Inverse Laplace Transform

If $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \phi(s)$

Then $f(t)$ is called inverse laplace transform of $\phi(s)$ given by $f(t) = L^{-1}[\phi(s)]$

Example

① $L(1) = 1/s$

$\therefore L^{-1}(1/s) = 1$

② $L(e^{at}) = \frac{1}{s-a}$

$\therefore L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

③ $L(\cos at) = \frac{s}{s^2+a^2}$

$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

④ $L(\sin at) = \frac{a}{s^2+a^2}$

$a L^{-1}\left(\frac{1}{s^2+a^2}\right) = \sin at$

⑤ $L(t^{n-1}) = \frac{\Gamma n}{s^n}$

$L^{-1}(1/s^n) = \frac{t^{n-1}}{\Gamma n}$

Linearity property

$$L^{-1}(a\,\phi_1(s) + b\,\phi_2(s)) = a\,L^{-1}(\phi_1(s)) + b\,L^{-1}(\phi_2(s))$$

eg. $$L^{-1}\left[\frac{2s-5}{4s^2+25} - \frac{4s-18}{9-s^2} + \frac{1}{s+1} + \frac{5}{s^4}\right]$$

$$= \frac{2}{4}L^{-1}\left(\frac{s}{s^2+\frac{25}{4}}\right] - \frac{5}{4}\left[\frac{1}{s^2+\frac{25}{4}}\right] + 4\left[\frac{s}{s^2-9}\right]$$

$$+ -18\,L^{-1}\left[\frac{1}{s^2-9}\right] + L^{-1}\left(\frac{1}{s+1}\right)$$

$$+\left[\frac{5}{s^4}\right]$$

$$= \frac{2}{4}\cos\frac{5}{2}t - \frac{\cancel{5}}{{}_2\cancel{4}}\,\frac{\cancel{2}}{\cancel{5}}\sin\frac{5}{2}t$$

$$+ 4\cosh 3t - \frac{18}{3}\sinh 3t + e^{-t} + \frac{5}{\Gamma 4}t^3$$

First shifting property

$$L[e^{at}f(t)] = \phi(s-a)$$

$$\therefore L^{-1}[\phi(s)] = f(t)$$

Then $L^{-1}[\phi(s-a)] = e^{at}f(t)$

eg. $L^{-1}\left[\dfrac{s+3}{(s+3)^2+16}\right]$

$$L^{-1}\left[\frac{s}{s^2+16}\right] = \cos 4t$$

$$L^{-1}\left[\frac{s+3}{(s+3)^2+16}\right] = e^{-3t}\cos 4t$$

1. Find Laplace inverse of $\dfrac{s^2}{(s-1)^3}$

$\rightarrow$ using first shifting property

If

$$L^{-1}\left(\frac{(s+1)^2}{s^3}\right) = f(t)$$

$$L^{-1}\left(\frac{s^2}{(s-1)^3}\right) = e^{t} f(t)$$

$$L^{-1}\left(\frac{s^2+2s+1}{s^3}\right)$$

$$= L^{-1}\left(\frac{1}{s} + \frac{2}{s^2} + \frac{1}{s^3}\right)$$

$$= \left(1 + \frac{2}{t} + \frac{1}{2t^2}\right)$$

Answer = $e^t \left(1 + 2t + \frac{t^2}{2}\right)$

Explanah $\frac{s^2}{(s-1)^3} = \frac{(s-1+1)^2}{(s-1)^3}$

If

$\left(\frac{s+1}{s}\right)^3 = \phi(s)$

$\phi(s-1) = \frac{s^2}{(s-1)^3} =$

$\therefore \phi(s-1) = e^t f(t)$

② $\mathcal{L}^{-1}\left(\frac{s}{s^4+s^2+1}\right)$

$$= \mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2 - s^2}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{s}{(s^2+s+1)(s^2-s+1)}\right)$$

$$= \frac{-1}{2}\mathcal{L}^{-1}\left(\frac{1}{s^2+s+1} - \frac{1}{s^2-s+1}\right)$$

$$= -\frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{(s+1/2)^2 - 3/4} - \frac{1}{(s-1/2)^2 + 3/4}\right)$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{(s-1/2)^2+3/4}\right) - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{(s+1/2)^2 - 3/4}\right)$$

