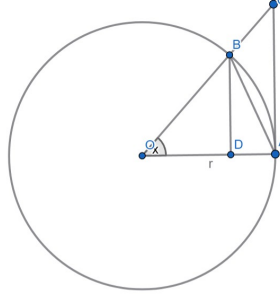


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## 1 Basel Problem(Cauchy's Solution)



Relating the areas of  $\triangle OAB$ , sector  $OAB$  and  $\triangle OAC$

$$\frac{r^2 \sin(x)}{2} < \frac{r^2 x}{2} < \frac{r^2 \tan(x)}{2}$$

Simplifying the inequality, we obtain,

$$\begin{aligned} \sin(x) &< x < \tan(x) \\ \cot^2(x) &< \frac{1}{x^2} < \csc^2(x) \\ \cot^2(x) &< \frac{1}{x^2} < 1 + \cot^2(x) \end{aligned}$$

Substituting  $x = \frac{n\pi}{2N+1}$  for  $1 \leq n \leq N$ ,

$$\cot^2\left(\frac{n\pi}{2N+1}\right) < \frac{(2N+1)^2}{n^2\pi^2} < 1 + \cot^2\left(\frac{n\pi}{2N+1}\right)$$

Now, multiply each term by  $\frac{\pi^2}{(2N+1)^2}$ ,

$$\frac{\pi^2}{(2N+1)^2} \cot^2\left(\frac{n\pi}{2N+1}\right) < \frac{1}{n^2} < \frac{\pi^2}{(2N+1)^2} + \frac{\pi^2}{(2N+1)^2} \cot^2\left(\frac{n\pi}{2N+1}\right)$$

Now, we sum up the terms from  $n$  to  $N$ ,

$$\sum_{n=1}^N \frac{\pi^2}{(2N+1)^2} \cot^2\left(\frac{n\pi}{2N+1}\right) < \sum_{n=1}^N \frac{1}{n^2} < \sum_{n=1}^N \left( \frac{\pi^2}{(2N+1)^2} + \frac{\pi^2}{(2N+1)^2} \cot^2\left(\frac{n\pi}{2N+1}\right) \right)$$

Simplifying further,

$$\frac{\pi^2}{(2N+1)^2} \sum_{n=1}^N \cot^2\left(\frac{n\pi}{2N+1}\right) < \sum_{n=1}^N \frac{1}{n^2} < \frac{N\pi^2}{(2N+1)^2} + \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^N \cot^2\left(\frac{n\pi}{2N+1}\right)$$

Now, we take the limit as  $N \rightarrow \infty$

$$\lim_{N \rightarrow \infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^N \cot^2\left(\frac{n\pi}{2N+1}\right) < \sum_{n=1}^{\infty} \frac{1}{n^2} < \lim_{N \rightarrow \infty} \frac{N\pi^2}{(2N+1)^2} + \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^N \cot^2\left(\frac{n\pi}{2N+1}\right)$$

Evaluating the limit of the term  $\frac{N\pi^2}{(2N+1)^2}$ , i.e

$$\lim_{N \rightarrow \infty} \frac{N\pi^2}{(2N+1)^2} = 0$$

Substituting back into the inequality,

$$\lim_{N \rightarrow \infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^N \cot^2\left(\frac{n\pi}{2N+1}\right) < \sum_{n=1}^{\infty} \frac{1}{n^2} < \lim_{N \rightarrow \infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^N \cot^2\left(\frac{n\pi}{2N+1}\right)$$

Now, the goal is to evaluate the limit and use **Squeeze Theorem**,

$$\lim_{N \rightarrow \infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^N \cot^2\left(\frac{n\pi}{2N+1}\right)$$

Using De Moivre's Theorem,

$$(\cos(x) + i \sin(x))^n = \cos(nx) + i \sin(nx)$$

Dividing both sides by  $\sin^n(x)$

$$(\cot(x) + i)^n = \frac{\cos(nx) + i \sin(nx)}{\sin^n(x)}$$

Using the Binomial Theorem on  $(\cot(x) + i)^n$  we get,

$$(\cot(x) + i)^n = \binom{n}{0} \cot^n(x) + \binom{n}{1} \cot^{n-1}(x) \cdot i + \dots + \binom{n}{n-1} \cot(x) \cdot i^{n-1} + \binom{n}{n} \cdot i^n$$

Equating the imaginary coefficients,

$$\frac{\sin(nx)}{\sin^n(x)} = \binom{n}{1} \cot^{n-1}(x) - \binom{n}{3} \cot^{n-3}(x) + \dots$$

Now, substituting  $x = \frac{n\pi}{2N+1}$  and  $n = (2N+1)$ , we obtain,  $\sin(nx) = 0$

Thus,

$$0 = \binom{2N+1}{1} \cot^{2N}(x) - \binom{2N+1}{3} \cot^{2N-2}(x) + \dots$$

Let  $t = \cot^2(x)$ ,

$$P(t) = \binom{2N+1}{1} t^N - \binom{2N+1}{3} t^{N-1} + \dots$$

$P(t)$  is polynomial.

Now, using Vieta's sum of roots of polynomial formula on  $P(t)$ ,

$$\begin{aligned} \sum_{n=1}^N t_n &= \frac{\binom{2N+1}{3}}{\binom{2N+1}{1}} \\ &= \frac{(2N+1)!}{(2N+1-3)! \cdot 3!} \cdot \frac{(2N+1-1)!}{(2N+1)!} \\ &= \frac{(2N)!}{(2N-2)! \cdot 6} \\ &= \frac{(2N)(2N-1)}{6} \end{aligned}$$

But we know  $t_n = \cot^2(\frac{n\pi}{2N+1})$  is the root of  $P(t)$ . So,

$$\sum_{n=1}^N \cot^2\left(\frac{n\pi}{2N+1}\right) = \frac{(2N)(2N-1)}{6}$$

Our goal was to evaluate the limit,

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^N \cot^2\left(\frac{n\pi}{2N+1}\right) &= \lim_{N \rightarrow \infty} \frac{\pi^2}{(2N+1)^2} \frac{(2N)(2N-1)}{6} \\ &= \frac{\pi^2}{6} \lim_{N \rightarrow \infty} \frac{4N^2 - 2N}{4N^2 + 4N + 1} \\ &= \frac{\pi^2}{6} \end{aligned}$$

