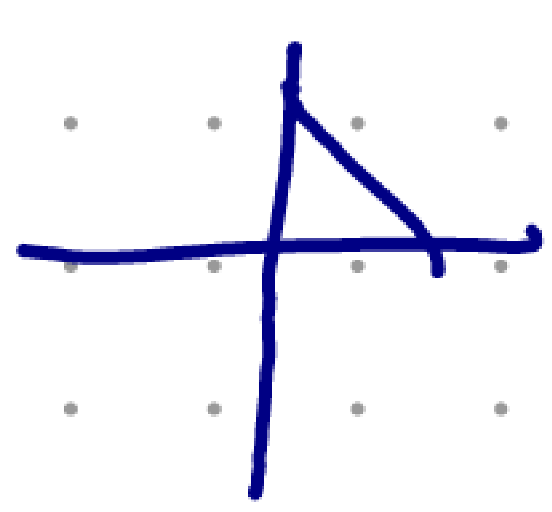
G: Anticlockwise



$x - y = 1$

$y : 0 : x-1$

$x : 0 : 1$



$x + y = 1$

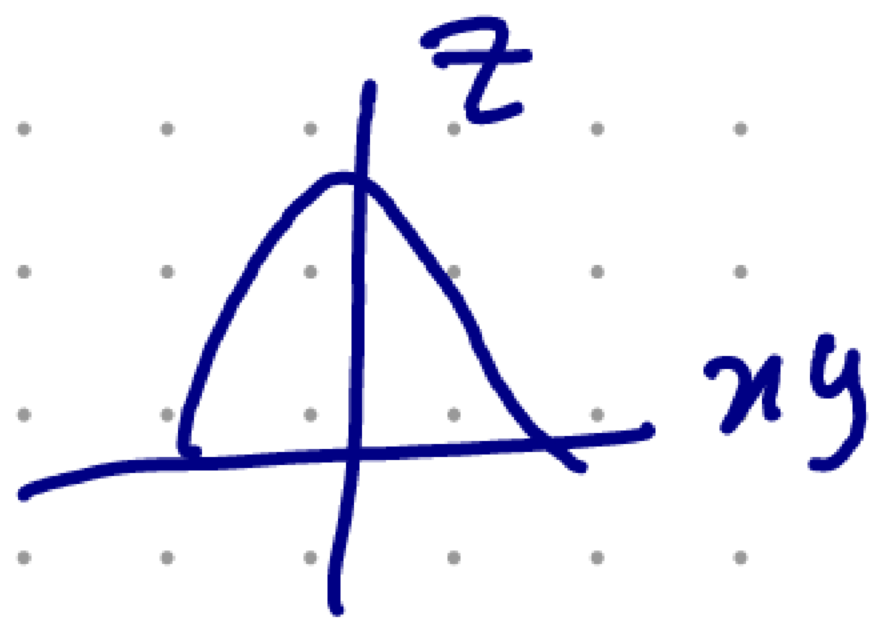
$y : 0 : 1-x$

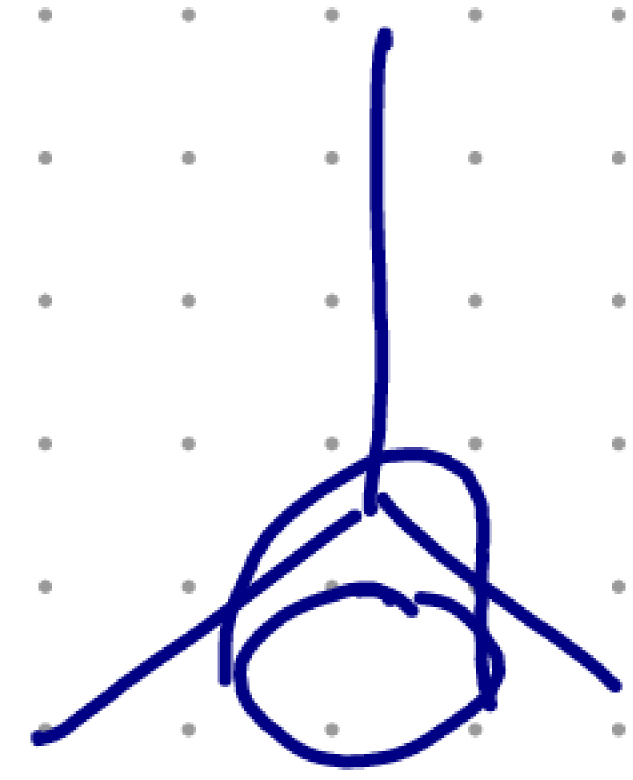
$x : 1 : 0$

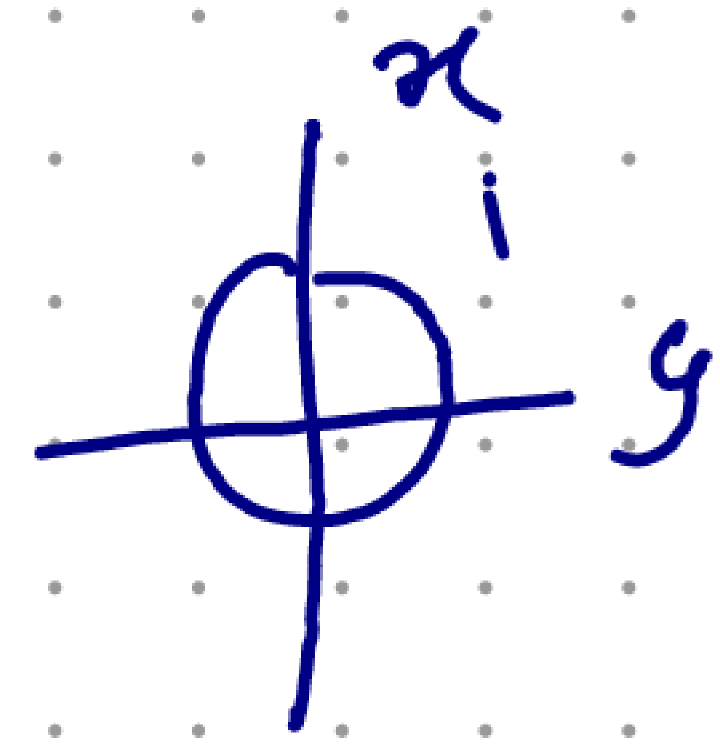
$\vdots$ for all points & integrate seperately

(11) $\int F \cdot d\bar{r}$ $\quad F = y\hat{i} + z\hat{j} + x\hat{k}$

C: $x^2 + y^2 = 1 - z \quad (z > 0)$







Paraboloid

Consider xy Projection

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

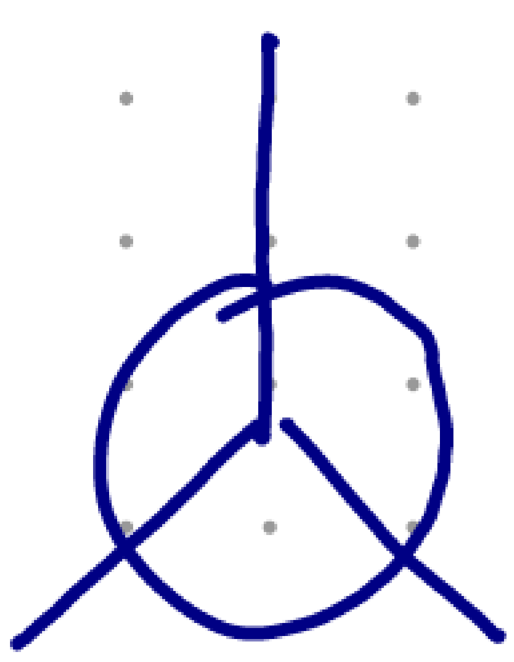
$$= -\hat{i} - \hat{j} - \hat{k}$$

$\hat{N} = -\hat{k}$: $dx\,dy = ds$

$(\hat{k}) \cdot (\nabla \times F) = -1$

$\therefore \iint_C dx\,dy = -\pi$ (Area of unit circle)

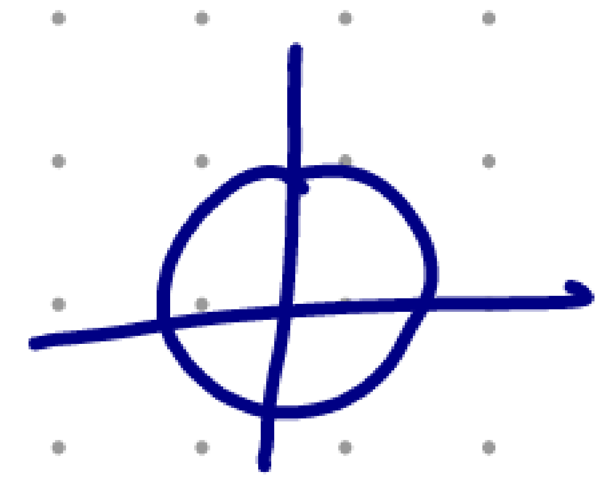
(12) $\int F \cdot dr$ $\quad F = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$



$x^2+y^2+z^2=a^2$

Projection on xy = $x^2+y^2=a^2$

$\hat{N} = \hat{k}$



$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x-y) & -yz^2 & -y^2z \end{vmatrix} = (-2yz+2yz) - 0 + -1\hat{k}$$

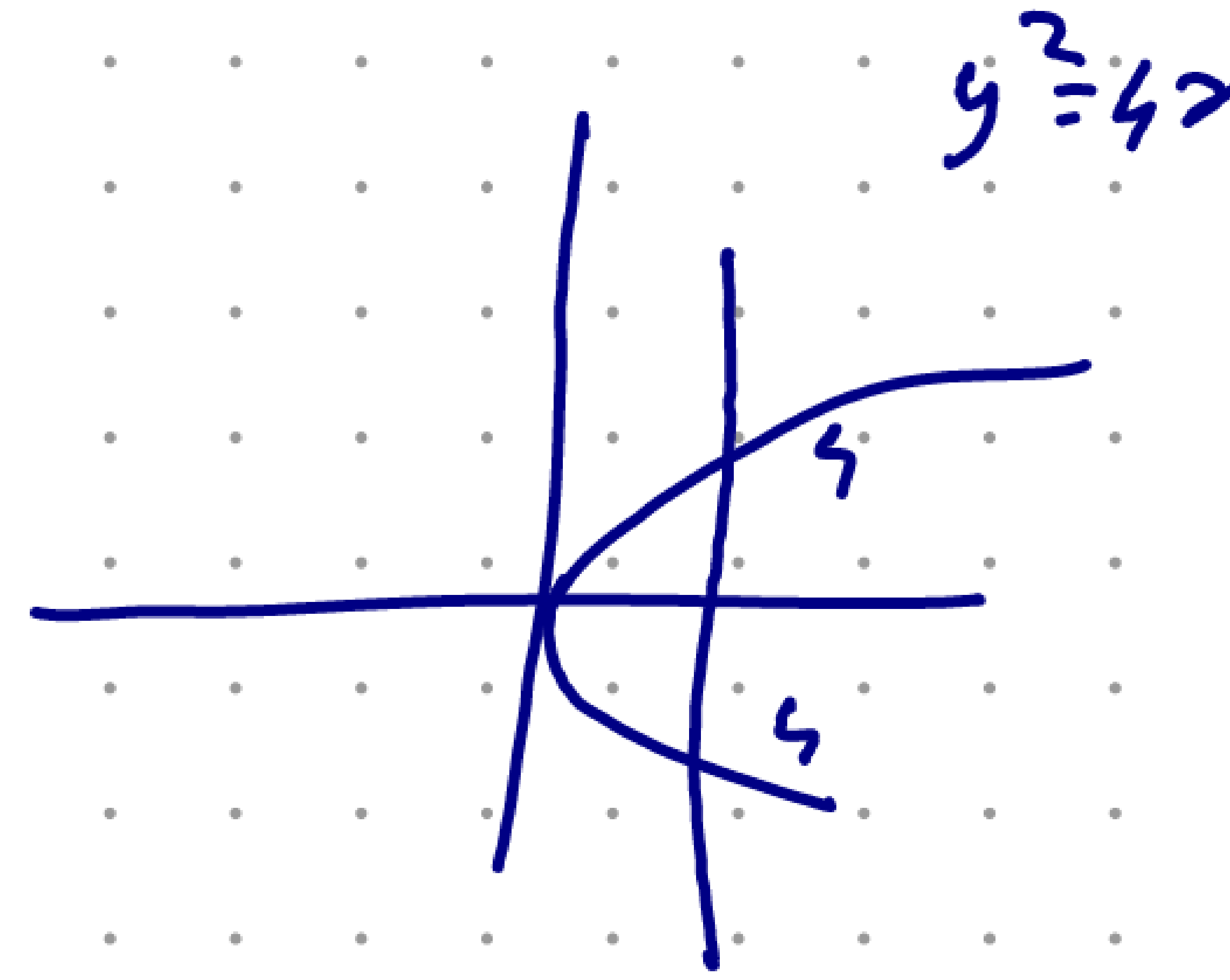
$\hat{N} \circ \bar{F} = -1$

$-1 \iint dx\,dy = \text{Area} = -\pi a^2$

(13) $\int F \cdot dr \qquad F = (x^2 + y^2)\hat{i} + 4xy\hat{j}$

$= \iint (4y - 2y)\, dx\, dy$

$= \iint 2y\, dx\, dy$



$= \int_{-4}^{4} \int_{y^2/4}^{4} 2y\, dx\, dy$

$x : \frac{y^2}{4} : 4$

$y : -4 : 4$

$= \int_{-4}^{4} 2y \left[4 - \frac{y^2}{4}\right] dy$

$= 0$

(14) $F = (x+y)\hat{i} + (y+z)\hat{j} - x\hat{k}$

$S: 2x+y+z=2$

$\phi = 2x+y+z-2$

$$\hat{N} = \frac{\nabla\phi}{|\nabla\phi|}$$

$\nabla\phi = 2\hat{i}+\hat{j}+\hat{k}$

$$\therefore \hat{N} = \frac{(2,1,1)}{\sqrt{6}} \quad (1)$$

$$ds = \frac{dx\,dy}{|\hat{N}\cdot\hat{k}|} = \sqrt{6}\,dx\,dy \quad (2)$$

$$\nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & y+z & -x \end{vmatrix} = -\hat{i}+\hat{j}+-\hat{k}$$

(3)