$s \to s+1/2$ $\qquad\qquad$ $s \to s-1/2$

$$= \frac{1}{2} e^{+1/2\, t} \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t - \frac{1}{2} e^{-1/2\, t} \frac{2}{\sqrt{3}} \sin \sqrt{3}/2\, t$$

③ $\mathcal{L}^{-1}\left(\dfrac{s^2+16s-24}{s^4+20s^2+64} \right)$

$$\mathcal{L}^{-1}\left(\frac{s^2+16s-24}{(s^2+10)^2-36} \right)$$

$$\mathcal{L}^{-1}\left(\frac{s^2+16s-24}{(s^2+10-6)(s^2+10+6)} \right)$$

$$\mathcal{L}^{-1}\left(\frac{s^2+16s-24}{(s^2+4)(s^2+16)} \right)$$

$$\frac{s^2+16s-24}{(s^2+4)(s^2+16)} = \frac{s^2-24}{(s^2+4)(s^2+16)} - s\left(\frac{16}{(s^2+4)(s^2+16)}\right)$$

Second shifting property

If $L[f(t)] = \phi(s)$

$$g(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$$

then $L[g(t)] = e^{-as}\,\phi(s)$

$\therefore$ If $f(t) = L^{-1}(\phi(s))$

Then $$L^{-1}[e^{-as}\phi(s)] = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$$

① Find $L^{-1}\left[\dfrac{e^{-2s}}{s^2+8s+25}\right]$

$$L^{-1}\left[\frac{1}{s^2+8s+25}\right] = \frac{1}{3}L^{-1}\left[\frac{3}{(s+4)^2+3^2}\right]$$

$$= \frac{1}{3}e^{-4t}\sin(3t)$$

$$\therefore L^{-1}[\quad] = \begin{cases} \frac{1}{3}e^{-4(t-2)}\sin(3t-6) & t > a \\ 0 & t < a \end{cases}$$

Inverse Laplace of derivative

$$L[t^n f(t)] = (-1)^n \frac{d^n \phi(s)}{dt^n}$$

$$\therefore \quad L^{-1}\left[\frac{d^n \phi(s)}{ds^n}\right] = (-1)^n t^n f(t)$$

① $L^{-1}\left[\frac{s+3}{(s^2+6s+10)^2}\right]$

$$\frac{1}{s^2+6s+10} \xrightarrow{d/ds} \frac{-2(s+3)}{(s^2+6s+10)^2}$$

$$\therefore = L^{-1}\left[\frac{1}{-2}\frac{d}{ds}\left(\frac{1}{s^2+6s+10}\right)\right] = -\frac{1}{2} L^{-1}\left[\frac{d}{ds}\quad\right]$$

$$= L^{-1}\left[\frac{1}{s^2+6s+10}\right] = L^{-1}\left[\frac{1}{(s+3)^2+1}\right]$$

$s \rightarrow s-3 \qquad = e^{-3t}\sin t$

$L[\quad] = \frac{t}{2} e^{-3t} \sin t$

Better way of writing

Let $\phi(s) = \dfrac{1}{s^2+6s+10}$

$$\phi'(s) = -\frac{(2s+6)}{(s^2+6s+10)^2}$$

$$-\frac{1}{2}\phi'(s) = \frac{(s+3)}{(s^2+6s+10)^2}$$

Taking Laplace inverse on both sides.

$$-\frac{1}{2}L^{-1}(\phi(s)) = L^{-1}\left(\frac{s+3}{(s^2+6s+10)^2}\right)$$

$$= -\frac{1}{2} - t\, L^{-1}(\phi(s))$$

$$= \frac{t}{2} L^{-1}\left[\frac{1}{s^2+6s+10}\right]$$

$$= \frac{t}{2} L^{-1}\left[\frac{1}{(s+3)^2+1}\right]$$

$$= \frac{t}{2} e^{-3t} L^{-1}\left(\frac{1}{s^2+1}\right)$$

first shifting

$$= \frac{t}{2} e^{-3t} \sin t$$

② $L^{-1}\left[\log\left(\frac{s^2+1}{s(s+1)}\right)\right]$

Dont Identify functions as derivative for hard functions

$$\phi(s) = \log(s^2+1) - \log s - \log(s+1)$$

$$\phi'(s) = \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}$$

$$L^{-1}(\phi'(s)) = -t \; L^{-1}(\phi(s))$$

$$\therefore \frac{L^{-1}\left[\frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right]}{-t} = L^{-1}(\phi(s))$$

