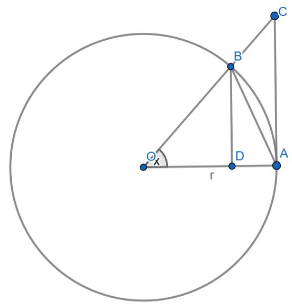
Contents

1 Basel Problem(Cauchy's Solution)



Relating the areas of $\triangle OAB$, sector OAB and $\triangle OAC$

$$\frac{r^2 \sin(x)}{2} < \frac{r^2 x}{2} < \frac{r^2 \tan(x)}{2}$$

Simplifying the inequality, we obtain,

$$\sin(x) < x < \tan(x)$$
$$\cot^2(x) < \frac{1}{x^2} < \csc^2(x)$$
$$\cot^2(x) < \frac{1}{x^2} < 1 + \cot^2(x)$$

Substituting $x = \frac{n\pi}{2N+1}$ for $1 \leq n \leq N$,

$$\cot^2\left(\frac{n\pi}{2N+1}\right) < \frac{(2N+1)^2}{n^2\pi^2} < 1 + \cot^2\left(\frac{n\pi}{2N+1}\right)$$

Now, multiply each term by $\frac{\pi^2}{(2N+1)^2}$,

$$\frac{\pi^2}{(2N+1)^2}\cot^2\left(\frac{n\pi}{2N+1}\right) < \frac{1}{n^2} < \frac{\pi^2}{(2N+1)^2} + \frac{\pi^2}{(2N+1)^2}\cot^2\left(\frac{n\pi}{2N+1}\right)$$

Now, we sum up the terms from n to N,

$$\sum_{n=1}^{N}\frac{\pi^2}{(2N+1)^2}\cot^2\left(\frac{n\pi}{2N+1}\right) < \sum_{n=1}^{N}\frac{1}{n^2} < \sum_{n=1}^{N}\left(\frac{\pi^2}{(2N+1)^2}+\frac{\pi^2}{(2N+1)^2}\cot^2\left(\frac{n\pi}{2N+1}\right)\right)$$

Simplifying further,

$$\frac{\pi^2}{(2N+1)^2}\sum_{n=1}^{N}\cot^2\left(\frac{n\pi}{2N+1}\right) < \sum_{n=1}^{N}\frac{1}{n^2} < \frac{N\pi^2}{(2N+1)^2}+\frac{\pi^2}{(2N+1)^2}\sum_{n=1}^{N}\cot^2\left(\frac{n\pi}{2N+1}\right)$$

Now, we take the limit as $N \to \infty$

$$\lim_{N\to\infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right) < \sum_{n=1}^{\infty} \frac{1}{n^2} < \lim_{N\to\infty} \frac{N\pi^2}{(2N+1)^2} + \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right)$$

Evaluating the limit of the term $\frac{N\pi^2}{(2N+1)^2}$, i.e

$$\lim_{N\to\infty} \frac{N\pi^2}{(2N+1)^2} = 0$$

Substituting back into the inequality,

$$\lim_{N\to\infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right) < \sum_{n=1}^{\infty} \frac{1}{n^2} < \lim_{N\to\infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right)$$

Now, the goal is to evaluate the limit and use **Squeeze Theorem**,

$$\lim_{N\to\infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right)$$

Using De Moivre's Theorem,

$$(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$$

Dividing both sides by $\sin^n(x)$

$$(\cot(x) + i)^n = \frac{\cos(nx) + i\sin(nx)}{\sin^n(x)}$$

Using the Binomial Theorem on $(\cot(x) + i)^n$ we get,

$$(\cot(x)+i)^n = \binom{n}{0}\cot^n(x) + \binom{n}{1}\cot^{n-1}(x)\cdot i + ... + \binom{n}{n-1}\cot(x)\cdot i^{n-1} + \binom{n}{n}\cdot i^n$$

Equating the imaginary coefficients,

$$\frac{\sin(nx)}{\sin^n(x)} = \binom{n}{1}\cot^{n-1}(x) - \binom{n}{3}\cot^{n-3}(x) + ...$$

Now, substituting $x = \frac{n\pi}{2N+1}$ and $n = (2N+1)$, we obtain, $\sin(nx) = 0$
Thus,

$$0 = \binom{2N+1}{1}\cot^{2N}(x) - \binom{2N+1}{3}\cot^{2N-2}(x) + ...$$

Let $t = \cot^2(x)$,

$$P(t) = \binom{2N+1}{1}t^N - \binom{2N+1}{3}t^{N-1} + ...$$

$P(t)$ is polynomial.

