① Scalar triple product

$$(\bar{a} \times \bar{b}) \cdot \bar{c} = \bar{a} \circ (\bar{b} \times \bar{c})$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$[\bar{a}\,\bar{b}\,\bar{c}] = [\bar{b}\,\bar{c}\,\bar{a}] = [\bar{c}\,\bar{a}\,\bar{b}]$$

$[\bar{a}\,\bar{b}\,\bar{c}] = -[\bar{a}\,\bar{c}\,\bar{b}]$ as order is changed

$$[\bar{a}\,\bar{a}\,\bar{b}] = 0$$

$\bar{a}\,\bar{b}\,\bar{c}$ are coplaner then $[\bar{a}\,\bar{b}\,\bar{c}] = 0$

Show that $(\bar{p} + \bar{q}) \circ [(\bar{q} + \bar{r}) \times (\bar{r} + \bar{p})]$

$$= 2[\bar{p}\,\bar{q}\,\bar{r}]$$

ans $(\bar{p} + \bar{q}) \circ ((\bar{q} + \bar{r}) \times (\bar{r} + \bar{p}))$

$$= [\bar{p}+\bar{q} \quad \bar{q}+\bar{r} \quad \bar{r}+\bar{p}] = [p\,q\,r] + [q\,r\,p] = 2[pqr]$$

② Vector triple product

$$(\bar{a} \times \bar{b}) \times \bar{c} = (a \circ c)\, \bar{b} - (b \circ c)\, \bar{a}$$

$$a \times (\bar{b} \times \bar{c}) = (a \circ c)\, \bar{b} - (\bar{a} \circ \bar{b})\, \bar{c}$$

If $\bar{a}, \bar{b}\ \bar{c}$ are non coplaner, P.T. $\bar{a} \times \bar{b}$, $\bar{b} \times \bar{c}$ & $\bar{c} \times \bar{a}$ are also non coplaner.

$\bar{a}\ \bar{b}\ \bar{c}$ are non coplaner

$\therefore\ [\bar{a}\ \bar{b}\ \bar{c}] \neq 0$

To prove $[a \times b \quad b \times c \quad c \times a] \neq 0$

$$= (a \times b) \circ [(b \times c) \times (c \times a)]$$

$$= (a \times b) \circ [(b \circ (c \times a))\, \bar{c} - (c \circ (c \times a))\, \bar{b}]$$

$$= (a \times b) \circ [\,[a\, b\, c]\, \bar{c}\,]$$

$$= [a b c]\, [a b c] = [a b c]^2 \neq 0$$

Hence Find value of $l\ m\ n$ such that

$$\bar{a} = l(\bar{b} \times \bar{c}) + m(\bar{c} \times \bar{a}) + n(\bar{a} \times \bar{b})$$

$$\bar{a} \cdot \bar{a} = l[a\ b\ c] + 0 + 0$$

$$l = \frac{\bar{a} \cdot \bar{a}}{[\bar{a}\ \bar{b}\ \bar{c}]}$$

Similarly $\quad m = \dfrac{\bar{a}\ \bar{b}}{[a\ b\ c]} \quad n = \dfrac{\bar{a} \cdot \bar{c}}{[a\ b\ c]}$

③ Scalar product of 4 Vectors

$$(\bar{a}\times\bar{b})\circ(\bar{c}\times\bar{d}) = \begin{vmatrix} \bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c} \\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d} \end{vmatrix}$$

Lagrange's Identity

P.T. $(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d}) + (\bar{b}\times\bar{c})\cdot(\bar{a}\times\bar{d})$
$+ (\bar{c}\times\bar{a})\cdot(\bar{b}\times\bar{d}) = 0$

$$\begin{vmatrix} \bar{a}\cdot\bar{c} & b\cdot c \\ \bar{a}\cdot\bar{d} & b\cdot d \end{vmatrix} + \begin{vmatrix} b\circ a & c\circ a \\ b\circ d & c\circ d \end{vmatrix}$$

$$+ \begin{vmatrix} c\circ b & a\circ b \\ c\circ d & a\circ d \end{vmatrix} = \begin{vmatrix} ac+ba+cb & bc+ca+cd \\ ad+bd+cd & bd+cd+ad \end{vmatrix}$$