from ① ② ③

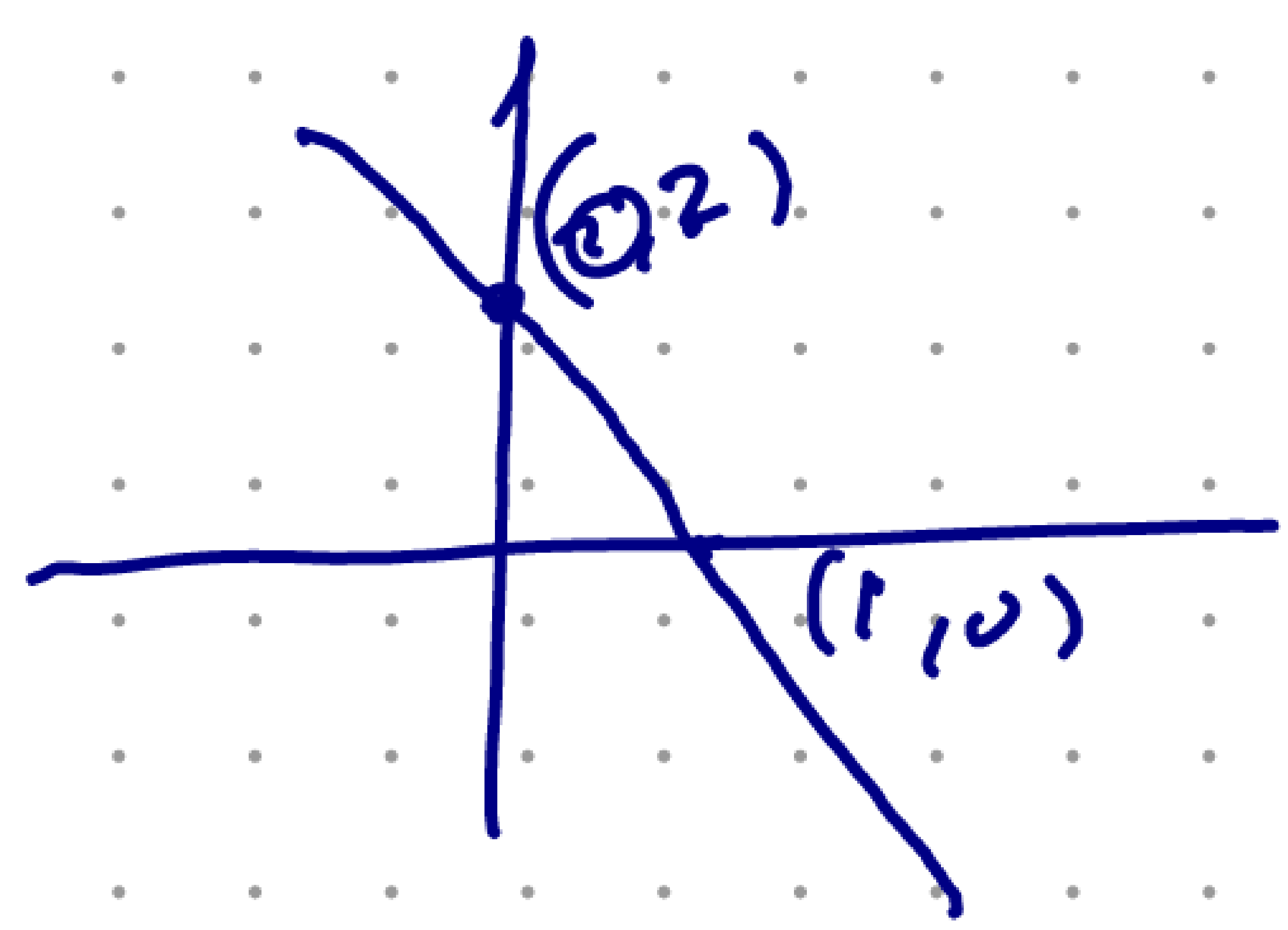
$\hat{N} \cdot (\nabla \times F) \, ds$

$= (-2 + 1 - 1) \, dx \, dy$

$\therefore \int F \cdot dr = -2 \iint dx \, dy$

In first quadrant, Projection

ie $2x + y = 2$



Area $= \frac{1}{2} \times 1 \times 2 = 1$

$\therefore$ Ans $= \underline{\underline{-2}}$

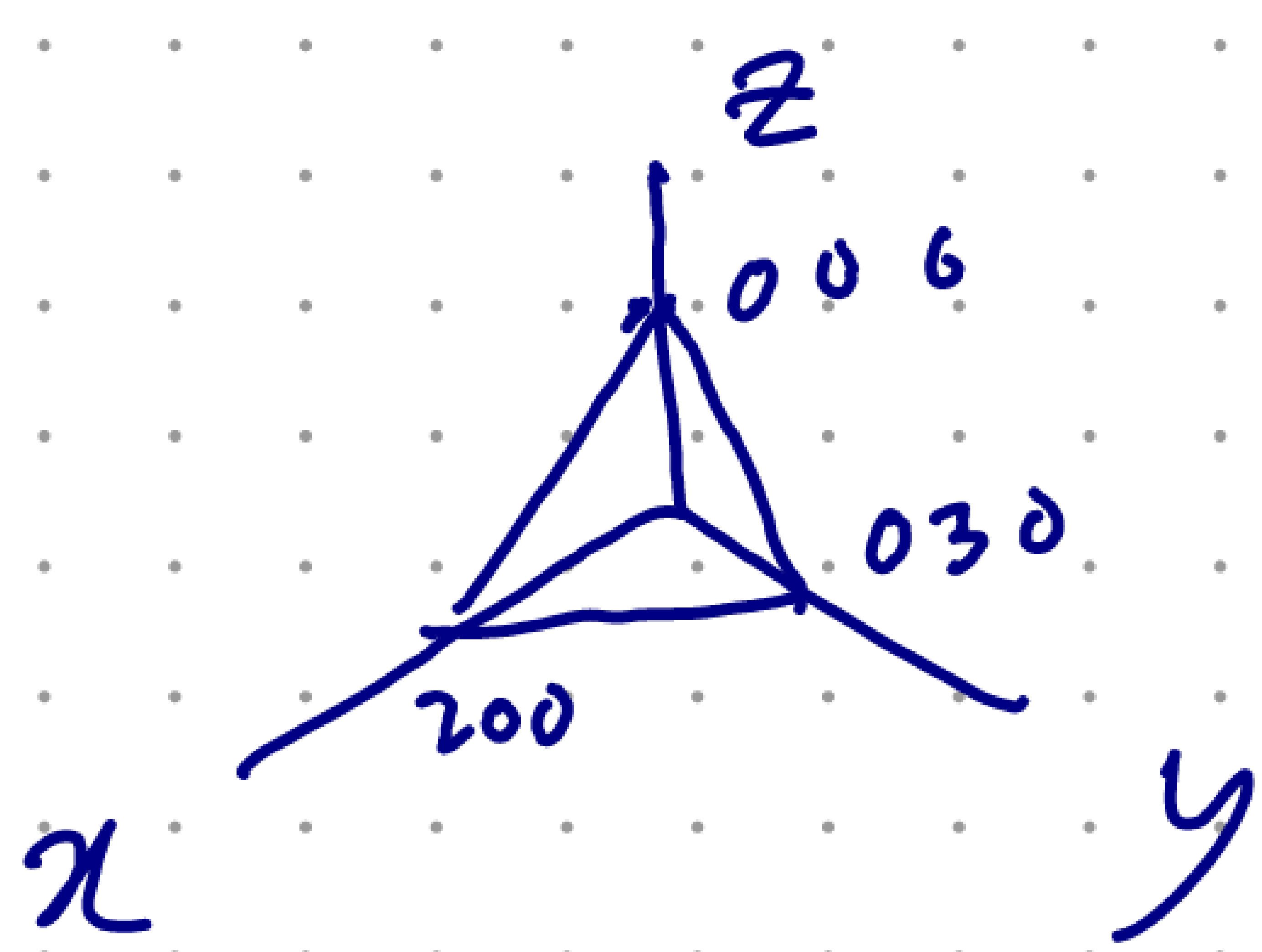
(15) Work done in moving particle once around perimeter of Δ $(2,0,0)$, $(0,3,0)$, $(0,0,6)$ under Force

$$F = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$$

$$(\nabla \times \bar{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= 2\hat{i} + \hat{k}$$

Surface



$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$$

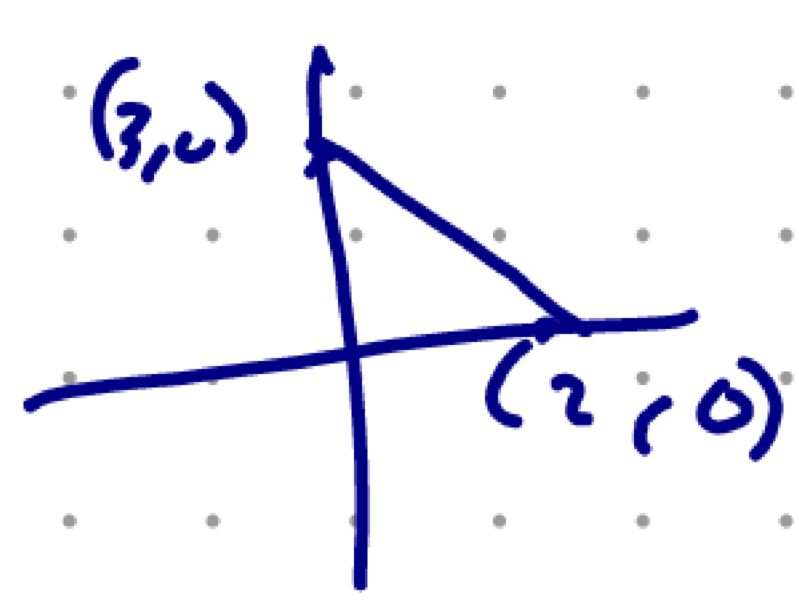
$$\nabla\phi = \frac{1}{2}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{6}\hat{k}$$

$$\frac{\nabla\phi}{|\nabla\phi|} = \frac{\frac{1}{2}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{6}\hat{k}}{\sqrt{\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{6^2}}} = \hat{N}$$

$$\hat{N}\cdot\hat{k} =$$

$$\iint (\nabla\times F)\cdot\frac{\hat{N}}{|\hat{N}\cdot\hat{k}|}$$

Area of xy projection

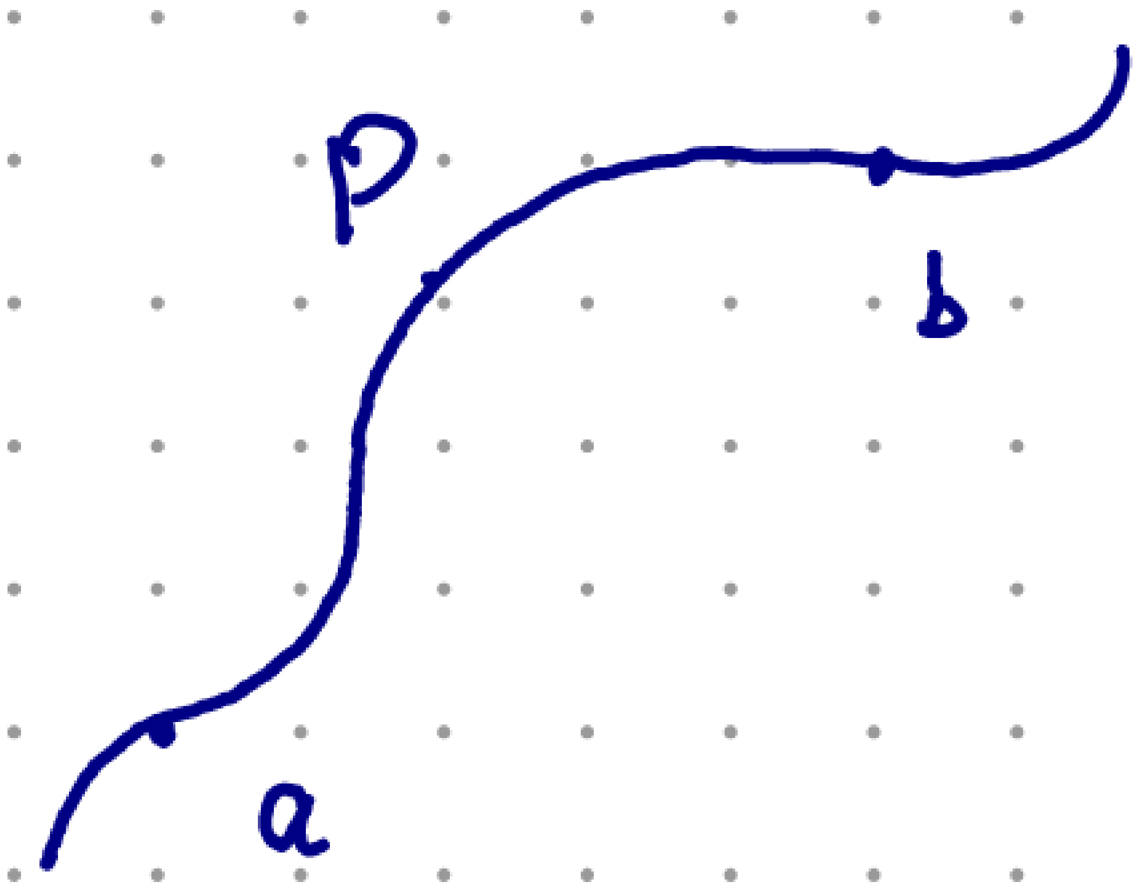


$$= \frac{1}{2}\times 2\times 3$$

$$= \underline{3}$$

Line Integral

Let $\bar{F}$ be a vector in some region R
Let c be any curve in this region
Let $\bar{R}$ be a position vector of point P on the curve



Then line Integral of $\bar{F}$ on the curve C

$$\bar{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$$

$$\int_C \bar{F} \cdot d\bar{r} = \int_c F_1 dx + F_2 dy + F_3 dz$$

$$dr = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$\int F \cdot dr = \int d\phi$ if F is conservative

Conservative feild

A vector field $\bar{F}$ is called conservative if there exists a scalar function such that

$$\bar{F} = \nabla\phi$$

Theorem → If $\bar{F}$ is a continuous field then the following statements are equivalent

(a) Line Integral $\int_c \bar{F} \cdot d\bar{r}$ is independant of the path joining the points A & b & only depends on end points of c

(b) $\bar{F}$ is a conservative field

$$\bar{F} = d\phi \text{ for some scalar } \phi$$

(c) For any closed curve C $\quad \oint_c \bar{F} \cdot d\bar{r} = 0$

$\oint$ = Closed contour

= Work done from moving a particle from A to B

① P.T. $\bar{F} = (y^2 \cos x + z^3)\hat{i} + (2y \sin x - 4)\hat{j} + (3xz^2 + 2)\hat{k}$

is a conservative field

find scalar potential for $\bar{F}$ & work done in moving an object from $(0, 1, -1)$ to $(\frac{\pi}{2}, -1, 2)$

Prove $\nabla \times \bar{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = -(3z^2 - 3z^2)\hat{j} + \left(\begin{matrix} 2y\cos x \\ - 2y\cos x \end{matrix}\right)\hat{k}$$

$= 0$

$$\frac{\partial \phi}{\partial x} = y^2 \cos x + z^3$$

$$\frac{\partial \phi}{\partial y} = 2y \sin x - 4$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 + 2$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$= (y^2 \cos x + z^3)\, dx + (2y \sin x - 4)\, dy$$

$$+ (3z^2 x + 2)\, dz$$

Combining Common terms

$$d\phi = d(y^2 \sin x) + d(xz^3) - d(4y)$$

$$+ d(2z)$$

$$\phi = y^2 \sin x + xz^3 - 4y + 2z + c$$

Work done

$\bar{F} = \nabla \phi$ $\therefore$ Line Integral is independant

$\bar{F} = \nabla \phi$

$\therefore$ work done $= \int_C \bar{F} \cdot d\bar{r}$

$= \int_C d\phi = \left[\phi\right]_{(0,1,-1)}^{(\frac{\pi}{2},-1,2)}$

Substitute limits, we get

$= 15 + 4\pi$

Green Theorem

Let $P(x,y)$ and $Q(x,y)$ be continuouse functions with $\frac{\partial P}{\partial y}$ & $\frac{\partial Q}{\partial x}$ which are also continuouse on closed region R in xy plane

Let c denote positively oriented boundry of the region R. Then

Anticlockwise

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Vector form of greens theorem

$\bar{F} = P\hat{i} + Q\hat{j}$ & $\bar{r} = x\hat{i} + y\hat{j}$

$\bar{F}\,d\bar{r} = P\,dx + Q\,dy$ $\quad \therefore \oint_c \bar{F}\,d\bar{r} = \iint_S \hat{k}\cdot(\nabla\times\bar{F})\,ds$

$\hat{k}$ = Unit Vector along z axis

① Verify Green's theorem for

$$\oint (y - \sin x)\, dx + \cos x \, dy$$

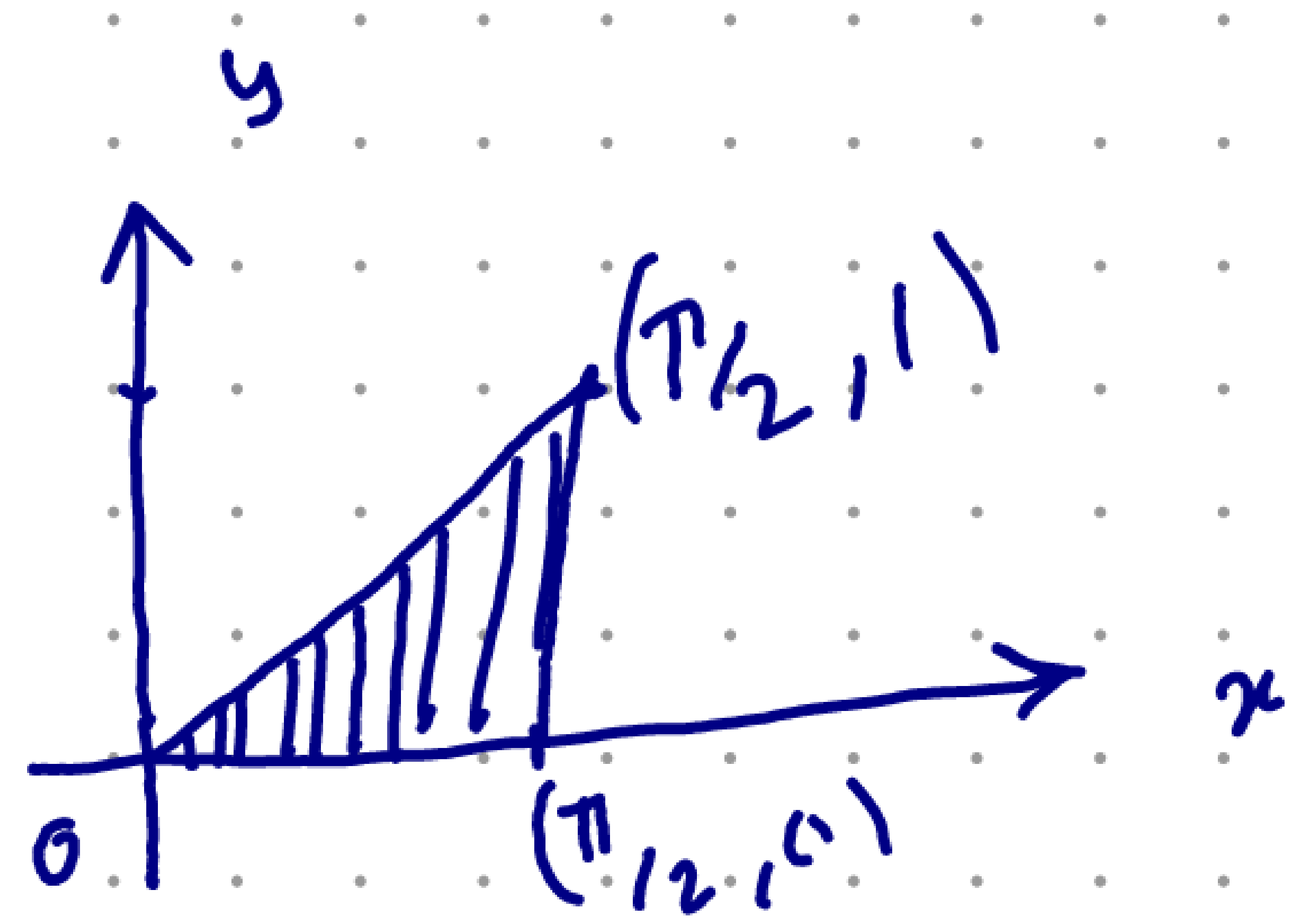
$c : \Delta AOB$

$$A = \left(\frac{\pi}{2}, 0\right)$$

$$O \equiv (0, 0)$$

$$B \equiv \left(\frac{\pi}{2}, 1\right)$$

C is Boundary of Δ O?P

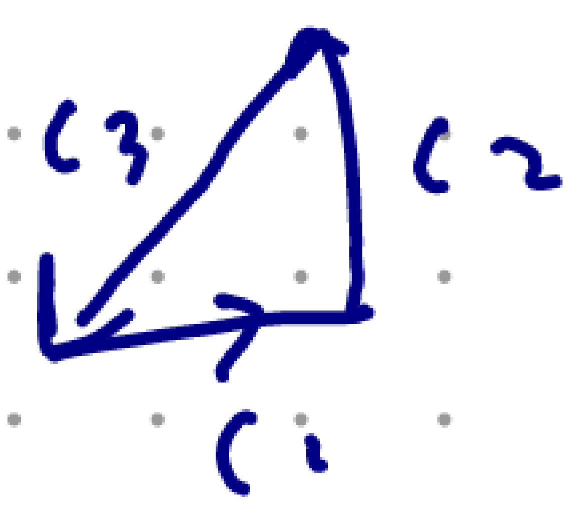


By greens theorem

Q is cos x Not y - sin x

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_R \cos x \quad dx\,dy \qquad - (1)$$