$$-\frac{1}{t}\left[2\cos t - 1 - e^{-t}\right]$$

③ $\tan^{-1}\left(\frac{2}{s^2}\right)$

$$\phi(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$$

$$\phi'(s) = \frac{\frac{-4}{s^3}}{1+\left(\frac{2}{s^2}\right)^2} = \frac{-4}{s^3\left(1+\frac{4}{s^4}\right)}$$

$$= \frac{-4s}{s^4+4}$$

$$L^{-1}(\phi'(s)) = L^{-1}\left(\frac{-4s}{s^4+4}\right)$$

$$= -4\,L^{-1}\left(\frac{s}{(s^2+2)^2-4s^2}\right)$$

$$= -\,L^{-1}\left(\frac{4s}{(s^2+2s+2)(s^2-2s+2)}\right)$$

$$= -L^{-1}\left(\frac{1}{s^2+2s+2} - \frac{1}{s^2-2s+2}\right)$$

$$= -L^{-1}\left[\frac{1}{(s+1)^2+1} - \frac{1}{(s-1)^2+1}\right]$$

$$= -\left[e^{-t}\sin t - e^{t}\sin t\right]$$

$$\frac{L[\phi'(s)]}{-t} = L[\phi(s)]$$

$$\therefore \text{Ans} = +\frac{1}{t}\left[e^{-t}\sin t - e^{t}\sin t\right]$$

$$= -\frac{2}{t}\sinh t \sin t$$

④ $2\tanh^{-1}(s)$

Let $\phi(s) = 2\tanh^{-1}(s)$

$$= \log\left[\frac{1+s}{1-s}\right]$$

$$= \log(1+s) - \log(1-s)$$

$$\phi'(s) = \frac{1}{1+s} + \frac{1}{1-s}$$

$$L^{-1}(\phi'(s)) = e^{-t} - e^{+t}$$

$$\frac{L^{-1}(\phi'(s))}{t} = L^{-1}(\phi(s))$$

$$\therefore \text{Ans} = \frac{e^{-t} - e^{t}}{-t}$$

Convolution Theorem

If $L^{-1}[\phi_1(s)] = f_1(u)$

$L^{-1}[\phi_2(s)] = f_2(u)$

$$L^{-1}(\phi_1(s)\,\phi_2(s)) = \int_0^t f_1(u)\, f_2(t-u)\,du$$

① $$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

$$= \frac{1}{(s^2+2s+5)} + \frac{1}{(s^2+2s+2)\,(s^2+2s+5)}$$

$$L^{-1}\left[\frac{1}{s^2+2s+5}\right] = L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= \frac{e^{-t}\sin 2t}{2}$$

$$L^{-1}\left[\frac{1}{s^2+2s+2}\right] = L^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t}\sin t$$

$$\therefore L^{-1}[\quad] = L^{-1}\left[\frac{1}{s^2+2s+2}\right]$$

$$+L^{-1}\left[\frac{1}{(s^2+2s+2)}\ \frac{1}{s^2+2s+5}\right]$$

Taking e^{t} out here is benificial

$$= \frac{1}{2}e^{-t}\sin 2t$$

$$+\int^{t} \frac{1}{2}e^{-u}\sin 2u \cdot e^{-u}\sin(t-u)\,du$$

$$L^{-1}\left[\frac{1}{(s+1)^2+1}\cdot\frac{1}{(s+1)^2+4}\right]$$

$$= e^{-t}L^{-1}\left[\frac{1}{s^2+1} * \frac{1}{(s^2+4)}\right]$$

$$= e^{-t}\int_0^t \sin(t-u)\frac{\sin 2u}{2}\,du$$

$$= \frac{e^{-t}}{4}\int_0^t \cos(3u-t) + \cos(t+u)$$

$$= \frac{e^{-t}}{2}\left(\frac{2}{3}\sin 2t - \frac{1}{3}\sin t\right)$$

$$An \rightarrow \quad e^{-t}\left[\frac{1}{3}\sin 2t + \frac{1}{3}\sin t\right]$$