Expand & prove

or

$$= (a+b+c)\begin{vmatrix} c & c \\ d & d \end{vmatrix} = 0$$

(4) Vector product of 4 vectors

$$(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d}) = [a\,c\,d]\,\bar{b}$$

$$- [b\,c\,d]\,\bar{a}$$

$$= [\bar{a}\,\bar{b}\,\bar{d}]\,\bar{c} - [\bar{a}\,\bar{b}\,\bar{c}]\,\bar{d}$$

Prove That

$$((\bar{a}\times\bar{b})\times(\bar{a}\times\bar{c}))\cdot\bar{d} = (a\cdot d)\,[\bar{a}\,\bar{b}\,\bar{c}]$$

$$([a\,a\,c]\,\bar{b} - [b\,a\,c]\,\bar{a})\cdot\bar{d}$$

0

$$= (a\cdot d)\,[a\,b\,c]$$

Prove that $\bar{d}\cdot(a\times(b\times(c\times d))) = (b\cdot\bar{d})\,[a\,c\,\bar{d}]$

$$\bar{d}\cdot(a\times((b\cdot d)\,c - (b\cdot c)\,\bar{d}))$$

$$\bar{d}\cdot(a\times c)(b\cdot d) - (b\cdot c)\,\bar{d}\cdot(\cancel{a}\times\bar{d})$$

0

$$(b\cdot d)\,[a\,c\,d]$$

QB

① Show that $[p+q,\ q+r,\ r+p] = (\bar{p}+\bar{q}) \cdot \left[(\bar{q}+\bar{r}) \times (\bar{r}+\bar{p})\right]$

$= 2(pqr)$

$[p+q \quad q+r \quad r+p] = [p\,q\,r] + [q\,r\,p]$ By Def

$= 2[p\,q\,r]$

$(\bar{p}+\bar{q}) \cdot \left[(\bar{q}+\bar{r}) \times (\bar{r}+\bar{p})\right] = [p+q \quad q+r \quad r+p]$ By def

Hence proved

② If $\bar{l}\ \bar{m}\ \bar{n}$ are three coplaner vectors

P.T.

$$[\bar{l}\,\bar{m}\,\bar{n}](\bar{a} \times \bar{b}) = \begin{vmatrix} \bar{l}\cdot\bar{a} & \bar{l}\cdot\bar{b} & \bar{l} \\ \bar{m}\cdot\bar{a} & \bar{m}\cdot\bar{b} & \bar{m} \\ \bar{n}\cdot\bar{a} & \bar{n}\cdot\bar{b} & \bar{n} \end{vmatrix}$$

$[\bar{i}\ \bar{m}\ \bar{n}]\ (\bar{a} \times \bar{b})$

$\bar{i}\,((m \times n) \cdot (\bar{a} \times \bar{b}))$

$p \times (\bar{a} \times \bar{b})$

p

③ P.T. points are colinear

- Take 3 points, matrix = 0, Take Next combination

(4) $i \times (\bar{a} \times i) + j \times (\bar{a} \times j) + k \times (\bar{a} \times \bar{k})$

$$\Big(\bar{i}\,(\bar{i} \circ \bar{a}) - \bar{a}\,(i \cdot i)$$

$$+ \bar{j}\,(j \circ \bar{a}) - \bar{a}$$

$$+ \bar{k}\,(\bar{k} \circ \bar{a}) - \bar{a} \Big) \times -1$$

$$= \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k} - 3a\right) \times -1$$

$$= +2\bar{a}$$

(5) p.7.

$$\bar{a} \times [\bar{b} \times (\bar{c} \times \bar{d})]$$

$$= \bar{a} \times [-\bar{d}[\bar{b} \cdot \bar{c}] + c[\bar{b} \cdot \bar{d}]]$$

$$= (a \times c)(b \cdot d) - (a \times d)(b \cdot c)$$

(6) $(b \times c \quad c \times a \quad a \times b)$

$$= (b \times c) \cdot ((c \times a) \times (a \times b))$$

$$b \times c \cdot \quad (a[cab]) - c[\overset{0}{\cancel{aab}}]$$

$$= (cab)(b \times c \cdot a) = [abc]^2$$

⑦ P7

$a \times (b \times c) + b \times (c \times a) + \bar{c} \times (\bar{a} \times \bar{b}) = 0$

$b(a \circ c) - c(a \circ b) + c(b \circ a) - a(b \circ c)$

$+ \bar{a}(\bar{c} \circ \bar{b}) - \bar{b}(\bar{c} \circ \bar{a}) = 0$

⑧ P7.

