Fourier Transform

Fourier transform for $f(x)$ is given by

$$F(s) = F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F^{-1}(F(s)) = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

Fourier cosine transform

$$F(s) = F[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos sx \, dx$$

$$F^{-1}(F(s)) = f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F(s) \cos sx \, ds$$

Fourier Sine Transform

$$F(s) = F[f(x)] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx \, dx$$

$$F^{-1}(F(s)) = f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F(s) \sin sx \, ds$$

Properties

① $F(f(x)\cos ax) = \frac{1}{2}\left[F(s+a) + F(s-a)\right]$

② $F_s(f(x)\cos ax) = \frac{1}{2}\left[F_s(s+a) + F_s(s-a)\right]$

③ $F_s\left[f(x)\sin ax\right) = \frac{1}{2}\left[F_c(s-a) - F_c(s+a)\right]$

④ $F_c\left[f(x)\sin ax\right] = \frac{1}{2}\left(F_s(s+a) - F_s(s-a)\right)$

⑤ $F_c\left[f(x)\cos a\xi\right) = \frac{1}{2}\left[F_c(s+a) + F_c(s-a)\right]$

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$$f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

find fourier sin

evaluate $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\,dx$

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{isx}\,dx$$

$$= \int_{-1}^{1} e^{isx}\,dx$$

$$= \left[\frac{e^{isx}}{is}\right]_{-1}^{1} = \frac{e^{is}}{is} - \frac{e^{-is}}{is}$$

$$= \frac{2\sin s}{s}$$

$$f(x) = F^{-1}(F(s)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\frac{\sin s}{s}\, e^{-isx}\,ds$$

even functi

at $x = 0$ $\displaystyle\int_{-\infty}^{\infty} \frac{\sin s}{s}\,dx = \frac{\pi}{2}$ $\therefore \displaystyle\int_0^{\infty} \frac{\sin s}{s}\,dx = \frac{\pi}{8}$

$f(0) = 1$

(2) $f(x) = \begin{cases} 1-x^2 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$

Hence prove

$$\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2}\, dx$$

$$F\left[f(x)\right] = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \int_{-1}^{1} (1-x^2)\, e^{isx}\, dx$$

$$= \left[(1-x^2)\frac{e^{isx}}{is} - +2x\frac{e^{isx}}{+s^2} + 2\frac{e^{isx}}{is^3} \right]_{-1}^{1}$$

$$= -2\frac{e^{is}}{s^2} - 2\frac{e^{-is}}{s^2} + 2\frac{(e^{is} - e^{is})}{is^3}$$

$$= -\frac{4}{S^2}\cos S$$

$$+ \quad 4\frac{\sin S}{S^3} \quad \rightarrow \text{even fun}$$

$$F^{-1}(F(s)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, e^{-isx}\, ds$$

at $x = 1/2$

$$\frac{3}{4} = \frac{1}{2\pi}\int_{-\infty}^{\infty} -\frac{4}{s^3}(s\cos s - \sin s)\, e^{isx}$$

at $x = 1/2$

$$\frac{3}{4} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\cos\frac{s}{2} + \frac{i}{2\pi}\int_{-\infty}^{\infty} f(s)\sin\frac{s}{2}$$

$$\therefore \int_0^{\infty} f(s)\cos\frac{s}{2} = \frac{-3\pi}{16}$$

$$\therefore \int_0^{\infty}\left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2} = -3\pi/16$$

Hence found.

Find fourier Sine transform

$f(x) = e^{-|x|}$ & Show that $\int_0^\infty \frac{x \sin mx}{1+x^2} dx$

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$= \frac{\pi e^{-m}}{2}, m > 0$

$$e^{-|x|} \begin{cases} e^{-x} & x > 0 \\ e^{x} & x < 0 \end{cases}$$

$$F_s(f(x)) = \int_0^\infty f(x) \sin xs \, dx$$

$$= \int_0^\infty e^{-x} \sin sx \, dx$$

$$F_s(x) = \frac{s}{1+s^2}$$

$$F_s^{-1}(F_s(x)) = \frac{2}{\pi} \int_0^\infty f(x) \sin xs \, dS = \frac{2}{\pi} \int_0^\infty \frac{S}{s^2+1} \sin sx$$