We Integrate along C_1 C_2 C_3 & Add to get along C

C_1 $\quad y = 0 : dy = 0$

$$\int_{C_1} \bar{F} \cdot d\bar{r} = \oint (y, -\sin x)\, dx + \cos x\, dy$$

$$= \int_0^{\pi/2} -\sin x\, dx$$

$$= \left[\cos x\right]_0^{\pi/2} = -1$$

$C_2 \quad x = \pi/2 \quad dx = 0 \quad \therefore \cos x = 0$

$$\int_{C_2} \bar{F} \cdot d\bar{r} = \int_{C_2} 0 = 0$$

$C_3 \quad y = mx + c$

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{2}{\pi}$$

$\therefore y = \frac{2}{\pi} x + c \qquad c = 0$ as line pass $(0,0)$

$\therefore \quad x = \frac{\pi}{2} y$

$\therefore \quad dx = \frac{\pi}{2} dy$

$$\therefore \int_{C_3} F \, dr = \oint \left(\frac{2}{\pi} x - \sin x \right) dx + \cos x \frac{2}{\pi} dx$$

$$= \int_{\pi/2}^{0} \left(\frac{2}{\pi} x - \sin x + \frac{2}{\pi} \cos x \right) dx$$

$$= \left(\frac{x^2}{\pi} + \cos x + \frac{2}{\pi} \sin x \right)_{\pi/2}^{0}$$

$$= 1 - \left(\frac{\pi^2}{4\pi} + \frac{2}{\pi}\right)$$

$$= 1 - \frac{\pi}{4} - \frac{2}{\pi}$$

$$C_1 + C_2 + C_3 = C$$

$$= -1 + 0 + 1 - \frac{\pi}{4} - \frac{2}{\pi}$$

$$= -\frac{\pi}{4} - \frac{2}{\pi} \quad — \quad 1$$

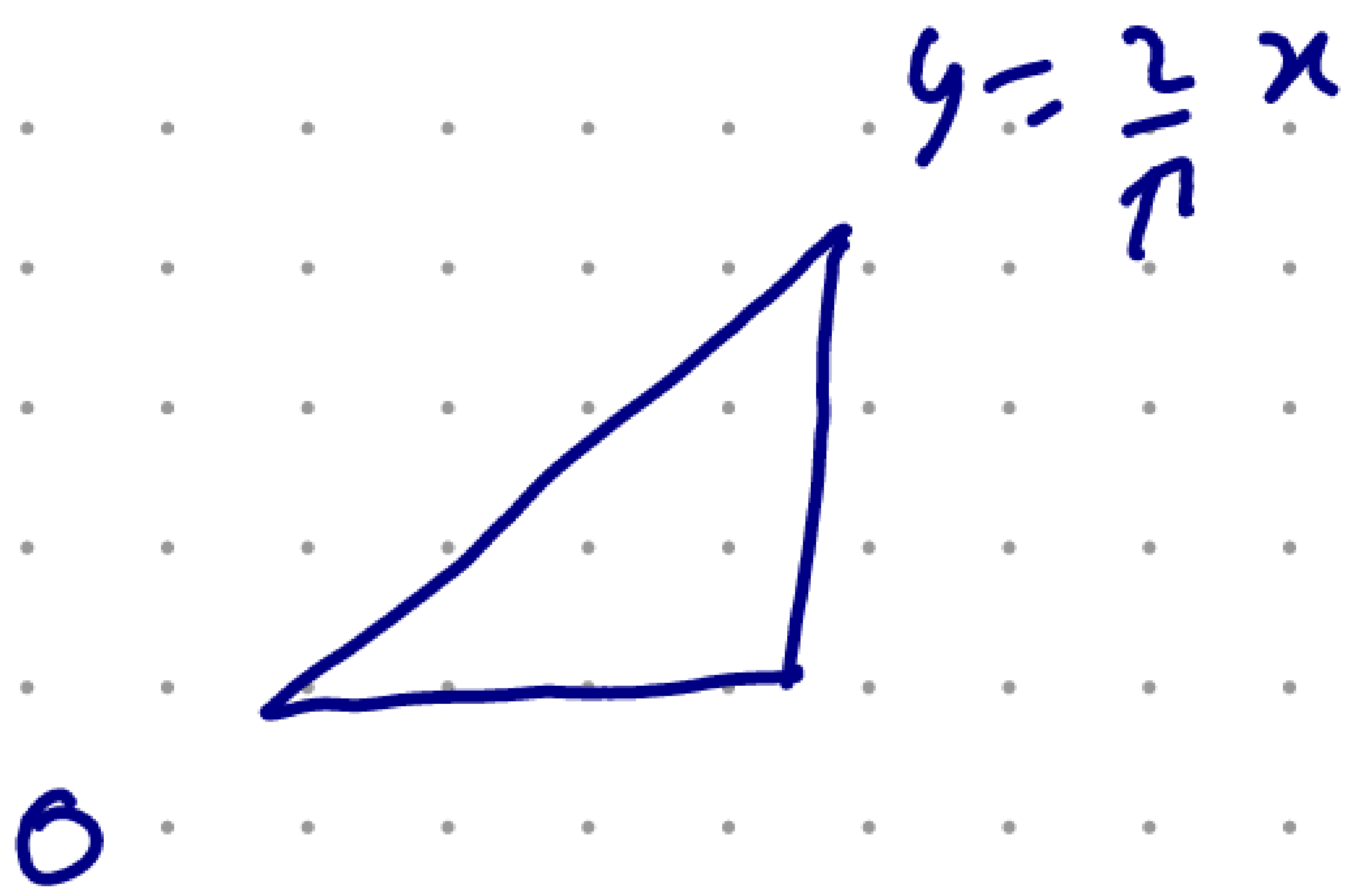
By green's thm

$Q = \cos x \qquad \frac{\partial Q}{\partial x} = -\sin x$

$P = y - \sin x \qquad \frac{\partial P}{\partial y} = +1$

$$\int_C P\,dx + Q\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$= -\int_{x=0}^{\pi/2}\int_{y=0}^{\frac{2}{\pi}x} (\sin x + 1)\, dx\, dy$$



other way of limits also possible

$$= -\int_{0}^{\pi/2} (\sin x + 1)\left[y\right]_0^{\frac{2}{\pi}x}\, dx$$

$$= -\int_{0}^{\pi/2} (1 + \sin x)\,\frac{2}{\pi}x\, dx$$

$$= \frac{2}{\pi}\left[x\,(x - \cos x - \frac{x^2}{2} - \sin x\right]_0^{\pi/2}$$

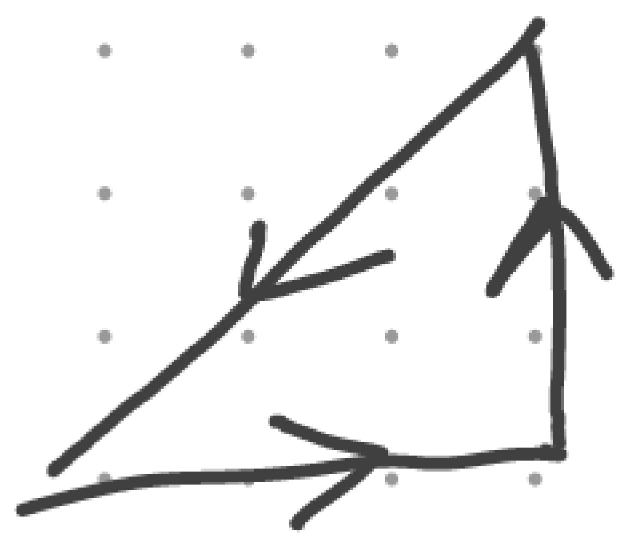
$$= -\frac{2}{\pi}\left[\frac{\pi}{2}\frac{\pi}{2} - \frac{\pi^2}{8}\right]$$

$$= -\frac{2}{\pi}\left[\frac{\pi^2}{8} + 1\right] = -\frac{\pi}{4} - \frac{2}{\pi}$$

2

By ① & ② Both are equal

hence verifies

Note  anticlockwise

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \quad \text{Not} +$$

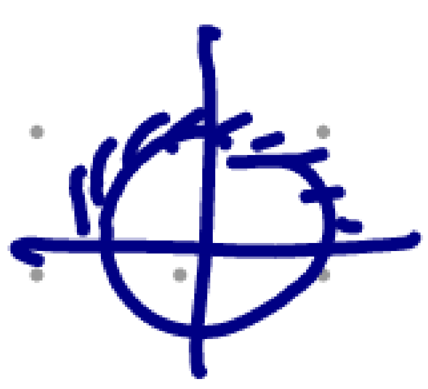
① Evaluate $\int \bar{F} \cdot d\bar{r}$ $\quad F = \cos y\,\hat{i} + x\sin y\,\hat{j}$

and c: $y = \sqrt{1-x^2}$ from $(1,0)$ to $(0,1)$

$$\int F \cdot dr = \int \cos y\,dx + x\sin y\,dy$$

$$= \int d(x\cos y)$$

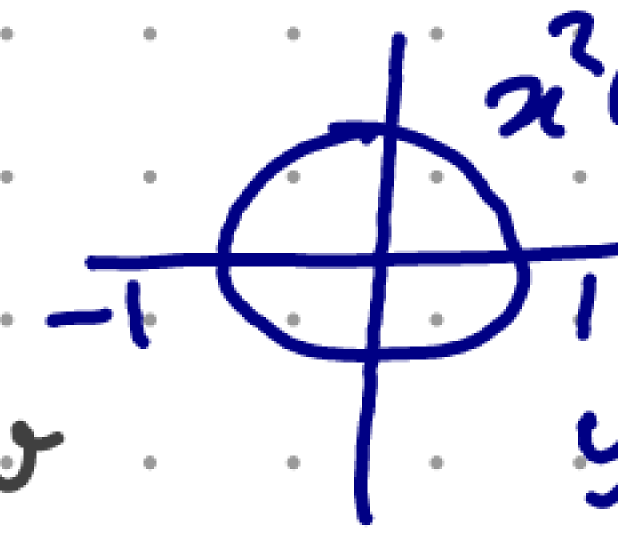
$$= \Big[x\cos y\Big]_{(1,0)}^{(0,1)} = -1$$



$y = \sqrt{1-x}$ $\quad \therefore x^2+y^2=1$ +ve (open path)

Here work done is independent of path)

Curve not closed



$x^2+y^2=1$

$y: 0\ 1$

$x: 0\ 1$

No green theorem as open curve

$$\int F\cdot dr = \iint\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= \iint \sin y - -\sin y = 2\iint \sin y\,dx\,dy$$

$$= 2\int_0^1 \sin y\,\sqrt{1-y^2}\,dy = \dots ?$$

③ Find work done in moving once round the circle $x^2+y^2=a^2$ $a \geq 0$ if $\bar{F} = \sin y\,\hat{i} + (x + x\cos y)\,\hat{j}$

→

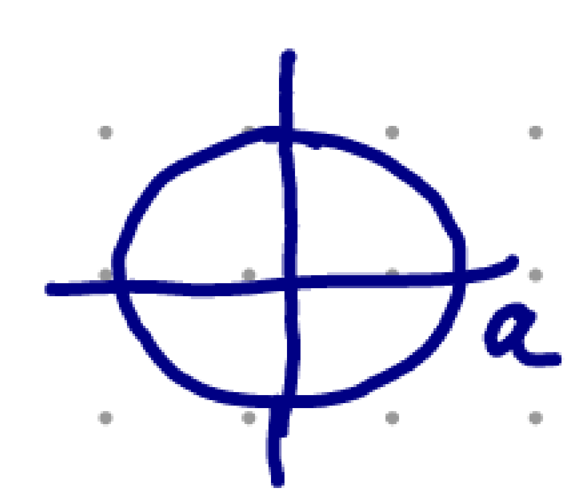
$$\int \bar{F}\cdot d\bar{r} = \int \sin y\,dx + (x + x\cos y)\,dy$$

$$\int_C d(x\sin y) + \int_C x\,dy$$

$x^2 + y^2 = a^2$

$x \to a\cos t$ $\qquad dy = a\cos t\,dt$

$y \to a\sin t$



$$\therefore \int x\,dy = \int_0^{2\pi} a\cos t\, a\cos t\,dt$$

$$= \pi a^2$$

$$\int_C d(x \sin y) = 0$$

Reason $\bar{F} = \sin y \,\hat{i} + \cos y \,\hat{i}$

$$\bar{F} \cdot dr = d(x \sin y)$$

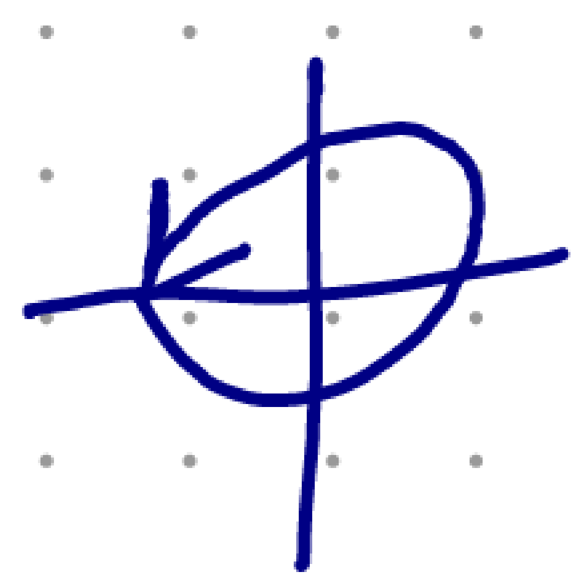
$$= d\phi$$

$$\bar{F} \cdot dr = \nabla \phi \cdot dr$$

$\bar{F} = \nabla \phi \quad \therefore \bar{F}$ is conservative (closed curve)

$$\int_C \bar{F} \cdot d\bar{r} = \int d\phi = 0$$

Work done is not independant at the path



Alternative → Green's theorem

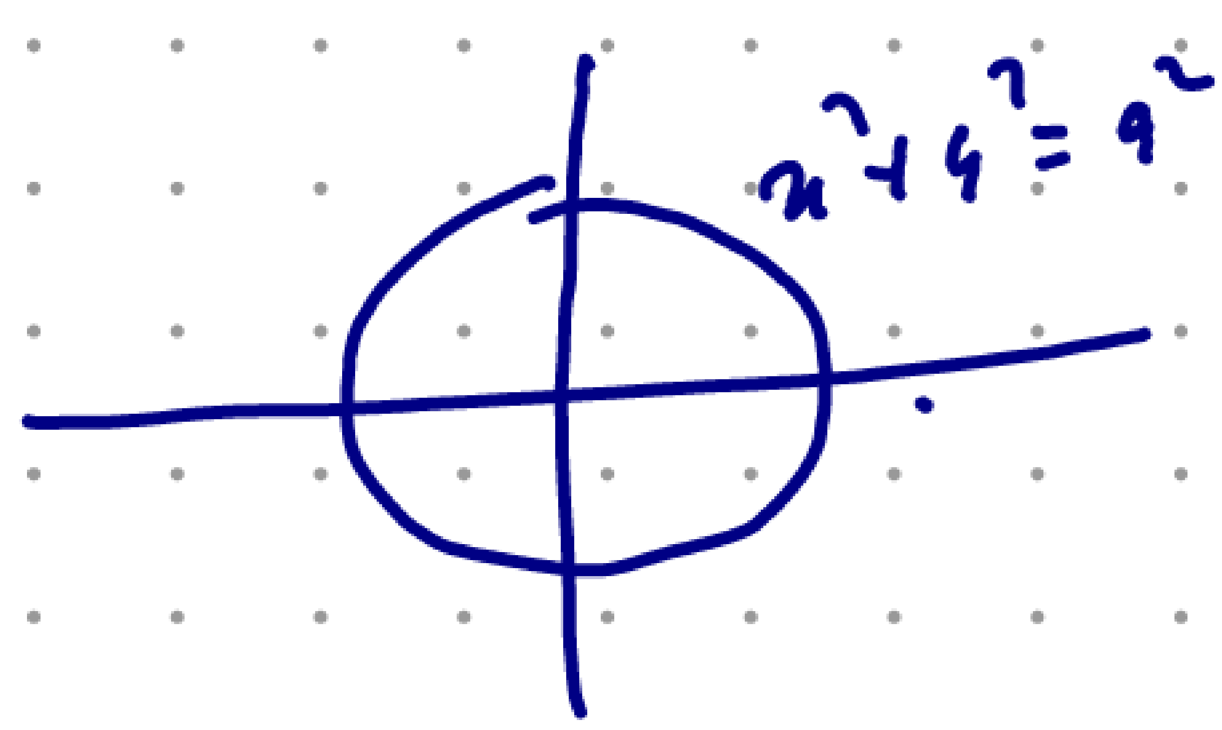
$P = \sin y$

$Q = x + x\cos y \, i$

$\frac{\partial P}{\partial y} = \cos y \qquad \frac{\partial Q}{\partial x} = 1 + \cos y$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 + \cos y - \cos y$