(2) $$\frac{s^2+s}{(s^2+1)(s^2+2s+2)} = \frac{s(s+1)}{(s^2+1)(s^2+2s+2)}$$

$$= \int_0^t L^{-1}\left[\frac{s}{s^2+1}\right] \cdot L^{-1}\left[\frac{s+1}{s^2+2s+2}\right]$$

$$= \int_0^t \cos(t-u) \cdot e^{-u}\cos(u)\, du$$

$$= \frac{1}{2}\int_0^t e^{-u}\left(\cos t + \cos(t-2u)\right) du$$

$$= \frac{1}{2}\cos t\left[\frac{e^{-u}}{-1}\right]_0^t + \frac{1}{2}\int_0^t e^{-u}\cos(t-2u)\, du$$

$$= \frac{1}{2}\left[-\cos t\,(e^{-t}-1) + \left[\frac{e^{-u}}{(1+4)}\left(-\cos(2u-t) + 2\sin(2u-t)\right)\right]_0^t\right]$$

$$= \frac{1}{2}\left[\cos t - e^{-t}\cos t + \frac{e^{-t}}{5}\left(-\cos(t) + \sin t\right) - \left[\frac{-\cos(t) - 2\sin t}{5}\right]\right]$$

$$= \frac{1}{2}\left[-e^{-t}\cos t\left[\frac{6}{5}\right] + \frac{6}{5}\cos t - 2\sin t + \frac{e^{-t}}{5}\sin t\right]$$

③ $\dfrac{(s+3)^2}{(s^2+6s+5)^2}$

$$= \frac{(s+3)^2}{\left((s+3)^2-2^2\right)^2}$$

$$= e^{-3t} L^{-1}\left[\left(\frac{s}{s^2-2^2}\right)\cdot\frac{s}{(s^2-2^2)}\right]$$

$$= e^{-3t}\int_0^t \cosh u \, \cosh(u-t)\, du$$

$$= e^{-3t}\int_0^t \left(\frac{e^{u}+e^{-u}}{2}\right)\left(\frac{e^{(u-t)}+e^{-(u-t)}}{2}\right)$$

$$= \frac{e^{-3t}}{4}\int_0^t e^{2u-t}+e^{t}+e^{-t}+e^{-(2u-t)}$$

$$= \frac{e^{-3t}}{4}\left[\frac{e^{2u-t}}{2}+e^{t}u+e^{-t}u+\frac{e^{-(2u-t)}}{-2}\right]$$

