# Chapter -3 Greedy Method

- ✓ **General Method**
- ✓ **Knapsack Problem**
- ✓ **Minimum Cost Spanning Tree –Kruskal and primal Algo**
- ✓ **Single Source Shorted Path**

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## General Method :

- An algorithm which always takes the best immediate, or local, solution while finding an answer. Greedy algorithms will always find the overall, or globally, *optimal solution* for some *optimization problems*, but may find less-than-optimal solutions for some instances of other problems.

### Example of Greedy Method

- *Prim's algorithm* and *Kruskal's algorithm* are greedy algorithms which find the globally optimal solution, a *minimum spanning tree*. In contrast, any known greedy algorithm to find an *Euler cycle* might not find the shortest path, that is, a solution to the *traveling salesman* problem.

- *Dijkstra's algorithm* for finding *shortest paths* is another example of a greedy algorithm which finds an optimal solution.

### Features

- Start with a solution to a small sub problem
- Build up to a solution to the whole problem
- Make choices that look good in the short term
- Disadvantage: Greedy algorithms don't always work ( Short term solutions can be disastrous in the long term). Hard to prove correct
- Advantage: Greedy algorithm work fast when they work. Simple algorithm, easy to implement

### Greedy Algorithm

```
Procedure GREEDY(A,n)
// A(1:n) contains the n inputs//
        solution ← ϕ //initialize the solution to empty//
for i ← 1 to n do
        x ← SELECT(A)
        if FEASIBLE(solution,x)
                then solution ← UNION(solution,x)
        end if
repeat
        return(solution)
end GREEDY
```

## Knapsack Problem

Greedy method is best suited to solve more complex problems such as a knapsack problem. In a knapsack problem there is a knapsack or a container of capacity M n items where, each

item I is of weight wi and is associated with a profit pi. The problem of knapsack is to fill the available items into the knapsack so that the knapsack gets filled up and yields a maximum profit. If a fraction xi of object i is placed into the knapsack, then a profit pi *xi is earned. The constrain is that all chosen objects should sum up to M.
OR

- **Problem definition**
  - Given n objects and a knapsack where object i has a weight $w_i$ and the knapsack has a capacity m
  - If a fraction $x_i$ of object i placed into knapsack, a profit $p_i x_i$ is earned
    The objective is to obtain a filling of knapsack maximizing the total profit
- **Problem formulation (Formula 4.1-4.3)**

$$maximize \sum_{1 \le i \le n} p_i x_i \qquad (4.1)$$

$$subject\ to \sum_{1 \le i \le n} w_i x_i \le m \qquad (4.2)$$

$$and\ 0 \le x_i \le 1, \quad 1 \le i \le n \qquad (4.3)$$

-
- **A *feasible solution* is any set satisfying (4.2) and (4.3)**
- **An *optimal solution* is a feasible solution for which (4.1) is maximized**
- Greedy selection policy: three natural possibilities
- Policy 1: Choose the lightest remaining item, and take as much of it as can fit.
- Policy 2: Choose the most profitable remaining item, and take as much of it as can fit.
- Policy 3: Choose the item with the highest price per unit weight (P[i]/W[i]), and take as much of it as can fit. =➔Policy 3 always gives an optimal solution.

**Illustration**
Consider a knapsack problem of finding the optimal solution where, M=15, (p1,p2,p3…p7) = (10, 5, 15, 7, 6, 18, 3) and (w1,w2, …., w7) = (2, 3, 5, 7, 1, 4, 1).In order to find the solution, one can follow three different strategies.
**Strategy 1**: non-increasing profit values(Largest Profit)
Let (a,b,c,d,e,f,g) represent the items with profit (10,5,15,7,6,18,3) then the sequence of objects with non increasing profit is (f,c,a,d,e,b,g).