Now, substituting this value in the inequality, we obtain,

$$\frac{\pi^2}{6} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{\pi^2}{6}$$

From Squeeze Theorem,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

## 2 Bernoulli Differential Equation

Let general equation be,

$$\frac{dy}{dx} + Py = Qy^n$$

Dividing both sides by  $y^n$  we obtain,

$$y^{-n} \frac{dy}{dx} + Py^{1-n} = Q$$

Let  $z = y^{1-n}$ ,  $\frac{dz}{dx} = (1-n)y^{-n} \frac{dy}{dx}$ . Multiplying above equation by  $(1-n)$  and substituting we obtain,

$$(1-n)y^{-n} \frac{dy}{dx} + (1-n)Py^{1-n} = Q(1-n)$$

$$\frac{dz}{dx} + (1-n)Pz = Q(1-n)$$

Now, this equation can be solved by using **Integrating Factor Method**.

Let  $(1-n)P = P_1$  and  $Q(1-n) = Q_1$ , also let I.F be Integrating Factor,  
I.F =  $e^{\int P_1 dx}$

Therefore,

$$\text{IF} \cdot z = \int Q_1 \cdot \text{IF} dx$$

We solve for  $z$  first and then use  $z = y^{1-n}$  relation to solve for  $y$ .

### 3 Epsilon Delta Definition of Limits

$\lim_{x \rightarrow a} f(x) = L$  means

$$\forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

## 4 Gamma Function

$$\begin{aligned}\int_0^1 \log(x) dx &= \left[ x \log(x) - 1 \cdot x \right]_0^1 = -1! \\ \int_0^1 \log^2(x) dx &= \left[ x \log^2(x) - 2x \log(x) + 2 \cdot 1 \cdot x \right]_0^1 = 2! \\ \int_0^1 \log^3(x) dx &= \left[ x \log^3(x) - 3x \log^2(x) - 2 \cdot 3x \log(x) - 3 \cdot 2 \cdot 1 \cdot x \right]_0^1 = -3!\end{aligned}$$

Observing the pattern, we can write,

$$\int_0^1 \log^n(x) dx = (-1)^n \cdot n!$$

Solving for  $n!$  we get,

$$\begin{aligned}n! &= \frac{1}{(-1)^n} \int_0^1 \log^n(x) dx \\ n! &= \int_0^1 \left( \frac{\log(x)}{-1} \right)^n dx \\ n! &= \int_0^1 (-\log(x))^n dx \\ n! &= \int_0^1 \left( \log\left(\frac{1}{x}\right) \right)^n dx\end{aligned}$$

Let  $u = \log\left(\frac{1}{x}\right)$ . Using the property of logarithms  $-u = \log(x)$ . Raising to the power of  $e$  on both sides, we get  $x = e^{-u}$  and thus,  $dx = -e^{-u} du$ . When  $x = 0$ ,  $u \rightarrow \infty$  and when  $x = 1$ ,  $u = 0$ . Therefore, the integral becomes,

$$n! = \int_0^\infty u^n (e^{-u}) du$$

Now, by the definition of gamma function,

$$\begin{aligned}\Gamma(n+1) &= n! = \int_0^\infty u^n (e^{-u}) du \\ \Gamma(n) &= (n-1)! = \int_0^\infty u^{n-1} e^{-u} du\end{aligned}$$

Replacing  $n$  by  $x$ ,

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$$

#### 4.1 $\Gamma\left(\frac{1}{2}\right)$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} u^{-\frac{1}{2}} e^{-u} du$$

Let  $t = u^{\frac{1}{2}}$  so that  $dt = \frac{1}{2} u^{-\frac{1}{2}} du$ . Making the substitution we get,

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^{\infty} e^{-t^2} dt$$

Since,  $e^{-t^2}$  is an even function we can write,

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^{\infty} e^{-t^2} dt$$

This is known as **Gaussian Integral**. Let's evaluate this Integral,

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$I^2 = \left( \int_{-\infty}^{\infty} e^{-x^2} dx \right)^2$$

$$I^2 = \left( \int_{-\infty}^{\infty} e^{-x^2} dx \right) \cdot \left( \int_{-\infty}^{\infty} e^{-y^2} dy \right)$$

$$I^2 = \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy$$

Now in polar coordinates,  $dx dy = dA = r dr d\theta$  and when  $x \rightarrow -\infty, r \rightarrow 0$  and  $x \rightarrow \infty, r \rightarrow \infty$  when  $y \rightarrow -\infty, \theta \rightarrow 0$  and  $y \rightarrow \infty, \theta \rightarrow 2\pi$

Substituting, we get,

$$I^2 = \int_0^{2\pi} \int_0^{\infty} r e^{-r^2} dr d\theta$$

$$I^2 = \int_0^{2\pi} \frac{1}{2} d\theta$$

$$I^2 = \frac{1}{2} [\theta]_0^{2\pi}$$

$$I^2 = \frac{1}{2} 2\pi$$

$$I^2 = \pi$$

$$I = \sqrt{\pi}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$



## 5 Integrating Factor Method for Linear Differential Equations

Consider general linear first order differential equation,

$$\frac{dy}{dx} + Py = Q$$

Let IF be **Integrating Factor** defined as  $IF = e^{\int P dx}$

Now, multiplying both sides with this integrating factor we obtain,

$$e^{\int P dx} \cdot \frac{dy}{dx} + P e^{\int P dx} y = e^{\int P dx} \cdot Q$$