$$= \frac{e^{-3t}}{4}\left[\frac{e^{t}}{2} + e^{t}t\right.$$

$$+ e^{-t}t + \frac{e^{-t}}{-2}$$

$$\left. - \frac{e^{-t}}{2} + \frac{e^{t}}{2}\right)$$

$$= \frac{e^{-3t}}{4}\left[\cosh t + t\,\frac{\sinh t}{2}\right]$$

④ $\frac{1}{s} \log\left(\frac{s+3}{s+2}\right)$

$\phi(s) = \log\left(\frac{s+3}{s+2}\right)$

$\phi(s) = \left(\log(s+3) - \log(s+2)\right)$

$\phi'(s) = \left(\frac{1}{s+3} + \frac{1}{s+2}\right)$

$L^{-1}\left(\phi'(s)\right) = e^{-3t} + e^{-2t}$

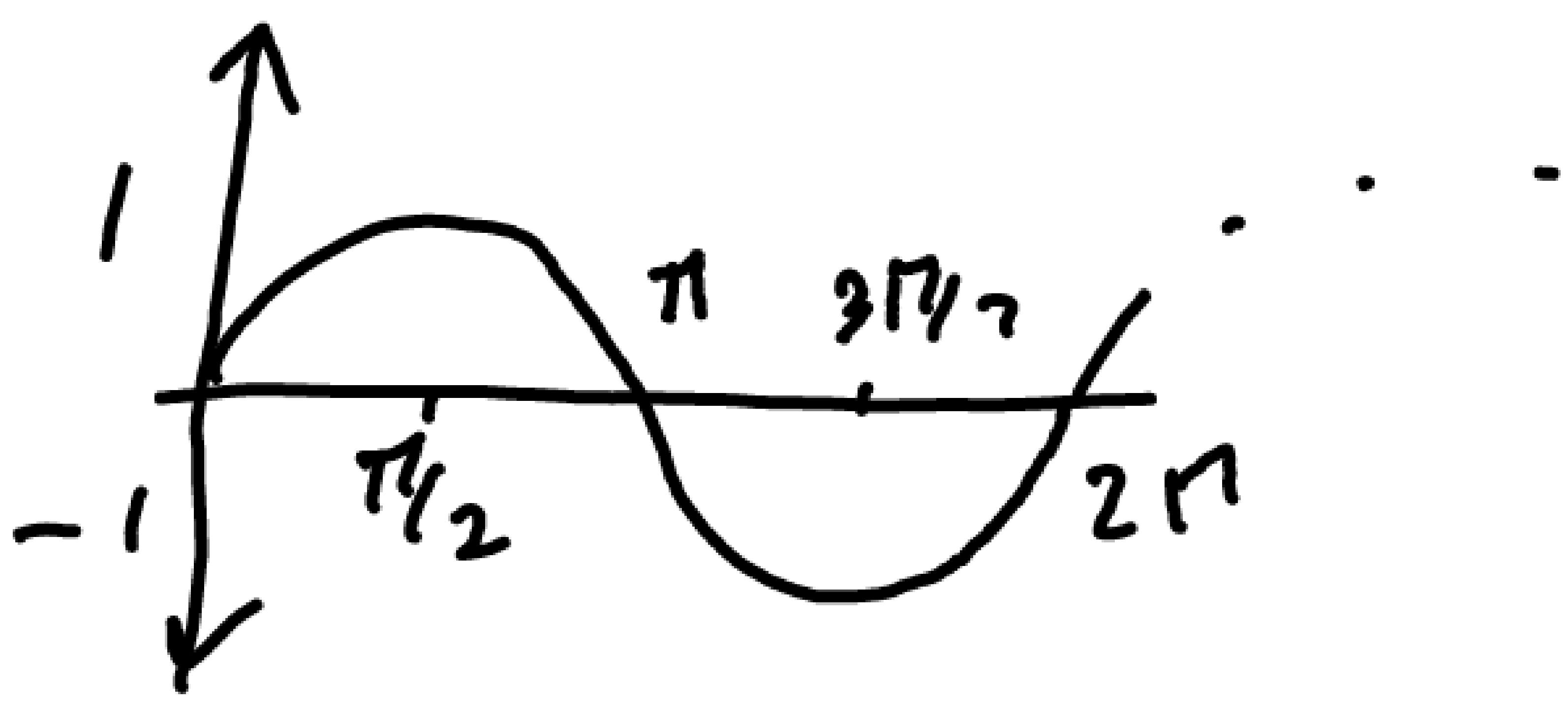
$L^{-1}\left(\phi'(s)\right) = L^{-1}\left(\frac{\phi(s)}{t}\right)$

$A \int_0^t L^{-1}\left[\frac{1}{s}\right] L^{-1}\left(\frac{\phi(s)}{t}\right) = \int_0^t \frac{e^{-2u} - e^{-3u}}{u}\, du$