$\iint_R 1 \, dx\,dy$



$x \to r\cos\theta$

$y \to r\sin\theta$

$r = a$

$dx\,dy = r\,dr\,d\theta$

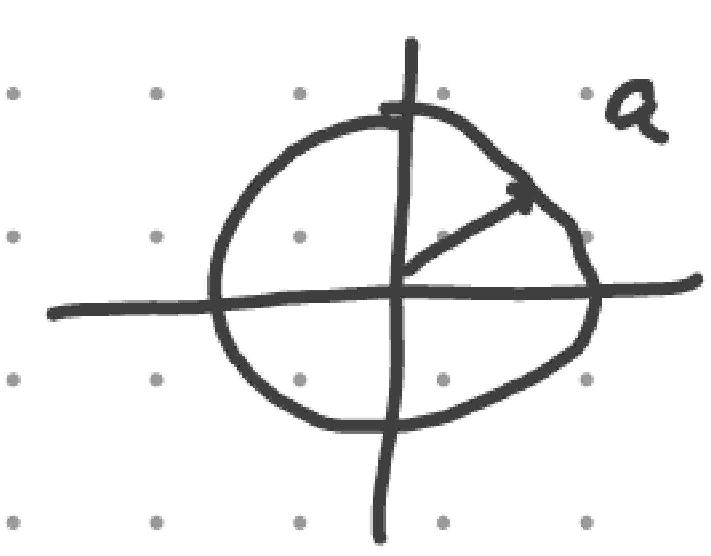
$\int_0^{2\pi}\int_0^{a} r \, dr\,d\theta$

$$\int_0^a r\,2\pi = 2\pi \frac{a^2}{2} = \underline{\underline{a^2\pi}}$$

Same result is obtained

$x \rightarrow r\cos\theta$

$y \rightarrow r\sin\theta$



$dx\,dy \rightarrow r\,dr\,d\theta$

$r:$ 0 to a

$\theta:$ 0 to 2π for full circle

(5) Evaluate using Green's theorem

$$\int (3x^2 - 8y)\, dx + (4y - 6xy)\, dy$$

$C : y = \sqrt{x}$ & $y = x$

$C :$ $y = \sqrt{x}$ & $y = x^2$

$P = 3x^2 - 8y^2$

$\frac{\partial P}{\partial x} = -16y$

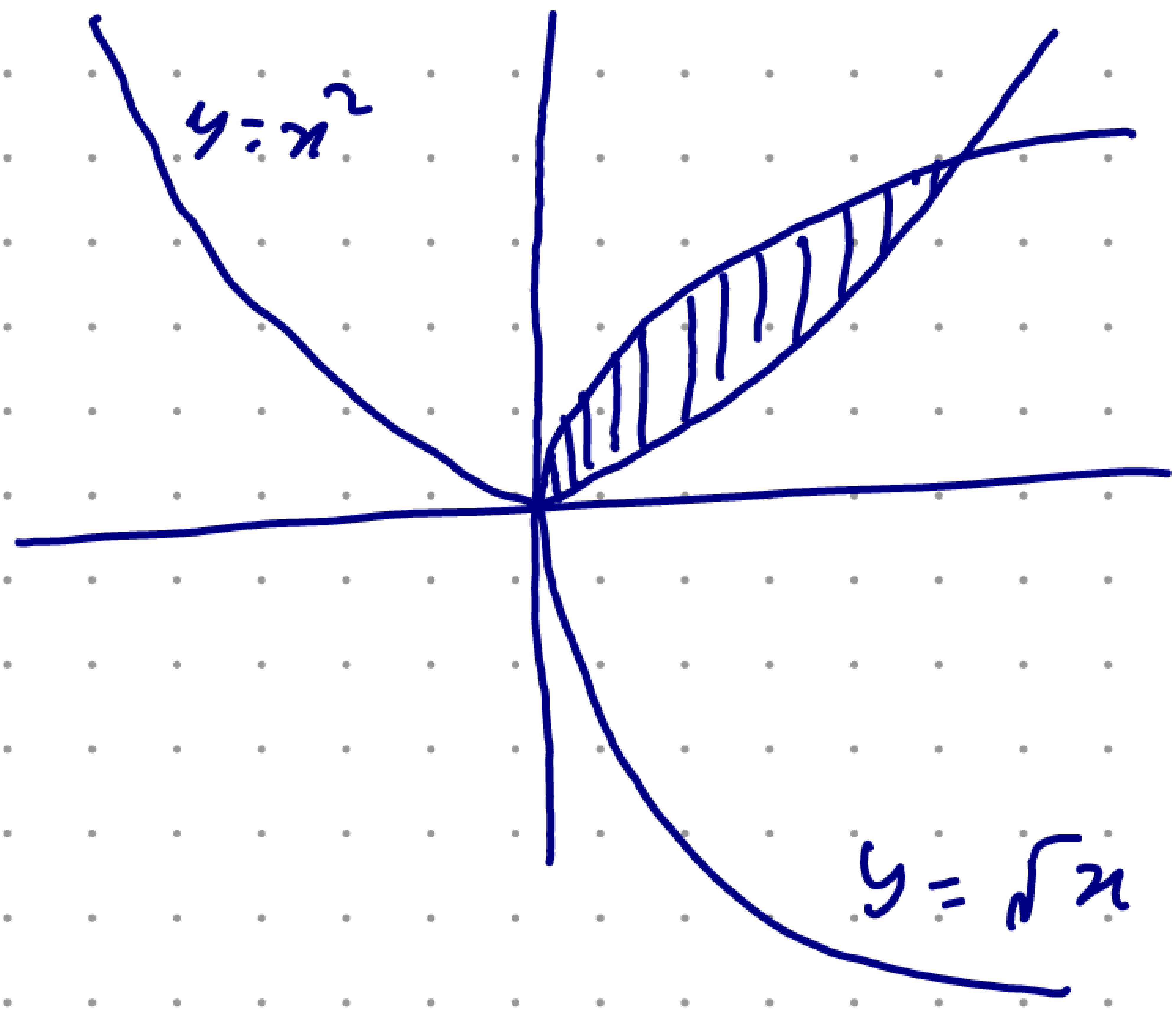
$Q = 4y - 6xy$

$\frac{\partial Q}{\partial x} = -6y$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 10y$$

$$\oint P\,dx + Q\,dy = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dx\,dy$$

$$= \iint 10y\, dx\,dy$$



$x \quad 0 \to 1$

$y: \quad \sqrt{x} \quad x^2$

$$= 10 \int_0^1 \int_{x^2}^{\sqrt{x}} y \, dy \, dx$$

$$= 5 \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = \frac{3}{2}$$

Stokes Theorem

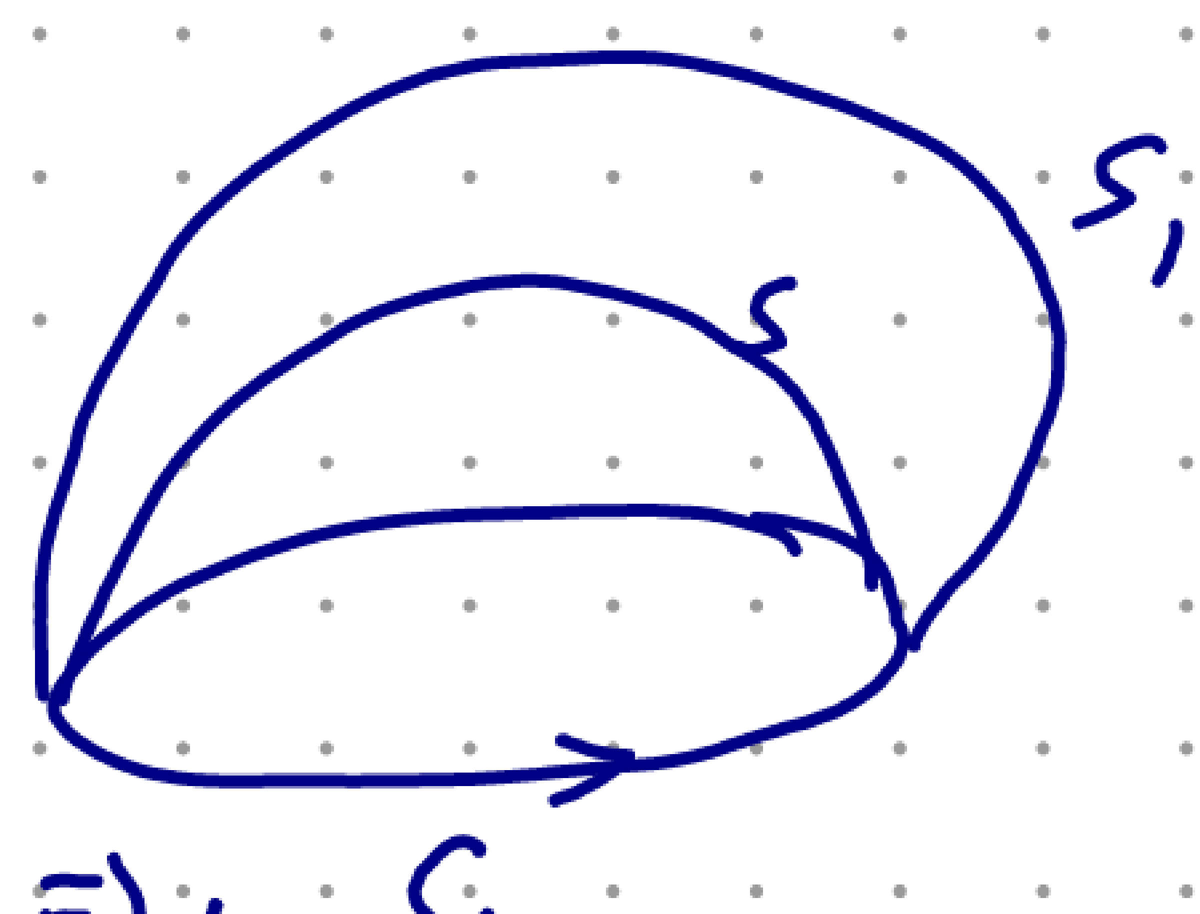
if $\bar{F}$ is a continuse vector field over S then

$$ds = \frac{dx\,dy}{|\hat{N} \cdot \hat{k}|}$$

$$\int_C F \cdot d\bar{r} = \iint_S \hat{N} \cdot (\nabla \times F)\, ds$$

$\hat{N}$ = unit outward normal vector $d\bar{s}$

If S & S_1 are two surfaces having same boundary C then



$$\iint_S \hat{N}(\nabla \times \bar{F})\, ds = \iint_{S_1} \hat{N}(\nabla \times \bar{F})\, ds_1$$

① $\int F \cdot d\bar{r} = y\hat{i} + z\hat{j} + x\hat{k}$

$S: x^2 + y^2 = 1 - z \; ; \; z \geq 0$

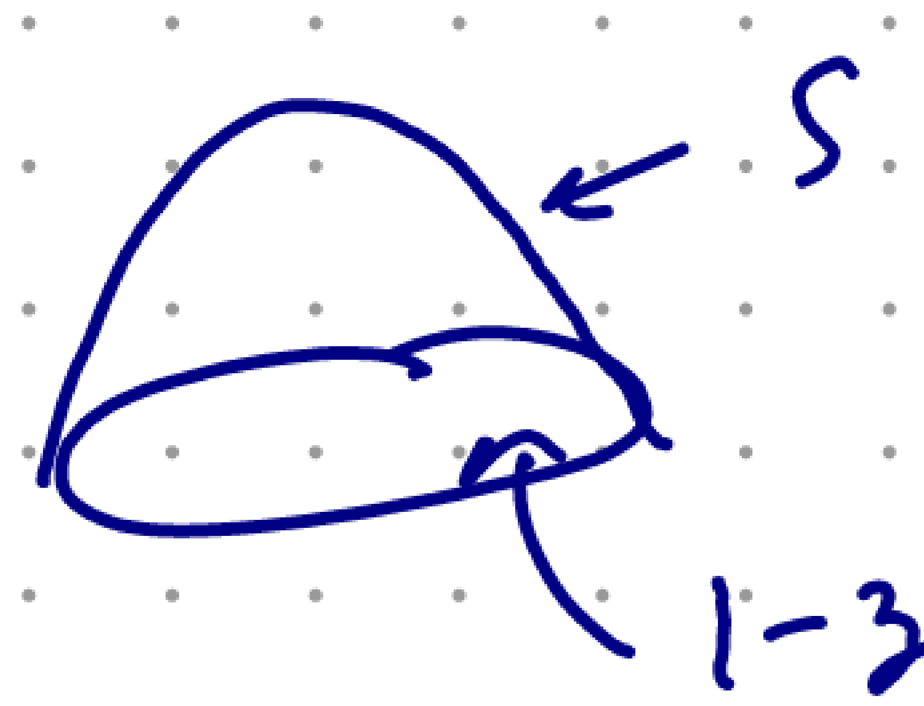
$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= -\hat{i} - \hat{j} - \hat{k}$$

$x^2 + y^2 = 1 - z \qquad z \geq 0$

$\therefore \; 1 - z \leq 0$

$\therefore x^2 + y^2 \leq 1$

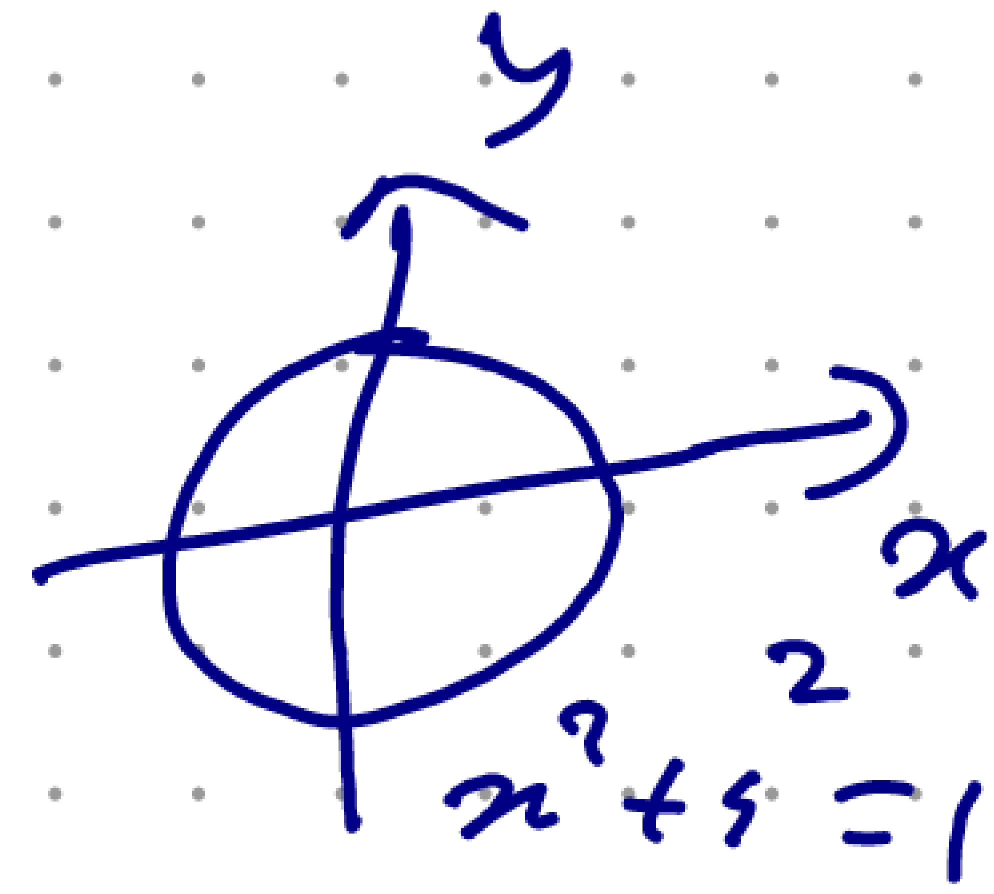


$\hat{N} = \hat{k}$

paraboloid passing through origin

$\therefore \iint_{S_1} \hat{N} \cdot (\nabla \times \bar{F})\, ds$, constant ∴ area $= \pi$

$$= \int_{-1}^{1} \int_{\sqrt{1-y^2}}^{-\sqrt{1-y^2}} -1 \, dx\, dy$$



$$= -\pi$$

② Apply stokes thm $\int \bar{F} \cdot d\bar{r}$

$$\bar{F} = (y^2 + z^2 - x^2)\hat{i} + (z^2 + x^2 - y^2)\hat{j} + (x^2 + y^2 - z^2)\hat{k}$$

