Probability distribution

Random variable : function which assigns a real number with the outcome of an experement

Sample space : Total number of possibilities of an experement

eg 3 coin tossal

$$S = \{ (HHH), (TTH), (HHT), (THT), (HTH), (THH), (HTT), (TTT) \}$$

$$n(S) = 8$$

eg Let x be random variable defined as no of tails

$x = 0$	1	$P(x=0)$	$= 1/8$
$x = 1$	3	$P(x=1)$	$= 3/8$
$x = 2$	3	$P(x=2)$	$= 3/8$
$x = 3$	1	$P(x=3)$	$= 1/8$

If x takes infinite values but countably infinite then x is discrete random variable

If x takes uncountably infinite values then x is continuouse random variable

Probability distribution of discrete random variable

Let X be discrete random variables taking values $x_1, x_2, x_3, \ldots$

$$P(x_i) \geq 0 \quad \forall i$$

$$\sum_{i=1}^{n} P(x_i) = 1$$

Then P is called probability mass function & $(x_i, P(x_i))$ is called probability distribution of x

① Find probability distribution of number of heads when a fair coin is tossed 4 times

Total no of outcomes $= 2^4 = 16$

$S = \{ (H\ H\ H\ H)$
$(H\ H\ H\ T)$
$\vdots$
$(T\ T\ T\ T) \}$

$x : 0, 1, 2, 3, 4$

$P(x=0) = 1/16$

$P(x=1) = 4/16$

$P(x=2) = 6/16$

$P(x=3) = 4/16$

$P(x=4) = 1/16$

② X is some of numbers appearing on toss of two unbaised dice find PD

Total outcoms $= 6^2 = 36$

$$S = \{ (1,1), (1,2), \ldots (1,6)$$
$$(2,1), \ldots (2,6)$$
$$\vdots \qquad \vdots$$
$$(6,1), \ldots (6,6) \}$$

X :		P
2	(1+1)	1/36
3		2/36
4		3/36
5		4/36
6		5/36
7		6/36
8		5/36
9		4/36
10		3/36
11		2/36
12	(6+6)	1/36

$$\therefore P(X = \text{odd}) = P(x=3) + P(x=5) + \cdots$$

$$\cdots + P(x=11)$$

$$= \frac{2}{36} + \frac{4}{36} + \cdots + \frac{2}{36}$$

$$= 1/2$$

(5) Random variable X has p.D.

X :	0	1	2	3	4	5	6	7
$p(x=x_i)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

find (i) k

(ii) $P\left(\frac{1.5 < x < 4.5}{x > 2}\right)$

(iii) Smallest value of λ for which

$p(x \leq \lambda) > 1/2$

$\sum P_i(x = x_i) = 1$

$\therefore \quad 10k^2 + 9k = 1$

On solving $k = 1/10 = 0.1$

$\therefore p(x=x_i)$: 0 0.1 0.2 0.2 0.3 + ...