Special function

$f(t)$ is a periodic function with period a

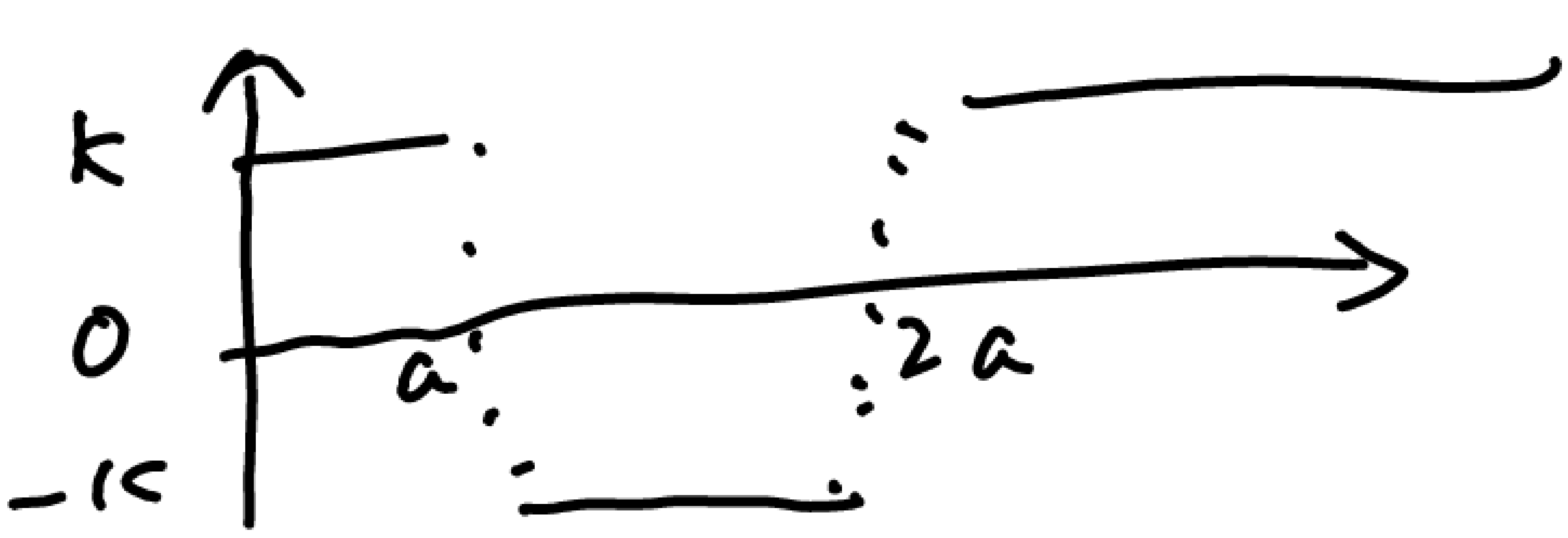
$f(t+a) = f(t)$



$$L[f(t)] = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\, dt$$

eg

① $f(t) \begin{cases} k & 0 < t < a \\ -k & a < t < 2a \end{cases}$ | 2a is period (given)



$$L[f(t)] = \frac{1}{1-e^{-as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$L[f(t)] = \frac{1}{1-e^{-2as}} \left[\int_0^{a} k e^{-st} dt + \int_a^{2a} -k e^{-st} dt \right]$$

$$= \frac{k}{1-e^{-2as}} \left[\left(\frac{e^{-st}}{-s} \right)_0^a - \left(\frac{e^{-st}}{-s} \right)_a^{2a} \right]$$

$$= \frac{k}{s(1-e^{-2as})} \left[-(e^{-as}-1) + (e^{-2as} - e^{-as}) \right)$$

$$= \frac{k}{s(1-e^{-2as})} \left[(e^{-as})^2 - 2aS + 1 \right.$$

$$= \frac{k}{s} \frac{(e^{-as}+1)^2}{(1-e^{-2as})} = \frac{k}{s} \frac{(1-e^{-as})}{(1-e^{-as})(1+e^{-as})}$$