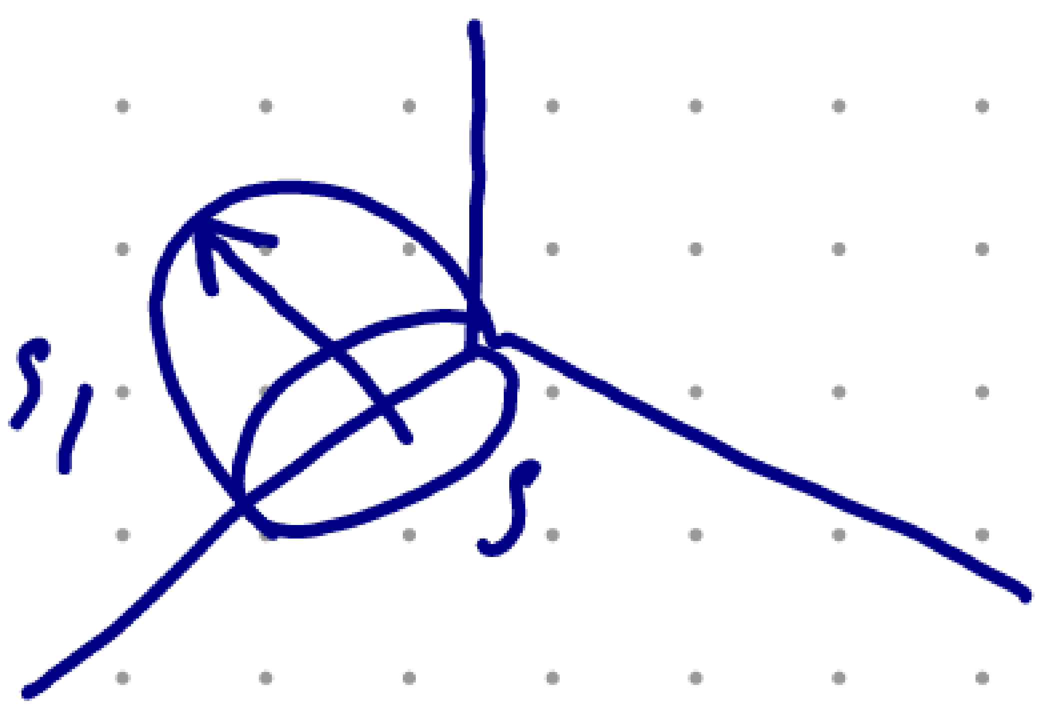
over surface $x^2+y^2-2ax+az=0$ above $z=0$

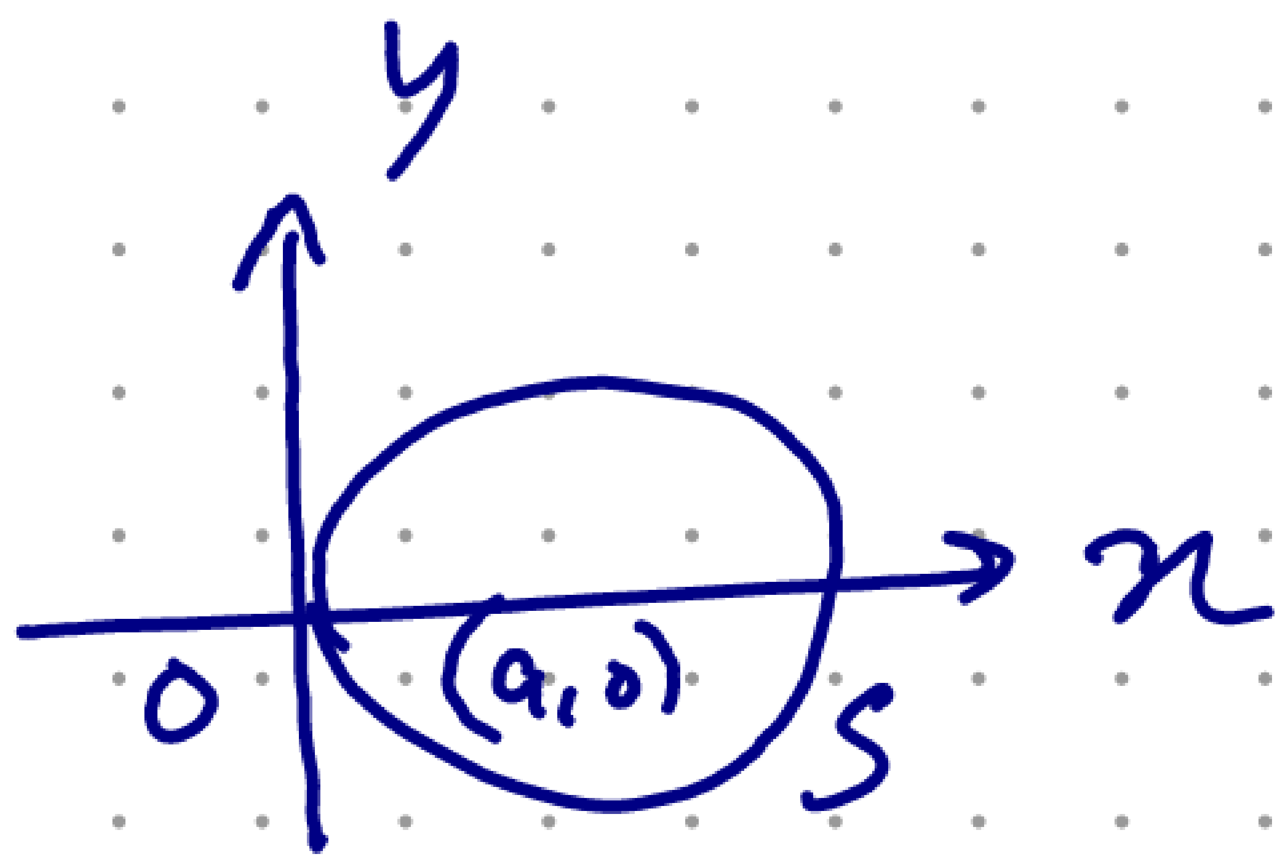
$$\nabla \times \bar{F} = (2y - 2z)\hat{i} + (2z - 2x)\hat{j} + (2x - 2y)\hat{k}$$

Surface $x^2 + y^2 - 2ax + az = 0$

$$(x-a)^2 + y^2 = -a(z-a)$$

Paraboloid passing through
(a 0 a)





We Integrate along any curve with surface S
∴ We chose circle along S

Limits

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dx\,dy = r\,dr\,d\theta$$

$$\therefore \int_C \bar{F} \cdot d\bar{r} = \iint 2r(\cos\theta - \sin\theta)\, r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{2a\cos\theta} 2r(\cos\theta - \sin\theta)\, r\,dr\,d\theta$$

$$= \frac{2}{3} \int_{-\pi/2}^{\pi/2} (\cos\theta - \sin\theta)\, 8a^3 \cos^3\theta \, d\theta$$

$$= \frac{16a^3}{3} \int_{-\pi/2}^{\pi/2} \left(\cos^4\theta - \sin\theta \cos^3\theta\right) d\theta$$

$$= \frac{32a^3}{3} \int_{0}^{\pi/2} \cos^4\theta \, d\theta$$

Beta Gamma

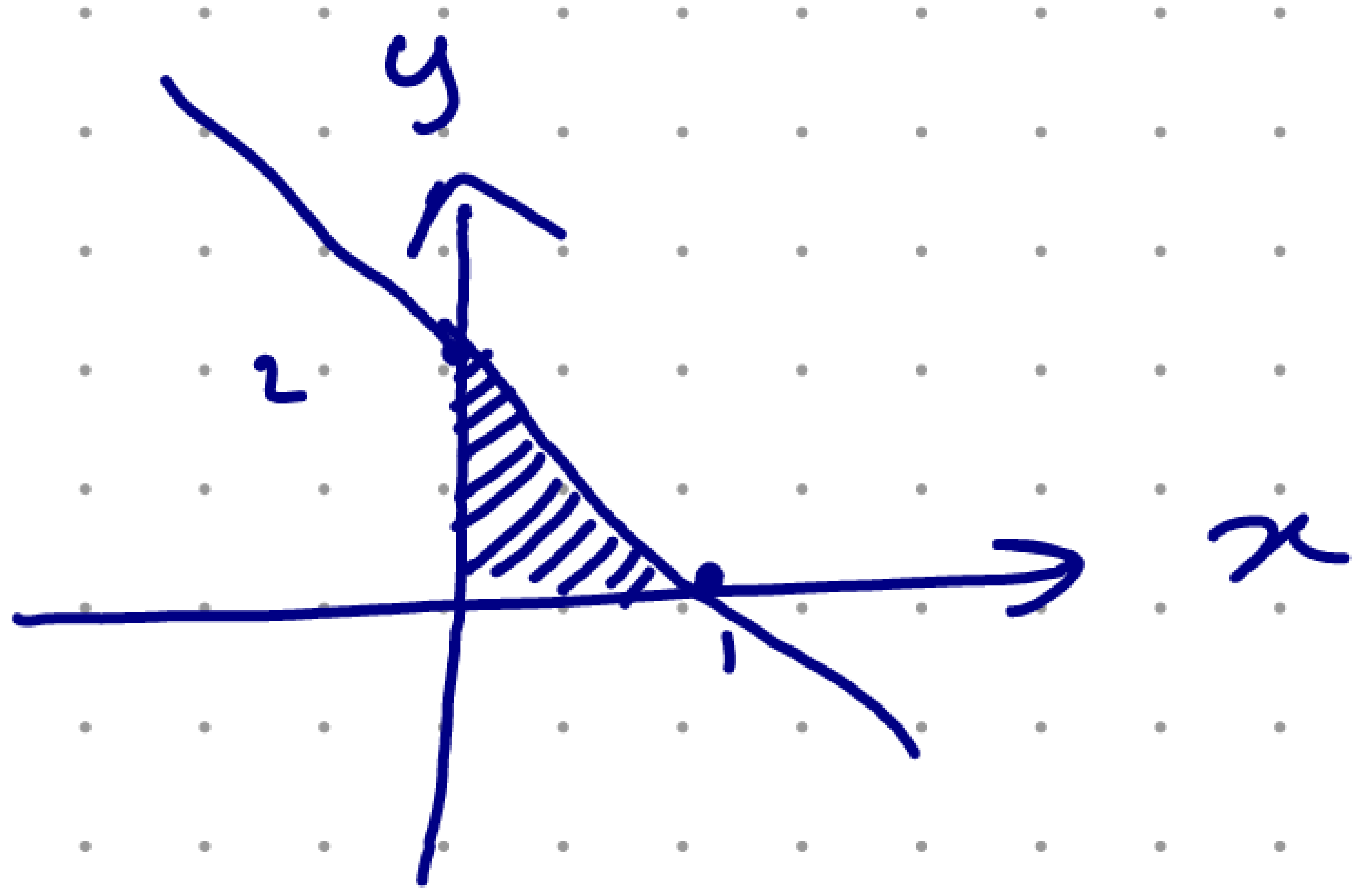
$$= \frac{32a^3}{3} \left(\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right)$$

③ Use stokes theorem $\bar{F} \cdot d\bar{r}$

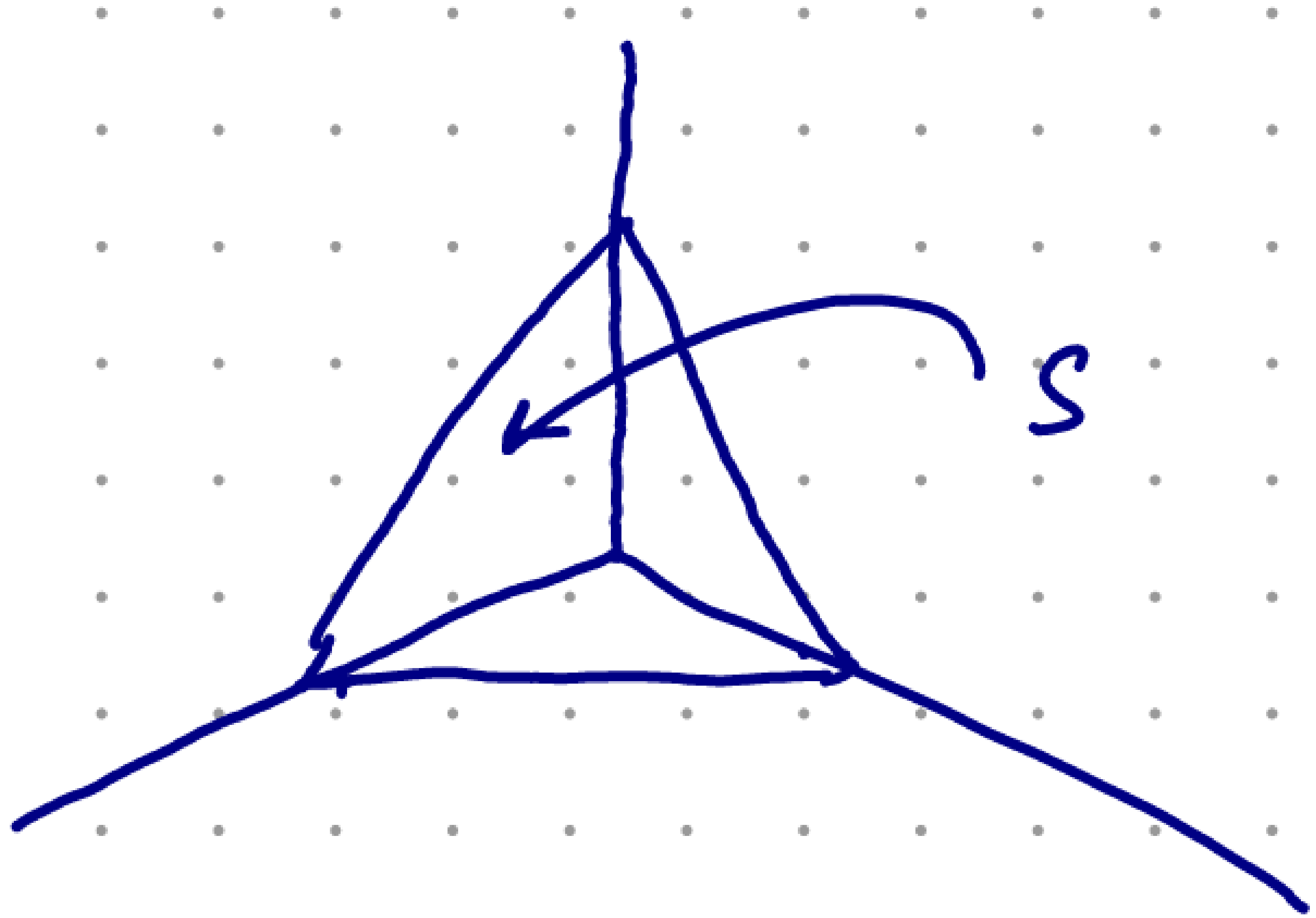
$\bar{F} = (x+y)\hat{i} + (y+z)\hat{j} - x\hat{k}$