$(a \times b) \cdot (c \times a) + (\bar{b} \times \bar{c})(\bar{a} \times \bar{d}) + (c \times \bar{a})(\bar{b} \times \bar{d}) = 0$

$$\begin{vmatrix} a \circ c & b \circ c \\ a \circ d & b \circ d \end{vmatrix} + \begin{vmatrix} b\ a & c\ a \\ b\ d & c\ d \end{vmatrix} + \begin{vmatrix} c\ b & a\ b \\ c\ d & a\ d \end{vmatrix}$$

$$= \begin{vmatrix} a \circ c + b \circ a + cb & ac + b\ a + cb \\ a \circ c + b \circ c + cb & ac + b \circ a + cb \end{vmatrix} = 0$$

(9) $(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{a}) = [a\,c\,d]\,\bar{b} - [b\,c\,a]\,\bar{a}$

Prove using i j k

Vector differentiation

Gradient of scalar point function f

$$\boxed{\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}}$$

$d\phi$ total differentiation of ϕ

$$= \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$= \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)(x, y, z)$$

$$\boxed{d\phi = \nabla\phi \; d\bar{r}}$$

Directional derivative of function f at point a in direction of unit vector $\bar{u}$ is $\boxed{D_{\hat{u}} f(a) = \nabla f(a) \cdot \hat{u}}$

$$= |\nabla f| \cos\theta$$

It is maximum if $\cos\theta = 1$ ie if $\theta = 0$

$\therefore$ $D_{\hat{u}} f(a)$ is maximum at direction of ∇f

① Find $\phi(r)$ such that $\nabla\phi = -\frac{\bar{r}}{r^5}$ &

$\phi(1) = 0$

$$\bar{r} = xi + yj + z\hat{k}$$

$$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$$\nabla\phi = -\frac{\bar{r}}{r^5} = \frac{-(xi+yj+z\hat{k})}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial\phi}{\partial x} = -\frac{x}{(x^2+y^2+z^2)^{5/2}}, \quad \frac{\partial\phi}{\partial z} = \frac{-z}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial\phi}{\partial y} = -\frac{y}{(x^2+y^2+z^2)}$$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= -\frac{(x\,dx + y\,dy + z\,dz)}{(x^2+y^2+z^2)^{5/2}}$$

Let $x^2+y^2+z^2=t$

$2x\,dx+2y\,dy+2z\,dz=dt$

$\therefore d\phi = \dfrac{-dt}{2t^{3/2}}$

$\therefore \phi = \dfrac{1}{2}\dfrac{t^{-3/2}}{-3/2} + c$

Subs $t \leftrightarrow r$

$\therefore \phi(r) = \dfrac{1}{3r^3} + c$

$\therefore \phi(1)=0 \quad c = -1/3$

$\therefore$

$\phi(r) = \dfrac{1}{3r^3} - \dfrac{1}{3}$

(2) $\nabla f(r) = f'(r)\frac{\bar{r}}{r}$ P.T.

Hence find f if $\nabla f = 2r^4\bar{r}$

$r^2 = x^2 + y^2 + z^2$

$\frac{\partial r}{\partial x} = \frac{x}{r}$ $\frac{\partial r}{\partial z} = \frac{z}{r}$