On the left hand side, we see the product rule of differentiation,

$$\frac{d}{dx} \left( y e^{\int P dx} \right) = Q e^{\int P dx}$$

Now, integrating with respect to  $x$  on both sides, we obtain,

$$y e^{\int P dx} = \int Q e^{\int P dx} dx$$

$$y \cdot IF = \int Q \cdot IF dx$$

## 6 Leibniz Theorem

Let  $u$  and  $v$  be functions of  $x$ . This theorem gives the closed form for the  $n^{th}$  derivative of product of two functions. That will be  $(uv)^{(n)}$  or  $\frac{d^n(uv)}{dx^n}$  for functions  $u$  and  $v$ .

for small values of  $n$ ,

$$\begin{aligned} n = 1, (uv)' &= u'v + uv' \\ n = 2, (uv)'' &= u''v + 2u'v' + uv'' \\ n = 3, (uv)''' &= u'''v + 3u''v' + 3u'v'' + uv''' \\ n = 4, (uv)^{(4)} &= u^{(4)}v + 4u^{(3)}v' + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + uv^{(4)} \end{aligned}$$

Looking at the pattern from above, coefficients are **binomial coefficients**. So, we can write general form as,

$$(uv)^{(n)} = \frac{d^n(uv)}{dx^n} = \sum_{r=0}^n C(n, r) u^{(n-r)} v^{(r)}$$

where  $C(n, r)$  is the binomial coefficient and  $C(n, r) = \frac{n!}{r!(n-r)!}$ .

### 6.1 Proof

Now, we prove this theorem by **induction**,

We assume for  $n=k$ ,  $(uv)^{(k)} = \frac{d^k(uv)}{dx^k} = \sum_{r=0}^k C(k, r) u^{(k-r)} v^{(r)}$  is true

And now for  $n=k+1$ , we prove,

$$(uv)^{(k+1)} = \frac{d^{k+1}(uv)}{dx^{k+1}} = \sum_{r=0}^{k+1} C(k+1, r) u^{(k+1-r)} v^{(r)}$$

By definition of successive differentiation,

$$\begin{aligned}
(uv)^{(k+1)} &= \frac{d^{k+1}(uv)}{dx^{k+1}} = \frac{d}{dx} \sum_{r=0}^k C(k, r) u^{(k-r)} v^{(r)} \\
&= \sum_{r=0}^k C(k, r) \left( u^{(k-r+1)} v^{(r)} + u^{(k-r)} v^{(r+1)} \right) \\
&= \sum_{r=0}^k C(k, r) u^{(k-r+1)} v^{(r)} + \sum_{r=0}^k C(k, r) u^{(k-r)} v^{(r+1)} \\
&= \sum_{r=0}^k C(k, r) u^{(k-r+1)} v^{(r)} + \sum_{r=1}^{k+1} C(k, r-1) u^{(k-r+1)} v^{(r)} \\
&= C(k, 0) u^{(k+1)} v^{(0)} + C(k, k) u^{(0)} v^{k+1} + \sum_{r=1}^k \left( C(k, r) + C(k, r-1) \right) u^{(k-r+1)} v^{(r)} \\
&= u^{(k+1)} v^{(0)} + u^{(0)} v^{k+1} + \sum_{r=1}^k C(k+1, r) u^{(k-r+1)} v^{(r)} \\
&= \sum_{r=0}^k C(k+1, r) u^{(k+1-r)} v^{(r)}
\end{aligned}$$

which is what we wanted to prove.

**7**  $\lim_{x \rightarrow \infty} \frac{x!}{x^x}$

Using the definition of factorial function and comparing it with the  $x^{x-1}$ ,

$$\begin{aligned} x! &= x(x-1)(x-2)\dots 3 \cdot 2 \cdot 1 \\ &\leq x \cdot x \cdot x \dots \cdot 1 = x^{x-1} \end{aligned}$$

Factorial of positive integer is greater than or equal to 1. So,

$$1 \leq x! \leq x^{x-1}$$

Dividing all terms by  $x^x$ ,

$$\begin{aligned} \frac{1}{x^x} &\leq \frac{x!}{x^x} \leq \frac{x^{x-1}}{x^x} \\ \frac{1}{x^x} &\leq \frac{x!}{x^x} \leq \frac{1}{x} \end{aligned}$$

Taking limit as  $x \rightarrow \infty$ ,

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{1}{x^x} &\leq \lim_{x \rightarrow \infty} \frac{x!}{x^x} \leq \lim_{x \rightarrow \infty} \frac{1}{x} \\ 0 &\leq \lim_{x \rightarrow \infty} \frac{x!}{x^x} \leq 0 \end{aligned}$$

Now, from **Squeeze Theorem**,

$$\lim_{x \rightarrow \infty} \frac{x!}{x^x} = 0$$

## 8 Maclaurin's Series

Let  $f(x)$  be function of  $x$ . The goal is to write the function as an infinite polynomial.

$$f(x) = a + bx + cx^2 + dx^3 + ex^4 + \dots$$

when  $x=0$ ,  $f(0) = a$  In order to find other coefficients we differentiate the function with respect to  $x$  and put  $x = 0$ .  $f'(0) = 1 \cdot b$   $f''(0) = 1 \cdot 2 \cdot b$   $f'''(0) = 1 \cdot 2 \cdot 3 \cdot c$  and so on. Looking at the pattern, the general form would be,

$$f^{(n)}(0) = n! \cdot (n^{th} coefficient)$$

Substituting in the above equation for coefficients, we get,

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

In Summation Notation,

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

$\sin(x)$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
$\cos(x)$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
$\tan(x)$	$x + \frac{x^3}{3} + \frac{2x^5}{15} - \dots$
$\sinh(x)$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$
$\cosh(x)$	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$
$e^x$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$
$\tan^{-1}(x)$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

Now, the more general expansion gives the **Taylor Series** which is shifted by  $h$ .

$$f(x+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x)$$

## 9 MIT Integration Bee

$$\begin{aligned} & \int \tan(x) \sqrt{2 + \sqrt{4 + \cos(x)}} dx \\ &= \int \frac{\sin(x)}{\cos(x)} \sqrt{2 + \sqrt{4 + \cos(x)}} dx \end{aligned}$$

