LCM - Linear Congruential Method

Most widely used technique for generating random numbers.

$$X_{i+1} = (a X_i + c) \mod m$$

multiplier increment modulus

$a = 13$
$m = 64$
$X_0 = 1234$

find period of generator

$X_0 = 1$

$X_1 = (13 \times 1) \bmod 64$

$X_1 = 13$

$X_2 = (13 \times 13) \bmod 64 = 41$

$X_3 = (41 \times 13) \bmod 64 = 21$

$X_4 = (21 \times 13) \bmod 64 = 17$

$X_5 = (17 \times 13) \bmod 64 = 29$

$X_5 = (29 \times 13) \bmod 64 = 57$

$X_6 = (57 \times 13) \bmod 64 = 37$

$X_7 = (37 \times 13) \bmod 64 = 33$

$X_8 = (33 \times 13) \bmod 64 = 45$

Test for Random Number Principle

Testing for uniformity. —

H_0 : Null Hypothesis $R_i \sim U[0,1]$

H_1 : Main Hypothesis $R_i \nsim U[0,1]$

failure to reject null hypothesis H_0 means that evidence of non uniformity cannot be detected.

Test for independance —

H_0 : $R_i \nsim$ independantly distributed

H_1 : $R_i \sim$ independantly distributed.

K S Test

$$D^+ = \max_{1 \le i \le N} \left\{ \frac{i}{N} - R_i \right\}$$

$$D^- = \max_{1 \le i \le N} \left\{ R_i - \frac{i-1}{N} \right\}$$

$$D = \max \left| F(x) - S_N(x) \right|$$

$$D = \max \left(D^+, D^- \right)$$

H_0: Both same distribution
H_1: Different distributions

$D_\alpha > D \rightarrow$ Not Reject ie random

$D > D_\alpha \longrightarrow$ Reject ie not random

Test for randomness K.S.

0.44, 0.81, 0.14, 0.05, 0.93

$N = 5$

i	1	2	3	4	5	
r	0.05	0.14	0.44	0.81	0.93	
i/N	0.2	0.4	0.6	0.8	1	
$(i-1)/N$	0	0.2	0.4	0.6	0.8	
$i/N - r$	0.15	0.26	0.24	-0.01	0.07	A
$r - (i-1)/N$	0.05	-0.06	0.04	0.21	-0.07	B

Max of row A & B is 0.26

$\therefore D = 0.26$ $\qquad \alpha \rightarrow 0.05$ confidence

$D < D_\alpha$ $\therefore$ in limits

$\therefore$ Can't reject null hypothesis

$\therefore$ Random no

χ^2 test

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

$\chi_0^2 > \chi_\alpha \rightarrow$ Reject ie. Not random

$\chi_\alpha < \chi_0^2 \rightarrow$ Cant reject ie- random.

0.34 0.13 0.17 0.55 0.43 0.8

for $n = 3$ intervals of equal length

0 – 0.33	III	3
0.33 – 0.66	II	2
0.66 – 1	I	1

Expected Uniform distribution (2 2 2)

O_i	E_i	$(O_i - E_i)^2 / E_i$
3	3	1/3
2	3	1/3
1	3	4/3
		2

$\chi_0 = 2$

at $\alpha =$ 95% confidence
$\chi = 1.96$

$\chi_0 > \chi_\alpha$

$\therefore$ Reject null Hypothesis

Not Random No

Runs test Up & down

Run is repetition of value

eg 0001101100

$$\left.\begin{array}{ll} 000 & \rightarrow \text{Run} \\ 11 & \rightarrow \text{Run} \\ 0 & \rightarrow \text{Run} \\ 11 & \rightarrow \text{Run} \\ 00 & \rightarrow \text{Run} \end{array}\right\} \text{5 runs}$$

Observed no of runs = 5

Expected no of runs in a sequence

$$\mu = \frac{2 n_1 n_2}{(n_1 + n_2)} + 1$$

$n_1 \rightarrow$ No of 0
$n_2 \rightarrow$ No of 1

Here $0 \rightarrow 6$
$1 \rightarrow 4$

$N \rightarrow n_1 + n_2 = 10$

$$\mu = \frac{2 \times 6 \times 4}{10} + 1 = 5.8$$

Variance $\bar{\bar{\sigma}} = \sqrt{\frac{(\mu-1)(\mu-2)}{N-1}}$

$$= \sqrt{\frac{(4.8) \times (3.8)}{9}} = \sqrt{2.02}$$

$= 1.42$

$$Z \text{ score} = \frac{\text{Observed} - \text{expected}}{\sqrt{\text{Variance}}}$$

$$= \frac{O_i - \mu}{\sigma}$$

$$= \frac{5 - 5.8}{1.42} = 0.56$$

0.57 0.64 0.34 0.43 0.22 0.01 0.23
0.78 0.88 0.1

Increment → +   0.57 < 0.64 → +
Decrement → −   0.64 > 0.34 → −

\+ − + − − + + + −

No of Runs

$$\left.\begin{matrix} + \\ - \\ + \\ - \; - \\ + \; + \; + \\ - \end{matrix}\right\} \text{6 Runs}$$

