Laplace transform

$f(t)$ be given function of t defined for $t \geq 0$ then laplace transform of $f(t)$ is given by $\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$

Q1 find laplace transform of $f(t)$ where

$$f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$$

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\pi e^{-st} \cos t + \int_\pi^\infty e^{-st} \sin t$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

$$= \left[\frac{e^{-st}}{s^2+1}\left[-s\cos t + \sin t\right]\right]_0^{\pi}$$

$$+ \left[\frac{e^{-st}}{s^2+1}\left[-s\sin t - \cos t\right]\right]_{\pi}^{\infty}$$

$$= e^{-s\pi} \frac{s}{s^2+1} - \frac{-s}{s^2+1} + -\frac{e^{-s\pi}}{s^2+1}[1] \quad \text{Ans}$$

① Linearity property

$$L\left[a f(t) + b g(t)\right] = a L[f(t)] + b L[g(t)]$$

Q2 $$L[e^{at}] = \int_0^\infty e^{-st} e^{at} \, dt$$

$$= \int_0^\infty e^{(-s+a)t} \, dt = \left[\frac{e^{(-s+a)t}}{(-s+a)}\right]_0^\infty$$

$s > a$

(s is real)

$$= \frac{-1}{-s+a} = \frac{1}{s-a}$$

Note →

if $a = 0$, $L[1] = 1/s$

Q3 $\mathcal{L}[\cos at]$

$$\mathcal{L}[e^{iat}] = \mathcal{L}\{\cos at + i\sin at\}$$

$$\frac{1}{s-ia} = \mathcal{L}[\cos at] + i\,\mathcal{L}[\sin at]$$

(s is real)

$$\frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + \frac{ia}{s^2+a^2}$$

$$\therefore \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

$$\therefore \mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$$

Q4 $\mathcal{L}[\cosh at] = \mathcal{L}\left[\frac{e^{at}+e^{-at}}{2}\right] = \frac{1}{2}\left(\mathcal{L}[e^{at}] + \mathcal{L}[e^{-at}]\right)$

$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right. = \frac{s}{s^2-a^2}$$

Similarly $\quad L[\sinh at] = \dfrac{a}{s^2 - a^2}$

Q5 Prove that

$$L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} \qquad n+1 > 0$$

$$= \frac{n!}{s^{n+1}} \qquad n = 0, 1, 2 \ldots$$

$\rightarrow$

$$L[t^n] = \int_0^\infty e^{-st} t^n \, dt$$

$st \longrightarrow m$

$dt \longrightarrow \dfrac{dm}{s}$

$$\therefore Z = \int_0^\infty e^{-m} \frac{m^n}{s^n} \frac{dm}{s} = \frac{\Gamma(n+1)}{s^{n+1}}$$

(1) These questions will not come in exam

Q1 find $L\left[t^2 - e^{-2t} + \cosh^2 3t + \sin 3t\right]$

By linearity

$$= L[t^2] - L[e^{-2t}] + L[\cosh^2 3t] + L[\sin 3t]$$

$$L[t^2] = \frac{2!}{s^3}$$

$$-L[e^{-2t}] = -\frac{1}{s+2}$$

$$L[\sin 3t] = \frac{3}{s^2+9}$$

$$\cosh^2 3x = \left(\frac{e^{3x}+e^{-3x}}{2}\right)^2 = \frac{e^{6x}+e^{-6x}}{4} + \frac{1}{2} = \frac{\cosh 6x + 1}{2}$$

$$L[\cosh^2 3t] = \frac{1}{2} L[\cosh 6t] + \frac{1}{2} L[1]$$

$$= \frac{1}{2} \frac{s}{s^2-36} + \frac{1}{2s}$$

Answer $\rightarrow$ $\dfrac{2}{s^3} - \dfrac{1}{s+2} + \dfrac{1}{2s} + \dfrac{s}{2(s^2-36)} + \dfrac{3}{s^2+9}$

(2) $L[\sin^5 t]$

$\sin t \rightarrow \dfrac{e^{ix} - e^{-ix}}{2i}$

$\sin^5 t \rightarrow \dfrac{(e^{ix} - e^{-ix})^5}{2^5 i}$

$= \dfrac{1}{i2^5}\Big[e^{i5xi} - 5 e^{4xi} e^{-xi} + 10 e^{3xi} e^{-2xi}$

$- 10 e^{2xi} e^{-3xi} + 5 e^{ix} e^{-4xi}$

$- e^{-5xi} \Big]$

$= \dfrac{1}{2^4}\Big[\sin 5x - 5\sin 3x + 10 \sin x \Big]$

$L[\sin^5 t] = \dfrac{1}{2^4}\Big[\dfrac{5}{s^2+25} - \dfrac{15}{s^2+9} + \dfrac{10}{s^2+1} \Big]$