Let  $u = 4 + \cos(x)$ ,  $du = -\sin(x)dx$

$$= \int \frac{-1}{u-4} \sqrt{2 + \sqrt{u}} du$$

Let  $v^2 = u$ ,  $2v dv = du$

$$\begin{aligned} &= \int \frac{-1}{v^2-4} \sqrt{2+v} \cdot 2v dv \\ &= \int \frac{-2v\sqrt{2+v}}{(v+2)(v-2)} dv \\ &= \int \frac{-2v}{\sqrt{(v+2)(v-2)}} dv \end{aligned}$$

Let  $w = \sqrt{v+2}$ ,  $v = w^2 - 2$ ,  $dv = 2w dw$

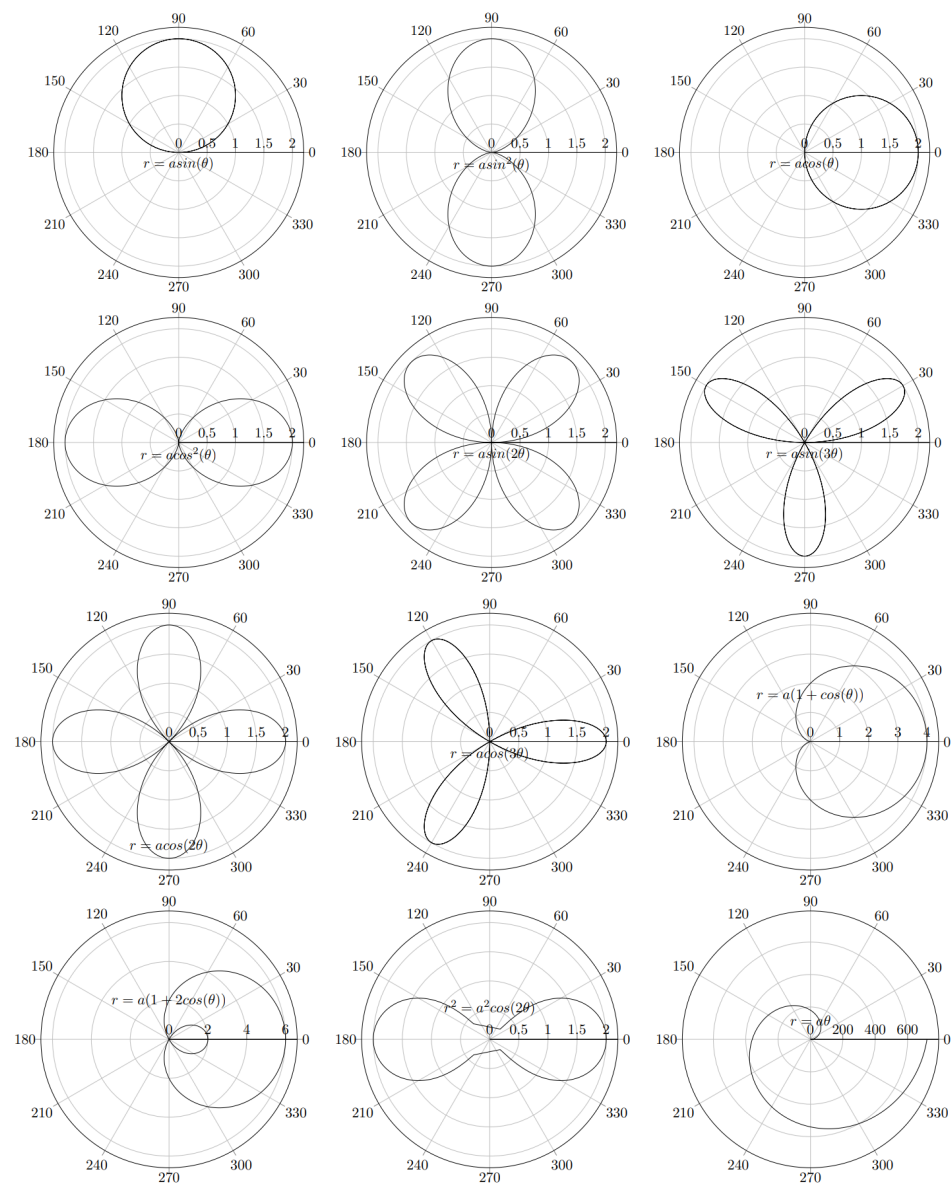
$$\begin{aligned} &= \int \frac{-2(w^2-2)}{w(w^2-4)} \cdot 2w dw \\ &= \int \frac{-4(w^2-2)}{w^2-4} dw \\ &= \int \frac{-4w^2+8}{w^2-4} dw \\ &= \int \frac{-4(w^2-4)-8}{w^2-4} dw \\ &= \int \left( -4 - \frac{8}{w^2-4} \right) dw \\ &= \int -4 dw - \int \frac{8}{(w+2)(w-2)} dw \\ &= -4w - \int \left( \frac{2}{w-2} - \frac{2}{w+2} \right) dw \end{aligned}$$

$$\begin{aligned}
&= -4w - 2 \ln |w - 2| + 2 \ln |w + 2| + c \\
&= -4w - 2 \ln \left| \frac{w + 2}{w - 2} \right| + c
\end{aligned}$$

Substituting back,

$$\begin{aligned}
&= -4\sqrt{v+2} - 2 \ln \left| \frac{\sqrt{v+2} + 2}{\sqrt{v+2} - 2} \right| + c \\
&= -4\sqrt{\sqrt{u}+2} - 2 \ln \left| \frac{\sqrt{\sqrt{u}+2} + 2}{\sqrt{\sqrt{u}+2} - 2} \right| + c \\
&= -4\sqrt{\sqrt{4+\cos(x)}+2} - 2 \ln \left| \frac{\sqrt{\sqrt{4+\cos(x)}+2} + 2}{\sqrt{\sqrt{4+\cos(x)}+2} - 2} \right| + c
\end{aligned}$$

## 10 Polar Curves





## 11 Second Order Differential Equations

Let's consider a general second order differential equation,

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$$

It can also be written as,

$$ay'' + by' + cy = f(x)$$

### 11.1 Homogeneous Equations

If  $f(x) = 0$ ,  $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$  is linear, constant coefficient, second order homogeneous differential equation.