$\frac{\partial r}{\partial y} = \frac{y}{r}$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$= \left(\frac{df}{dr}\frac{\partial r}{\partial x}, \frac{df}{dr}\frac{\partial r}{\partial x}, \frac{df}{dr}\frac{\partial r}{\partial z}\right)$$

$$= \frac{df}{dr}\left(\frac{x}{r} + \frac{y}{r} + \frac{z}{r}\right)$$

$$= f'(r)\frac{\bar{r}}{r}$$

$$\boxed{\nabla f(r) = \frac{f'(r)\,\bar{r}}{r}}$$

③ ⓐ P.T. $\nabla r^n = n r^{n-2} \bar{r}$

$f(r) = r^n \quad f'(r) = n r^{n-1}$

$\therefore \nabla r^n = n r^{n-2} \bar{r}$

ⓑ P.T. $\nabla(e^{r^2}) = 2e^{r^2} \bar{r}$

Same method use $\nabla f(r) = f'(r) \frac{\bar{r}}{r}$ formula

④ P.T. $\nabla \frac{(\bar{a} \cdot \bar{r})}{r^n} = \frac{\bar{a}}{r^n} - \frac{n(\bar{a} \cdot \bar{r})\bar{r}}{r^{n+2}}$

$\bar{a} = (a_1, a_2, a_3)$

$\bar{r} = (x, y, z)$

$\frac{\bar{a} \cdot \bar{r}}{r^n} = \frac{a_1 x + a_2 y + a_3 z}{r^n}$

$$\nabla\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right) = \frac{\partial\left(\frac{\bar{a}\cdot\bar{r}}{r^n}\right)}{\partial x}\hat{i} + \frac{\partial(\ )}{\partial y}\hat{j} + \frac{\partial(\ )}{\partial z}\hat{k}$$

$$= \frac{a_1 r^n - (a_1 x + a_2 y + a_3 z)\, n r^{n-1}\frac{\partial r}{\partial x}}{r^{2n}}\hat{i}$$

But $\frac{\partial r}{\partial x} = \frac{x}{r}$

$\frac{\partial r}{\partial y} = \ldots$

$$+ \vdots\ (\ )\hat{j} + (\ )\hat{k}$$

$$= \frac{a_1\hat{i} + a_2\hat{j} + a_3\hat{k}}{r^n}$$

$$- \frac{(\bar{a}\cdot\bar{r})}{r^{n+2}}\ (x, y, z)\ n$$

$$= \frac{\bar{a}}{r^n} - \frac{(\bar{a}\cdot\bar{r})}{r^{n+2}}\,\bar{r}\, n$$

① Find Directional Derivative of

$\phi = x^4+y^4+z^4$ at $A=(1,-2,1)$

$B=(2,6,-1)$ in direction of AB

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\nabla \phi = 4x^3\hat{i} + 4y^3\hat{j} + 4z^3\hat{k}$$

at $A = (1,-2,1)$

$$= 4\hat{i} - 32\hat{j} + 4\hat{k}$$

In $\overline{AB}$ direction ie $\overline{B}-\overline{A} = (1,8,-2)$

$$\hat{u} = \frac{(1,8,-2)}{\sqrt{69}}$$

$$DD = \frac{4 \quad -8\times 32 \quad -8}{\sqrt{69}} = -\frac{260}{\sqrt{69}}$$

② Find D.D. of $\phi = x^2 + y^2 + z^2$ at 1 2 3

in Dir of $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$

$$\nabla\phi = (2x, 2y, 2z) \text{ at } 123 = (2, 4, 6)$$

$$\hat{u} = \frac{(3, 4, 5)}{\sqrt{5^2 + 4^2 + 3^2}}$$

Ans $$\frac{6 + 16 + 30}{\sqrt{5^2 + 4^2 + 3^2}} = \frac{52}{\sqrt{50}}$$

③ $\phi = e^{2x}\cos yz$ at $(0,0,0)$ in dir of tangent to curve $x = a\sin t$ $y = a\cos t$ $z = at$ at $t = \pi/4$

$$\nabla\phi = \left(2e^{2x}\cos yz,\ -e^{2x} z \sin yz,\ -e^{2x} y \sin yz\right)$$

$$\text{at } (0,0,0) = (2,0,0)$$

$$r = x\hat{i} + y\hat{j} + z\hat{k} = (a\sin t,\ a\cos t,\ at)$$

$$\text{Tangent} = \frac{dr}{dt} = (a\cos t,\ -a\sin t\ \ a)$$

$$\text{at } t = \pi/4 = \left(\frac{a}{\sqrt{2}},\ \frac{-a}{\sqrt{2}},\ a\right)$$

$$\hat{u} = \frac{\left(\frac{1}{\sqrt{2}},\ \frac{-1}{\sqrt{2}},\ 1\right)}{\sqrt{2}}$$

$$DD = \frac{2}{2} = 1$$

Find Angle between surfaces.