③ $L[\cos t \cos 2t \cos 3t]$

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$\cos x \cos 2x = \frac{1}{2}\cos 3x + \frac{1}{2}\cos x$$

$$\cos x \cos 2x \cos 3x = \frac{1}{2}\cos 3x \cos 3x + \frac{1}{2}\cos x \cos 3x$$

$$= \frac{1}{4}\cos 6x + \frac{1}{4} + \frac{1}{4}\cos 4x + \frac{1}{4}\cos 2x$$

$$L[\cdot] = \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{1}{s} + \frac{s}{s^2+16} + \frac{s}{s^2+4}\right]$$

(4) $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$ Imp

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$x \to \sqrt{t}$$

$$\frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \cdots$$

$$L[\ \ ] = \frac{\Gamma(-1/2+1)}{s^{-1/2+1}} - \frac{1}{2!}\frac{\Gamma(1/2+1)}{s^{1/2+1}} + \frac{\Gamma(3/2+1)}{4!\, s^{3/2+1}} + \cdots$$

$$= \frac{\Gamma 1/2}{s^{1/2}} - \frac{1}{2!}\frac{\Gamma 3/2}{s^{3/2}} + \frac{1}{4!}\frac{\Gamma 5/2}{s^{5/2}} + \cdots$$

$$= \frac{\Gamma 1/2}{s^{1/2}} - \frac{1}{2}\,\frac{1}{2}\,\frac{\Gamma 1/2}{s^{3/2}} + \frac{1}{4!} \times \frac{3}{2} \times \frac{1}{2} + \cdots$$

$$= \frac{\sqrt{\pi}_{1/2}}{s^{1/2}}\left[1 - \frac{1}{2}\frac{1}{2s} + \frac{\frac{3}{2}\,\frac{1}{2}}{4!\, s^2} - \frac{1}{6!}\frac{\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}}{s^3} + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{s^{1/2}}\left[1 - \frac{1}{4s} + \frac{1}{2}\frac{3\times 1}{2\times 2\times 4\times 3\times 2}\frac{1}{s^2}\right.$$

$$\left. - \frac{5\times 3}{2\times 2\times 2\times 6\times 5\times 4\times 3\times 2}\frac{1}{s^3}\right]$$

$$= \frac{\sqrt{\pi}}{s^{1/2}}\left[1 - \frac{1}{4s} + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}\left(e^{-1/4s}\right)$$

$$L\left[\frac{\sin\sqrt{t}}{\sqrt{t}}\right]$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$$

$$\frac{\sin\sqrt{t}}{\sqrt{t}} = \frac{1}{\sqrt{t}} \sum_{n=1}^{\infty} \frac{t^n}{\sqrt{t}\,(2n-1)!}$$

$$= \sum_{n=1}^{\infty} \frac{t^{n-1}}{(2n-1)!}$$

$$L[\quad] = \sum_{n=1}^{\infty} \frac{L[t^{n-1}]}{(2n-1)!}$$

$$= \sum_{n=1}^{\infty} \frac{\Gamma n}{s^n\,(2n-1)!}$$

Change of scale

$$L[f(t)] = \phi(s)$$

$$L[f(at)] = \frac{1}{a}\,\phi\left[\frac{s}{a}\right]$$

① If $L[\operatorname{erf}\sqrt{t}] = \dfrac{1}{s\sqrt{s+1}}$

find $L[\operatorname{erf} 2\sqrt{t}]$

$$L[\operatorname{erf} 2\sqrt{t}] = L[\operatorname{erf}\sqrt{4t}] = \frac{1}{4\,\frac{s}{4}\sqrt{\frac{s}{4}+1}}$$

$$= \frac{2}{s\sqrt{s+4}}$$

First shifting property

If $L[f(t)] = \phi(s)$

$$L[e^{at} f(t)] = \phi(s-a)$$

① Show that $L\left[\sinh \frac{t}{2} \cdot \sin \frac{\sqrt{3}}{2}t\right] = \frac{\sqrt{3}}{2} \frac{s}{s^4+s^2+1}$

$$L\left[\sin \frac{\sqrt{3}}{2} t\right] = \frac{\sqrt{3}/2}{s^2+\frac{3}{4}}$$

$$L\left[e^{t/2} \sin \frac{\sqrt{3}}{2} t\right] = \frac{\sqrt{3}/2}{(s-1/2)^2+3/4} = \frac{\sqrt{3}/2}{s^2-s+\frac{1}{4}+3/4}$$

$$= \frac{\sqrt{3}/2}{s^2-s+1}$$

$$L\left[e^{-t/2} \sin \frac{\sqrt{3}}{2} t\right] = \frac{\sqrt{3}/2}{(s+1/2)^2+3/4}$$

$$= \frac{\sqrt{3}/2}{s^2+s+1}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

