LPP

(I) Canonical form :→ Max $Z = \sum_{i=1}^{n} c_i x_i$ Subject To $\sum_{j=1}^{n} a_{ij} x_j \le b_i$ $i = 1, 2, 3 \ldots n$

$x_{ij} \ge 0$ $j = 1, 2, \ldots n$

characteristics are as follows →

(a) If Objective function is of minimization type $\Sigma a x \ge b_i$

(b) maximization $\Sigma a x \le b_i$

(c) If the constraint are of equality type then write in form of inequality

$a_1 x_1 + a_2 x_2 = b_1$ then $a_1 x_1 + a_2 x_2 \le b_1$ & $a_2 x_2 + a_1 x_1 \ge b_1$

(d) If any variable is unrestricted eg x_j then write $x_j' - x_j'' = x_j$ where $x_j', x_j'' \ge 0$

(II) Standard form :→ In standard form introduce slack variables & express objective function (z) as well as all constraints in form of equalities

$$Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n + 0 s_1 + 0 s_2 + \ldots \; 0 s_m$$

Subject to $a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n + s_1 + 0 s_2 + 0 s_3 + \ldots 0 s_j = b_i$

ie $\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i$

All the conditions are expressed in equations using slack variables

Right hand side of all constraint must be non-negative (ie $\le$ else multiply by -1)

~~Opt~~ z must be of maximization

$x_1, x_2, \ldots x_n, s_1, \ldots s_m \ge 0$

$\longrightarrow n$

$\downarrow m$

① $z = -3x_1 + 2x_2 - x_3$

$x_1 - 3x_2 + 2x_3 \geq -6$

$3x_1 + 4x_3 \leq 3$

$-3x_1 + 5x_2 \leq 4$

$x_1, x_2 \geq 0$, x_3 unrestricted

Since x_3 is unrestricted

$x_3 = x_3' - x_3''$ $\quad x_3, x_3', x_3'' \geq 0$

Maximize $z' = 3x_1 - 2x_2 + x_3 + 0S_1 + 0S_2 + 0S_3$

subject to

$-x_1 + 3x_2 - 2x_3 + S_1 + 0S_2 + 0S_3 = -6$

$3x_1 + 0x_2 + 4x_3 + 0S_1 + S_2 + 0S_3 = 3$

$-3x_1 + 5x_2 + 0x_3 + 0S_1 + 0S_2 + S_3 = 4$

$x_i, S_i \geq 0$

→ Definations →

i) set of values which satisfy constraint is solution

ii) Solution of LPP which satisfy non-negativity condition is feasable solution

iii) feasable solution which optimizes ~~feasable~~ objective function is called optimal feasable solution.

iv) Basic ~~feasable~~ solution is solution obtained by putting any n variables to zero & finding values of remaining m variables then variables which are not equal to zero are called basic ~~solution~~ variables & other n zero valued are non basic variables.

No of Basic solution is ${}^{n+m}C_m$

v) Basic feasable solution → satisfy non negative condition & basic solution (BFS)

Two types of BFS

i) Non degenerate BFS → if in basic feasable solution, all m values of basic ~~solution~~ variable are +ve, non zero then it is non degenerate basic feasable solution

ii) Degenerate BFS → if one or more m basic variables are 0 then degenerate BFS

1. Which of the following are degenerate, basic, etc. solution? optimal

Maximize $z = x_1 + 3x_2 + 3x_3$

$$x_1 + 2x_2 + 3x_3 = 4$$

$$2x_1 + 3x_2 + 5x_3 = 7$$

→ No of Variables 3, No of constraint 2

∴ No of basic solution can be obtained by $3-2=1$ variable to 0

(1) No of Basic Solution (2) NonBasic equal to 0 (3) Basic Variables (4) Equation (5) Values

(6) Is feasible (7) Degenerate (8) Value of Z (9) Is optimal

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	$x_3=0$	$x_1 x_2$	$x_1+2x_2=4$, $2x_1+3x_2=7$	$x_1=2$, $x_2=1$	yes	No $x_1\neq0$ $x_2\neq0$	5
2	$x_2=0$	$x_1 x_3$	$x_1+3x_3=4$, $2x_1+5x_3=7$	$x_1=1$, $x_3=1$	yes	No $x_1\neq0$ $x_3\neq0$	4
3	$x_1=0$	$x_2 x_3$	$2x_2+3x_3=4$, $3x_2+5x_3=7$	$x_2=-1$, $x_3=2$	No $x_2<0$	No $x_2\neq0$ $x_3\neq0$	2

