NLPP

There are 3 types of problems

1) NLPP with no constrainst
2) NLPP with linear equality constrainst
3) NLPP with linear inequality constrainst

1) No Constrainst

Object function is of form $Z = a_{11} x_1^2 + a_{22} x_2^2 + \dots a_{m} x_n^2$
$a_{12} x_1 x_2 + a_{13} x_1 x_3 + \dots - a_{1n} x_1 x_n$
$+ C_1 x_1 + C_2 x_2 + \dots + C_n x_n$

step

i) Put $\frac{\partial f}{\partial x_i} = 0 \quad 1 \le i \le n$

& find $x_0 (x_1 \, x_2 \dots\dots x_n)$

ii) Define a Hessian Matrix as follows

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \dots\dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \dots\dots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \dots\dots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Find H at X_0

If

i) All Principle minors of H at X_0 are +ve then minima at X_0

ii) If principle minors $D_1 \, D_3 \, D_5$ are Negative and $D_2 \, D_4 \, D_6$ are positive then there is A maxima at X_0

iii) In general if H is indefinate pattern at X_0, then X_0 saddle point (Neither Maxima nor minima

Max value is $f(X_0)$

① Optimize $z = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 8x_2 - 10x_3$

② Optimize $z = x_1 + 2x_3 + x_2x_3 - x_1^2 - x_2^2 - x_3^2$

z is $f(x_1, x_2, x_3)$

$\frac{\partial f}{\partial x_1} = 0 \Rightarrow 1 - 2x_1 = 0$

$\therefore x_1 = 1/2$

$\frac{\partial f}{\partial x_2} = 0 \Rightarrow x_3 - 2x_2 = 0 \quad$ —①

$\frac{\partial f}{\partial x_3} = 0 \Rightarrow 2 + x_2 - 2x_3 = 0 \quad$ —②

from ① & ② $\quad x_2 = 2/3$

$x_3 = 4/3$

$x_0 \equiv (1/2, 2/3, 4/3)$ is a stationary point

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \text{ At } x_0 (1/2, 2/3, 4/3)$$

Principle Minors are $D_1 = |-2|$ $\quad D_3 = |H|$

$D_2 = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix}$

$D_1 = -2 \quad D_2 = 4 \quad D_3 = -6 \quad$ D_1 & D_3 are -ve, D_2 +ve $\therefore$ Maxima

Hence Maximum value of Z is $f(1/3, 2/3, 4/3) = 1.588$

② NLPP with equality constraints

ⓐ With one equality constraint

Consider the following NLPP $\rightarrow z = f(x_1, x_2, x_3 \dots x_n)$

Subject to $g(x_1, x_2, x_3 \cdot x_n) = b \qquad x_1, x_2, x_3 \dots x_n \geq 0$

Step i) Let $g(x_1, x_2, x_3 \dots x_n) - b = h(x_1, x_2 \dots x_n)$

ii) Construct new function called Lagrangian function using multiplier called Lagrangian multiplier

① $L(x_1, x_2, x_3 \dots x_n, \lambda) = f(x_1, x_2, x_3 \dots x_n) - \lambda \cdot h(x_1, x_2 \dots x_n)$

Necessary condition for Maxima & Minima subject to $h(x_1, x_2 \dots x_n) = 0$ are

$$\frac{\partial L}{\partial x_i} = 0 = \frac{\partial f}{\partial x_i} - \lambda \frac{\partial h}{\partial x_i} \qquad 0 \leq i \leq \dots n$$

$$\frac{\partial L}{\partial \lambda} = 0 \dots h(x_1, x_2, x_3 \dots x_n)$$

Solving $n+1$ conditions find x_1, x_2, x_n & λ

Find value of determinant of order $n+1$ at $x_0 = (x_1, x_2 \dots \lambda)$

$$\Delta_{n+1} = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_n} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_n} \\ \frac{\partial h}{\partial x_2} & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \frac{\partial h}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_n \partial x_1} & \cdots & \cdots & \frac{\partial^2 f}{\partial x_n^2} - \lambda \frac{\partial^2 h}{\partial x_n^2} \end{vmatrix}$$

If signs of principle minor $\Delta_3, \Delta_4, \Delta_5 \dots$ are alternately positive & Negative x_0 Maxima & all are negative then minima. This is Lagrange Multiplier Method.

① Optimize $Z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$

subject to $x_1 + x_2 + x_3 = 20$ where $x_1, x_2, x_3 \geq 0$

$\rightarrow$

$$f(x_1, x_2 \dots x_3) = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$$

$$L(x_1, x_2, x_3, \lambda) = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100 - \lambda(x_1 + x_2 + x_3 - 20)$$

Lagranges conditions $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$, $\frac{\partial L}{\partial x_3} = 0$, $\frac{\partial L}{\partial \lambda} = 0$