(4) $x \log z + 1 - y^2 = 0$ & $x^2 y + z = 2$ at $(1,1,1)$

→ Normal to any surface is $\nabla \phi$

$\phi = x \log z \quad - y^2$

$\nabla \phi = \left(\log z, -2y, \frac{x}{z} \right)$ at 111 $(0, -2, 1)$

$\psi = x^2 y + z = 2$

$\nabla \psi \ (2xy, x^2, 1)$ at 111 $(2, 1, 1)$

Angle between $\nabla \phi$ & $\nabla \psi$

$$\cos\theta = \frac{\nabla \phi \cdot \nabla \psi}{|\nabla \phi| |\nabla \psi|} = \frac{-1}{\sqrt{5}\sqrt{6}}$$

$$\cos\theta = \frac{-1}{\sqrt{30}}$$

$\theta = \cos^{-1}(-1/\sqrt{30})$

(5) Find values of a, b, c if DD of $\phi = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has maximum magnitude 64 in direction ∥ to z axis

$$\nabla\phi = (3x^2z^2c + ay^2, 2yx + bz, by + 2czx^3)$$

$$\nabla\phi = (3c + 4a, 2a - b, 2b - 2c)$$

$$\hat{u} = (0, 0, 1)$$

Maximum is in dir of $\hat{u}$

$$3c + 4a = 0k$$

$$2a - b = 0k$$

$$2b - 2c = k$$

But $\hat{u} \cdot \nabla\phi = 64 \therefore 2b - 2c = 64 \therefore k = 64$

$$\therefore \left.\begin{aligned} 3c + 4a &= 0 \\ 2a - b &= 0 \\ 2b - 2c &= 64 \end{aligned}\right\} \text{Solve} \longrightarrow \begin{aligned} a &= 6 \\ b &= 24 \\ c &= -8 \end{aligned}$$

1 QB

$$\frac{d\bar{a}}{dt} = a \times a$$

p.7 $$\frac{d}{dt}[a \times \bar{b}] = a \times (\bar{a} \times \bar{b})$$

$$\frac{db}{dt} = \bar{a} \times \bar{b}$$

② Find $\nabla\phi$ if $\phi = 3x^2y - y^3z^2$ at $(1,-2,1)$

$$\nabla\phi = (6xy,\ 3x^2 - 3y^2z^2,\ -2y^3z)$$

at $(1,-2,1)$

$$= (-12,\ -9,\ +16)$$

③ Find D.D. of $\phi = x^2y + y^2z + z^2x^2$ at $(1,2,1)$ in direction Normal to $x^2 + y^2 - z^3x = 1$ at $Q = (1,1,1)$

$$\nabla\phi = (2xy + 2xz^2,\ x^2 + 2zy,\ y^2 + 2zx^2)$$

at 121 =

$$\nabla\phi = (2x + z^3,\ 2y,\ 3xz^2) \text{ at } 111$$

$$= (3,\ 2,\ 3) \text{ is Norml}$$

$$\hat{n} = \frac{1}{\sqrt{3^2+2^2+3^2}}(3\ 2\ 3)$$

(4) $\phi = (x^4 + y^4 + z^4)$ $A\ (1, -2, 1)$

$B = (2, 6, -1)$

$\nabla \phi = (4x^3, 4y^3, 4z^3)$ at $1, -2, 1$

$Dir = B - A = (1, 8, -2)$

$$\hat{u} = \frac{(1\ 8\ -2)}{\sqrt{1^2 + 8^2 + 2^2}}$$

(5)

Divergence & Curl

Let $\bar{f} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}$

$$\nabla \cdot \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$ Divergence

