Complex form of fourier series

$$f(x) = \sum_{n=0}^{\infty} C_n e^{\frac{in\pi x}{L}}$$

$$C_n = \frac{1}{2L} \int_{a}^{a+2L} f(x)\, e^{\frac{-in\pi x}{L}}\, dx$$

$$= \frac{a_n - i b_n}{2}$$

$$C_{-n} = \frac{a_n + i b_n}{2}$$

$$C_0 = \frac{a_0}{2}$$

① Obtain complex form of fourier series

$$f(x) = \begin{cases} a & 0 < a < L \\ -a & L < x < 2L \end{cases}$$

Hence deduce corresponding Trigonometri Series

$$C_n = \frac{1}{2L} \int_a^{a+2L} f(x)\, e^{\frac{-in\pi x}{L}}\, dx$$

$$= \frac{1}{2L} \int_0^L a\, e^{\frac{-in\pi x}{L}} - \int_L^{2L} a\, e^{\frac{-in\pi x}{L}}\, dx$$

$$= \frac{a}{2L} \left[\left[\frac{e^{\frac{-in\pi x}{L}}}{\frac{-in\pi}{L}} \right]_0^L - \left[\frac{e^{\frac{-in\pi x}{L}}}{\frac{-in\pi}{L}} \right] \right)$$

$$= \frac{a}{2L} \checkmark -\frac{L}{in\pi} \left((e^{-in\pi} - 1) - e^{-i2n\pi} - e^{-in\pi} \right]$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-in\pi} = \cos n\pi - i\sin n\pi$$

$$= (-1)^n$$

$$\therefore c_n = \frac{ai}{2n\pi}\left[(-1)^n - 1\right]$$

$$c_0 = \frac{a_0}{2} = \frac{1}{2}\,\frac{1}{L}\int_0^{2L} f(x)\,dx$$

$$= \frac{a}{2L}\left[L - L\right] = 0$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}} = \frac{ai}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{n}\left[(-1)^n - 1\right]$$

$$a_n = c_n + c_{-n} = \frac{ai}{n\pi}\left((-1)^n - 1\right) - \frac{ai}{n\pi}\left((-1)^n - 1\right)$$

$$b_n = \frac{c_n - c_{-n}}{-i}$$

$$= \frac{-1}{i}\left(\frac{ai}{n\pi}\left((-1)^n - 1\right) - \frac{(-ai)}{n\pi}\left((-1)^n - 1\right)\right)$$

$$= \frac{-2a}{n\pi}\left[(-1)^n - 1\right]$$

$$f(x) = \frac{-2a}{\pi} \sum_{n=1}^{\infty} \left(\frac{(-1)^n - 1}{n}\right) \sin \frac{n\pi x}{L}$$

Find Complex form of Fourier series

$$f(x) = \cosh 2x + \sinh 2x \quad \text{in } (-5,5)$$

$$f(x) = \left(\frac{e^{2x} + e^{-2x}}{2}\right) + \left(\frac{e^{2x} - e^{-2x}}{2}\right)$$

$$= e^{2x}$$

$$C_n = \frac{1}{10}\int_{-5}^{5} e^{2x}\, e^{-\frac{in\pi x}{5}}\, dx$$

$$= \frac{1}{10}\int_{-5}^{5} e^{\left(2 - \frac{in\pi}{5}\right)x}\, dx$$

$$= \frac{1}{10}\left[\frac{e^{\left(\frac{10 - in\pi}{5}\right)}}{\frac{10 - in\pi}{5}}\right]_{-5}^{5}$$

$$= \frac{1}{2(10 - in\pi)}\left[e^{10} \times e^{-in\pi} - e^{10} \times e^{in\pi}\right)$$