Basic Optimal Solution $x_3=0$

Simplex Method

step 1. Write in standard form
step 2. Write initial basic solution
step 3. Make initial simplex table
step 4. ~~The~~ Are all entries in Z row non negative?

i) If Yes then current solution is optimal basic feasable solution. stop here if Z has more than S zeros, else continue

ii) If No select most negative or smallest entry in Z row. The corresponding column is key column or pivot column & the corresponding ~~(x)~~ variable enters in the basic

step 5. Obtain replacement ratio by dividing solution column by key column

step 6. Are all ratios infinite/And/or negative?

i) If Yes LPP has unbounded solution. Stop Here

ii) If No select minimum finite non Negative ratio Corresponding row is called key row & corresponding variable leaves the basic. In case of a tie select any value.

If any ratio is zero then consider that ratio which has denominator positive in pivot column

step 7. ~~Make~~ Mark the key element as intersection of key row & key column

step 8. Make New simplex method by elementary row transformation as

i) Make key element 1 by dividing key row by key element

ii) Make all other elements of key column zero.

step 9. Go to step 4 & continue

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Batch: ____ Roll No.: ____
Name: ____
Course: ____
Experiment / assignment / tutorial No. ____
Grade: ____ Signature of the Faculty with date

① Maximize $Z = 3x_1 + 2x_2$ subject to $3x_1 + 2x_2 \le 18$
$0 \le x_1 \le 4$
$0 \le x_2 \le 6$

→ $Z - 3x_1 - 2x_2 + 0S_1 + 0S_2 + 0S_3 = 0$

S.T.

$3x_1 + 2x_2 + S_1 \underset{+0S_2+0S_3}{=} 18$

$x_1 + 0S_1 + S_2 + 0S_3 = 4$

$x_2 + 0S_1 + 0S_2 + S_3 = 6$

$x_1, x_2, S_1 S_2 S_3 \ge 0$

Initial basic feasible solution Let $x_1 = 0$, $x_2 = 0$
$S_1 = 18$, $S_2 = 4$, $S_3 = 6$
$Z = 0$

Iteration	Basic Var	x_1	x_2	S_1	S_2	S_3	RHS Sol.	Ratio
0	Z	-3	-2	0	0	0	0	0/-3 = 0
	S_1	3	2	1	0	0	18	18/3 = 6
	S_2	1	0	0	1	0	4	4/1 = 4 →
	S_3	0	1	0	0	1	6	6/0 = ∞

All entries Not Non Negative. Choose Most small entry
∴ x_1 is entering basis, column is key column
∴ Write ratio

All are Not Unbounded ∴ ~~row~~ S_2 select smaller i.e. 4 (Not 0)
S_2 leaves key element is intersection of key row & key column i.e. 4

1	Z	0	-2	0	3	0	12	
	S_1	2	2	-1	-3	0	6	6/2 = 3
	x_1	1	0	0	1	0	4	4/0 = ∞
	S_3	0	1	0	0	1	6	6/1 = 6

Key element already 1. Make column rows 0 $R_1 \to 3R_2 + R_1$
$R_2 \to R_2 - 3R_1$
$R_1 - R_3$

Repeat procedure. x_2 enters S_1 leaves.
key element 2 ∴ Need to make as one.

$R_2 \to R_2/2$ $R_1 \to R_1 + 2R_2$ $R_3 \to R_3 - R_2$

z	0	0	1	0	0	18
x_2	0	1	$1/2$	$-3/2$	0	3
x_1	1	0	0	1	0	4
s_3	0	0	$-1/2$	$3/2$	1	3

All entries Non Negative hence achieved optimal solution
basic feasible
$x_1 = 4$
$x_2 = 3$
$S_1 = 0$
$S_2 = 0$
$S_3 = 3$