$$P(A/B) = \frac{n(A \cap B)}{n(B)}$$

$$P\left(\frac{1.5 < x < 4.5}{x > 2}\right)$$

$$= \frac{P(2 < x < 4.5)}{P(x > 2)}$$

$$= \frac{0.5}{0.7} = 5/7$$

$P(x \le \lambda) > \underline{0.5}$

$\lambda =$ 0 ✗

1 ✗

2 ✗

3 $\longrightarrow 0.5 \not> 0.5$

4 ✓

x:	1	2	3	4	5	6	7
$P(x=x)$	k	$2k$	$3k$	k^2	k^2+k	$2k^2$	$4k^2$

① Find k

② $P\left(\dfrac{x<5}{2<x<6}\right)$ $\qquad$ $P\left(\dfrac{x=4}{3<x<5}\right)$

$= 1/34$

④ $E(x)$ $\qquad$ ⑤ $Var(x)$

① $k = 1/8$ $\quad (\sum k = 1)$

② $\dfrac{P((x<3) \cap (2<x\leq 6))}{P(x\leq 6)}$

$= \dfrac{P(2<x<5)}{P(x\leq 6)} = \dfrac{3k+k^3}{4k+4k^2}$

$= 0.695$

④ $E(x) = \sum x_i p_i$

$= 49k^2 + 19k$

⑤ $Var(x) = E(x^2) - [E(x)]^2$

$= \sum x_i^2 p_i - (\sum(x_i p_i))^2$

$= \frac{149}{16}$

Distribution function of Discrete random variable

Consider $F(x_i) = P(x \leq x_i)$

$F(x_1) = P(x = x_1)$

$F(x_2) = P(x_1) + P(x_2)$

$F(x_n) = P(x_1) + P(x_2) + \dots$

Cumulative distribution

$x_i, F(x_i)$ = Cumulative Probability distribution

① $2P(x=1) = 3(P(x=2) = P(x=3)$

$= 5P(x=4)$

Find PDF & CDF

Let $= k$

$P(x=1) = \frac{k}{2}$

$P(x=2) = \frac{k}{3}$

$= k$

$= \frac{k}{5}$

$\sum P x_i = 1$

$\therefore k = 0.49$

$$F(x_1) = k/2 = 0.245$$

$$F(x_2) = \frac{k}{2} + \frac{k}{3} = 0.409$$

$$F(x_3) = 0.409 + k = 0.901$$

$$F(x_4) = 0.901 + \frac{k}{5} = 1$$

Probability density function

$\sum P_i = 1$

$P_i \geq 0$

A continuous function $y = f(x)$ is PDF if

① $f(x)$ is integrable

② $f(x) \geq 0$

③ $\int_a^b f(x) = 1$

$\forall x \in (a, b)$

④ $\int_\alpha^\beta f(x)\,dx = P(\alpha \leq x \leq \beta)$

$a < \alpha < \beta < b$

① $f(x) = k(x^2)$ $\quad 0 \leq x \leq 2$

find k, $P(0.2 \leq x \leq 0.5)$,

find $P(x \geq 3/4)$ given that $x \geq 1/2$

① $\int_0^2 kx^2 = 1 \therefore k\left(\frac{8}{3} - 0\right) = 1$

$\therefore k = 3/8$

② $\int_{0.2}^{0.5} k \cdot x^2 = \frac{3}{8}\left(\frac{(0.5)^3}{3} - \frac{(0.2)^3}{3}\right)$

$= 0.0146$

③ $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{P(x > 3/4)}{P(x \geq 1/2)}$

$$\Rightarrow \frac{\int_{0.75}^{2} k x^2}{\int_{0.5}^{2} k x^2} \doteq \frac{2.841}{2.9333}$$

$$= 0.962$$

CDF of Continuous Variable

If X is random var having PDF $f(x)$

then $X = P(x \leq x)$

$$= \int_{-\infty}^{x} f(t)\,dt$$

$(-\infty \leq x < \infty)$ is called CDF for

continuous random variable x

Properties of CDF

① $0 < X < 1$

② $x_1 \le x_2,\ X_1 \le X_2$

③ $F'(x) = f(x)$

④ $P(a \le x \le b) = F(b) - F(a)$

(2) CDF of continuous random variable X is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

Find PDF, $P(1/2 \le x \le 4/5)$

$f(x) = F'(x)$

$$\therefore f(x) = \begin{cases} 0 & x < 0 \\ 2x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

$$P(1/2 \le x \le 4/5) = F(4/5) - F(1/2)$$

$$= \left(\frac{4}{5}\right)^2 - (1/2)^2 = 0.39$$

③ find Distribution function

$$f(x) = \tfrac{1}{2}\, x^2 e^{-x} \qquad 0 \le x \le \infty$$

else 0

$\rightarrow$

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

$$= \int_{0}^{x} \tfrac{1}{2}\, t^2 e^{-t}\, dt$$

$$= t^2 \frac{e^{-t}}{-1} - 2t\, \underline{e^{-t}}$$

$$+ 2 \frac{e^{-t}}{-1}$$

$$= \Big(-e^{-t}\,(t^2 + 2t + 2)\Big)_0^x$$

$$= -e^{-x}(x^2 + 2x + 2)$$

$$+ 2 \quad + c$$

$$f(x) = \int_{-\infty}^{0} 0 \, dx + c$$

$$= c$$

$$F(x) \begin{cases} 1 - \dfrac{e^{-x}}{2}(x^2 + 2x + 2) \\ c \end{cases}$$

Expectation & variance

Discrete

$$E(x) = \sum p_i x_i$$

$$Var(x) = E(x^2) - \left[E(x)\right]^2$$

$$= \sum x_i^2 p_i - \left(\sum x_i p_i\right)^2$$

Continuous

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

$$V(x) = E(x^2) - \left[E(x)\right]^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x) - \left(\int_{-\infty}^{\infty} x f(x)\right)^2$$