| Item chosen for inclusion | Quantity of item included | Remaining space in M | $P_iX_i$ |
|---|---|---|---|
| f | 1 full unit | 15-4=11 | 18*1=18 |
| C | 1 full unit | 11-5=6 | 15*1=15 |
| A | 1 full unit | 6-2=4 | 10*1=10 |
| d | 4/7 unit | 4-4=0 | 4/7*7=04 |

Profit= 47 units The solution set is (1, 0, 1, 4/7, 0, 1, 0).
**Strategy 2**: non-decreasing weights(Smallest Wight)
The sequence of objects with non-decreasing weights is (e,g,a,b,f,c,d).

| Item chosen for inclusion | Quantity of item included | Remaining space in M | $P_iX_I$ |
|---|---|---|---|
| E | 1 full unit | 15-1=14 | 6*1=6 |
| G | 1 full unit | 14-1=13 | 3*1=3 |
| A | 1 full unit | 13-2=11 | 10*1=10 |
| b | 1 full unit | 11-3=8 | 5*1=05 |
| f | 1 full unit | 8-4=4 | 18*1=18 |
| c | 4/5 unit | 4-4=0 | 4/5*15=12 |

**Profit= 54 units The solution set is (1,1,4/5,0,1,1,1).**

**Strategy 3**: maximum profit per unit of capacity used (This means that the objects are considered in decreasing order of the ratio Pi/wI)
a: P1/w1 =10/2 = 5 b: P2/w2 =5/3=1.66 c: P3/w3 =15/5 = 3 d: P4/w4 =7/7=1 e: P5/w5 =6/1=6 f: P6/w6 =18/4 = 4.5 g: P7/w7 =3/1=3
Hence, the sequence is (e, a, f, c, g, b, d)

| Item chosen for inclusion | Quantity of item included | Remaining space in M | $P_iX_I$ |
|---|---|---|---|
| E | 1 full unit | 15-1=14 | 6*1=6 |
| A | 1 full unit | 14-2=12 | 10*1=10 |
| F | 1 full unit | 12-4=8 | 18*1=18 |
| C | 1 full unit | 8-5=3 | 15*1=15 |
| g | 1 full unit | 3-1=2 | 3*1=3 |
| b | 2/3 unit | 2-2=0 | 2/3*5=3.33 |

**Profit= 55.33 units The solution set is (1,2/3,1,0,1,1,1).**

**Example2.** n = 3, M = 20, $(p_1, p_2, p_3) = (25, 24, 15)$ $(w_1, w_2, w_3) = (18, 15, 10)$

Sol: $p_1/w_1 = 25/18 = 1.32$

$p_2/w_2 = 24/15 = 1.6$

$p_3/w_3 = 15/10 = 1.5$

Optimal solution: $x_1 = 0$, $x_2 = 1$, $x_3 = 1/2$

total profit = 24 + 7.5 = 31.5

**Algorithm GREEDY_KNAPSACK (P,W,M,X,n)**

//P(1:n) and W(1:n) contain the profit and weights respectively of the n objects ordered so that P(i)/W(i) >=P(i+1)/W(i+1).M is the knapsack size and X(1:n) is the solution vector

```
Real P(1:n),W(1:n),X(1:n) ,M, cu;
Integer I,n;
X← 0              //initialize solution to Zero
Cu←M          // cu is remaining knapsack capacity
for  i← 1 to n do
      if(W(i)  >cu )  then  exit  endif
      X(i)←1
     Cu← cu-W(i)
Repeat

If (i<=n ) then  X(i) ← cu/W(i) endif

End
```

**Time complexity**

- Sorting: O(n log n) using fast sorting algorithm like merge sort
- GreedyKnapsack: O(n)
- So, total time is O(n log n)

**Minimum Cost Spanning Tree –Kruskal and Prim's Algo**

## Tree:

– A tree is a graph with the followingproperties:
• The graph is connected (can go from anywhere to anywhere)
• There are no cycle

## Spanning Tree

• A spanning tree is a tree that spans all the nodes  Thus, if there are n nodes in the network, a tree spanning this network will have n-1 arcs that go through all the nodes.