S is $2x+y+z=2$ in first octant

$2x+y=2$



Surface
Not $\perp$ to xy



Normal to surface

$\phi = 2x+y+z-2$

is $\nabla\phi$

$\nabla\phi = 2\hat{i}+\hat{j}+\hat{k}$

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & y+z & -x \end{vmatrix}$$

$$= -\hat{i} + \hat{j} - \hat{k}$$

$$\hat{N} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}}$$

$$ds = \frac{dx\,dy}{|\hat{N} \cdot \hat{k}|} = \sqrt{6}\, dx\,dy$$

$$\hat{N} \cdot \nabla \times \bar{F} \; ds = -2$$

$$= -2 \times \text{Area of } \Delta$$

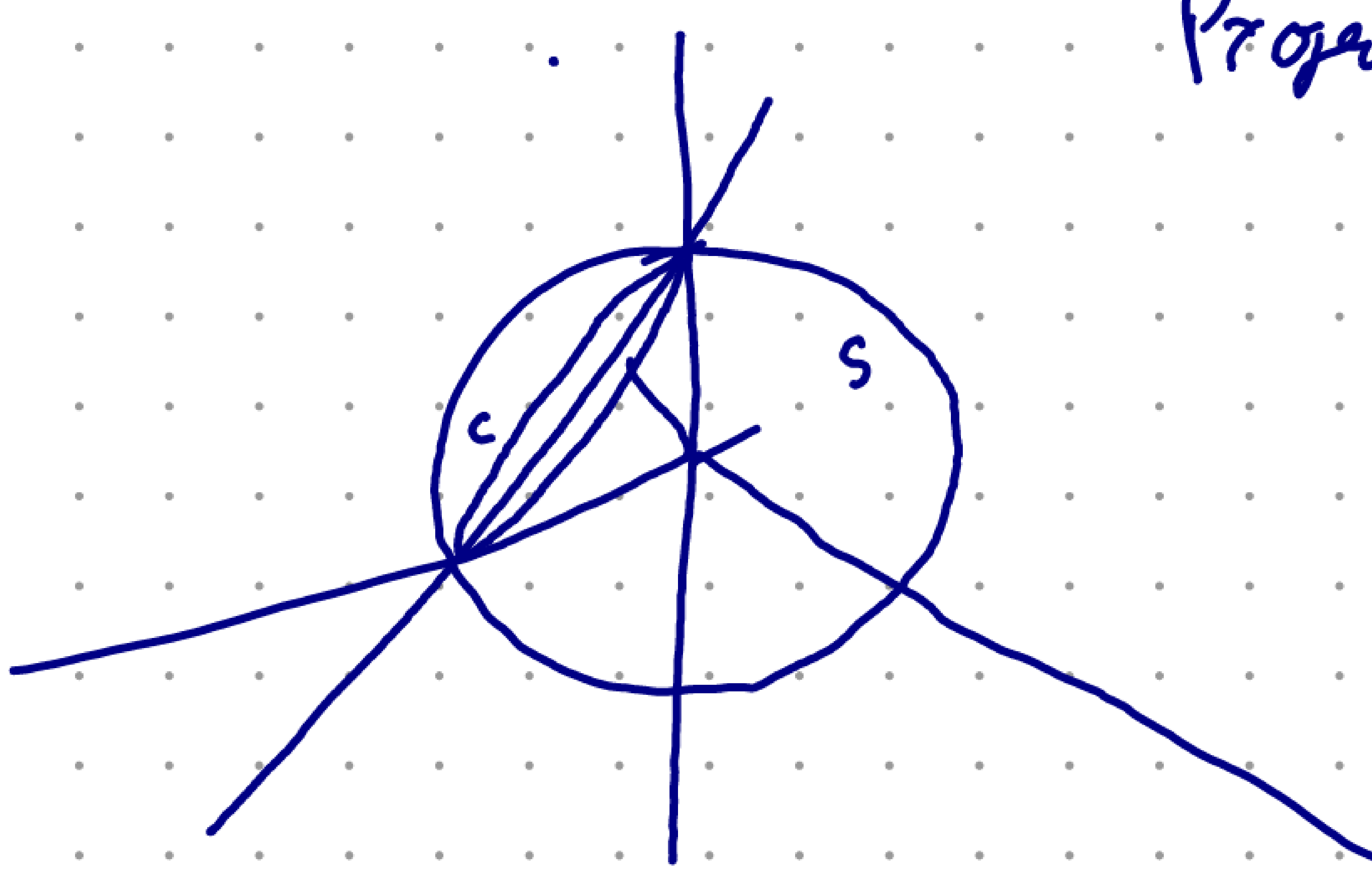
$$= -2 \times \frac{1}{2} \times 1 \times 2 = -2$$

④ Apply stokes theorem

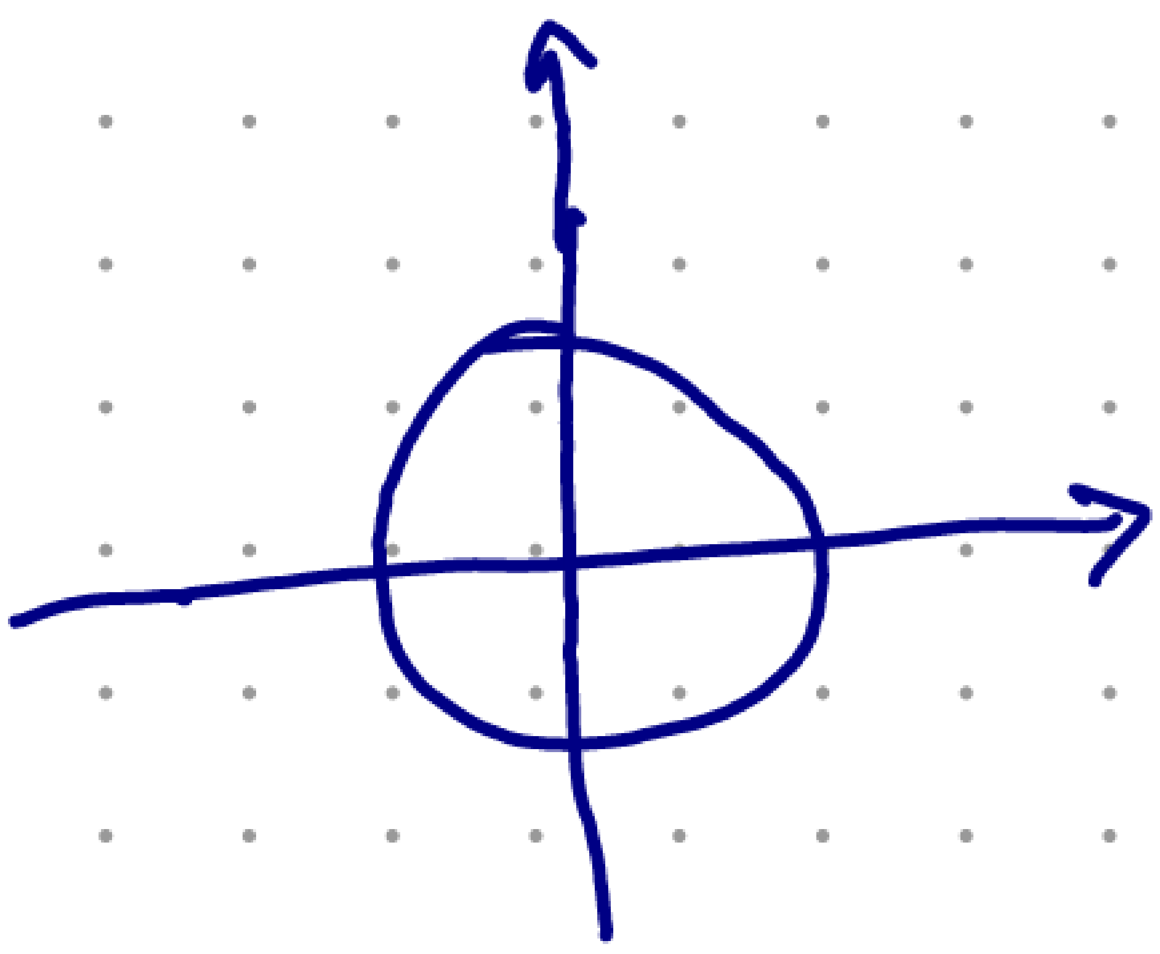
$$\int_C y\,dx + z\,dy + x\,dz$$

C : Curve of Intersection of

$x^2+y^2+z^2 = a^2$ & $x+z = a$



Projection will be circle on plane $x+z=a$



$F = y\hat{i} + 3\hat{j} + x\hat{k}$

$\nabla \times \bar{F} = -\hat{i} - \hat{j} - \hat{k}$

$\hat{N} = \nabla \phi$ $\quad\quad$ ϕ: plane / surface

$\therefore \phi = x + z - a$

$\therefore$ $\quad\quad$ $\nabla \phi = \hat{i} + \hat{k}$

$$\therefore \hat{N} = \frac{\hat{i} + \hat{k}}{\sqrt{2}}$$

$\hat{N} \cdot (\nabla \times \bar{F}) = -\sqrt{2}$

$$ds = \frac{dx\,dy}{(1/\sqrt{2})}$$

$$\therefore \iint_R -2 \quad dx\,dy \quad = -2 \iint dx\,dy$$

note: Here we cannot take projection on

xz plane as it will be straight line

---

finding C projection on xy

Solve $x^2+y^2+z^2=a^2$

$x+z=a$

$x^2+y^2(a-x)^2=a^2$

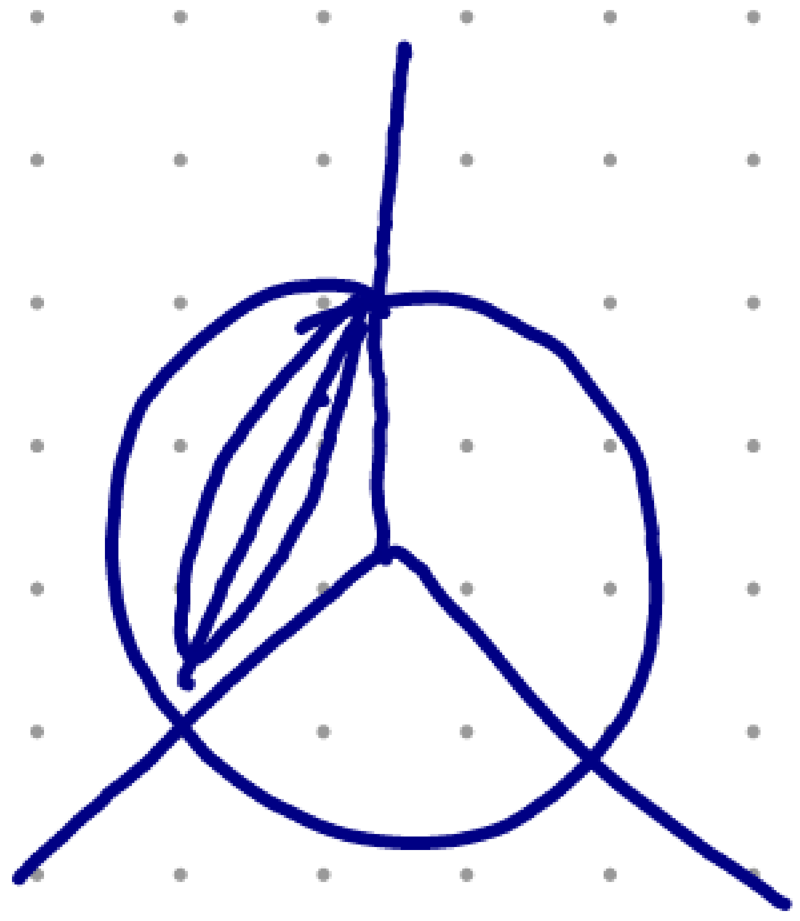
$\therefore 2(x-a/2)^2+y^2=a^2/2$

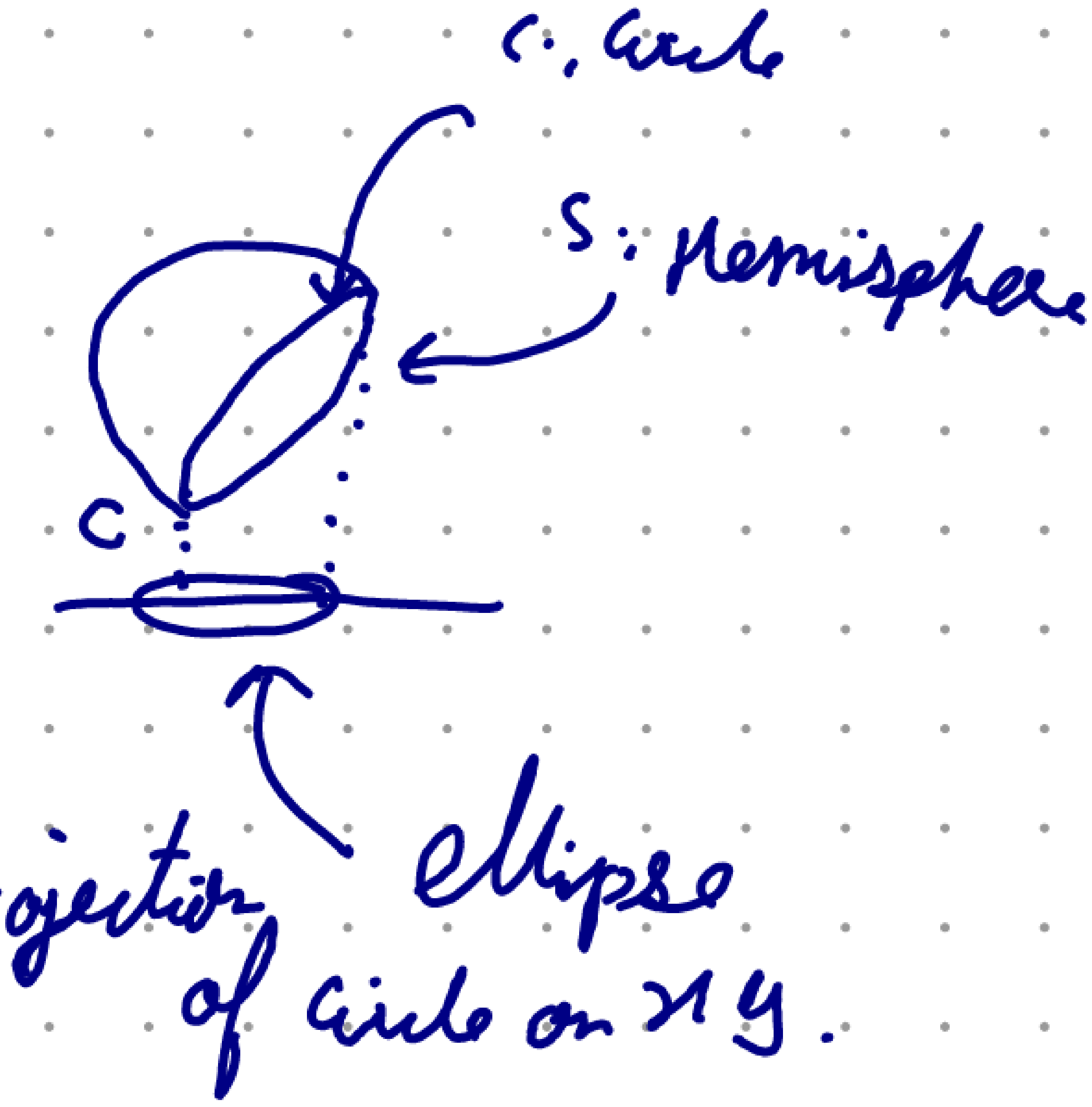
$$\therefore \frac{(x-a/2)^2}{(a/2)^2}+\frac{y^2}{(a/\sqrt{2})^2}=1$$

It is ellipse

Note C is circle

Projection of C on xy plane is ellipse





$$= -2 \iint_S dx\,dy$$

$\boxed{\text{Area of ellipse } \pi ab}$

$$S: \frac{(x-a/2)^2}{(a/2)^2} + \frac{y^2}{(a/\sqrt{2})^2} = 1$$

$$= -2 \times \pi \times \frac{a}{2} \times \frac{a}{\sqrt{2}} = -\frac{\pi a^2}{\sqrt{2}}$$

Apply stokes theorem

$\int 3y\,dx + 4z\,dy + 6y\,dz$ Along C

C is curve of Intersection $x^2+y^2+z^2 = 83$ & $z = x+4$

$F = 3y\hat{i} + 4z\hat{k} + 6y\hat{k}$

H.W.