Why is S_2 extra?
cus S_2 is Not in variable

Entry for non basic variable S_2 is 0. Hence LPP has alternate solution.
The alternate optimum solution is obtained by taking S_2 as entering variable
∴ S_2 enters

							ratio
z	0	0	1	0	0	18	
x_2	0	1	$1/2$	$-3/2$	0	3	$3/{-3/2} = -2$
x_1	1	0	0	01	0	4	$4/1 = 4$
s_3	0	0	$-1/2$	$+3/2$	1	3	$3/{3/2} = 2$

All entries not non-negative. Minimum finite is +2 ∴ S_3 leaves

$R_3 \to 2/3 R_3$ to make key element 1
$R_1 \to \frac{3}{2} R_1 + R_3$ $R_2 \to R_2 - R_3$ to make key column 0

	x_1	x_2	S_1	S_2	S_3	RHS	ratio
z	0	0	1	0	0	18	
x_2	0	1	0	0	1	6	
x_1	1	0	1/3	0	-2/3	2	
S_2	0	0	-1/3	1	2/3	2	

All entries in z row Non Negative

Here Values

$x_1 = 2$ $\quad$ $S_1 = 0$

$x_2 = 6$ $\quad$ $S_3 = 0$

$S_2 = 2$

$\therefore$ ∞ No of nonbasic optimal feasible solution are $X = \lambda x_1 + (1-\lambda) x_2$ where $\lambda \in (0,1)$

X

similarly

$$X_1 = \begin{bmatrix} x_1 = 4 \\ x_2 = 3 \\ x_3 = 6 \\ S_1 = 0 \\ S_2 = 0 \\ S_3 = 3 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 2 \\ 6 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} 2+2\lambda \\ 6-3\lambda \\ 0 \\ 2-2\lambda \\ 3\lambda \end{bmatrix}$$

Since X has only 1 zero $\therefore$ Non basic solution As it does not have m zeros i.e 2 zeros

② Minimize $Z = x_1 - 3x_2 + 3x_3$ S.t. $3x_1 - x_2 + 2x_3 \le 7$

$2x_1 + 4x_2 \ge -12$

$-4x_1 + 3x_2 + 8x_3 \le 10$

standard form $3x_1 - x_2 + 2x_3 + S_1 + 0S_2 + 0S_3 = 7$

$-2x_1 + -4x_2 - 0x_3 + 0S_1 + S_2 + 0S_3 = 12$

$-4x_1 + 3x_2 + 8x_3 + 0S_1 + 0S_2 + S_3 = 10$

(Max Z') $\quad Z' + x_1 + -3x_2 + 3x_3 + 0S_1 + 0S_2 + 0S_3 = 0$

$x_i, S_i \ge 0$

Initial basic feasible solution $x_1 = 0$ $x_2 = 0$ $x_3 = 0$

$\therefore S_1 = 7, \ S_2 = 12, \ S_3 = 10$

zero value

① Basic	x_1	x_2	x_3	S_1	S_2	S_3	RHS	ratio
Z	1	-3	+3	0	0	0	0	-0
S_1	3	-1	2	1	0	0	7	-7
S_2	-2	-4	0	0	1	0	12	-3
S_3	-4	3	8	0	0	1	10	$\boxed{10/3}$

key

x_2 enter
S_3 leave

Z	-3	0	11	0	0	1	10	37/5
S_1	5/3	0	14/3	0	0	1/3	37/3	-37/22
S_3	-22/3	0	32/3	0	1	4/3	46/3	-46/22
x_2	-4/3	1	8/3	0	0	1/3	10/3	-10/4

x_1 enter
S_1 leave

Z	0	0	97/5	9/5	0	8/5	143/5
x_1	1	0	14/5	3/5	0	1/3	37/5
S_2	0	0	468/15	66/15	1	164/15	1062/15
x_2	0	1	96/15	12/15	0	9/15	174/15

Thus we have optimal feasible solution as all rows non negative

$x_1 = 37/5$ $\quad x_3 = 0$ $\quad S_2 = 1062/15$ $\quad Z_{max} = 143/5$
$x_2 = 174/15$ $\quad S_1 = 0$ $\quad S_3 = 0$

$Z_{min} = -Z_{max}$
$\therefore Z_{min} = -143/5$

S.F.