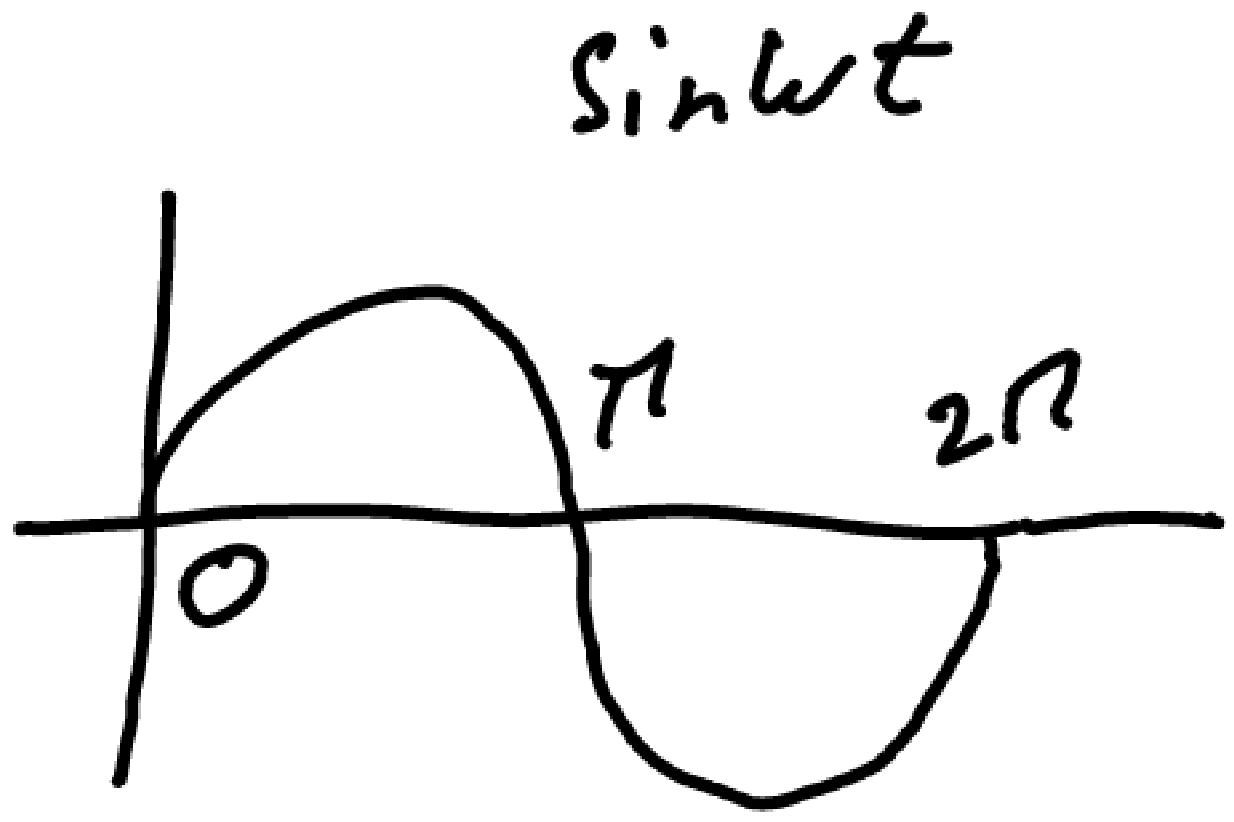
$$= \frac{k}{s} \frac{(1-e^{-as})}{(1+e^{-as})}$$