① A fair coin is tossed till head appears find E(x)

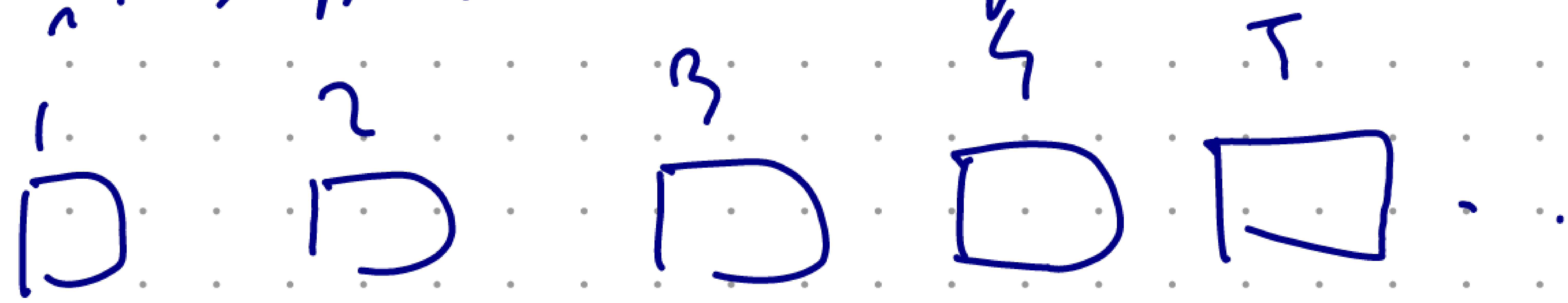
Event	H	TH	TTH . . .
P(x)	$1/2$	$1/2 \times 1/2$	$\frac{1}{2} \times \frac{1}{2} \times 1/2$
x: No of toss	1	2	3

$$\Sigma P(x) = 1 \times \frac{1}{2} + 2 \times \frac{1}{2^2} + 3 \times \frac{1}{2^3} = S$$

$$\frac{1}{2} S = \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots$$

$$S - \frac{1}{2} S = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

(a) $E(x)$, sum of & prod of Nos appear on throw of n dice



x_i : 1 2 3 4 5 6

p_i = $1/6$ $1/6$ $1/6$ . . .

$$E(x_i) = \sum p_i x_i = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \cdots$$

$$= 7/2$$

$$E(x_i) = 7/2$$

$$\sum E(x_i) = \frac{7}{2} n$$

$$E\left(\prod_{i=1}^{n} x_i\right) = \left(\frac{7}{2}\right)^n$$

③ Find expectant of failure (q) preceding first sucess (p)

	p	qp	qqp	...	
x (No of failure)	0	1	2	3	...
$P(x)$	p	qp	qqp	...	

$$E(x) = p + qp + q^2p + q^3p + \dots$$

$$= qp(1 + 2q + 3q^2 + \dots)$$

$$= qp \frac{1}{(1-q)^2} = \frac{q}{p}$$

(4) A cont random variable has PDF

$$f(x) = \begin{cases} k x^2 (1-x^3) & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

find k, $P(0 \le x \le 1/2)$

mean, Var

$$\int_0^1 f(x) = 1 \quad (\text{PDF})$$

$$= k \int_0^1 x^2 (1-x^3) = 1$$

$$1 = k \left[\frac{1}{6} (1-x^3)^2 \right]_0^1$$

$k = 6$

$$P(0 \le x \le 1/2) = \int_0^{1/2} f(x)\, dx$$

$$= \int_0^{1/2} 6(x^2 - x^5)\, dx$$

$$= \frac{15}{64}$$

$$\text{Mean} = E(X) = \int_0^1 f(x) \cdot x\, dx$$

$$= 6\left[\frac{x^4}{4} - \frac{x^7}{7}\right]_0^1$$

$$= 1$$

$$\text{Var} = (x) \quad \int_0^1 x^2 f(x) = 6\left[\frac{x^5}{5} - \frac{x^8}{8}\right]_0^1$$

$$= 6\left(\frac{1}{5} - \frac{1}{8}\right) = 0.45$$

$$V(x) = E(x^2) - (E(x))^2$$

$$= 0.45 - (0.65)^2$$

$$= 0.0167$$

PMF , CDF $\rightarrow$ discrete

PDF , CDF $\rightarrow$ cont