③ $z = 2x_1 - 2x_2 + 4x_3 - 5x_4 + \cancel{0 S_1} \cancel{+ 0 S_2}$

③ $x_1 + 2x_2 + 3x_3 + \cancel{S_1} \cancel{+ 0 S_2} = 7$

$3x_1 + 4x_2 + 6x_3 + \cancel{0 S_1} \cancel{+ 1 S_2} = 15$

No of variables = 3 No of constraint = 2

∴ No of basic solution canbe obtained by 3 − 2 = 1 Variable to 0

$x_1 = 0$, $\left.\begin{array}{l} 2x_2 + 3x_3 = 7 \\ 4x_2 + 6x_3 = 13 \end{array}\right\} \begin{array}{l} x_2 = \\ x_3 = \end{array}$ Math erro ie. unbounded solution

$x_2 = 0$, $\left.\begin{array}{l} x_1 + 3x_3 = 7 \\ 3x_1 + 6x_3 = 15 \end{array}\right\} \begin{array}{l} x_1 = 1 \\ x_3 = 2 \end{array}$

$x_3 = 0$, $\left.\begin{array}{l} x_1 + 2x_2 = 7 \\ 3x_1 + 4x_2 = 15 \end{array}\right\} \begin{array}{l} x_1 = 1 \\ x_2 = 3 \end{array}$

~~No of Variables =~~

④ Max $z = 107x_1 + x_2 + 2x_3$

$14x_1 + x_2 - 6x_3 + 3x_4 = 7$

$16x_1 + 1/2\, x_2 - 6x_3 \le 5$

$3x_1 - x_2 - x_3 \le 0$

$x_4, x_1, x_2, x_3 \ge 0$

S.F. → $z - 107x_1 - x_2 - 2x_3 = 0$

$14/3\, x_1 + 1/3\, x_2 - 2x_3 + 1x_4 = 7/3$ Here x_4 is slack variable

$16x_1 + 1/2\, x_2 - 6x_3 + S_1 = 5$

$3x_1 - x_2 - x_3 + S_2 = 0$

$x_i, S_i > 0$

Initial basic feasable solution $x_1, x_2, x_3 = 0$

$x_4 = 7/3$

$S_1 = 5$

$S_2 = 0$

Iteration	Basis	x_1	x_2	x_3	x_4	s_1	s_2	~~s_3~~	RHS	Ratio
0	Z	-107	-1	-2	0	0	0		0	–
x_1 enters	x_4	$14/3$	$1/3$	-2	1	0	0		$7/3$	$1/2$
s_2 leaves	~~x_1~~ s_1	16	$1/2$	-6	0	1	0		5	$5/16$
	s_2	3	-1	-1	0	0	1		0	0
	Z	0	$-110/3$	$-113/3$	0	0	$107/3$		0	–
x_3 enter	x_4	0	$17/9$	$-4/9$	1	0	$-14/9$		$7/3$	[illegible]
No leaves	s_1	0	$35/6$	$-2/3$	0	1	$-16/3$		5	[illegible]
	x_1	1	$-1/3$	$-1/3$	0	0	$1/3$		0	[illegible]

All ratios $-ve$ $\therefore$ unbounded solution

No Variable will leave basis, x_3 cannot enter hence unbounded solution

Big M (Penalty Method)

① Maximize $Z = 6x_1 + 4x_2$

$$2x_1 + 3x_2 \leq 30$$
$$3x_1 + 2x_2 \leq 24$$
$$x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

Is the solution unique? if not find another solution

$\rightarrow$ Max $Z = 6x_1 + 4x_2 - MA_3$

$$2x_1 + 3x_2 + S_1 = 30$$
$$3x_1 + 2x_2 + S_2 = 24$$
$$x_1 + x_2 - S_3 + A_3 = 3$$
$$x_1, x_2, x_3, S_1, S_2, S_3, A_3 \geq 0$$