$\nabla \cdot \bar{f} = 0$, $\bar{f}$ is <u>solenoidal</u>

$$\nabla \times \bar{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$ Curl

If $\nabla \times \bar{f} = \bar{0}$ $\bar{f}$ is <u>irrotational</u>

∇f Gradient of Scalar is Vector

$\nabla \cdot f$ Divergence of vector is scalar

$\nabla \times f$ Curl of vector is Vector

Scalar $\rightarrow$ scalar function eg xyz^2

Vector $\rightarrow$ Vector function eg $x\hat{i} + xy\hat{j} + z^2\hat{k}$

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla\cdot(\phi\bar{f}) = \phi(\nabla\cdot\bar{f}) + \bar{f}\cdot(\nabla\phi)$$

$$\nabla\cdot(\bar{f}\times\bar{g}) = \bar{g}\cdot(\nabla\times\bar{f}) - \bar{f}\cdot(\nabla\times\bar{g})$$

$$\nabla\times(\phi\bar{f}) = \phi(\nabla\times\bar{f}) + (\nabla\phi)\times\bar{f}$$

$$\nabla(f(\bar{r})) = f'(r)\frac{\bar{r}}{r}$$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Imp $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\nabla \cdot \bar{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1 + 1 + 1 = \underline{\underline{3}}$$

$$\nabla \times \bar{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)$$

$$- \hat{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right)$$

$$+ k\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right.$$

$$= \underline{\underline{0\hat{i} + 0\hat{j} + 0\hat{k}}}$$

① If $\bar{a}$ is a constant vector such that $|a| = a$, prove that

$$\nabla \cdot \{(\bar{a} \cdot \bar{r})\, \bar{a}\} = a^2$$

$$\bar{a} \cdot \bar{r} = \phi \;(\text{Scalar}) = a_1 x + a_2 y + a_3 z$$

$$\nabla(\phi \cdot \bar{a}) = \phi(\nabla \cdot \bar{a}) + a(\nabla \phi)$$

as a is constant $\nabla \cdot \bar{a} = \bar{0}$

$$\therefore \nabla(\phi \cdot \bar{a}) = \bar{a} \cdot (\nabla \phi)$$

$$\nabla \phi = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) = \bar{a}$$

$$\therefore \text{Ans} = \bar{a} \cdot \bar{a} = |a|^2 =$$

② Prove that $\nabla \left\{ \nabla \circ \frac{\bar{r}}{r} \right\} = -\frac{2}{r^3}\bar{r}$

$$\nabla \circ \left(\frac{1}{r} \circ \bar{r}\right) = \frac{1}{r}(\nabla \circ \bar{r}) + \bar{r}\,\nabla(1/r)$$

$$\nabla \frac{1}{r} = \frac{-1}{r^2}\frac{\bar{r}}{r} = -\frac{\bar{r}}{r^3}$$

$$\nabla \circ \bar{r} = 3$$

$$\left(\frac{3}{r} - \frac{r^2}{r^3}\right) = \frac{2}{r}$$

$$\nabla \frac{2}{r} = \frac{-2}{r^2}\frac{\bar{r}}{r} = -\frac{2\bar{r}}{r^3}$$

③ Prove that $\nabla \cdot \left(r \nabla \frac{1}{r^n} \right) = \frac{n(n-2)}{r^{n+1}}$

$$\nabla \frac{1}{r^n} = \frac{n}{r^{n+1}} \frac{\bar{r}}{r}$$

$$r \cdot \nabla \frac{1}{r^n} = \frac{n \bar{r}}{-r^{n+1}}$$

$$\nabla \cdot \left(\bar{r} \cdot -\frac{n}{r^{n+1}} \right) = -\frac{n}{r^{n+1}} (\nabla \cdot \bar{r})$$

$$+ \cancel{\bar{r}} + \frac{n(n+1)}{r^{n+2}} \frac{\cancel{\bar{r}}}{\cancel{r}}$$

$$\nabla \cdot \bar{r} = -\frac{3n}{r^{n+1}} + \frac{(n)(n+1)}{r^{n+1}}$$

$$= \frac{n(n-2)}{r^{n+1}}$$

4. Prove that

$$\nabla \circ \left\{ \frac{f(r)}{r} \bar{r} \right\} = \frac{1}{r^2} \frac{d}{dr} \left[r^2 f(r) \right]$$