Let  $u(x)$  and  $v(x)$  be two solutions of the equation, It satisfies above equation,

$$a \frac{d^2 u}{dx^2} + b \frac{du}{dx} + cu = 0$$

$$a \frac{d^2 v}{dx^2} + b \frac{dv}{dx} + cv = 0$$

Adding these two we get,

$$a \left( \frac{d^2 u}{dx^2} + \frac{d^2 v}{dx^2} \right) + b \left( \frac{du}{dx} + \frac{dv}{dx} \right) + c(u + v) = 0$$

$$a \frac{d^2 (u + v)}{dx^2} + b \frac{d(u + v)}{dx} + c(u + v) = 0$$

Thus,  $y = (u + v)$  is also solution.

If  $a = 0$ , This becomes first order differential equation with  $b$  and  $c$  as constant coefficients.

$$b \frac{dy}{dx} + cy = 0$$

$$b \frac{dy}{dx} = -cy$$

$$\text{Let } \frac{c}{b} = k,$$

$$\int \frac{dy}{y} = \int -k dx$$

$$\ln |y| = -kx + c$$

$$y = e^{-kx+c}$$

$$y = e^{-kx} \cdot e^c$$

$$\text{Let } A = e^c$$

$$y = Ae^{-kx}$$

$$\text{Let } -k = m$$

$$y = Ae^{mx}$$

Using this, we guess  $y = Ae^{mx}$  is solution to the second order differential equation as well,  $\frac{dy}{dx} = Ame^{mx}$  and  $\frac{d^2y}{dx^2} = Am^2e^{mx}$ . Substituting these derivatives,

$$\begin{aligned} aAm^2e^{mx} + bAme^{mx} + cAe^{mx} &= 0 \\ am^2 + bm + c &= 0 \end{aligned}$$

This quadratic equation is known as **Auxiliary Equation**. Let  $m_1$  and  $m_2$  be two roots of the Auxiliary Equation. Thus, two solutions to the differential equation are,  $y = Ae^{m_1x}$  and  $y = Be^{m_2x}$  if these two are solutions, then sum of these two solutions is also solution.

$$y = Ae^{m_1x} + Be^{m_2x}$$

A and B are two necessary arbitrary constants for a second-order differential equations. So, there are no further solutions.

### 11.1.1 Types of Solution

Solution depends on roots of auxiliary equation,

$$am^2 + bm + c = 0$$

#### Real and Different Roots

If  $m_1$  and  $m_2$  are two **distinct real roots**,

$$y = Ae^{m_1x} + Be^{m_2x}$$

## Real and Equal Roots

If  $m_1 = m_2 = m$  are **real and equal roots**,

$$y = Ae^{mx} + Be^{mx}$$

$$y = e^{mx}(A + B)$$

But this has only one arbitrary constant  $(A + B)$ .

Thus,  $Be^{mx}x$  can be checked and it is solution, So, solution is,

$$y = Ae^{mx} + Bxe^{mx}$$

## Complex Roots

Let  $m_1 = \alpha + i\beta$  and  $m_2 = \alpha - i\beta$

$$\begin{aligned} y &= Ce^{(\alpha+i\beta)x} + De^{(\alpha-i\beta)x} \\ &= e^{\alpha x}(Ce^{i\beta x} + De^{-i\beta x}) \end{aligned}$$

Substituting  $e^{i\beta x} = \cos(\beta x) + i\sin(\beta x)$  and  $e^{-i\beta x} = \cos(\beta x) - i\sin(\beta x)$  and writing  $A = C + D$  and  $B = i(C - D)$  we obtain,

$$y = e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x))$$

### 11.1.2 Special Case

Let  $b = 0$  and  $\frac{c}{a} = \pm n^2$ , we obtain,

$$\frac{d^2y}{dx^2} \pm n^2y = 0$$

If constant coefficient of  $y$  is  $+n^2$ ,

$$\frac{d^2y}{dx^2} + n^2y = 0$$

Auxiliary Equation is,

$$m^2 + n^2 = 0$$

$$m^2 = -n^2$$

$$m = \pm ni$$

Thus, Solution is,

$$y = A \cos(nx) + B \sin(nx)$$

If constant coefficient of y is  $-n^2$ ,

$$\frac{d^2y}{dx^2} - n^2y = 0$$

Auxiliary Equation is,

$$m^2 - n^2 = 0$$

$$m^2 = n^2$$

$$m = \pm n$$

Thus, Solution is,

$$y = Ce^{nx} + De^{-nx}$$

Using definitions of  $\sinh(x)$  and  $\cosh(x)$  we obtain,  $\sinh(nx) = \frac{e^{nx} - e^{-nx}}{2}$   
 $\cosh(nx) = \frac{e^{nx} + e^{-nx}}{2}$ .

Now, adding and subtracting we obtain following identities,

$$e^{nx} = \cosh(nx) + \sinh(nx)$$

$$e^{-nx} = \cosh(nx) - \sinh(nx)$$

Substituting in above obtained solution,

$$y = C \cosh(nx) + Ci \sinh(nx) + D \cosh(nx) - Di \sinh(nx)$$

$$y = (C + D) \cosh(nx) + i(C - D) \sinh(nx)$$

Writing  $A = C + D$  and  $B = i(C - D)$ ,

$$y = A \cosh(nx) + B \sinh(nx)$$

## 12 Infinite factors of $\sin(x)$

We know, the half angle formula for  $\sin(x)$  is

$$\sin(x) = 2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) = 2 \sin\left(\frac{x}{2}\right) \sin\left(\frac{\pi}{2} + \frac{x}{2}\right)$$

Now, we find  $\sin(\frac{x}{2})$  and  $\sin(\frac{\pi}{2} + \frac{x}{2})$  and substitute them back into the above identity.