## Minimum Spanning tree

• It is the shortest spanning tree (length of a tree is equal to the sum of the length of the arcs on the tree).
• Very important
– Practice (eg. communication)
– Theory (eg. basis)
– Algorithms (as a sub problem)

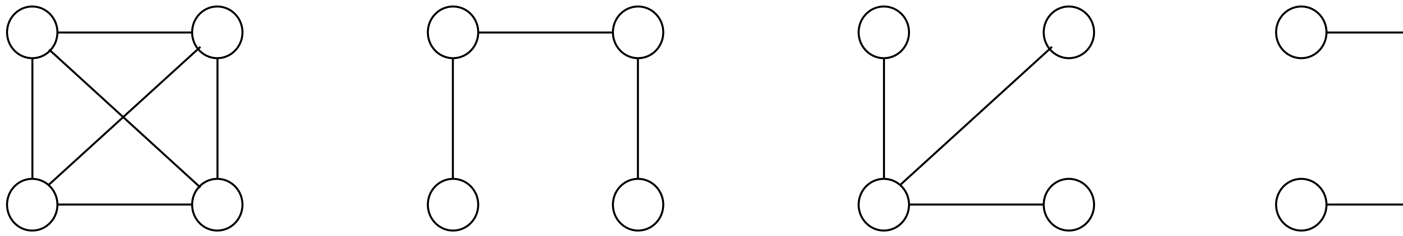
Or

A **minimum spanning tree** (MST) or **minimum weight spanning tree** is then a spanning tree with weight less than or equal to the weight of every other spanning tree.

Or

- **Definition Let *G=(V, E)* be at undirected connected graph. A subgraph *t=(V, E')* of *G* is a *spanning tree* of *G* iff t is a tree.**

**Example**



## Algorithm for a Spanning Tree

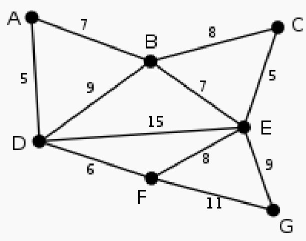
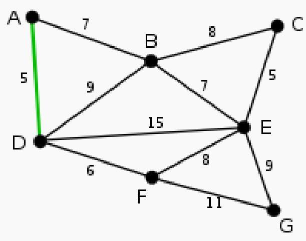
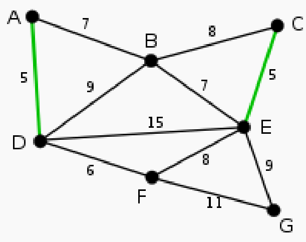
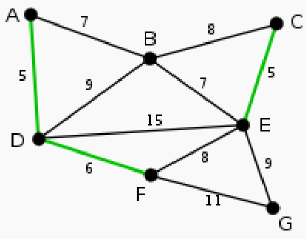
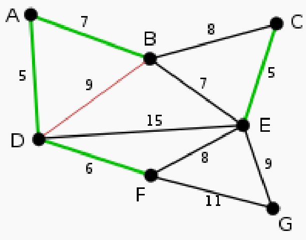
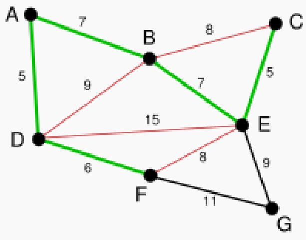
• Two basic algorithms exists – Kruskal (by arc) – Prim (by sub-tree)
• Both are greedy
• May have different complexity (efficiency) depending on the topology (eg. density) of the network

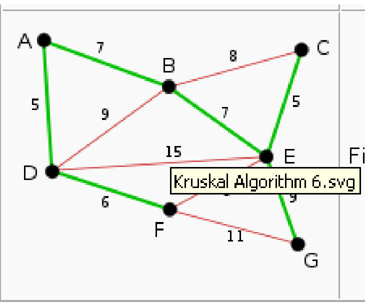
## Kruskal Algorithm

• Append new arcs to the tree in increasing order of the arc length (making sure cycles are not created)

Kruskal's algorithm is an algorithm in graph theory that finds a minimum spanning tree for a connected weighted graph. This means it finds a subset of the edges that forms a tree that includes every vertex, where the total weight of all the edges in the tree is minimized. If the graph is not connected, then it finds a minimum spanning forest (a minimum spanning tree for each connected component). Kruskal's algorithm is an example of a greedy algorithm.