Ans = $40\pi\sqrt{2}$

$$F = 3y\hat{i} + 4z\hat{j} + 6y\hat{k}$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 3y & 4z & 6y \end{vmatrix}$$

$$= (6-4)\hat{i} - 0 + -3\hat{k} = 2\hat{i} - 3\hat{k}$$

$$\cancel{\phi = x^2 + y^2 + z^2 - 8z}$$

$$\cancel{\nabla\phi = 2x\,\hat{i} + 2y\,\hat{j} + (2z-8)\,\hat{k}}$$

$$\cancel{|\nabla\phi|}$$

ϕ can be plane also as $\perp$ is same.

$$\phi = x + y - z$$

$$\nabla\phi = \hat{i} + -\hat{k}$$

$$\hat{N} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{(1, 0, -1)}{\sqrt{2}}$$

$$ds = \frac{dx\,dy}{|\hat{N}\cdot\hat{k}|} = \sqrt{2}\,dx\,dy$$

$$\hat{N}\cdot(\nabla\times\vec{F})\,ds = \frac{(2+3)}{\sqrt{2}} \times \sqrt{2}\,dx\,dy = 5\,dx\,dy$$

Curve of intersection (x y) projection

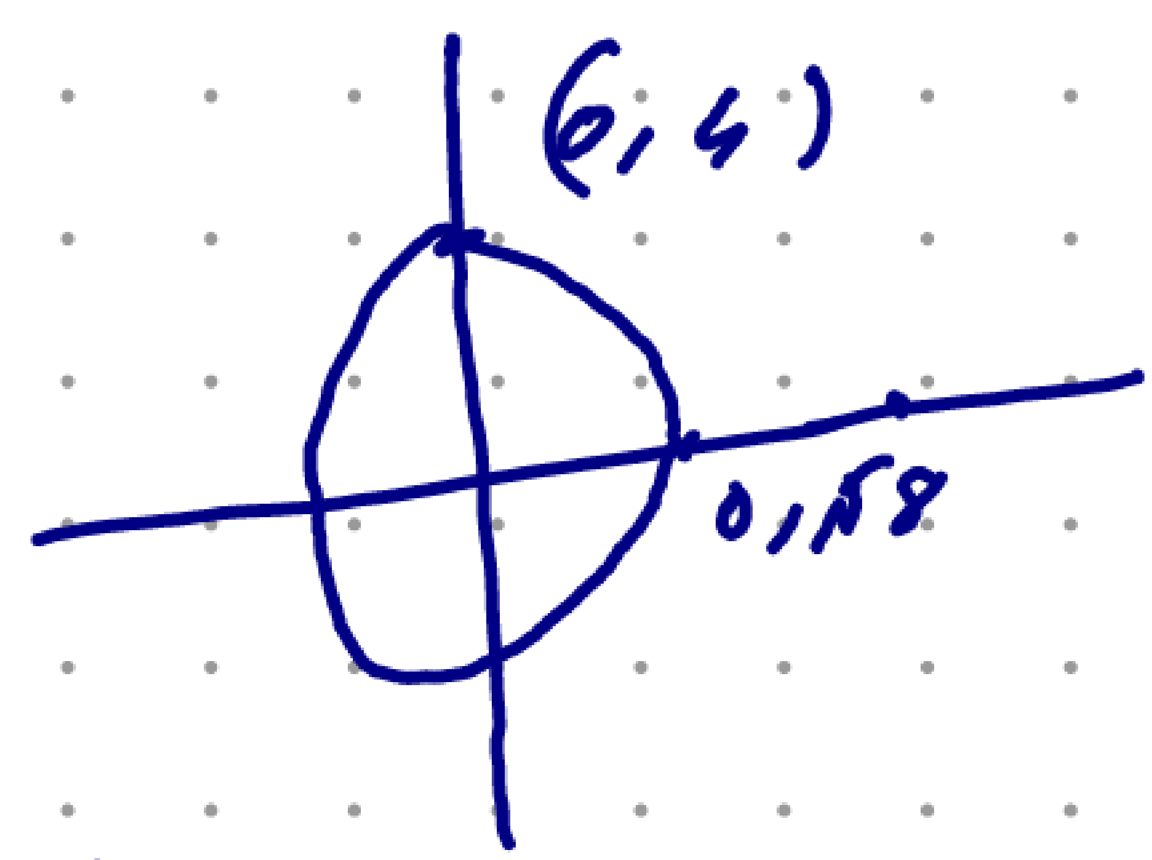
$$x^2+y^2+z^2 = 8z$$

$$z = x+4$$

$$\therefore\ x^2+y^2+x^2+\cancel{8x}+16 = \cancel{8x}+32$$

$$\therefore\ 2x^2+y^2 = 16$$

$$\therefore\ \frac{x^2}{8}+\frac{y^2}{16} = 1$$



ellipse

$a = \sqrt{8}$

$b = 4$

area $= 4\pi\sqrt{8} = 8\pi\sqrt{2}$

$$.5\iint dx\,dy = 40\pi\sqrt{2}$$

Stokes theorem

① find $\hat{N}$ $\qquad\qquad$ $\phi \rightarrow$ Plane

$$\hat{N} = \frac{\nabla \phi}{|\nabla \phi|}$$

$$ds = \frac{dx\,dy}{|\hat{N} \circ \hat{k}|}$$

$$\therefore \iint \frac{\nabla \phi\,(\nabla \times \bar{F})}{(\hat{N} \circ \hat{k})}\,dx\,dy$$

② Find Curl

$\nabla \times F$

(3) Find Limits

Drar

Gauss Divergance Theorem

If $\bar{F}$ is continuse vector field then

$$\oiint_S \hat{N} \circ \bar{F} \, ds = \iiint_V \bar{\nabla} \circ \bar{F} \, dv$$

(Verification wont be asked)

① Use divergance theorem for $\bar{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ over cylinder $x^2+y^2 = a^2$ & $z=0$, $z=b$

$\nabla \cdot \bar{F} = 4 + -4y + 2z$

$x \to r\cos\theta$
$y \to r\sin\theta$

$r = a$, $\theta \to 0$ to 2π , $z: 0b$

$dx\,dy\,dz : r\,dr\,d\theta\,dz$

$$\int_0^b\int_0^a\int_0^{2\pi} (4 - 4y + 2z)\, r\,dr\,d\theta\,dz$$

$$\int\int\int_0^{2\pi} (4r - 4r^2\sin\theta + 2zr)$$

$$= \int\int_0^a 8r\pi + 4zr\pi \; dr = \int_0^b \left[8\pi\frac{a^2}{2} + 4z\pi\frac{a^2}{2}\right]$$

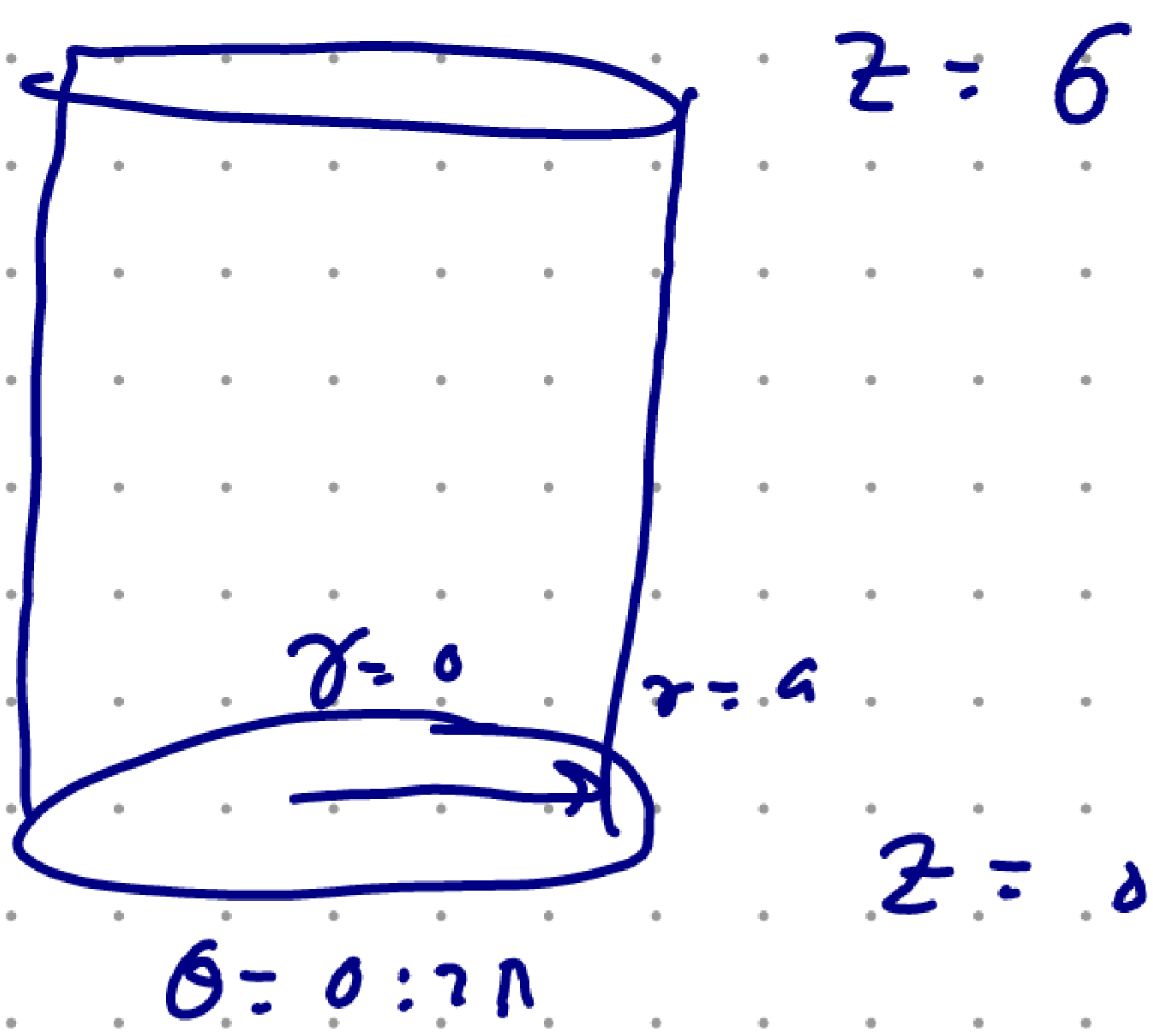
$$= 4\pi a^2 b + \frac{a^2 b^2}{2} \times 2\pi$$