$$\sin\left(\frac{x}{2}\right) = 2 \sin\left(\frac{x}{2^2}\right) \sin\left(\frac{\pi}{2} + \frac{x}{2^2}\right) = 2 \sin\left(\frac{x}{2^2}\right) \sin\left(\frac{2\pi + x}{2^2}\right)$$

$$\sin\left(\frac{\pi}{2} + \frac{x}{2}\right) = 2 \sin\left(\frac{\pi}{2^2} + \frac{x}{2^2}\right) \sin\left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{x}{2^2}\right) = 2 \sin\left(\frac{\pi}{2^2} + \frac{x}{2^2}\right) \sin\left(\frac{3\pi + x}{2^2}\right)$$

After substitution,

$$\sin(x) = 2^3 \sin\left(\frac{x}{2^2}\right) \sin\left(\frac{\pi + x}{2^2}\right) \sin\left(\frac{2\pi + x}{2^2}\right) \sin\left(\frac{3\pi + x}{2^2}\right)$$

Repeat for each sine function that we obtain,

$$\sin(x) = 2^7 \sin\left(\frac{x}{2^3}\right) \sin\left(\frac{\pi + x}{2^3}\right) \dots \sin\left(\frac{7\pi + x}{2^3}\right)$$

from the pattern, for any  $p = \text{power of } 2$ ,

$$2^{p-1} \sin\left(\frac{x}{p}\right) \sin\left(\frac{\pi + x}{p}\right) \dots \sin\left(\frac{(p-1)\pi + x}{p}\right)$$

Last factor is,

$$\begin{aligned} \sin\left(\frac{(p-1)\pi + x}{p}\right) &= \sin\left(\pi - \frac{\pi - x}{p}\right) \\ &= \sin\left(\frac{\pi - x}{p}\right) \end{aligned}$$

The factor, one before the last factor,

$$\begin{aligned} \sin\left(\frac{(p-2)\pi + x}{p}\right) &= \sin\left(\pi - \frac{2\pi - x}{p}\right) \\ &= \sin\left(\frac{2\pi - x}{p}\right) \end{aligned}$$

Now, grouping the factors,

$$\sin(x) = 2^{p-1} \sin\left(\frac{x}{p}\right) \left[ \sin\left(\frac{\pi+x}{p}\right) \sin\left(\frac{\pi-x}{p}\right) \right] \left[ \sin\left(\frac{2\pi+x}{p}\right) \sin\left(\frac{2\pi-x}{p}\right) \right] \dots$$

The middle factor has been left out in the above grouping, which is,

$$\sin\left(\frac{\frac{p}{2}\pi + x}{p}\right) = \sin\left(\frac{\pi}{2} + \frac{x}{p}\right) = \cos(x)$$

Including this middle factor as well, we get,

$$\sin(x) = 2^{p-1} \sin\left(\frac{x}{p}\right) \left[ \sin\left(\frac{\pi+x}{p}\right) \sin\left(\frac{\pi-x}{p}\right) \right] \dots \left[ \sin\left(\frac{\frac{p}{2}\pi + x}{p}\right) \sin\left(\frac{\frac{p}{2}\pi - x}{p}\right) \right] \cos\left(\frac{x}{p}\right)$$

Using the formula,  $\sin(a+b) \cdot \sin(a-b) = \sin^2(a) - \sin^2(b)$ , which can be verified by substituting the expansions for  $\sin(a+b)$  and  $\sin(a-b)$ .

Since,

$$\lim_{x \rightarrow 0} \left[ \frac{\sin(x)}{\sin\left(\frac{x}{p}\right)} \right] = \lim_{x \rightarrow 0} \left[ p \frac{\sin(x)}{x} \cdot \frac{x/p}{\sin(x/p)} \right] = p$$

Substituting for  $\frac{\sin(x)}{\sin(x/p)}$ , and taking the limit as  $x$  approaches zero, we get,

$$p = 2^{2p-1} \sin^2\left(\frac{\pi}{p}\right) \sin^2\left(\frac{2\pi}{p}\right) \dots \sin^2\left(\frac{(\frac{p}{2}-1)\pi}{p}\right)$$

Now, if we take the ratio of  $\sin(x)$  to  $p$ , we get,

$$\frac{\sin(x)}{p} = \sin\left(\frac{x}{p}\right) \left[ 1 - \frac{\sin^2(x/p)}{\sin^2(\pi/p)} \right] \dots \left[ 1 - \frac{\sin^2(x/p)}{\sin^2\left(\frac{(\frac{p}{2}-1)\pi}{p}\right)} \right] \cos(x/p)$$

We want to express  $\sin(x)$  as infinite product. So we let  $p \rightarrow \infty$ . Thus, as  $p \rightarrow \infty$ ,  $p \sin(x/p) = x$  and  $\frac{\sin^2(x/p)}{\sin^2(\pi/p)} = \frac{x^2}{\pi^2}$

Substituting, we get:

$$\sin(x) = x \left( 1 - \frac{x^2}{\pi^2} \right) \left( 1 - \frac{x^2}{2^2\pi^2} \right) \left( 1 - \frac{x^2}{3^2\pi^2} \right) \dots$$

$$\sin(x) = x \prod_{r=1}^{\infty} \left( 1 - \frac{x^2}{r^2\pi^2} \right)$$

### 12.1 Basel Problem (Euler's Solution)

Using the infinite product formula and the infinite sum formula for  $\frac{\sin(x)}{x}$ .

$$\frac{\sin(x)}{x} = \left(1 - \frac{x^2}{1^2\pi^2}\right) \left(1 - \frac{x^2}{2^2\pi^2}\right) \left(1 - \frac{x^2}{3^2\pi^2}\right) \dots$$

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots$$

Multiplying the infinite product formula to obtain the  $x^2$  term and comparing it with the  $x^2$  term of the infinite sum formula, we get:

$$\frac{-x^2}{\pi^2} \left( \frac{1}{1^2} + \frac{1}{2^2} + \dots \right) = \frac{-x^2}{3!}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \dots = \frac{\pi^2}{6}$$