**Example**

| Image | Description |
|---|---|
|   | This is our original graph. The numbers near the arcs indicate their weight. None of the arcs are highlighted. |
|   | **AD** and **CE** are the shortest arcs, with length 5, and **AD** has been arbitrarily chosen, so it is highlighted. |
|   | **CE** is now the shortest arc that does not form a cycle, with length 5, so it is highlighted as the second arc. |
|   | The next arc, **DF** with length 6, is highlighted using much the same method. |
|   | The next-shortest arcs are **AB** and **BE**, both with length 7. **AB** is chosen arbitrarily, and is highlighted. The arc **BD** has been highlighted in red, because there already exists a path (in green) between **B** and **D**, so it would form a cycle (**ABD**) if it were chosen. |
|   | The process continues to highlight the next-smallest arc, **BE** with length 7. Many more arcs are highlighted in red at this stage: **BC** because it would form the loop **BCE**, **DE** because it would form the loop **DEBA**, and **FE** because it would form **FEBAD**. |




Finally, the process finishes with the arc **EG** of length 9, and the minimum spanning tree is found

**<u>Algorithm :</u>**

KRUSKAL ( E,Cost,n,T,mincost)
//E is the set of edges in G.
//G has n vertices.
//Cost(U,v) is the cost of edge (u,v).
//T is the set of edes in the minimum spanning tree and mincost is its cost

1. Real min cost, cost(1:n,1:n)
2. Integer PARENT(1:n) ,T(1:n-1,2) ,n construct a heap out of the edge costs using HEAPIFY
3. Parent← 1 // each vertex is in a different set
4. I←mincost ← 0
5. While i←n-1 and heap not empty do
6. Delete a minimum cost edge (u,v) from the heap and reheapify using ADJUST
7. J← FIND (u) ;K← FIND(V)
8. If j != k then i← i+1
9. T(I,1) ← u ;T(I,2) ← v
10. Mincost ←mincost +cost (u,v)
11. Endif
12. Repeat
13. If i< > n-2 then print (“no spanning tree) end if
14. Return
15. End Kruskal

**How to implement -**

**Two functions should be considered**

- **Determining an edge with minimum cost**
- **Deleting this edge**

**<u>Analysis of Algorithm</u>**

Where *E* is the number of edges in the graph and *V* is the number of vertices, Kruskal's algorithm can be shown to run in $O(E \log E)$ time, or equivalently, $O(E \log V)$ time, all with simple data structures. These running times are equivalent because:

- $E$ is at most $V^2$ and $\log V^2 = 2\log V$ is $O(\log V)$.
- If we ignore isolated vertices, which will each be their own component of the minimum spanning forest, $V \leq E+1$, so $\log V$ is $O(\log E)$.

We can achieve this bound as follows: first sort the edges by weight using a comparison sort in $O(E \log E)$ time; this allows the step "remove an edge with minimum weight from $S$" to operate in constant time. Next, we use a disjoint-set data structure to keep track of which vertices are in which components. We need to perform $O(E)$ operations, two 'find' operations and possibly one union for each edge. Even a simple disjoint-set data structure such as disjoint-set forests with union by rank can perform $O(E)$ operations in $O(E \log V)$ time. Thus the total time is $O(E \log E) = O(E \log V)$.

**Or**

Edge set E.
Operations are:

- Is E empty?
- Select and remove a least-cost edge.