No of + → 5
− → 4

$$\mu = 2\frac{n_1 * n_2}{(n_1+n_2)} + 1 = \frac{2\times5\times4}{9} + 1$$

$$= 5.44$$

$$\sigma = \sqrt{\frac{(\mu-1)(\mu-2)}{\cancel{N}-1}} = \sqrt{\frac{(4.44)(3.44)}{8}}$$

$$\sqrt{1.91} = 1.38$$

$$Z = \frac{O-E}{\sigma} = \frac{(6-5.44)}{1.38}$$

$$= 0.405$$

$Z_\alpha \rightarrow 1.96$

Runs Above & Below Mean

$x_1 \ldots$
$\bar{x}$

$x_i > \bar{x} \rightarrow +$
$x_i < \bar{x} \rightarrow -$

$+ + - + - - \ldots N$

No of run = $\underline{a}$

Total Sample $\rightarrow \underline{N}$

$$\mu = \frac{2N-1}{3}$$

$$\sigma = \sqrt{\frac{16N-29}{90}}$$

$$z = \frac{a-\mu}{\sigma}$$

$z \leq z'$: Accept
$z > z'$: Reject

Inverse transform technique, can be used to sample from the distributions

step 1: Compute CDF

step 2: Set $F(x) = R$

step 3: Solve $F(x) = R$ for X in terms of R

step 4: Generate random numbers R_i and compute desired random variates.

$$X_i = F^{-1}(R_i)$$

for exp distributions

$$F(x) = 1 - e^{-\lambda x} = R$$

$$-\frac{\ln(1-R)}{\lambda} = x$$

for Uniform

$$F(x) = \frac{x-a}{b-a}$$

$$x = (b-a)R + a$$

for Weibull

$$x_i = \alpha\left[-\ln(1-R)\right]^{1/\beta}$$

for Empirical

$$X_i = X_{i-1} + a_i\left(R - \frac{(i-1)}{n}\right)$$

$$a_i = n(x_i - x_{i-1})$$

For exam Variate generation

Exponential
Uniform
Weibull
empirical

Continuous	Discrete
Derivation + Numerical	Numerical

Acception Rejection $\longrightarrow$ No derivation lifetime of a computer chip in hours Weibull $\alpha = 0.2$

$\beta = 0.5$

$\gamma = 0$

Generate 2 random lifetime of computer chip for this

Use $R_1 = 0.6173$

$R_2 = 0.4829$

$$\rightarrow \quad X_i = \alpha\left[-\ln(1-R_i)\right]^{1/\beta}$$

$$X_1 = 0.2\left(-\ln(1-0.6173)\right)^{1/0.5}$$

$$= 0.1843$$

$$X_2 = 0.2\left(-\ln(1-0.4829)\right)^{1/0.5}$$

$$= 0.08699$$

5 observations of firebrigade station response times to incomming alarms have been collected and are used in simulation to investigate alternate staffing and firebrigade van scheduling

Response time
1.60
0.55
1.12
2.20
1.94

Set up table for generating response time by table lookup method (Empirical Distribution) using uniform random no.

0.5426 & 0.1524   n = 5

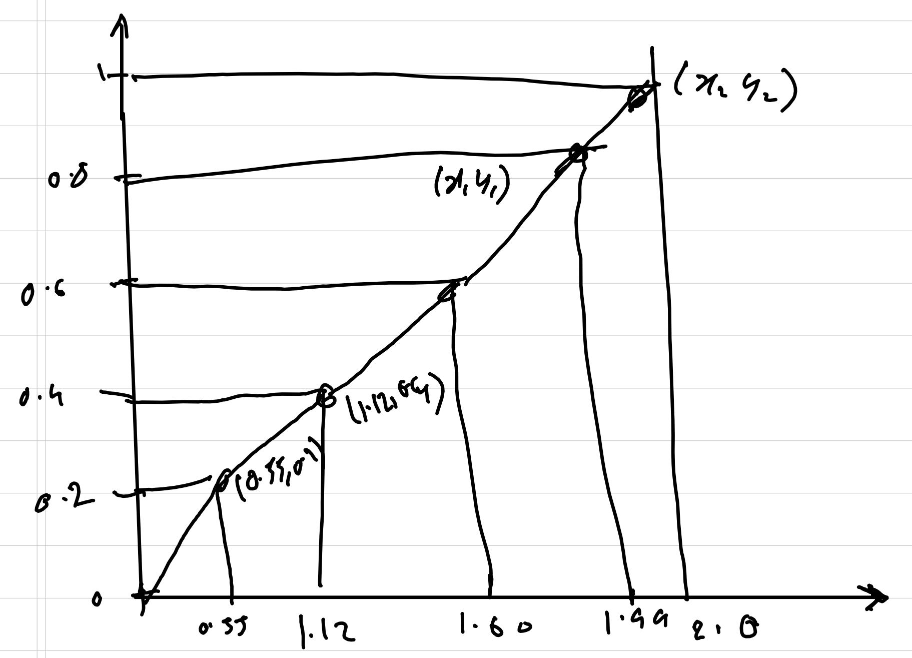
i	Interval	p	CDF	Slope $n(x_i - x_{i-1})$
1	$0 < x < 0.55$	0.2	0.2	$5(0.55-0) = 2.75$
2	$0.55 < x < 1.12$	0.2	0.4	$5(1.12-0.55) = 2.85$
3	$1.12 < x < 1.60$	0.2	0.6	2.4
4	$1.60 < x < 1.94$	0.2	0.8	1.7
5	$1.94 < x < 2.20$	0.2	1	1.3