$$\therefore \quad C_n = \frac{(10+in\pi)\sinh(10-in\pi)}{100+n^2\pi^2}$$

$$f(x) = e^{2x} = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{5}}$$

Find Complex form of fourier series

$$f(x) = e^{ax} \text{ in } (-\pi, \pi)$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\frac{in\pi x}{\pi}} e^{ax} \, dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{(-in+a)x}}{-\frac{in\pi}{\pi}+a}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\frac{e^{(-in+a)\pi} - e^{(-in\pi+a)-\pi}}{(-in+a)}$$

$$= \frac{1}{\pi}\frac{\operatorname{Sinh}(-n\pi+a)\pi}{(-in+a)}$$

④ $f(x) = \sin ax \qquad (-\pi, \pi), \; a \notin \mathbb{Z}$

$$c_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$$

$$c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin ax \, e^{-inx}$$

$$= \frac{1}{\pi}\left[\frac{e^{-inx}}{-n^2+a^2}\left[-in \sin ax - a\cos ax\right]\right]_{-\pi}^{\pi}$$

$2x$

$$\frac{2}{n}\int_0^{2\pi} x\, e^{-inx}$$

$$\left. + \frac{e^{-inx}}{n^2} \right]_0^{2\pi}$$

$$\frac{i}{n}\left[x \frac{e^{-inx}}{-in} \right.$$

$$\left. + \cancel{\frac{1}{n^2}} \right) e^{inx}$$

$$+\frac{2i}{n}\left(\frac{2\pi}{-in} - \cancel{\frac{1}{n^2}} \right)$$

$$\frac{}{\pi}$$

Fourier Integral

If $f(x)$ satisfies Dirichlet's conditions in finite Interval $-L \le x \le L$ & it is integrable in the interval $(-\infty, \infty)$ then fourier integral of $f(x)$ is

$$f(x) = \frac{1}{\pi} \int_{w=0}^{\infty} \int_{s=-\infty}^{\infty} f(s) \cos w(s-x)\, ds\, dw$$

$$f(x) = \frac{1}{\pi} \int_{w=0}^{\infty} \int_{s=-\infty}^{\infty} f(s) \cos sw \cos wx\, ds\, dw$$

$$+ \frac{1}{\pi} \int_{w=0}^{\infty} \int_{s=-\infty}^{\infty} f(s) \sin sw \sin wx\, ds\, dw$$

If $f(x)$ is even. $\because$ $f(s)$ is even.

$f(s) \sin ws$ is odd

$\therefore$ Second Integral $= 0$

$f(s) \cos ws$ is even

$\therefore$ first Integral from 0 to ∞ $\times 2$

(fourier cosine Integral representation)

If $f(x)$ is odd $\therefore$ $f(s)$ is odd

$\because$ $f(s) \sin ws$ is even

first Integral $= 0$

$f(s) \cos ws$ is odd

$\therefore$ Second Integral from 0 to ∞ $\times 2$

(fourier Sine representation)

① Express the function $f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$

as fourier integral

Hence evaluate $\int_0^\infty \frac{\sin w \cos wx}{w} \, dw$

$f(x)$ is even $\therefore$

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(s) \cos ws \cos wx \, dw \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \int_0^1 \cos ws \cos wx \, dw \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \cos wx \left[\frac{\sin ws}{w} \right]_0^1$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\cos wx \sin w}{w} \, dw$$

leave here

$$\cancel{= \frac{1}{\pi} \int_0^\infty \frac{\sin((x+1)w) + \sin((x-1)w)}{2w} \, dw}$$

$$\int_0^\infty \frac{\sin w \cos wx}{x}\, dx = \frac{\pi}{2} f(x)$$

$$= \begin{cases} \pi/2 & |x| < 1 \\ \pi/4 & x = 1 \\ 0 & |x| < 1 \end{cases}$$

at $x = 1$, f is discontinouse

$$\therefore f(1) = \frac{1}{2}\left\{ \lim_{x \to 1^-} f(x) + \lim_{x \to 1^+} f(x) \right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + 0\right] = \frac{\pi}{4}$$