Applying linearity property

$$L\left[\sinh(t/2)\sin\left(\frac{\sqrt{3}}{2}t\right)\right] = \frac{1}{2}L\left[e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

$$- \frac{1}{2}L\left[e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2(s^2+s+1)} - \frac{\sqrt{3}}{2(s^2-s+1)}\right)$$

$$= \frac{1}{2}\,\frac{\sqrt{3}}{2}\left(\frac{s^2-s+1-s^2-s+1}{((s^2+1)-s)((s^2+1)+s)}\right))$$

$$= \frac{\sqrt{3}}{4}\,\frac{2s}{((s^2+1)^2-s^2)} = \frac{\sqrt{3}}{2}\,\frac{s}{s^4+s^2+1}$$

② find $\mathcal{L}[t e^{-4t} \sin 3t]$

$$\mathcal{L}[t \sin 3t] = \mathcal{L}\left[t \left(\frac{e^{i3t} - e^{-i3t}}{2i}\right)\right]$$

$$= \frac{1}{2i}\mathcal{L}[t e^{i3t}] - \frac{1}{2i}\mathcal{L}[t e^{-i3t}]$$

$(\mathcal{L}(t) = 1/s^2)$

$$= \frac{1}{2i}\left(\frac{1}{(s-i3)^2} - \frac{1}{(s+i3)^2}\right)$$

$$= \frac{1}{2i} \frac{\cancel{s^2} + 6is - \cancel{9} - \cancel{s^2} + 6is + \cancel{9}}{((s-i3)(s+i3))^2}$$

$$= \frac{1}{2\cancel{i}} \frac{12\cancel{i}s}{(s^2+9)^2} = \frac{6s}{(s^2+9)^2}$$

$$\mathcal{L}[e^{-4t} t \sin 3t] = \frac{6(s+4)}{((s+4)^2+9)^2}$$

Second Shifting property

If $L(f(t)) = \phi(s)$

& $g(t) \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$

then $L(g(t)) = e^{-as}\phi(s)$

① find $L[g(t)]$ where $g(t) = \begin{cases} \cos(t - 2\pi/3) & t > 2\pi/3 \\ 0 & t < 2\pi/3 \end{cases}$

$$L[g(t)] = e^{-as}\phi(s)$$

$$= e^{-2\pi/3\, s} L[\cos t]$$

$$= e^{-2\pi/3\, s} \frac{s}{s^2+1}$$

Multiptication by t property

If $L(f(t)) = \phi(s)$

$$L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n}[\phi(s)]$$

eg

$$L[t f(t)] = -\phi'(s)$$

① $L[t e^{-4t} \sin 3t)$

$$L[e^{-4t} t \sin 3t] = \phi_1(s+4)$$

$$L[t \sin 3t] = -\phi_2'(s)$$

$$f(t) = \sin 3t$$

$$\phi_2(s) = \frac{3}{s^2+9}$$

$$-\phi'(s) = \frac{6s}{(s^2+9)^2}$$

$$\therefore \quad \phi_1(s) = \frac{6s}{(s^2+9)^2}$$

$$\phi(s+4) = \frac{6(s+4)}{((s+4)^2+9)^2}$$

② $\mathcal{L}\left[e^{3t}\, t\sqrt{1+\sin t}\,\right]$

$$1+\sin 2\theta = (\cos\theta + \sin\theta)^2$$

$$\therefore \sqrt{1+\sin t} = (\cos t + \sin t)$$

$$\mathcal{L}\left[e^{3t}\, t\,(\cos t_{/2} + \sin t_{/2})\right]$$

let

$$f(t) = t\left(\cos\frac{t}{2} + \sin\frac{t}{2}\right)$$

$$= \frac{-d}{ds}\left(\frac{s}{s^2+1/4} + \frac{1/2}{s^2+1/4}\right) = \frac{-d}{ds}\left(\frac{s+1/2}{s^2+1/4}\right)$$

$$= \quad -\frac{s}{(s^2+1/4)^2} + \frac{s^2+1/4}{(s^2+1/4)^2} - \frac{2s^2}{(s^2+1/4)^2} \Big)$$