$MA_3 \rightarrow$ Penalty

Remove Penalty by M ③ + ①

$\therefore Z = (6+M)x_1 + (4+M)x_2 \cancel{+ S_3}$

$-MS_3 - 3M$

i.e. $Z - (6+M)x_1 - (4+M)x_2 + MS_3 = -3M$

Initial BFS Put $6-3=3$ variable 0

$x_1 = 0 \quad \therefore S_1 = 30$

$x_2 = 0 \quad \therefore S_2 = 24$

$x_3 = 0 \quad \therefore A_3 = 3$

M is very big No

Basic	x_1	x_2	S_1	S_2	S_3	A_3	RHS	ratio
Z	$-6-M$	$-4-M$	0	0	M	0	$-3M$	
S_1	2	3	1	0	0	0	30	15
S_2	3	2	0	1	0	0	24	8
← A_3	1	1	0	0	-1	1	3	3

x_1 enters
A_3 leaves
∴ leaves

Z	0	2	0	0	-6	—	18	~~12~~
S_1	0	1	1	0	2	—	24	12
S_2	0	-1	0	1	3	~~3~~ —	15	+5
x_1	1	1	0	0	-1	~~1~~ —	3	-3

S_3 enters
S_2 leaves

Z	0	0	0	2	0	—	48	~~48~~
S_1	0	5/3	1	$-2/3$	0	—	14	~~+42/5~~
S_3	0	$-1/3$	0	1/3	1	—	15/3 = 5	~~-15~~
x_1	1	2/3	0	1/3	0	—	8	12

Optimal solution as all solutions are non-negative.

$x_1 = 8$ $\quad S_1 = 14$ $\quad Z_{max} = 48$

$x_2 = 0$ $\quad S_2 = 0$

~~x_3~~ $\quad S_3 = 5$

There are 3 basic solution, $S_1\, x_1\, S_3$ must be 0 in Z row but x_2 is also 0
Hence there is an alternating solution. x_2 enters ∴ S_1 leaves
∴ Infinitely many solutions $\quad x_1 = \begin{bmatrix} 8 \\ 0 \\ 14 \\ 0 \\ 5 \end{bmatrix}$

Z	0	0	0	2	0	—	48
x_2	0	1	3/5	$-2/5$	0	—	42/5
S_3	0	0	1/5	1/5	1	—	39/5
x_1	1	0	$-2/5$	3/5	0	—	12/5

Alternate solution is $X_2 = \begin{bmatrix} 12/5 \\ 42/5 \\ 0 \\ 0 \\ 39/5 \end{bmatrix}$

$X = \lambda X_1 + (1-\lambda) X_2 = \begin{bmatrix} 8\lambda + (1-\lambda)12/5 \\ (1-\lambda)42/5 \\ 14\lambda \\ 0 \\ 5\lambda + (1-\lambda)39/5 \end{bmatrix}$ Non Basic solution
as $n-m = 6-3 = 3$ zeros not present

② Minimize $z = x_1 + 2x_2 + x_3$

$x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 \leq 1$

$\frac{3}{2}x_1 + 2x_2 + x_3 \geq 8$

$x_1, x_2, x_3 \geq 0$

S.F. →

Max $z' = -x_1 - 2x_2 - x_3 - MA_2$

Sub. to $x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 + S_1 = 1$

$\frac{3}{2}x_1 + 2x_2 + x_3 + - S_2 + A_2 = 8$

$x_1, x_2, x_3, S_1, S_2, A_2 \geq 0$

Max $z \to z'$

$z' - (-1 + \frac{3}{2}M)x_1 - (-2 + 2M)x_2 - (1 - M)x_3 - MS_2 - 8M$

Initial Basic solution $6 - 2 = 4$ Variable to 0

$x_1, x_2, x_3, S_2 = 0$

$\therefore S_1 = 1 \quad A_2 = 8$

Basic	x_1	x_2	x_3	S_1	S_2	A_2	RHS	ratio
z'	$1 - \frac{3}{2}M$	$2 - 2M$	$1 - M$	0	M	0	$-8M$	
S_1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	0	0	1	2
A_2	$\frac{3}{2}$	2	1	0	-1	1	8	4

x_2 enter, S_1 leave

Basic	x_1	x_2	x_3	S_1	S_2	A_2	RHS
z	$-3 + \frac{5}{2}M$	0	$-1 + M$	$-4 + 4M$	M	0	$-4 - 4M$
x_2	2	1	1	2	0	0	2
A_2	$-\frac{5}{2}$	0	$\cancel{0} -1$	-4	-1	1	4

All entries Non Negative but A_2 remains in basis with positive value

$A_2 > 0 \quad \therefore$ Pseudo solution