Use a min heap of edges.

- Initialize. O(e) time.
- Remove and return least-cost edge. O(log e) time.

Set of selected edges T.
Operations are:

- Does T have n - 1 edges?
- Does the addition of an edge (u, v) to T result in a cycle?

Add an edge to T.

Use an array linear list for the edges of T.

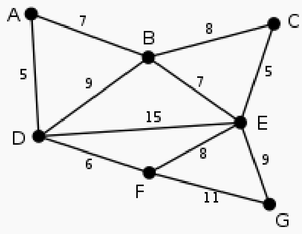
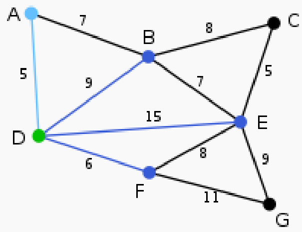
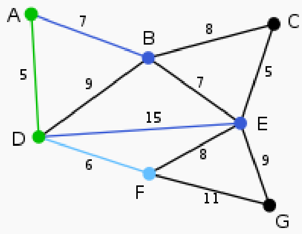
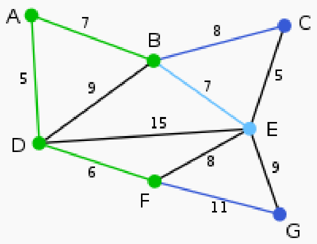
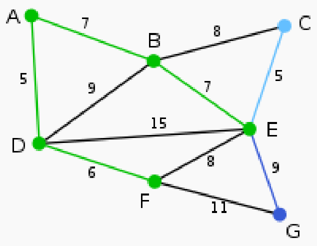
- Does T have n - 1 edges?
  - Check size of linear list. O(1) time.
- Does the addition of an edge (u, v) to T result in a cycle?
  - Not easy.
- Add an edge to T.
  - Add at right end of linear list. O(1) time.

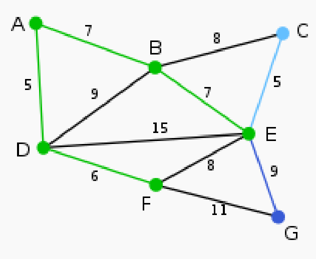
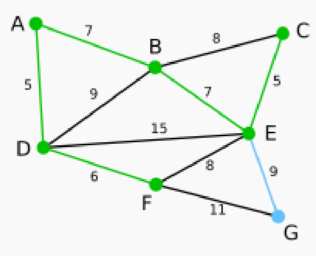
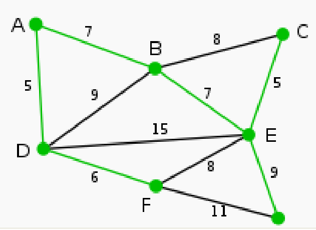
Just use an array rather than ArrayLinearList

- Use FastUnionFind.
- Initialize.
  - O(n) time.
- At most 2e finds and n-1 unions.
  - Very close to O(n + e).
- Min heap operations to get edges in increasing order of cost take O(e log e).
- Overall complexity of Kruskal's method is O(n + e log e).

**Prim's Algorithm**

• Similar, except that we always have a sub-tree as a partial solution: the new arc we add connects a node in the existing sub-tree to a node not yet in the sub-tree.
Example

<table>
<tr><th>Image</th><th>Description</th></tr>
<tr><td>
</td><td>This is our original weighted graph. The numbers near the arcs indicate their weight.</td></tr>
<tr><td>
</td><td>Vertex D has been arbitrarily chosen as a starting point. Vertices A, B, E and F are connected to D through a single edge. A is the vertex nearest to D and will be chosen as the second vertex along with the edge AD.</td></tr>
<tr><td>
</td><td>The next vertex chosen is the vertex nearest to either D or A. B is 9 away from D and 7 away from A, E is 15, and F is 6. F is the smallest distance away, so we highlight the vertex F and the arc DF.</td></tr>
<tr><td>
</td><td>In this case, we can choose between C, E, and G. C is 8 away from B, E is 7 away from B, and G 11 away from F. E is nearest, so we highlight the vertex E and the arc BE.</td></tr>
<tr><td>
</td><td>Here, the only vertices available are C and G. C is 5 away from E, and G is 9 away from E. C is chosen, so it is highlighted along with the arc EC.</td></tr>
</table>