Hence or otherwise prove $\nabla \circ (r^n \bar{r}) = (n+3) r^n$

$$\frac{f(r)}{r} = \phi$$

$$\nabla \circ (\phi \bar{r}) = \phi \nabla \circ \bar{r} + \bar{r} \nabla \phi$$

$$= \frac{f(r)}{r} \nabla \circ \bar{r} + \bar{r} \left[\frac{r f'(r) - f(r)}{r^2} \right] \frac{\bar{r}}{r}$$

$$= \frac{f(r)}{r} \times 3 + f'(r) - \frac{f(r)}{r}$$

$$= f'(r) + 2\frac{f(r)}{r}$$

$$\frac{1}{r^2} \frac{d}{dr} (r^2 f(r)) = \frac{1}{r^2} \left(2r f(r) + r^2 f'(r) \right) = f'(r) + 2\frac{f(r)}{r}$$

Hence prov

If $f(r) = r^{n+1}$

$$\nabla \cdot (r^n \bar{r}) = \frac{1}{r^2} \frac{d r^{n+3}}{dr}$$

$$= (n+3) \frac{r^{n+2}}{r^2}$$

$$= (n+3) r^n$$

Hence proved.

⑤ Find $f(r)$ so that $f(r)\bar{r}$ is both solenoidal and irrotational

$$\nabla \cdot f(r)\bar{r} = 0$$

$$0 = \bar{r} \cdot \nabla f(r) + f(r) \cdot \nabla \cdot \bar{r}$$

$$f(r)\, \nabla \cdot \bar{r} = \bar{r} \cdot \nabla f(r)$$

But $\nabla \cdot \bar{r} = 3$

$$\therefore \quad 3 f(r) = \bar{r} \cdot \nabla f(r)$$

$$-3f(r) = \bar{r} \cdot \frac{f'(r)\,\bar{r}}{r}$$

$$\therefore \; 3 f(r)\, r = f'(r)$$

$$\therefore \; \frac{-3}{r} = \frac{f'(r)}{f(r)}$$

$$\therefore \; -3\log r = \log f(r) \qquad \therefore \; K r^{-3} = f(r)$$

$$\therefore \; f(r) = \frac{K}{r^3}$$

⑥ P.T. $\bar{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j}$
$+ (4x+cy+2z)\hat{k}$

P.T. $\bar{F}$ is solenoidal & determine abc such that $\bar{F}$ is irrotational.

$\nabla \cdot \bar{F} =$

$1 + -3 + 2 = 0$

$\therefore$ solenoidal.

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$i\,(c+1) - j\,(4-a) + k\,(b-2) = \bar{0}$

$\therefore c = -1$

$a = 4$

$b = 2$

(2) $\bar{F} = (y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2z^3)\hat{j}$

$+ (6z^3 - 3x^2yz^2)\hat{k}$

Find scalar potential function ϕ such that

$\bar{F} = \nabla\phi \quad \phi(1\,0\,1) = 8$

$\phi =$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}k$$

$$\frac{\partial\phi}{\partial x} = \cancel{y^2 - 2xyz^3}$$

$$\therefore \phi = \cancel{y^2x - x^2yz^3}$$

$\rightarrow$ wrong way

or Integrate & take terms out common

Find $d\phi$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$\phi = \int \; dx + \int \; dy + \int \; dz$$

$$= xy^2 + yz^3x^2$$

$$+ 3y + xy^2 + x^2z^3y$$

$$+ 6\frac{z^4}{4} + x^2yz^3$$

Take terms out common

$$= d(xy^2) - d(x^2yz^3)$$

$$+ d(zy) + d(6\frac{z^4}{4})$$

$$\therefore \phi = xy^2 - x^2yz^3 + 3y + \frac{3}{2}z^4 + c$$

$$\phi(1,0,1) = 8 \quad \therefore c = \frac{13}{2}$$

$$\phi = xy^2 - x^2yz^3 + 3y + \frac{3}{2}z^4 + \frac{13}{2}$$