② $f(x) = \begin{cases} 1-x^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$ | Homework

even function ∴ Second term 0

$$f(x) = \frac{2}{\pi}\int_0^\infty \int_{-\infty}^{\infty} \cos wx \cos sw (1-s^2)\, ds\, dw$$

$$= \frac{2}{\pi}\int_0^\infty \cos wx \int_{-1}^{1} \cos sw (1-s^2)$$

$$= \frac{2}{\pi}\int_0^\infty \cos wx \left[+ \frac{\sin sw}{w}(1-s^2) - 2s\frac{\cos sw}{w^2} + 2\frac{\sin sw}{w^3} \right]_{-1}^{1}$$

$$= \int_0^\infty \cos wx \Big[-2\cos w + 2\sin w + 2\cos w + 2\sin w \Big] = \int_0^\infty \cos wx \frac{\sin w}{w^3}$$

③ $f(x) = \frac{e^{-ax}}{x}$

$$f(x) = \frac{2}{\pi} \int_{w=0}^{\infty} \int_{s=0}^{\infty} f(s) \sin wx \sin ws \, dw \, ds$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin wx \int_0^{\infty} \frac{e^{-as}}{s} \sin ws \, dw$$

DUIS

$$I = \int_0^{\infty} \frac{e^{-as}}{s} \sin ws \, ds$$

$$\frac{dI}{dw} = \frac{d}{dw} \int_0^{\infty} \frac{e^{-as}}{s} \sin ws \, ds$$

$$= \int_0^{\infty} \frac{\partial}{\partial w} \left(\frac{e^{-as}}{s} \sin ws \, ds \right.$$

$$\frac{dI}{dw} = \int_0^{\infty} \frac{e^{-as}}{\cancel{s}} \cancel{s} \sin ws \, ds = \frac{a}{w^2 + a^2}$$

$$\therefore I = \int \frac{a^2}{w^2 + a^2} \, dw = \tan^{-1}(w/a) + c$$

at $w = 0$ $I = 0$ $\quad \tan^{-1}(w/a) + c = 0$

$\therefore c = 0$

After DUIS dont forget the value of constant.

$\therefore I(w) = \tan^{-1}\left(\frac{w}{a}\right)$

Substituting

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \sin wx \, \tan^{-1}(w/a) \, dw$$

Ans

(3)

$$f(x) \begin{cases} -e^{kx} & x < 0 \\ e^{-kx} & x > 0 \end{cases}$$

$$f(-x) \begin{cases} -e^{-kx} & x > 0 \\ e^{kx} & x < 0 \end{cases}$$

$f(-x) = -f(x)$ ∴ odd function

$$f(x) = \frac{1}{\pi}\int_{\omega=0}^{\infty}\int_{s=-\infty}^{\infty} f(s)\cos sx \cos \omega x \, ds \, d\omega$$

$$+\frac{1}{\pi}\int_{\omega=0}^{\infty}\int_{s=-\infty}^{\infty} f(s)\sin sx \sin \omega x \, ds \, d\omega$$

Since $f(x)$ is odd funct

$$= \frac{2}{\pi}\int_{\omega=0}^{\infty}\int_{s=0}^{\infty} f(s)\sin sx \sin \omega x \, ds \, d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \sin wx \int_0^\infty e^{-ks} \sin ws \; ds\, dw$$

$$= \frac{2}{\pi}\int_0^\infty \sin w x \; \frac{w}{k^2+w^2} \; dw$$

$$\therefore \int_0^\infty \sin ws \; \frac{w}{k^2+w^2} = \frac{\pi}{2} e^{-kx} \quad \begin{matrix} x>0 \\ k>0 \end{matrix}$$