| | |
|---|---|
|  | Here, the only vertices available are **C** and **G**. **C** is 5 away from **E**, and **G** is 9 away from **E**. **C** is chosen, so it is highlighted along with the arc **EC**. |
|  | Vertex **G** is the only remaining vertex. It is 11 away from **F**, and 9 away from **E**. **E** is nearer, so we highlight it and the arc **EG**. |
|  | Now all the vertices have been selected and the minimum spanning tree is shown in green. In this case, it has weight 39. |

**<u>Algorithm</u>**

**<u>PRIME(E,COST,nT,mincost)</u>**

//E is the set of edges in G
//COST (n,n) is the cost adjacency matrix of an n vertex graph such that COST(I,j) is either a positive real number + infinity. If no edge exists. A minimum spanning tree is computed and stored as set of edges in the array T(1:n-1,2). T(1,1)T(I,2) is an edge in the min-cost spanning tree .The final cost is assigned to mincost

1. real COST( n,n) ,mincost ;
2. integer NEAR(n) ,n,I,j,k,l ,T(1:n-1,2);
3. (K,l) ← edge with minimum cost
4. mincost = cost(k,l);
5. T(1,1),T(1,2) ←(k,l)
6. for I ←1 to n do // Initialize near.
7. If COST(i ,l)< COST (i, k) then NEAR (i) ← l
8. Else NEAR(i) ← k endif
9. Repeat
10. NEAR (k) ← NEAR (l) ← 0
11. for I ← 2 to n-1 do //find n-2 additional edges for T.
12. // Let j be an index such that NEAR (J)!= 0 and COST (j ,NEAR(j)) is minimum
13. (T(i,1) ,T(i,2)) ← (J ,NEAR (j))
14. mincost ←mincost +COST (j,NEAR(j))
15. NEAR (j) ← 0
16. for K ←1 to n do //update NEAR
17. if NEAR(K) != 0 and COST (K ,NEAR (k)) > COST(K,j) then

18. NEAR(K) ← j
19. End if
20. Repeat
21. Repeat
22. If mincost >= infinity then print (“no spanning tree )
23. End PRIM

**Time complexity**

| Minimum edge weight data structure | Time complexity (total) |
|---|---|
| adjacency matrix, searching | O(VE) |