Now, using Vieta's sum of roots of polynomial formula on $P(t)$,

$$\begin{aligned}
\sum_{n=1}^{N} t_n &= \frac{\binom{2N+1}{3}}{\binom{2N+1}{1}} \\
&= \frac{(2N+1)!}{(2N+1-3)! \cdot 3!} \cdot \frac{(2N+1-1)!1!}{(2N+1)!} \\
&= \frac{(2N)!}{(2N-2)! \cdot 6} \\
&= \frac{(2N)(2N-1)}{6}
\end{aligned}$$

But we know $t_n = \cot^2(\frac{n\pi}{2N+1})$ is the root of $P(t)$. So,

$$\sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right) = \frac{(2N)(2N-1)}{6}$$

Our goal was to evaluate the limit,

$$\begin{aligned}
\lim_{N\to\infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right) &= \lim_{N\to\infty} \frac{\pi^2}{(2N+1)^2} \frac{(2N)(2N-1)}{6} \\
&= \frac{\pi^2}{6} \lim_{N\to\infty} \frac{4N^2-2N}{4N^2+4N+1} \\
&= \frac{\pi^2}{6}
\end{aligned}$$

Now, substituting this value in the inequality, we obtain,

$$\frac{\pi^2}{6} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{\pi^2}{6}$$

From Squeeze Theorem,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

2 Gamma Function

$$\int_0^1 log(x)dx = \Big[xlog(x) - 1 \cdot x\Big]_0^1 = -1!$$

$$\int_0^1 log^2(x)dx = \Big[xlog^2(x) - 2xlog(x) + 2 \cdot 1 \cdot x\Big]_0^1 = 2!$$

$$\int_0^1 log^3(x)dx = \Big[xlog^3(x) - 3xlog^2(x) - 2 \cdot 3xlog(x) - 3 \cdot 2 \cdot 1 \cdot x\Big]_0^1 = -3!$$

Observing the pattern, we can write,

$$\int_0^1 log^n(x)dx = (-1)^n \cdot n!$$

Solving for $n!$ we get,

$$n! = \frac{1}{(-1)^n} \int_0^1 log^n(x)dx$$

$$n! = \int_0^1 \left(\frac{log(x)}{-1}\right)^n dx$$

$$n! = \int_0^1 \big(-log(x)\big)^n dx$$

$$n! = \int_0^1 \left(log\left(\frac{1}{x}\right)\right)^n dx$$

Let $u = log\big(\frac{1}{x}\big)$. Using the property of logarithms $-u = log(x)$. Raising to the power of e on both sides, we get $x = e^{-u}$ and thus, $dx = -e^{-u}du$. When $x = 0$, $u \to \infty$ and when $x = 1$, $u = 0$. Therefore, the integral becomes,

$$n! = \int_0^\infty u^n(e^{-u})du$$

Now, by the definition of gamma function,

$$\Gamma(n+1) = n! = \int_0^\infty u^n(e^{-u})du$$

$$\Gamma(n) = (n-1)! = \int_0^\infty u^{n-1}e^{-u}du$$

Replacing n by x,

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du$$

2.1 $\Gamma(\frac{1}{2})$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty u^{-\frac{1}{2}} e^{-u} du$$

Let $t = u^{\frac{1}{2}}$ so that $dt = \frac{1}{2} u^{\frac{-1}{2}} du$. Making the substitution we get,

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-t^2} dt$$

Since, e^{-t^2} is an even function we can write,

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^\infty e^{-t^2} dt$$

This is know as **Gaussian Integral**. Let's evaluate this Integral,

$$I = \int_{-\infty}^\infty e^{-x^2} dx$$

$$I^2 = \left(\int_{-\infty}^\infty e^{-x^2} dx\right)^2$$

$$I^2 = \left(\int_{-\infty}^\infty e^{-x^2} dx\right) \cdot \left(\int_{-\infty}^\infty e^{-y^2} dy\right)$$

$$I^2 = \int_{-\infty}^\infty e^{-(x^2+y^2)} dxdy$$

Now in polar coordinates, $dxdy = dA = rdrd\theta$ and when $x \to -\infty, r \to 0$ and $x \to \infty, r \to \infty$ when $y \to -\infty, \theta \to 0$ and $y \to \infty, \theta \to 2\pi$