④ $e^{-x} \cos x = f(x)$ → Neither odd nor even

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(s) \cos ws \cos wx \, ds \, dw$$

$$+ \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(s) \sin ws \sin wx \, ds \, dw$$

fourier cosine asked & not fourier sine

$$= \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty e^{-s} \cos s \cos ws \cos wx \, ds \, dw$$

$$+ \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty e^{-s} \cos s \sin ws \sin wx \, ds \, dw$$

$$= \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty e^{-s} \cos wx \left(\cos(w+1)s + \cos(w-1)s \right)$$

$$+ \frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty e^{-s} \sin wx \left(\sin(w+1)s + \sin(w-1)s \right)$$

for cosine Integral, consider only cos part

$$f(x) = \frac{1}{\pi} \int_0^\infty \cos wx \int_0^\infty e^{-s} \left(\cos(w+1)s + \cos(w-1)s \right)$$

$$= \frac{1}{\pi} \int_0^\infty \cos wx \left[\frac{1}{(w+1)^2+1} + \frac{1}{(w-1)^2+1} \right]$$

$$= \frac{1}{\pi} \int_0^\infty \cos wx \left[\frac{w^2-2w+2+w^2+2w+2}{(w^2+2)^2-(2w)^2} \right]$$

$$= \frac{2}{\pi} \int_0^\infty \cos wx \left(\frac{w^2+2}{w^4+4} \right)$$

Hence proved.

(3) $f(x) = \begin{cases} \sin x & 0 < x \leq \pi \\ 0 & x < 0, x > \pi \end{cases}$

Fourier Sine Integral

$$f(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(s) \sin wx \sin ws \, ds \, dw$$

$$= \frac{2}{\pi}\int_0^\infty \int_0^\pi \sin s \sin ws \sin wx \, ds \, dw$$

$$= \frac{2}{\pi}\int_0^\infty \sin wx \int_0^\pi \left(\cos (w-1)s - \cos(w+1)s\right)$$

$$= \frac{2}{\pi}\int_0^\infty \sin wx \left(\frac{\sin(w-1)\pi}{(w-1)} - \frac{\sin(w+1)\pi}{w+1}\right)$$

(6) $f(x) = e^{-kx}$ $(k>0)$ as Fourier Sine & Cosine Integral

$$\int_0^\infty \frac{\omega \sin \omega x}{k^2 + \omega^2}\, d\omega$$

$$\frac{2}{\pi} \int_0^\infty \int_0^\infty e^{-ks} \sin \omega x \sin \omega s \, ds\, d\omega$$

$$\frac{2}{\pi} \int_0^\infty \sin \omega x \int_0^\infty \sin \omega s\, e^{-ks} \, ds\, d\omega$$

$$\int_0^\infty \frac{\omega \sin \omega x}{\omega^2 + k^2} \, d\omega = \frac{\pi}{2} e^{-kx}$$

Similarly for Cosine

$$\frac{2}{\pi} \int_0^\infty \cos \omega x \int_0^\infty \cos \omega s \, e^{-ks} \, ds\, d\omega$$

$$= \frac{2}{\pi} \int_0^\infty \cos \omega x \; \frac{k}{\omega^2 + k^2} \, d\omega$$

$$f(x) \begin{cases} 1-x^2 & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

$$\frac{2}{\pi} \int_0^\infty \int_0^1 (1-x^2) \cos xw \cos sw \, dw \, ds$$

$$\frac{2}{\pi} \int_0^\infty \cos xw \left[(1-s^2) \frac{\sin sw}{+w} + 2s \frac{\cos sw}{-w^2} + 2 \frac{\sin sw}{+w^3} \right]_0^1 ds \, dw$$

$$\frac{2}{\pi} \int_0^\infty \cos xw \left[2 \frac{\cos w}{-w^2} + 2 \frac{\sin w}{w^3} \right]$$

at $x = 1/2$

$$1 - \frac{1}{4} = \frac{2}{\pi} \int_0^\infty \cos \frac{w}{2} \left[2 \frac{\cos w}{-w^2} + 2 \frac{\sin w}{w^3} \right)$$