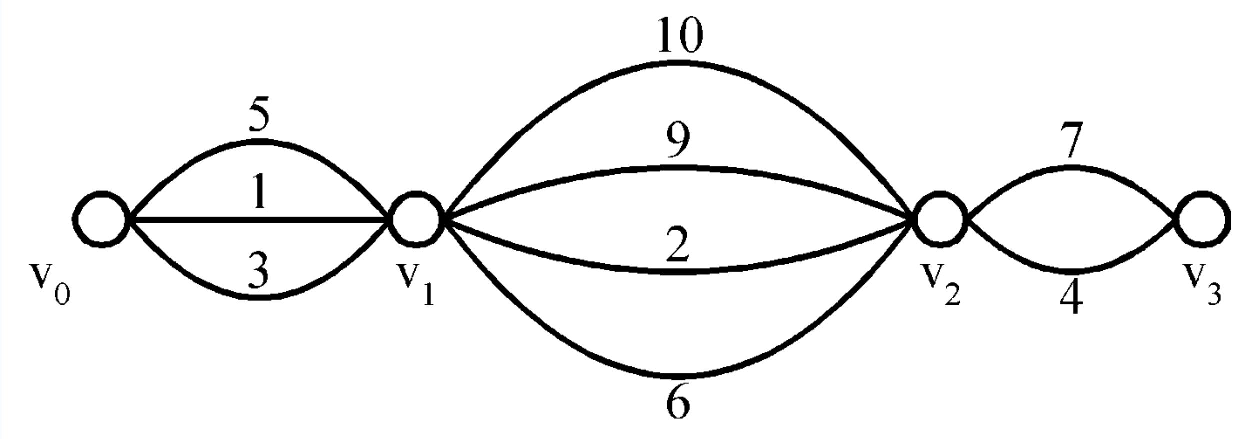
**Shortest Path :**

In graph theory, the **shortest path problem** is the problem of finding a path between two vertices (or nodes) such that the sum of the weights of its constituent edges is minimized

- Directed weighted graph.
- Path length is sum of weights of edges on path.
- The vertex at which the path begins is the source vertex.
- The vertex at which the path ends is the destination vertex.

**Example**

Finding the quickest way to get from one location to another on a road map; in this case, the vertices represent locations and the edges represent segments of road and are weighted by the time needed to travel that segment.




- Problem: Find a shortest path from $v_0$ to $v_3$.
- The greedy method can solve this problem.
- The shortest path: 1 + 2 + 4 = 7.

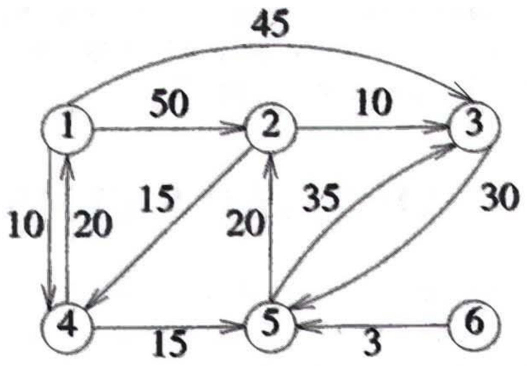
The problem is also sometimes called the **single-pair shortest path problem**, to distinguish it from the following generalizations:

- The **single-source shortest path problem**, in which we have to find shortest paths from a source vertex *v* to all other vertices in the graph.
- The **single-destination shortest path problem**, in which we have to find shortest paths from all vertices in the graph to a single destination vertex *v*. This can be reduced to the single-source shortest path problem by reversing the edges in the graph.
- The **all-pairs shortest path problem**, in which we have to find shortest paths between every pair of vertices *v*, *v'* in the graph.

**<u>Single Source Shortest Path :</u>**

- **<u>Design of greedy algorithm</u>**

Building the shortest paths one by one, in non decreasing order of path lengths
e.g., in Figure 4.15
1→4: 10
1→4→5: 25
…
We need to determine 1) the next vertex to which a shortest path must be generated and 2) a shortest path to this vertex
**Three observations**
If the next shortest path is to vertex *u*, then the path begins at $v_0$, ends at *u*, and goes through only those vertices that are in *S*.
The destination of the next path generated must be that of vertex *u* which has the minimum distance, *dist(u),* among all vertices not in *S*.
Having selected a vertex *u* as in observation 2 and generated the shortest $v_0$ to *u* path, vertex *u* becomes a member of *S*.



(a) Graph

| | Path | Length |
|---|---|---|
| 1) | 1, 4 | 10 |
| 2) | 1, 4, 5 | 25 |
| 3) | 1, 4, 5, 2 | 45 |
| 4) | 1, 3 | 45 |