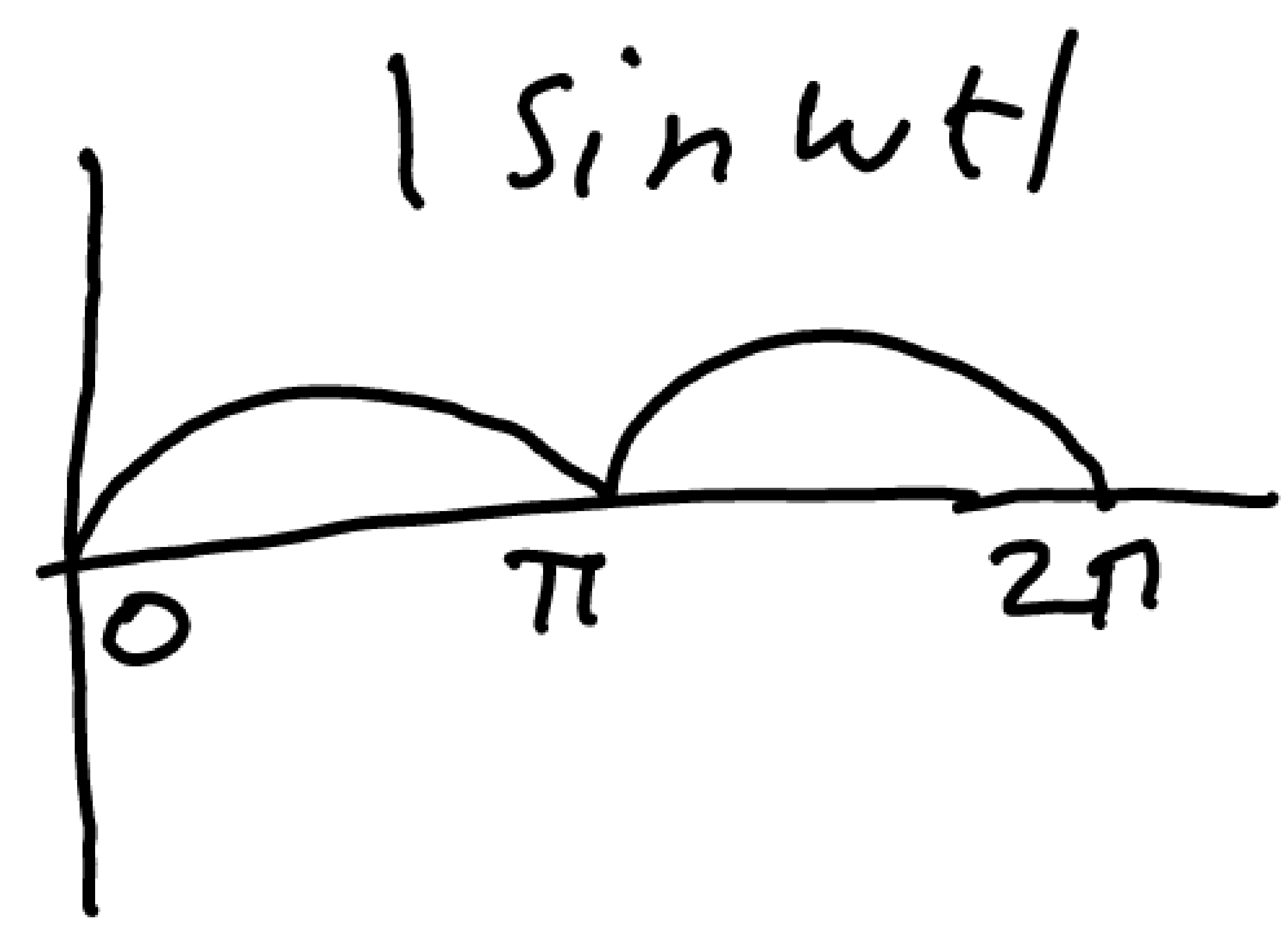
② find Laplace transform of full wave rectifier $f(t) = |\sin wt|$ $t \geq 0$

→

period $= \pi/w$ by observation

we observe





$$f\left(t + \frac{\pi}{\omega}\right) = \left| \sin w(t + \pi/\omega) \right|$$

$$= | \sin(\omega t + \pi)$$

$$= | - \sin(\omega t| = | \sin \omega t|$$

$$= f(t)$$

$$L\left[\,|\sin wt|\,\right] = \frac{1}{1-e^{-\frac{\pi}{w}s}} \int_0^{\pi/w} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-\pi s}} \int_0^{\pi/w} e^{-st} \sin wt \; dt$$

$$= \frac{1}{1-e^{-\pi/s}} \left[\frac{e^{-st}}{s^2+w^2} \left(-s\sin wt - w\cos wt \right) \right]_0^{\pi/w}$$

$$= \frac{1}{1-e^{-\pi/s}} \left[\frac{e^{-s\pi/w}}{(s^2+w^2)} \left(-s\sin\pi - w\cos\pi \right) - \frac{w}{s^2+w^2} \right]$$

$$= \frac{1}{1-e^{\pi/s}} \left[\frac{(e^{-s\pi/w}-1)w}{s^2+w^2} \right).$$

$$L^{-1}\left(\frac{e^{-as}}{s}\right) = H(t-a)$$

If $a = 0$ then $L[H(t)] = \frac{1}{s}$

$$\therefore L^{-1}[1/s] = H(t)$$

$s \to s + a$

$f(t) \to e^{at} f(t)$

$$L^{-1}[\phi_1(s)\phi_2(s)] = \int_0^t L^{-1}(\phi_1(u))\, L^{-1}(\phi_2(t-u))\, du$$