## 13 Stirling's Approximation

Using **Gamma Function**,

$$\Gamma(n+1) = \int_0^{\infty} x^n e^{-x} dx = n!$$

Substituting  $x = nz$  and  $dx = ndz$

$$\begin{aligned} n! &= \int_0^{\infty} (nz)^n e^{-nz} ndz \\ &= n^{n+1} \int_0^{\infty} z^n e^{-nz} dz \\ &= n^{n+1} \int_0^{\infty} e^{\ln(z^n)} e^{-nz} dz \\ &= n^{n+1} \int_0^{\infty} e^{n \ln(z)} e^{-nz} dz \\ &= n^{n+1} \int_0^{\infty} e^{n(\ln(z)-z)} dz \end{aligned}$$

Now, writing the **Taylor Series** expansion for  $\ln(z) - z$  centered at  $z = 1$ ,

$$f(z) = \ln(z) - z = f(1) + f'(1)(z-1) + \frac{1}{2!}f''(1)(z-1)^2 \dots$$

For first three term approximation of  $f(z)$ , Substituting  $f(1) = -1$ ,  $f'(1) = 0$ ,  $f''(1) = -1$ ,

$$f(z) = \ln(z) - z \approx -1 - \frac{1}{2!}(z-1)^2$$



Substituting this approximation in above expression,

$$\begin{aligned} n! &\approx n^{n+1} \int_0^\infty e^{n(-1-\frac{1}{2i}(z-1)^2)} dz \\ &= n^{n+1} e^{-n} \int_0^\infty e^{\frac{-n}{2}(z-1)^2} dx \end{aligned}$$

For large values of  $n$ ,

$$\begin{aligned} &= n^{n+1} e^{-n} \sqrt{\frac{2}{n}} \pi \\ &= \frac{n^{n+1}}{n^{1/2} n^{1/2}} e^{-n} \sqrt{2\pi n} \\ &= n^n e^{-n} \sqrt{2\pi n} \\ &= \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \end{aligned}$$

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

## 14 Vieta's Formula for $\pi$

We take the formula for  $\sin(x) = 2 \sin(\frac{x}{2}) \cos(\frac{x}{2})$  and apply this formula to itself repeatedly for the sine function as follows:

$$\begin{aligned}\sin(x) &= 2 \sin(x/2) \cos(x/2) \\ &= 2 \cdot 2 \sin(x/4) \cos(x/4) \cos(x/2) \\ &= 2 \cdot 2 \cdot 2 \sin(x/8) \cos(x/8) \cos(x/4) \cos(x/2)\end{aligned}$$

Repeating this process for n-times,

$$\sin(x) = 2^n \sin\left(\frac{x}{2^n}\right) \prod_{k=1}^n \cos\left(\frac{x}{2^k}\right)$$

Now, we take the limit of  $n \rightarrow \infty$  and  $\sin(x/2^n) \rightarrow x/2^n$  because as  $n$  gets larger,  $x/2^n \rightarrow 0$

$$\lim_{n \rightarrow \infty} \sin(x) = \lim_{n \rightarrow \infty} 2^n \sin\left(\frac{x}{2^n}\right) \prod_{k=1}^n \cos\left(\frac{x}{2^k}\right)$$

$$\begin{aligned}\sin(x) &= \lim_{n \rightarrow \infty} 2^n \left(\frac{x}{2^n}\right) \prod_{k=1}^n \cos\left(\frac{x}{2^k}\right) \\ &= \lim_{n \rightarrow \infty} x \prod_{k=1}^n \cos\left(\frac{x}{2^k}\right)\end{aligned}$$

Thus,

$$\frac{\sin(x)}{x} = \prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right)$$

When we substitute  $x = \frac{\pi}{2}$ ,

$$\frac{2}{\pi} = \cos\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{8}\right) \cos\left(\frac{\pi}{16}\right) \dots$$

we know the value for  $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ . So, in order to obtain all the other values for which the angles are just half of the previous angle, we use the identity,  $\cos(x/2) = 1/2\sqrt{2 + 2\cos(x)}$

$$\begin{aligned}
\cos(\pi/8) &= 1/2\sqrt{2+2\cos(\pi/4)} \\
&= 1/2\sqrt{2+2\sqrt{2}} \\
\cos(\pi/16) &= 1/2\sqrt{2+2\cos(\pi/8)} \\
&= 1/2\sqrt{2+\sqrt{2+2\sqrt{2}}}
\end{aligned}$$

Using this pattern, we obtain,

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \dots$$

## 15 Vieta's Formula For Polynomials

Consider the quadratic equation,

$$x^2 + \frac{a_1}{a_2}x + \frac{a_0}{a_2} = (x - r_1)(x - r_2)$$

where  $r_1$  and  $r_2$  are the roots of the polynomial. Expanding the right hand side, we get,

$$x^2 + \frac{a_1}{a_2}x + \frac{a_0}{a_2} = x^2 - (r_1 + r_2)x + r_1r_2$$

Now, comparing the coefficients of the like terms on both sides, we obtain,

$$r_1 + r_2 = \frac{-a_1}{a_2}$$

$$r_1r_2 = \frac{a_0}{a_2}$$

Consider the cubic equation,

$$x^3 + \frac{a_2}{a_3}x^2 + \frac{a_1}{a_3}x + \frac{a_0}{a_3} = (x - r_1)(x - r_2)(x - r_3)$$

where  $r_1$ ,  $r_2$  and  $r_3$  are the roots of the polynomial. Expanding the right hand side, we get,

$$x^3 + \frac{a_2}{a_3}x^2 + \frac{a_1}{a_3}x + \frac{a_0}{a_3} = x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_2r_3 + r_3r_1)x - r_1r_2r_3$$