(b) Shortest paths from 1

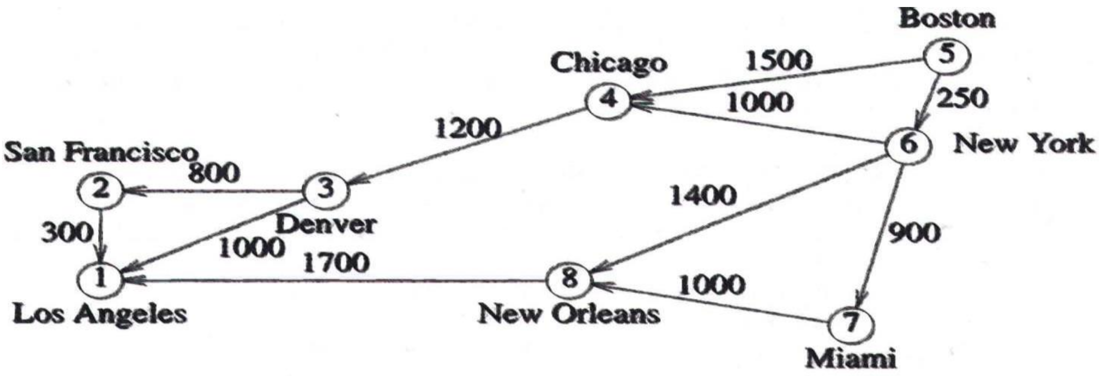
**Algorithm :Greedy algorithm ( Dijkstra's algorithm)**

ShortestPaths(v, cost, dist, n)

//DIST(j) ,1<=j<=n is set to the length of the shortest path from vertex v to vertex j in a diagraph G with n vertices. DIST(v) is set to zero. G is represented by its cost adjacency matrix ,COST(n,n)

```
Boolean S(1:n); real COST(1:n,1:n) DIST(1:n)
Integer u,v, n,num ,l,w
For I ← to n do  //initialize set S to empty
  S(i)←0 ; DIST(i) ← COST(V,i)
Repeat
S(V) ←1 ;DIST(v) ← 0  //put vertex v in set  S
For  num ← 2 to n-1 do    //determine n-1 paths from vertex v //
     Choose u such that DIST (u) =min {DIST(w)}
    S(w)=0
    S(u)←1    //put vertex u in set s
    For all W with S(w) =0 do
        DIST(w) ← min(DIST(w) ,DIST( u) +COST(u,w))
    Repeat
Repeat
End SHORTEST-PATHS.
```

**Time Complexity :- O($n^2$)**




(a) Digraph

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | | | | | | | |
| 2 | 300 | 0 | | | | | | |
| 3 | 100 | 800 | 0 | | | | | |
| 4 | | | 1200 | 0 | | | | |
| 5 | | | | 1500 | 0 | 250 | | |
| 6 | | | | 1000 | | 0 | 900 | 1400 |
| 7 | | | | | | | 0 | 1000 |
| 8 | 1700 | | | | | | | 0 |

(b) Length-adjacency matrix

| Iteration | S | Vertex selected | Distance | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | LA [1] | SF [2] | DEN [3] | CHI [4] | BOST [5] | NY [6] | MIA [7] | NO [8] |
| Initial | -- | ---- | +∞ | +∞ | +∞ | 1500 | 0 | 250 | +∞ | +∞ |
| 1 | {5} | 6 | +∞ | +∞ | +∞ | 1250 | 0 | 250 | 1150 | 1650 |
| 2 | {5,6} | 7 | +∞ | +∞ | +∞ | 1250 | 0 | 250 | 1150 | 1650 |
| 3 | {5,6,7} | 4 | +∞ | +∞ | 2450 | 1250 | 0 | 250 | 1150 | 1650 |
| 4 | {5,6,7,4} | 8 | 3350 | +∞ | 2450 | 1250 | 0 | 250 | 1150 | 1650 |
| 5 | {5,6,7,4,8} | 3 | 3350 | 3250 | 2450 | 1250 | 0 | 250 | 1150 | 1650 |
| 6 | {5,6,7,4,8,3} | 2 | 3350 | 3250 | 2450 | 1250 | 0 | 250 | 1150 | 1650 |
| | {5,6,7,4,8,3,2} | | | | | | | | | |