$$\therefore \phi(s) = -\left[\frac{-s^2 - s + 1/4}{(s^2+1/4)^2} \right]$$

$$\mathcal{L}[\quad] = \phi(s-3)$$

$$= -\left[\frac{-(s-3)^2 - 2(s-3) + 1/4}{\left((s-3)^2+1/4\right)^2} \right]$$

(3) $L[t^3 \cos t] = -\frac{d^3}{ds^3}\left(\frac{s}{s^2+1}\right)$

(4) Evaluate $\int_0^\infty e^{-3t} t^3 \cos t \, dt$

→ Comparing with $\int_0^\infty e^{-st} f(t)\, dt$

$s = 3$

$f(t) = t^3 \cos t$

$\therefore I = L[t^3 \cos t]_{s=3}$

$= -\frac{d^3}{ds^3}\left(\frac{s}{s^2+1}\right)$

$\phi(s)$ $\qquad \dfrac{s}{s^2+1}$

$\phi'(s)$ $\qquad \dfrac{s^2+1}{(s^2+1)^2} - \dfrac{2s^2}{(s^2+1)^2} = \dfrac{1-s^2}{(s^2+1)^2}$

$\phi''(s)$ $\qquad \dfrac{-2s(s^2+1) - 4s(1-s^2)}{(s^2+1)^4}$

$$= \frac{-2s^3-2s-2s-2s^3}{s^2+1}$$

$$= \frac{2s^3-6s}{(s^2+1)^3}$$

$$\phi'''(s) = \frac{(s^2+1)(6s^2-6)-6s(2s^3-6s)}{(s^2+1)^4} = \frac{71}{1250}$$

Division by t

If $L[f(t)] = \phi(s)$

then $L\left[\dfrac{f(t)}{t}\right] = \int_s^\infty \phi(s)\,ds$

$$L\left[\frac{f(t)}{t^2}\right] = \int_s^\infty \int_s^\infty \phi(s)\,ds\,ds$$

eg. 1 $L\left[\dfrac{e^{-2t}\sin 2t \cosh t}{t}\right]$

Let $f(t) = e^{-2t}\dfrac{(e^{t}+e^{-t})}{2}\sin 2t$

$= \dfrac{e^{-t}}{2}\sin 2t + \dfrac{e^{-3t}}{2}\sin 2t$

$L[f(t)] = \dfrac{1}{2}\dfrac{2}{(s+1)^2+4} + \dfrac{1}{2}\dfrac{2}{(s+3)^2+4}$

$$L[f(t)] = \frac{1}{s^2+2s+5} + \frac{1}{s^2+6s+13}$$

$$L[\quad] = \int_s^\infty L[f(t)]\,ds$$

$$= \int_s^\infty \left(\frac{1}{s^2+2s+5} + \frac{1}{s^2+6s+13} \right) ds$$

$$= \int_s^\infty \frac{1}{(s+1)^2+4} + \frac{1}{(s+3)^2+4}$$

$$= \left[\frac{1}{2}\tan^{-1}\left(\frac{s+1}{2}\right) + \frac{1}{2}\tan^{-1}\left(\frac{s+3}{2}\right) \right]_s^\infty$$

$$= \frac{1}{2}\left[\pi/2 + \pi/2\right] - \frac{1}{2}\tan^{-1}\left(\frac{s+1}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{s+3}{2}\right)$$

(2) Find $L\left[\frac{\sin^2 t}{t}\right]$ Hence prove that

$$\int_0^\infty e^{-t}\,\frac{\sin^2 t}{t}\,dt = \frac{1}{4}\ln 5$$

$$\sin^2 t = \frac{1-\cos 2t}{2}$$

$$L[\sin^2 t] = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$\phi(s) = -1. -$$

$$\therefore L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\int_s^\infty \frac{1}{s} - \frac{s}{s^2+4}$$

$$= \frac{1}{2}\left[\ln s - \frac{\ln(s^2+4)}{2}\right]_s^\infty$$

$$= \frac{1}{4}\ln\left[\frac{s^2}{s^2+4}\right]_s^\infty$$

$$= \frac{-1}{4}\ln\left[\frac{s^2}{s^2+4}\right]$$

at $s=1$

$$L\left[\frac{\sin^2 t}{t}\right] = \int_0^\infty e^{-t}\,\frac{\sin^2 t}{t}\,dt = -\frac{1}{4}\log\frac{1}{5}$$

$$= \frac{1}{4}\log 5$$

③ Evaluate $\displaystyle\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt$

$$= \int_0^\infty e^{-0t}\,\frac{(\cos 6t - \cos 4t)}{t}$$

Let $f(t) = (\cos 6t - \cos 4t)$

$$L[\cos 6t - \cos 4t] = \frac{s}{s^2+36} - \frac{s}{s^2+16}$$

$$L\left[\frac{\cos 6t - \cos 4t}{t}\right] = \int_s^\infty \frac{s}{s^2+36} - \frac{s}{s^2+16} = \frac{1}{2}\left[\ln\left[\frac{s^2+36}{s^2+16}\right]\right]_s^\infty$$