Now, comparing the coefficients of the like terms on both sides, we obtain,

$$r_1 + r_2 + r_3 = \frac{-a_2}{a_3}$$

$$r_1r_2 + r_2r_3 + r_3r_1 = \frac{a_1}{a_3}$$

$$r_1r_2r_3 = \frac{-a_0}{a_3}$$

Now, we consider the general polynomial,

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_2x^2 + a_1x + a_0$$

Dividing both sides by  $a_n$ ,

$$\frac{p(x)}{a_n} = x^n + \frac{a_{n-1}}{a_n}x^{n-1} + \dots + \frac{a_2}{a_n}x^2 + \frac{a_1}{a_n}x + \frac{a_0}{a_n}$$

and consider  $r_1, r_2, \dots, r_n$  be the roots of this general polynomial. So,

$$p(x) = a_n(x - r_1)(x - r_2)\dots(x - r_n)$$

$$\frac{p(x)}{a_n} = (x - r_1)(x - r_2)\dots(x - r_n)$$

Expanding we obtain,

$$\frac{p(x)}{a_n} = x^n - (r_1 + r_2 + \dots + r_n)x^{n-1} + (r_1r_2 + r_2r_3 + \dots + r_{n-1}r_n)x^{n-2} - \dots + (r_1r_2r_3\dots r_n)$$

These coefficients are **Elementary Symmetric Polynomials**

### 15.1 Elementary Symmetric Polynomials

These polynomials can be written in the form,

$$s_k = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n} r_{i_1}r_{i_2}r_{i_3}\dots r_{i_k}$$

So, above expansion can be written as,

$$\frac{p(x)}{a_n} = x^n - s_1x^{n-1} + s_2x^{n-2} - \dots + s_n$$

Now, comparing the coefficients of the like terms we obtain,

$$\frac{a_{n-1}}{a_n} = -s_1$$

$$\frac{a_{n-2}}{a_n} = s_2$$

$$\frac{a_{n-3}}{a_n} = -s_3$$

The general form is,

$$\frac{a_{n-k}}{a_n} = (-1)^k s_k$$

Substituting the definition of  $s_k$ ,

$$\frac{a_{n-k}}{a_n} = (-1)^k \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n} r_{i_1}r_{i_2}r_{i_3}\dots r_{i_k}$$

## 16 Wallis Formula

Let's find reduction formulas for  $\int \sin^n(x)dx$  and  $\int \cos^n(x)dx$ . Let  $I_n = \int \sin^n(x)dx$

$$\begin{aligned}
 I_n &= \int \sin^n(x) = \int \sin^{n-1}(x) \sin(x)dx \\
 &= \int \sin^{n-1} d(-\cos(x)) \\
 &= \sin^{n-1}(x)(-\cos(x)) + (n-1) \int \cos(x) \cdot \sin^{n-2}(x) \cdot \cos(x)dx \\
 &= -\sin^{n-1}(x) \cdot \cos(x) + (n-1) \int (1 - \sin^2(x)) \cdot \sin^{n-2} dx \\
 &= -\sin^{n-1}(x) \cdot \cos(x) + (n-1) \left[ \int \sin^{n-2} dx - \int \sin^n(x)dx \right] \\
 I_n &= -\sin^{n-1}(x) \cdot \cos(x) + (n-1)[I_{n-2} - I_n] \\
 n \cdot I_n &= -\sin^{n-1}(x) \cdot \cos(x) + (n-1)I_{n-2} \\
 I_n &= -\frac{1}{n} \sin^{n-1}(x) \cdot \cos(x) + \frac{n-1}{n} I_{n-2}
 \end{aligned}$$

Using same method for evaluating  $\int \cos^n(x)dx$  we get following results

$$\begin{aligned}
 \int \sin^n(x)dx &= -\frac{1}{n} \sin^{n-1}(x) \cdot \cos(x) + \frac{n-1}{n} I_{n-2} \\
 \int \cos^n(x)dx &= \frac{1}{n} \cos^{n-1}(x) \cdot \sin(x) + \frac{n-1}{n} I_{n-2}
 \end{aligned}$$

Now, evaluating these integrals from 0 to  $\frac{\pi}{2}$  we get,

$$\int_0^{\frac{\pi}{2}} \cos^n(x)dx = \int_0^{\frac{\pi}{2}} \sin^n(x)dx = I_n = \frac{n-1}{n} I_{n-2}$$

### 16.1 Wallis Product Formula For $\pi$

Taking the ratios of integrals,

$$\frac{I_n}{I_{n-2}} = \frac{n-1}{n}$$

for **even** numbers,  $I_0 = \pi, I_2 = \frac{1}{2}\frac{\pi}{2}, I_4 = \frac{3}{4}\frac{1}{2}\frac{\pi}{2}$

$$I_{2n} = \frac{\pi}{2} \prod_{k=1}^n \frac{2k-1}{2k}$$

for **odd** numbers,  $I_1 = 1, I_3 = \frac{2}{3}, I_5 = \frac{4}{5}\frac{2}{3}$

$$I_{2n+1} = 1 \prod_{k=1}^n \frac{2k}{2k+1}$$

since, when  $0 < x < \frac{\pi}{2}$ ,  $0 < \sin(x) < 1$  which means,  $\sin^{2n+1}(x) \leq \sin^{2n}(x) \leq \sin^{2n-1}(x)$ . Thus,  $I_{2n+1} \leq I_{2n} \leq I_{2n-1}$ .

Dividing by  $I_{2n+1}$  we get,

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{I_{2n-1}}{I_{2n+1}}$$

Using the iterative relation from above,

$$\frac{I_{2n-1}}{I_{2n+1}} = \frac{2n}{2n+1}$$

Now, by substitution,

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{2n}{2n+1}$$

As  $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} 1 \leq \lim_{n \rightarrow \infty} \frac{I_{2n}}{I_{2n+1}} \leq \lim_{n \rightarrow \infty} \frac{2n}{2n+1}$$

$$1 \leq \lim_{n \rightarrow \infty} \frac{I_{2n}}{I_{2n+1}} \leq 1$$

Thus, by squeeze theorem,

$$\lim_{n \rightarrow \infty} \frac{I_{2n}}{I_{2n+1}} = 1$$

$$\frac{\pi}{2} \prod_{k=1}^{\infty} \frac{2k-1}{2k} \cdot \frac{2k+1}{2k} = 1$$

$$\frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{2k}{2k-1} \cdot \frac{2k}{2k+1}$$

This is known as **Wallis product**.

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \dots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \dots}$$