$$= \frac{2}{\pi} \int_0^\infty \frac{x}{1+x^2} \sin mx = e^{-|m|}$$

$m > 0$

$$\int_0^\infty \frac{x \sin mx}{1+x^2} = \frac{\pi e^{-m}}{2}$$

Find fourier transform of $f(x) = e^{-\frac{x^2}{2}}$

$$F(f(x)) = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-ixs}\, dx$$

(4)

$$= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \cos sx + i \cancel{\int_{-\infty}^{\infty} e^{-x^2/2} \sin xs}$$

odd

0

$$I = 2\int_0^{\infty} e^{-x^2/2} \cos sx \, dx$$

DUIS

$$\frac{dI}{ds} = 2\int_0^{\infty} -e^{-x^2/2} \sin sx \; x \, dx$$

$$\frac{dI}{ds} = 2 \sin sx \left[e^{-x^2/2}\right]_0^{\infty} + 2s\int_0^{\infty} e^{-x^2/2} \cos sx$$

$$\frac{dI}{ds} = 2sI \quad \therefore \frac{dI}{I} = 2s\, ds$$

$$\therefore \ln I = s^2$$

$$I = e^{s^2} c$$

at $s = 0$

$$I = 2\int_0^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

$$\therefore c = \sqrt{2\pi}$$

$$\frac{x^2}{2} = t \qquad 2\int_0^{\infty} e^{-t} \frac{1}{\sqrt{2t}}\, dt$$

method ① Gamma $\Gamma 1/2$

or

Laplace

$$L\left[\frac{1}{\sqrt{t}}\right] \longrightarrow \text{method ② } t^n \to \frac{\Gamma 1/2}{s^{1/2}}$$

Method ③ Laplace $\rightarrow$

$$= 2L\left[\frac{d}{dx}\sqrt{t}\right] = 2\,\frac{\Gamma 3/2}{s^{3/2}} - \cancel{f(0)}$$

$$= 2 \times \frac{1}{2}\Gamma 1/2 = \sqrt{\pi}$$

Find fourier sin & cosine transform of

5 1 x^{n-1}

2 $\frac{1}{\sqrt{x}}$

$$f(x^{n-1}) = \int_0^\infty x^{n-1} e^{-is \cdot x} \, dx$$

$$= \frac{\Gamma n}{(is)^n}$$

$$\frac{\Gamma n}{s^n} (i)^{-n} = F_c(x^{n-1}) - i F_s(x^{n-1})$$

$$\frac{\Gamma n}{s^n} \left(\cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2} \right)$$

$$\therefore F_c = \frac{\Gamma n}{s^n} \cos \frac{n\pi}{2} \quad F_s = \frac{\Gamma n}{s^n} \sin \frac{n\pi}{2}$$

$$F\left(\frac{1}{\sqrt{x}}\right) = \int_0^\infty \frac{1}{\sqrt{x}} e^{-isx} \, dx$$

$$= \frac{\Gamma 1/2}{(is)^{1/2}}$$

$$= \frac{1}{\sqrt{i}} \frac{\Gamma 1/2}{\sqrt{s}}$$

$$F_c = \frac{\Gamma 1/2}{\sqrt{s}} \cos \frac{\pi}{4} = \sqrt{\frac{\pi}{2s}}$$

$$F_s = \frac{\Gamma 1/2}{\sqrt{s}} \sin \frac{\pi}{4} = \sqrt{\frac{\pi}{2s}}$$

⑥

$$f(x) \begin{cases} x & |x| \le a \\ 0 & |x| > a \end{cases}$$

$$F(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \int_{-a}^{a} x\, e^{isx}\, dx$$