$$= \frac{1}{2} \ln \left(\frac{s^2+16}{s^2+36} \right)$$

at $s = 0$, e^0 vanishes

$$\mathcal{L}[\;] = \int_0^\infty e^{-st} \frac{f(t)}{t}$$

$$= \int_0^\infty \frac{f(t)}{t} = \ln\left(\frac{16}{36}\right)$$

$$= \ln(2/3)$$

Laplace transform of derivative

$$L\left[\frac{d^n}{dt^n} f(t)\right] = s^n L[f(t)] - s^{n-1} f(0)$$
$$- s^{n-3} f'(0) \ldots\ldots$$
$$- f^{(n-1)}(0)$$

eg $n=1$ $\quad L[f'(t)] = s L[f(t)] - f(0)$

$n=2$ $\quad L[f''(t)] = s^2 L[f(t)] - s f(0) - f'(0)$

$n=3$ $\quad L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - s f'(0) - f''(0)$

Using $L[\cos at]$ find $L[\sin at]$

$$L\{\sin at) = a L\left[\frac{d}{dt} \cos at\right)$$
$$= \frac{-1}{a}\left(s L[\cos at] - \cos 0\right)$$
$$= \frac{-1}{a}\left(\frac{s^2}{s^2+a^2} - 1\right) = \frac{a}{s^2+a^2}$$

Laplace of Integral

If $L[f(t)] = \phi(s)$

$$L\left[\int_0^t f(u)\,du\right] = \frac{\phi(s)}{s}$$

① $erf_c\sqrt{t} = 1 - erf\sqrt{t}$, $erf\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du$

find $L[erf_c\sqrt{t}]$

$$L\{erf_c\sqrt{t}\} = L\{1\} - L\{erf\sqrt{t}\}$$

Let $u^2 = v$

$2u\,du = dv$

$$\therefore \; erf\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^t \frac{e^{-v}}{2\sqrt{v}}\,dv$$

Let $f(v) = \frac{e^{-v}}{\sqrt{v}}$

$$L\left[v^{1/2}\right] = \frac{\Gamma 1/2}{s^{1/2}}$$

$$L\left[e^{-v} v^{1/2}\right] = \frac{\Gamma 1/2}{(s+1)^{1/2}}$$

$$\frac{L}{\sqrt{\pi}}\left[\int_0^t e^{-v} v^{1/2}\, dv\right] = \frac{1}{\sqrt{\pi}} \frac{\sqrt{\pi}}{(s+1)^{1/2}\, s}$$

$$L\left[\operatorname{erf}_c \sqrt{t}\right] = \frac{1}{s} - \frac{1}{s\sqrt{s+1}}$$

(2) $$L\left[\cosh t \int_0^t e^{u} \cosh u \, du\right]$$

$$L\left[e^{u} \cosh u\right] = L\left[\frac{e^{u} e^{u}}{2} - \frac{e^{u} e^{-u}}{2}\right]$$

$$= L\left[\frac{e^{2u}}{2} - \frac{1}{2}\right] = \frac{1}{2}\left[\frac{1}{s-2} - \frac{1}{s}\right]$$

$$L\left[\int_0^t e^{u}\cosh u\right] = \frac{1}{2}\left(\frac{1}{s(s-2)} + \frac{1}{s^2}\right)$$

$$L\left\{\cosh t\left[\int_0^t e^{u}\cosh u\right]\right\} = \frac{1}{2}L\left\{(e^{u}+e^{-u})f(t)\right\}$$

By first shifting property

$$= \frac{1}{4}\left(\frac{1}{(s-1)(s-3)} + \frac{1}{(s-1)^2}\right)$$

$$+ \frac{1}{4}\left(\frac{1}{(s+1)(s-1)} + \frac{1}{(s+1)^2}\right)$$

③ $L\left[\int_t^{\infty} \frac{\cos u}{u}\, du\right]$

Limits from ∞ to t must be shifted to t to 0

Let $u = vt$

$du = t\,dv$

$$f(t) = \int_1^{\infty} \frac{\cos vt}{vt}\, t\, dv$$

$$= \int_1^{\infty} \frac{\cos vt}{v}\, dv$$

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$= \int_0^{\infty} e^{-st} \int_1^{\infty} \frac{\cos vt}{v}\, dv\, dt$$