$$= 4\pi a^2 b + \pi a^2 b^2$$

Note order of Integration

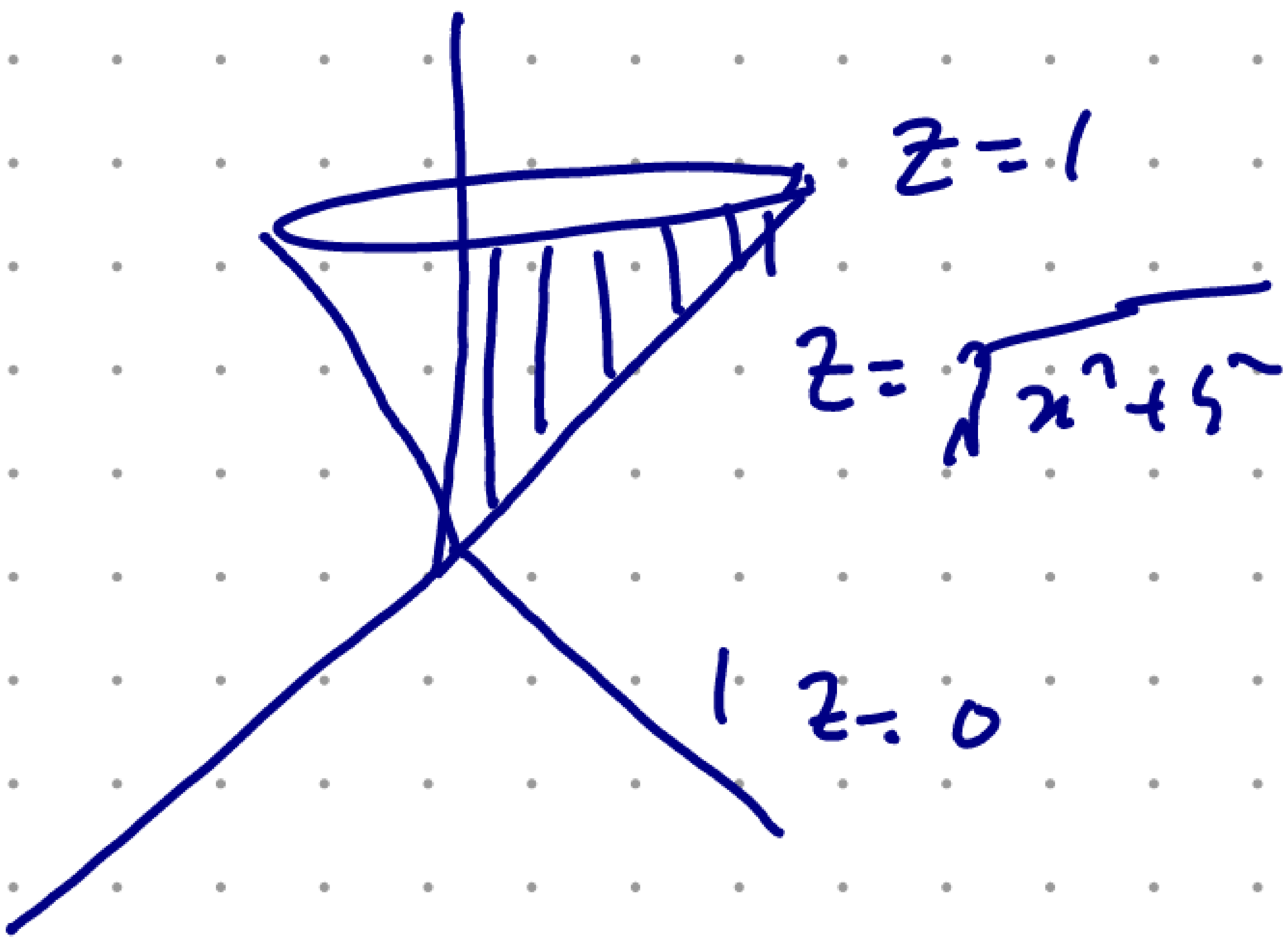
z, θ, r

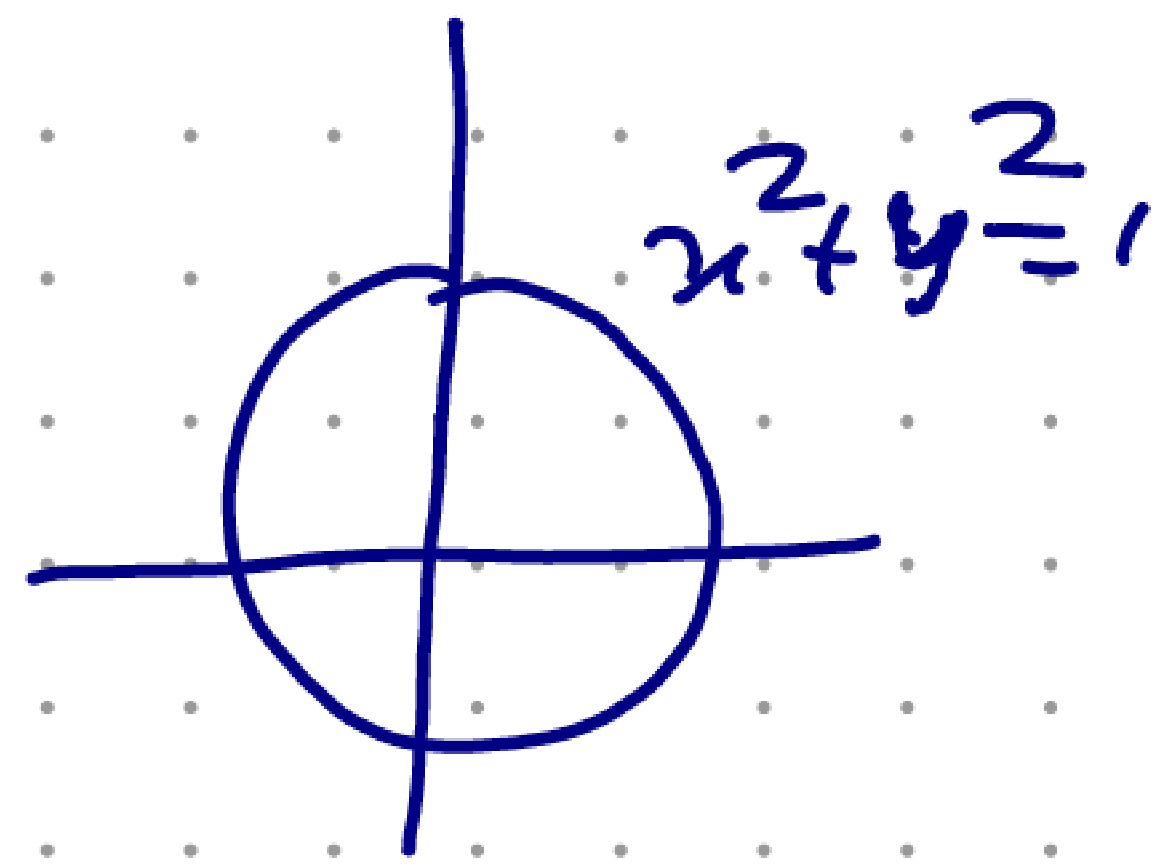
will be easier



② $F = x\hat{i} + y\hat{j} + z^2\hat{k}$ cone

$x^2 + y^2 = z^2$ & plane $z = 1$





$$\nabla \cdot F = (1 + 1 + 2z) = (2 + 2z)$$

$$\iiint_V 2 + 2z$$

$z \to z$

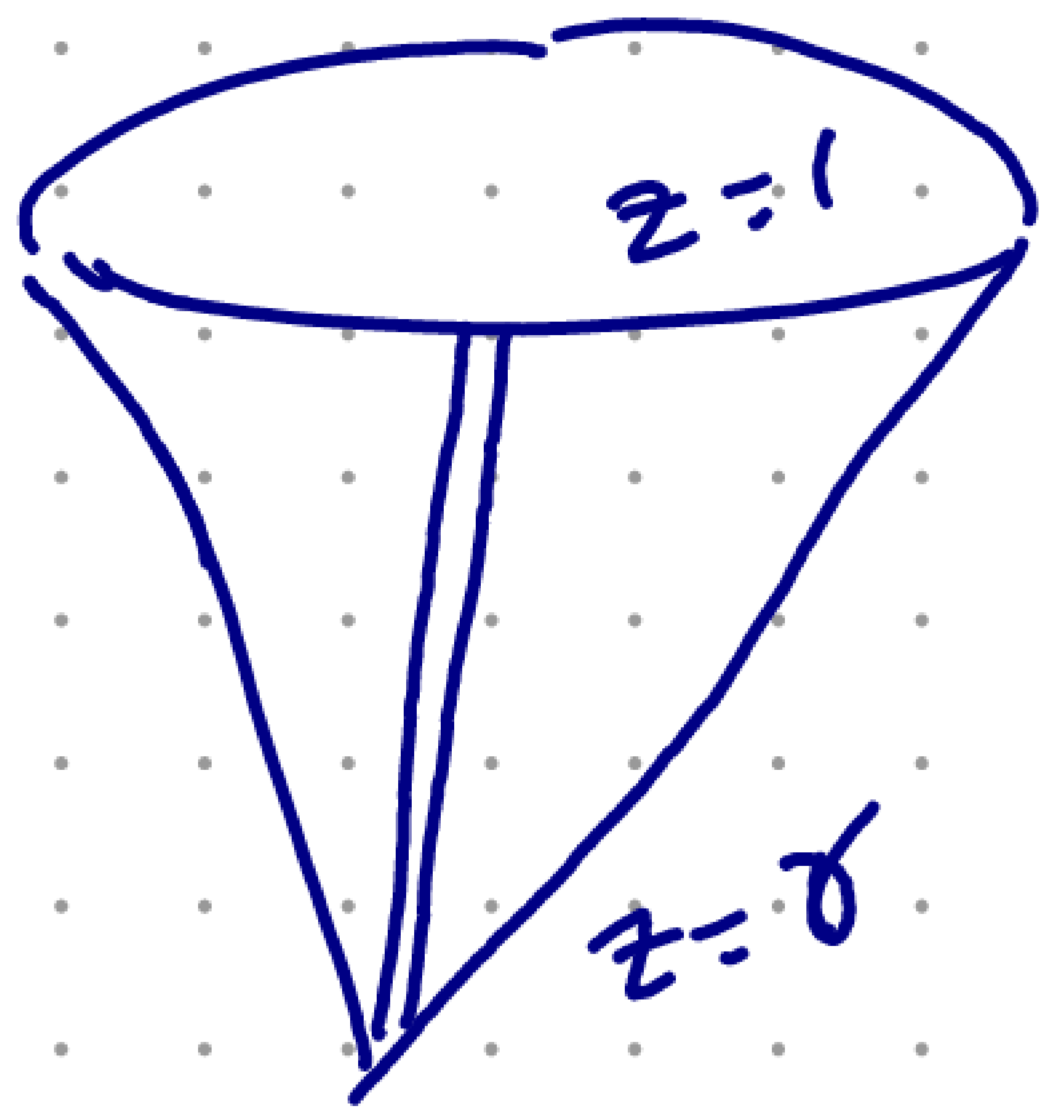
$x \to r\cos\theta$

$y \to r\sin\theta$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

$$\iiint (2 + 2z)r \; dz \, d\theta \, dr$$

$$= \iint r \left[2z + z^2\right]_r^1 \; dr\, d\theta$$



$z: \quad 1 : r$

$r: \quad 0 : 1$

$\theta: \quad 0 \; 2\pi$

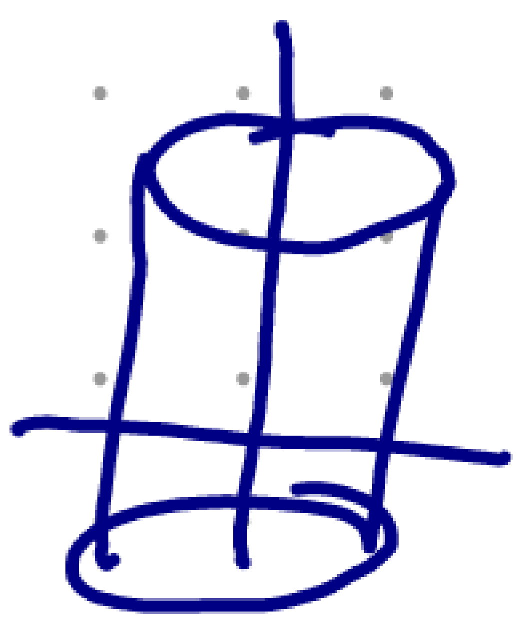
$$\int_0^{2\pi}\int_0^1\int_1^r ☺ \, dz\, dr\, d\theta$$

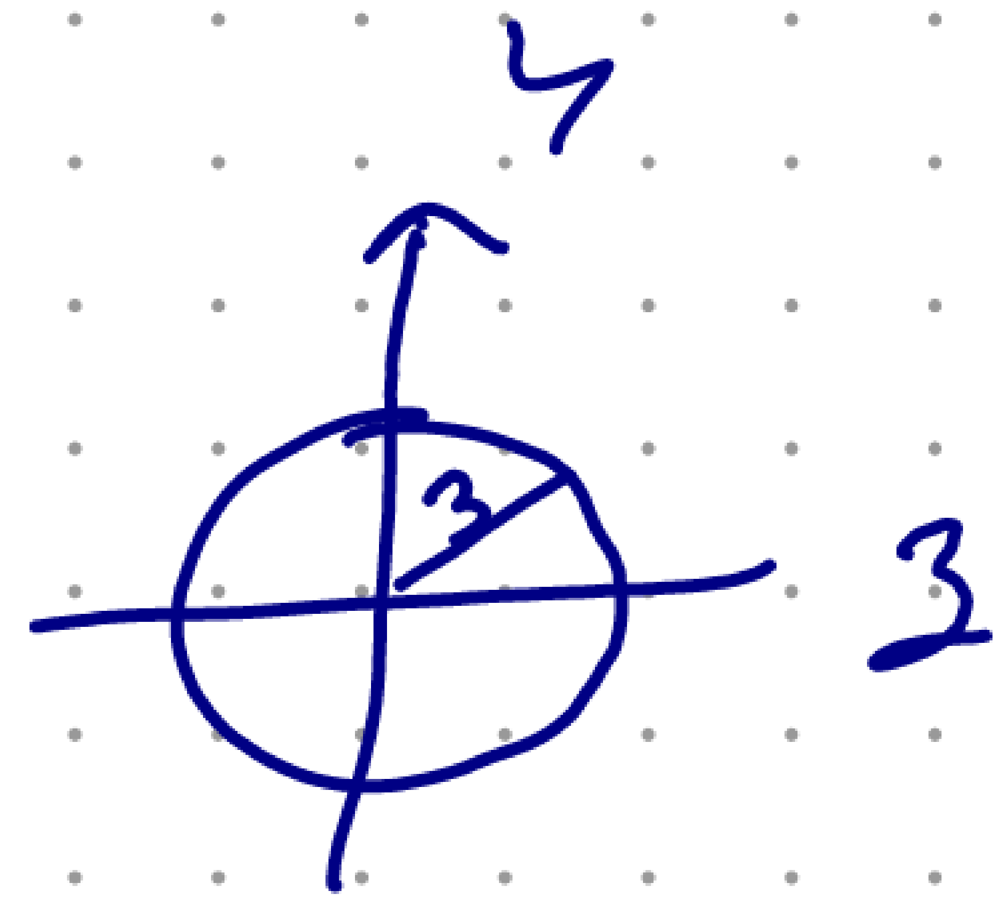
$$= \int_0^{2\pi} \left[\frac{3}{2} r^2 - \frac{2}{3} r^3 - \frac{r^4}{4} \right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{3}{2} - \frac{2}{3} - \frac{1}{4} \right) = \frac{7}{12} 2\pi = \frac{7\pi}{6}$$

③ $\bar{F} = 2x^2y\,\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$

$R:\ y^2+z^2=9\ \&\ x=2$ in first octant





$$\int_0^{\pi/2}\int_0^3\int_0^2 (\nabla\cdot F)\, r\, dx\, dr\, d\theta$$

Here $r = z^2+y^2$

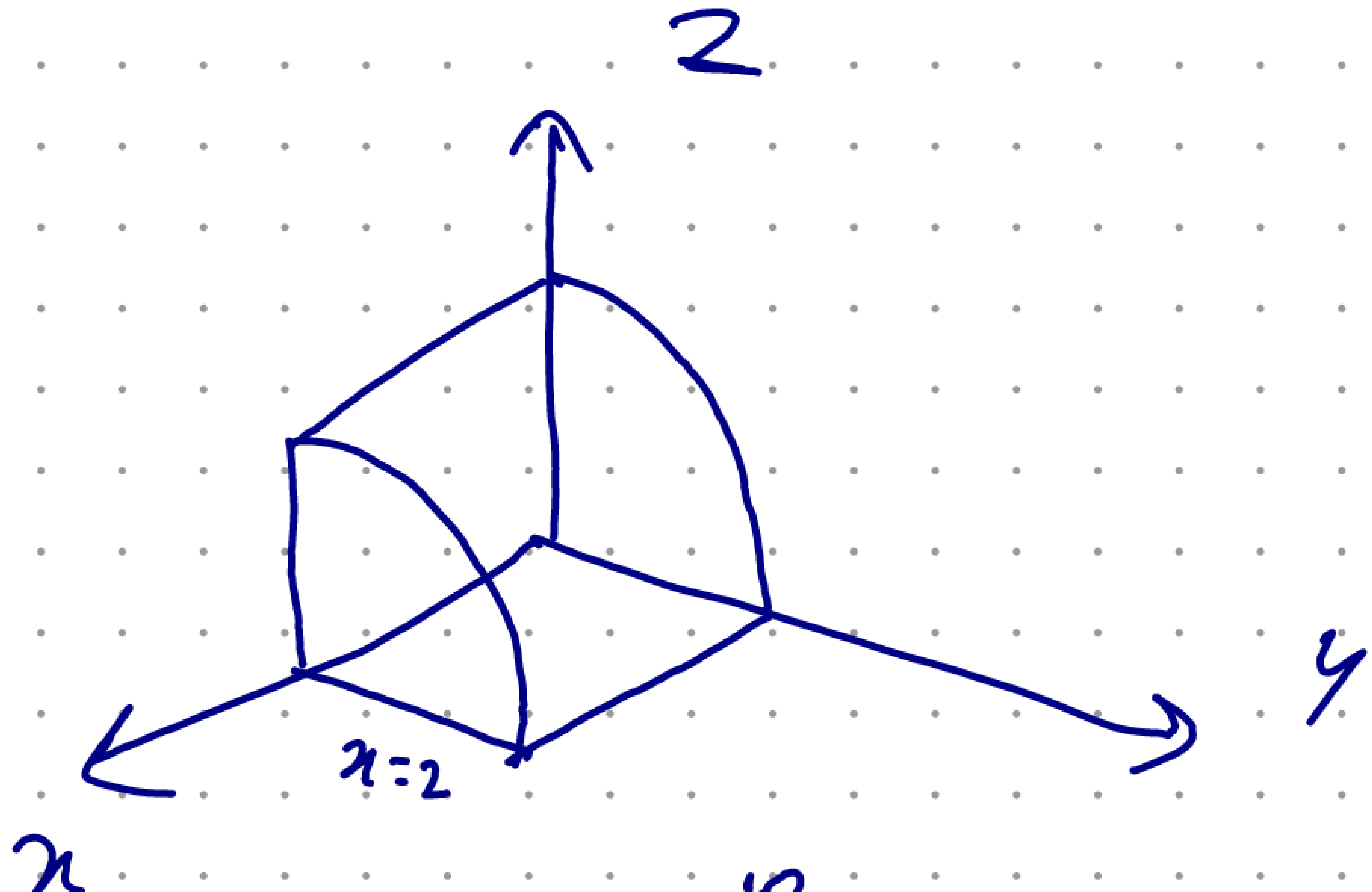
$$\nabla\cdot F = (4xy - 2y + 8xz)$$

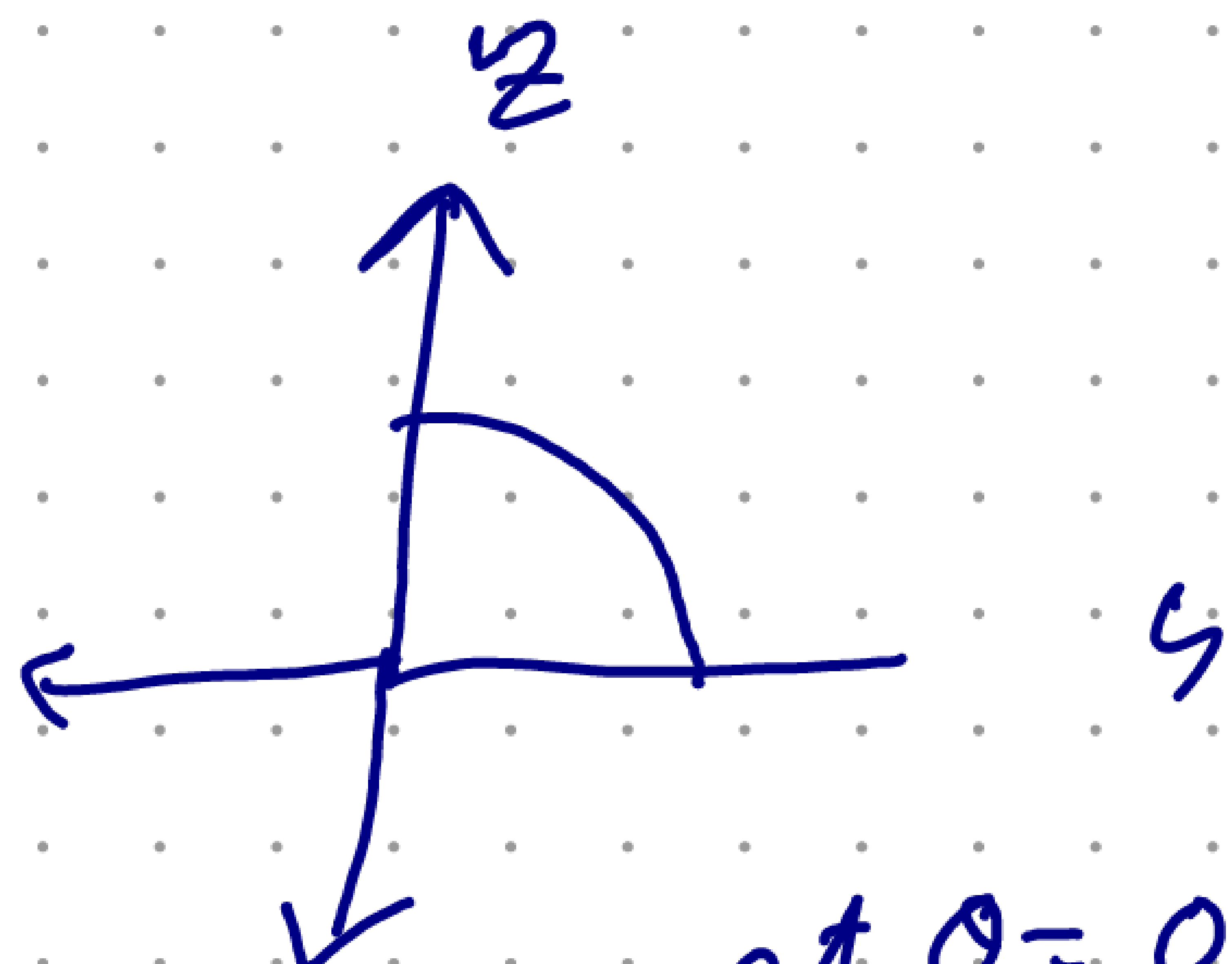
$$= \int_0^{\pi/2}\int_0^3 r\left(4y\frac{x^2}{2} - 2yx + 8z\frac{x^2}{2}\right)_0^2$$

$$= \int_0^{\pi/2}\int_0^3 r\,(8r\cos\theta - 4r\cos\theta + r\sin\theta\cdot 16)$$

$$= \int_0^{\pi/2} (\qquad 4\cos\theta + 16\sin\theta)\left[\frac{r^3}{3}\right]_0^3 = 9(16+4)$$

$$= 180$$





at $\theta = 0$, $z = 0$

$\therefore z = r\sin\theta$

& $y = r\cos\theta$

Alternative → Don't use cylinder coordinates

② For spherical coordinates

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

		First octant
r	: 0 to a	0 to a
θ	: 0 to 2π	$0 \to \pi/2$
ϕ	: 0 to 2π	$0 \to \pi/2$

Angle with z axis