$R_1 = 0.5326$ → Interval 3 $0.4 < x < 0.6$

$$X = x_i + a_i \left(R_i - \frac{(i-1)}{n} \right)$$

$$= 1.12 + 2.4 \left(0.5326 - \frac{(3-1)}{5} \right)$$

$$= 1.462$$



$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

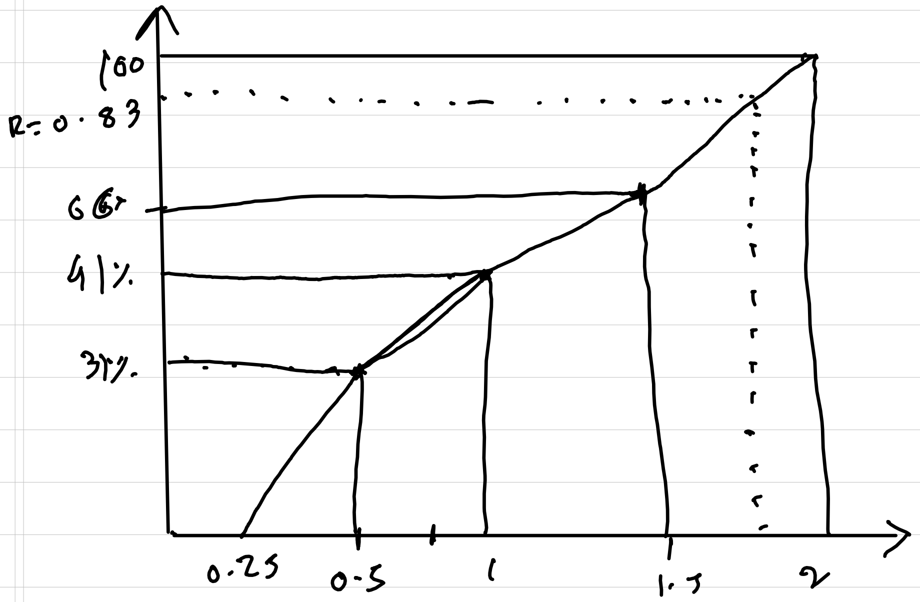
Suppose 100 machine items have been collected for repair. Time required to repair is given in table →

Time (hrs)	No of Items
$0.25 < x < 0.5$	31
$0.5 < x < 1$	10
$1 < x < 1.5$	25
$1.5 < x < 2.0$	34

Calculate random number for above empirical distribution using $R = 0.83$

→

	Prob	C.P.
0.5	31 %	0 – 31 %
1	10 %	31 – 41 %
1.5	25 %	41 – 66 %
2	34 %	66 – 100 %



$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{0.83 - 0.66}{1 - 0.66} = \frac{x - 1.5}{1.5 - 2}$$

$= 1.75$

Acceptance Rejection Technique

Method used to produce samples from complex probability distributions

Used when direct inverse transform is not feasable

step 1 choose proposal distribution $g(x)$

step 2 choose target distribution $f(x)$

step 3 Decide threshold $\frac{f(x)}{M\,g(x)}$

step 4 Generate R

step 5 if $R \leq \frac{f(x)}{M\,g(x)}$ Accept

else reject

Useful for a large number of distributions

Acceptance Rejection for NSPP

step 1 set $P = 1$

step 2 Generate R

step 3 Set $P = P \times R$

step 4 If $P \leq e^{-\lambda}$ Accept, end

step 5 else Go to step 2

Generate 3 random number for poisson variate
$\lambda = 0.2$

R : 0.4357, 0.4146, 0.8353, 0.9952, 0.8004

→ $e^{-\lambda} = e^{-0.2} = 0.818$

P	R	$P = P \times R$	$P < e^{-\lambda}$?
1	0.4357	0.4357	Yes, Accept
1	0.4146	0.4146	Yes Accept
1	0.8353	0.8333	No, Reject
0.8353	0.9952	0.8312	No, Reject
0.8312	0.8004	0.6653	Yes Accept

R : 0.4357, 0.4146, 0.6653

Random Number	Random Variate
Number generated by random number generator	Number transformed to follow a specific probability distribution
Usually between 0-1	Depends on target distribution
Uniform	Matches Target
Serves as base for other distributions	Simulations & models requiring specific patterns