$$= \int_1^{\infty} \frac{1}{v} \int_0^{\infty} e^{-st} \cos vt\, dt\, dv$$

$$= \int_1^{\infty} \frac{d\nu}{\nu} \; L[\cos \nu t]$$

$$= \int_1^{\infty} \frac{s}{\nu(s^2+\nu^2)} \, d\nu$$

$$= \frac{1}{s} \int_1^{\infty} \left(\frac{1}{\nu} - \frac{\nu}{s^2+\nu^2} \right)$$

$$= \frac{1}{s} \left[\log \nu - \frac{1}{2} \ln (s^2+\nu^2) \right]_1^{\infty}$$

$$= \frac{1}{s} \left[\log \left(\frac{\nu}{\sqrt{s^2+\nu^2}} \right) \right]_1^{\infty}$$

$$= \frac{-1}{s} \left[\log \left[\frac{1}{\sqrt{s^2+1}} \right] \right]$$

$$= \frac{1}{s} \log \left(\sqrt{s^2+1} \right)$$

④

$$L\left[\int_0^t u\,e^{-3u}\cos^2 2u\,du\right]$$

$$\cos^2 2u = \frac{1-\cos 4u}{2}$$

$$L\left[\cos^2 2u\right] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4^2}\right]$$

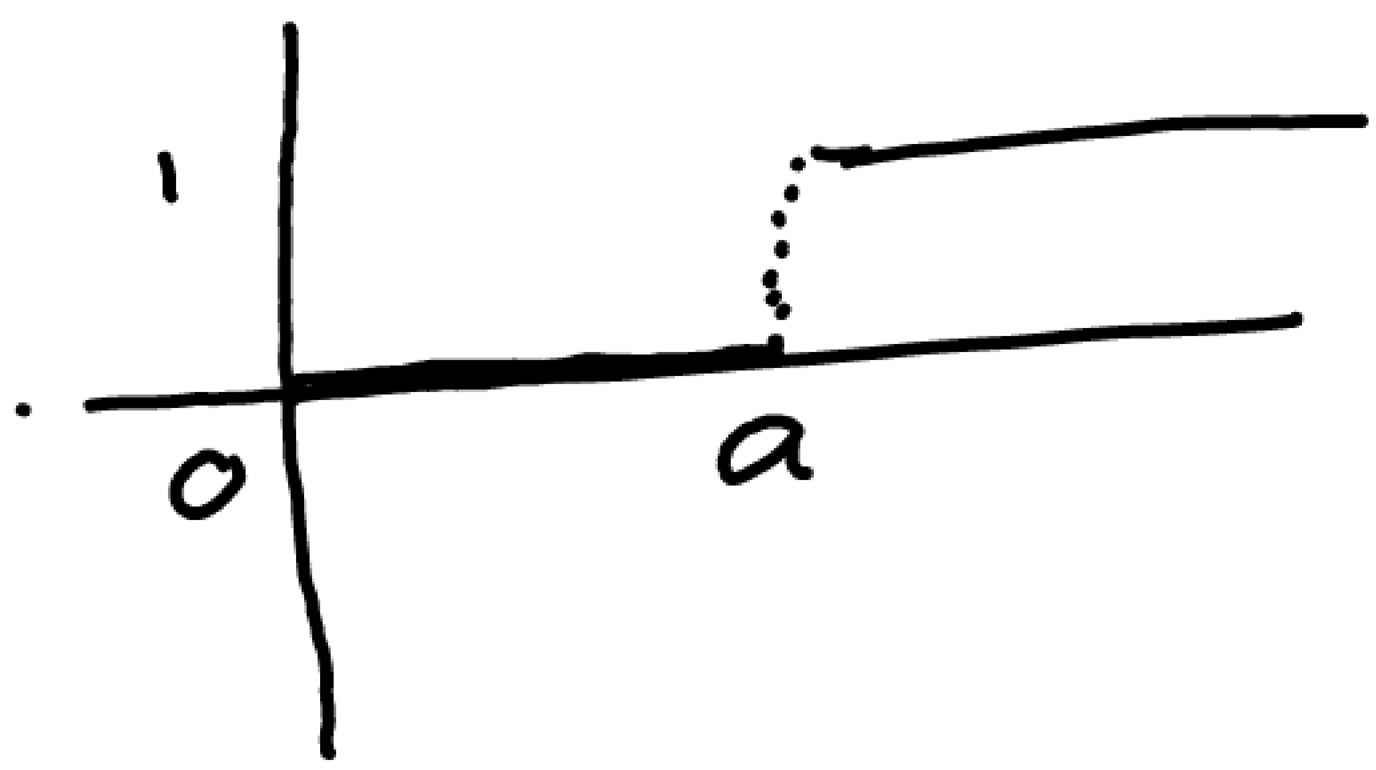
$$L\left[u\cos^2 2u\right] = \frac{1}{2}\left[\frac{-1}{s^2} - \frac{s^2+4^2-2s^2}{(s^2+4^2)^2}\right]$$