$4x_1 + 10 - \lambda = 0$ — ①

$2x_2 + 8 - \lambda = 0$ — ②

$6x_3 + 6 - \lambda = 0$ — ③

$-(x_1 + x_2 + x_3 - 20) = 0$

$x_1 + x_2 + x_3 = 20$ — ④

Solving

$$\frac{10 - \lambda}{4} + \frac{8 - \lambda}{2} + \frac{6 - \lambda}{6} = -20$$

$\therefore \lambda =$ ~~[illegible]20~~ 30

$\therefore x_1 = 5$

$\therefore x_2 = 11$

$\therefore x_3 =$ ~~3~~ 4

$\therefore$ ~~[illegible]~~ $\therefore X_0 = (5, 11, 4)$ is a stationary point

$$\Delta_{n+1} = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ \frac{\partial h}{\partial x_1} & \frac{\partial f}{\partial x_1^2} - \lambda\frac{\partial h}{\partial x_1^2} & \frac{\partial f}{\partial x_1 x_2} - \lambda\frac{\partial h}{\partial x_1 x_2} & \frac{\partial f}{\partial x_1 x_3} - \lambda\frac{\partial h}{\partial x_1 x_3} \\ \frac{\partial h}{\partial x_2} & \frac{\partial f}{\partial x_1 x_2} - \lambda\frac{\partial h}{\partial x_1 x_2} & \frac{\partial f}{\partial x_2^2} - \lambda\frac{\partial h}{\partial x_2^2} & \frac{\partial f}{\partial x_2 x_3} - \lambda\frac{\partial h}{\partial x_2 x_3} \\ \frac{\partial h}{\partial x_3} & \frac{\partial f}{\partial x_3 x_1} - \lambda\frac{\partial h}{\partial x_3 x_1} & \frac{\partial f}{\partial x_3 x_2} - \lambda\frac{\partial h}{\partial x_3 x_2} & \frac{\partial f}{\partial x_3^2} - \lambda\frac{\partial h}{\partial x_3^2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 6 \end{vmatrix}$$

$\Delta_3 = -6$

$\Delta_4 =$ ~~[illegible]~~ -44

$\therefore$ Minimum

NLPP with one inequality Constraint

Consider : Maximize $z = f(x_1, x_2 \dots x_n)$ s.t. $g(x_1, x_2 \dots x_n) \le b$ $x_1, x_2 \dots x_n > 0$

Introducing slack variable s^2 we get constraint as $g(x_1, x_2 \dots x_n) - b \Rightarrow h(x_1, x_2 \dots x_n) + s^2 = 0$

There are $n+1$ variables & one equality constraint.

Construct lagrangian function $L(x_1, x_2 \dots x_n, \lambda, s) = f(x_1, x_2, x_3) - \lambda[h(x_1, x_2 \dots x_n) + s^2]$

Necessary condition $\rightarrow$ $\frac{\partial f}{\partial x_i} - \lambda \frac{\partial h}{\partial x_i} = 0 \quad \forall i$ $h(x_1, x_2, x_n) + s^2 = 0$, $\lambda s = 0$

$\therefore s = 0$ or $\lambda = 0$

If $s = 0$, then $h(x_1, x_2 \dots x_n)$ must be 0

Necessary conditions for Maximization are $\frac{\partial f}{\partial x_i} - \lambda \frac{\partial h}{\partial x_i} = 0 \quad \forall i$ $\therefore$ n conditions

$\lambda h(x_1, x_2, x_n) = 0$ $\lambda \ge 0$, $h(x_1, x_2, x_n) \le 0$

These are Kuhn-tucker conditions for Maximization

for Minimization also $\cancel{\lambda \ge 0}$ $h(x_1, x_2, x_n) \ge 0$

① Solve: Maximize $Z = 2x_1^2 - 7x_2^2 + 12x_1x_2$ subject to $2x_1 + 5x_2 \le 98$
$x_1, x_2 \ge 0$

$\rightarrow$ $f(x_1, x_2) = 2x_1^2 - 7x_2^2 + 12x_1x_2$

$h(x_1, x_2) = 2x_1 + 5x - 98$

$\frac{\partial f}{\partial x_1} - \lambda\left(\frac{\partial h}{\partial x_1}\right) = 0 = 4x_1 - \cancel{12}x_2 - \lambda(2)$ ①

$\frac{\partial f}{\partial x_2} - \lambda\left(\frac{\partial h}{\partial x_2}\right) = 0 = -14x_2 + 12x_1 - \lambda(5)$ ②

$\lambda h(x_1, x_2 \dots x_n) = 0$

$\therefore$ $(2x_1 + 5x - 98)\lambda = 0$

Case 1 if $\lambda = 0$, solve ① & ① $x_1 = 0$ $x_2 = 0$

$\therefore h(x_1, x_2, x_3) = -98$ But if $x_1 = 0$, $x_2 = 0$, $z = 0$ $\therefore$ Reject

Case 2 if $h(x_1, x_2, x_3) = 0$ $2x_1 + 5x_2 - 98 = 0$ - ③ solve ① ② ③

~~we~~ $\lambda = 100$

$x_1 = 44$

$x_2 = 2$

} Acceptable values as $\lambda \ge 0$

Values satisfy all necessary conditions

$\therefore X = (44, 2)$ $\lambda = 100$

$\therefore$ Maximum $x_1 = 44$, $x_2 = 2$ $\therefore Z = 4900$

(5) Use Kuhn Tucker to solve the NLPP

Maximize $z = 8x_1 + 10x_2 - x_1^2 - x_2^2$, $3x_1 + 2x_2 \leq 6$

$f(x) = 8x_1 + 10x_2 - x_1^2 - x_2^2$

$h(x) = 3x_1 + 2x_2 - 6$

$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 8 - 2x_1 - \lambda(3) = 0$ (1)

$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 10 - 2x_2 - \lambda(2) = 0$ (2)

$\lambda h = 0$

$\lambda = 0$ $8 - 2x_1 = 0$, $10 - 2x_2 = 0$ $\therefore$ $x_1 = 4$, $x_2 = 5$

If $x_1 = 4$, $x_2 = 5$, $3x_1 + 2x_2 > 6$ .. reject

/

$h = 0$ $3x_1 + 2x_2 - 6 = 0$ - (3)

from (1) (2) (3)

$\lambda = 2.46$

$x_1 = 0.307$

$x_2 = 2.538$

$z = \frac{277}{13}$