$$= \left[x \frac{e^{isx}}{is} - \frac{e^{isx}}{-s^2x^2} \right]_{-a}^{a}$$

$$= \frac{e^{isa}}{s^2a^2} - \frac{e^{-isa}}{s^2a^2}$$

$$= \frac{2}{s^2a^2} \sin sa$$

⑦ $f(x) \begin{cases} 1-|x| & , |x| \le 1 \\ 0 & , |x| > 1 \end{cases}$

$$F(f(x)) = \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$= \int_{-1}^{1} (1-|x|) e^{isx}\, dx$$

$$= \int_{0}^{1} (1-x) e^{isx}\, dx + \int_{-1}^{0} (1+x) e^{isx}$$

$$= \left[(1-x)\frac{e^{isx}}{is} + \frac{e^{isx}}{s^2} \right]_0^1 = \frac{e^{is}}{s^2} - \frac{1}{is}$$

$$+ \left[(1+x)\frac{e^{isx}}{is} - \frac{e^{isx}}{s^2} \right]_{-1}^{0} \quad - \frac{1}{s^2} + \frac{1}{is} - \frac{1}{s} + \frac{e^{-is}}{s}$$

$$= \frac{2\cos s - 1}{s^2}$$

$$= \frac{\sin^2 s/2}{s^2}$$

$$F^{-1}(F(f(x))) = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin^2 s/2}{s} e^{-isx}\, ds$$

at $x = 0$

$$1 = \frac{1}{\pi}\int_0^{\infty} \frac{\sin^2 s/2}{s^2}\, ds$$

$s/2 \to t$

$ds = 2t\,dt$

$$2\pi = \int_0^{\infty} \frac{\sin^2 s}{s^2}\, ds$$

(8) Find Inverse fourier transform of

$$\phi(s) = \begin{cases} 1+s^2 & |s| \le 1 \\ 0 & |s| > 1 \end{cases}$$

$$F^{-1}(\phi(s)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, e^{-isx}\, ds$$

$$= \frac{1}{2\pi}\int_{-1}^{1} 1+s^2\, e^{-isx}\, ds$$

$$= \frac{1}{\pi}\int_{0}^{1} 1+s^2 \left(\frac{e^{-isx} + e^{isx}}{2}\right)$$

$$= \frac{1}{\pi}\int_{0}^{1} \left((1+s^2)\right) \cos sx$$

$$= \frac{1}{\pi}\left[(1+s^2)\frac{\sin sx}{x} + 2s\frac{\cos sx}{x^2} - 2\frac{\sin sx}{x^3}\right]_0^1$$

$$= 2\frac{\sin x}{x} + 2\frac{\cos x}{x^2} + -2\frac{\sin x}{x^3}$$

(9) Find fourier Cosine transform e^{-ax}

$$f(x) = \int_0^\infty f(x) \cos sx \, dx$$

$$= \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \frac{a}{s^2 + a^2}$$

(10) Fourier Sine Transform e^{-ax}

$$= \int_0^\infty f(x) \sin sx \, dx$$

$$= \frac{s}{s^2 + a^2}$$

$$F_s(x e^{-ax}) = \frac{-2as}{s^2+a^2}$$

$$F_s\left(\frac{e^{-ax}}{x}\right) = \tan^{-1}\left(\frac{a}{s}\right)$$

$$F^{-1}(F_s) = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+a^2} \sin sx$$

at $a=0$, $$\frac{\pi}{2} = \int_0^\infty \frac{\sin sx}{x}$$

(11) $f(x) = \begin{cases} \cos x & 0 < x < a \\ 0 & x > a \end{cases}$

$$f(x) = \int_0^\infty \cos x \cos sx$$

$$= \int_0^a \cos x \cos sx \, dx = = \left[\cos(s+1)x + \cos(s-1)x \right]_0^a$$

$$= \cos(s+1)a + \cos(s-1)a$$

(12) $f(x)$ if $\int_0^\infty f(x) \sin sx \, dx = e^{-as}$

$$\therefore g(x) = e^{-as} \, 2\pi$$

$$f(x) = F[g(x)] = 2\pi \int_0^\infty e^{-as} \sin sx \, ds$$

$$= 2\pi \frac{x}{a^2 + x^2}$$