$$L\left[e^{-3u}\,u\cos^2 2u\right] = -\frac{1}{2}\left[+\frac{1}{(s+3)^2} + \frac{(s+3)^2-16}{((s+3)^2+16)^2}\right]$$

$$L\left[\int_0^t e^{-3u}\,u\cos^2 2u\right] = \frac{-1}{2s}\left[\frac{1}{(s+3)^2} + \frac{(s+3)^2-16}{((s+3)^2+16)^2}\right]$$

Heaverside unit step function

function defined by $H(t-a)\begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$



$$L[H(t-a)] = \frac{e^{-as}}{s}$$

$$L[f(t)\ H(t-a)] = e^{-as} L[f(t+a)]$$

Find $\mathcal{L}\left[\sin t\left\{u(t-\pi/2)-u(t-3\pi/2)\right\}\right]$

$$= \mathcal{L}\{\sin t\, u(t-\pi/2)\} - \mathcal{L}\{\sin t\, u(t-3\pi/2)\}$$

$$= e^{-\pi/2 s}\,\mathcal{L}[\sin(t+\pi/2)] - e^{-3\pi/2 s}(\sin(t+3\pi/2))$$

$$= e^{-\pi/2 s}\,\mathcal{L}[\cos t] + e^{-\frac{3\pi}{2}s}(\cos t)$$

$$= e^{-\pi/2 s}\,\frac{s}{s^2+1} + e^{-3\pi/2 s}\left(\frac{s}{s^2+1}\right)$$

$$f(t) = \begin{cases} f(t), & a < t < b \\ f_2(t), & b < t < c \\ f_3(t) & t > c \end{cases}$$

Then $f(t) = f_1(t)\left[u(t-a) - u(t-b)\right]$

$+ f_2(t)\left[u(t-b) - u(t-c)\right]$

$+ f_3(t)\left[u(t-c)\right]$

$$L\left[f(t)\,u(t-a)\right] = e^{-as} L\left[f(t+a)\right]$$

① $f(t) = t^2$ for $0 < t < 5$

$= 4t$ for $t > 5$

find $L[f(t)]$

$$f(t) = t^2\left(u(t) - u(t-1)\right) + 4t\left(u(t-1)\right)$$

$$\mathcal{L}\left[f(t)\right] = \mathcal{L}\left[u(t)t^2\right] - \mathcal{L}\left[u(t-1)t^2\right]$$

$$+ 4\mathcal{L}\left[t\, u(t-1)\right]$$

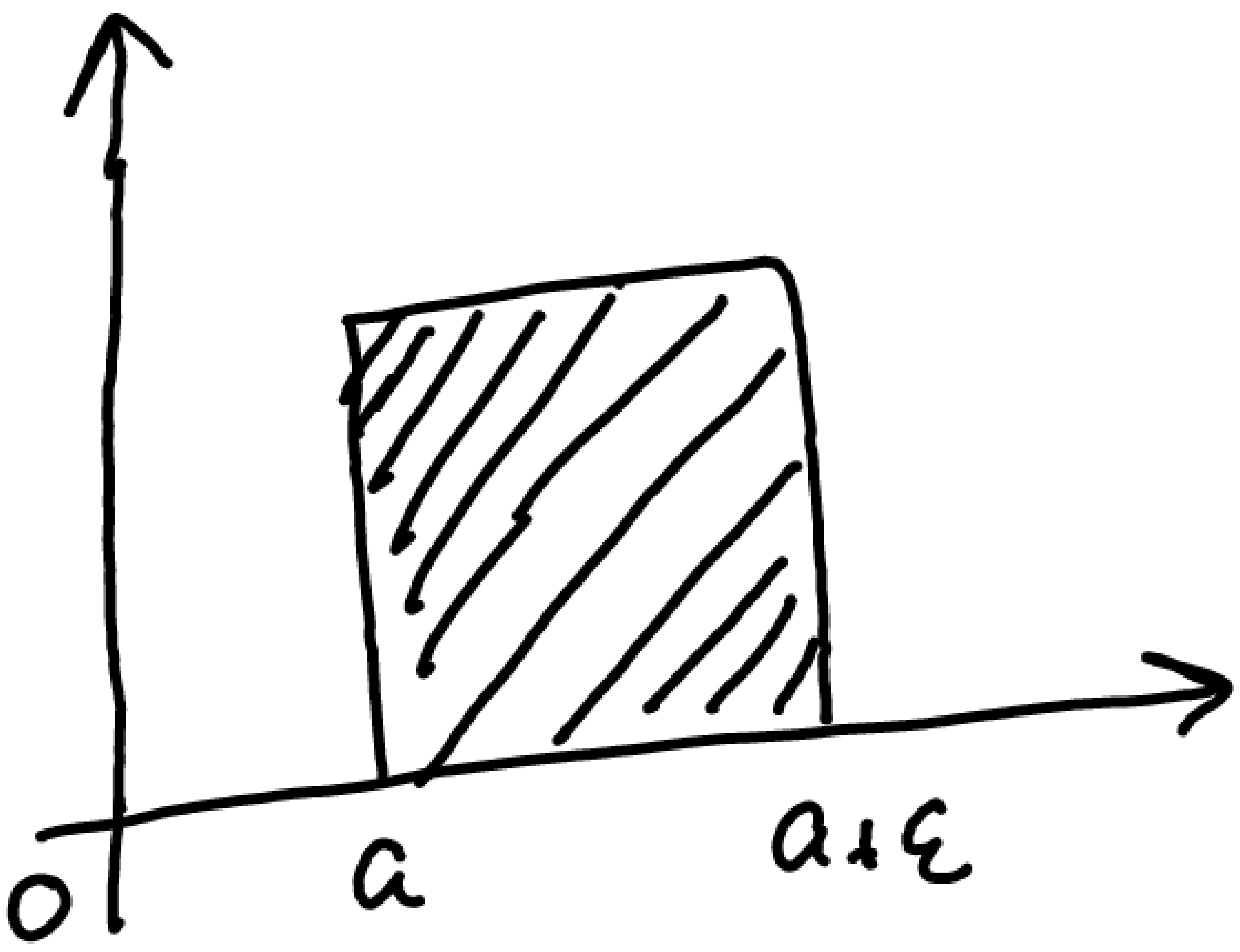
$$= \mathcal{L}\left[t^2\right]$$

$$- e^{s}\ \mathcal{L}\left[(t+1)^2\right]$$

$$+ 4e^{s}\ \mathcal{L}\left[t+1\right]$$

$$= \frac{2}{s^3} - \frac{e^{2s}}{s.} + 4\frac{e^{2s}}{s.}$$

Dirac Delta function



$$f(t) \begin{cases} 1/\varepsilon & \text{if } a < t < a+\varepsilon \\ 0 & \text{if otherwise} \end{cases}$$

$$\lim_{\varepsilon \to 0} [\delta(t-a)] = \delta(t-a)$$

$$\mathcal{L}[\delta(t-a)] = e^{-as}$$

If $a = 0$ Then $\mathcal{L}[\delta(t)] = 1$

$t\, H(t-a) + t^2 \delta(t-a)$

$L[t\, H(t-a)] + L[t^2 \delta(t-a)]$

$\therefore\ L[f(t)\, H(t-a)] = e^{-as} L[f(t+a)]$

$e^{-as} L[(t+a)] + e^{-as} f(a)\ (f(t)+t^2)$

$= e^{-as}\left[\frac{1}{s^2} + \frac{a}{s}\right] + e^{-as} a^2$

Evaluate $\int_0^\infty t\, e^{2t} \sin 3t\, \delta(t-2)\, dt$

$= L[t \sin 3t\, \delta(t-2)]$

$= e^{-2s}\ 2 \sin 6$

at $s=-2$ $= 2 e^4 \sin 6$

1	$1/s$
e^{at}	$\frac{1}{s-a}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
t^m	$\frac{\Gamma(m+1)}{s^{m+1}}$
t	$1/s^2$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$

$$\operatorname{erf}\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2}\,du$$

$$L[u(t-a)] = \frac{e^{-as}}{s}$$

$$L[f(t-a)] = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)$$

$$L[\delta(t-a)\ f(t)] = e^{-as} f(a)$$

$$L[f(at)] = \frac{1}{a}\phi(s/a)$$

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\phi(s)]$$

$$L[e^{at} f(t)] = \phi(s-a)$$

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \phi(s)\,ds$$

$$g(t)\begin{cases} f(t-a) & t>a \\ 0 & t<a \end{cases}$$

$$L\left[\frac{d^n f(t)}{dt}\right] = s^n L[f(t)] - \sum_{i=0}^{n} s^{n-i}\frac{d^i f(0)}{dt^i}$$

$$L[g(t)] = e^{-as} L[f(t)]$$

$$L\left[\int_0^t f(t)\right] = \frac{\phi(s)}{s}$$