Substituting, we get,

$$I^2 = \int_0^{2\pi} \int_0^\infty re^{-r^2} drd\theta$$

$$I^2 = \int_0^{2\pi} \frac{1}{2} d\theta$$

$$I^2 = \frac{1}{2} \left[\theta\right]_0^{2\pi}$$

$$I^2 = \frac{1}{2} 2\pi$$

$$I^2 = \pi$$

$$I = \sqrt{\pi}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

3 Leibniz Theorem

Let u and v be functions of x. This theorem gives the closed form for the n^{th} derivative of product of two functions. That will be $(uv)^{(n)}$ or $\frac{d^n(u \cdot v)}{dx^n}$ for functions u and v.

for small values of n,

$$
\begin{aligned}
n = 1, (uv)' &= u'v + uv' \\
n = 2, (uv)'' &= u''v + 2u'v' + uv'' \\
n = 3, (uv)''' &= u'''v + 3u''v' + 3u'v'' + uv''' \\
n = 4, (uv)^{(4)} &= u^{(4)}v + 4u^{(3)}v' + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + uv^{(4)}
\end{aligned}
$$

Looking at the pattern from above, coefficients are **binomial coefficients**. So, we can write general form as,

$$(uv)^{(n)} = \frac{d^n(uv)}{dx^n} = \sum_{r=0}^{n} C(n,r)u^{(n-r)}v^{(r)}$$

where $C(n,r)$ is the binomial coefficient and $C(n,r) = \frac{n!}{r!(n-r)!}$.

3.1 Proof

Now, we prove this theorem by **induction**,
We assume for n=k, $(uv)^{(k)} = \frac{d^k(uv)}{dx^k} = \sum_{r=0}^{k} C(k,r)u^{(k-r)}v^{(r)}$ is true

And now for n=k+1, we prove,

$$(uv)^{(k+1)} = \frac{d^{k+1}(uv)}{dx^{k+1}} = \sum_{r=0}^{k+1} C(k+1,r)u^{(k+1-r)}v^{(r)}$$

By definition of successive differentiation,

$$
\begin{aligned}
(uv)^{(k+1)} = \frac{d^{k+1}(uv)}{dx^{k+1}} &= \frac{d}{dx}\sum_{r=0}^{k} C(k,r)u^{(k-r)}v^{(r)} \\
&= \sum_{r=0}^{k} C(k,r)\left(u^{(k-r+1)}v^{(r)} + u^{(k-r)}v^{(r+1)}\right) \\
&= \sum_{r=0}^{k} C(k,r)u^{(k-r+1)}v^{(r)} + \sum_{r=0}^{k} C(k,r)u^{(k-r)}v^{(r+1)} \\
&= \sum_{r=0}^{k} C(k,r)u^{(k-r+1)}v^{(r)} + \sum_{r=1}^{k+1} C(k,r-1)u^{(k-r+1)}v^{(r)}
\end{aligned}
$$

$$
\begin{aligned}
&= C(k,0)u^{(k+1)}v^{(0)} + C(k,k)u^{(0)}v^{k+1} + \sum_{r=1}^{k}\left(C(k,r) + C(k,r-1)\right)u^{(k-r+1)}v^{(r)} \\
&= u^{(k+1)}v^{(0)} + u^{(0)}v^{k+1} + \sum_{r=1}^{k} C(k+1,r)u^{(k-r+1)}v^{(r)} \\
&= \sum_{r=0}^{k} C(k+1,r)u^{(k+1-r)}v^{(r)}
\end{aligned}
$$

which is what we wanted to prove.

4 Maclaurin's Series

Let $f(x)$ be function of x. The goal is to write the function as an infinite polynomial.

$$f(x) = a + bx + cx^2 + dx^3 + ex^4 + ...$$

when x=0, $f(0) = a$ In order to find other coefficients we differentiate the function with respect to x and put $x = 0$. $f'(0) = 1 \cdot b$ $f''(0) = 1 \cdot 2 \cdot b$ $f'''(0) = 1 \cdot 2 \cdot 3 \cdot c$ and so on. Looking at the pattern, the general form would be,

$$f^{(n)}(0) = n! \cdot (n^{th} coefficient)$$

Substituting in the above equation for coefficients, we get,

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$$

In Summation Notation,

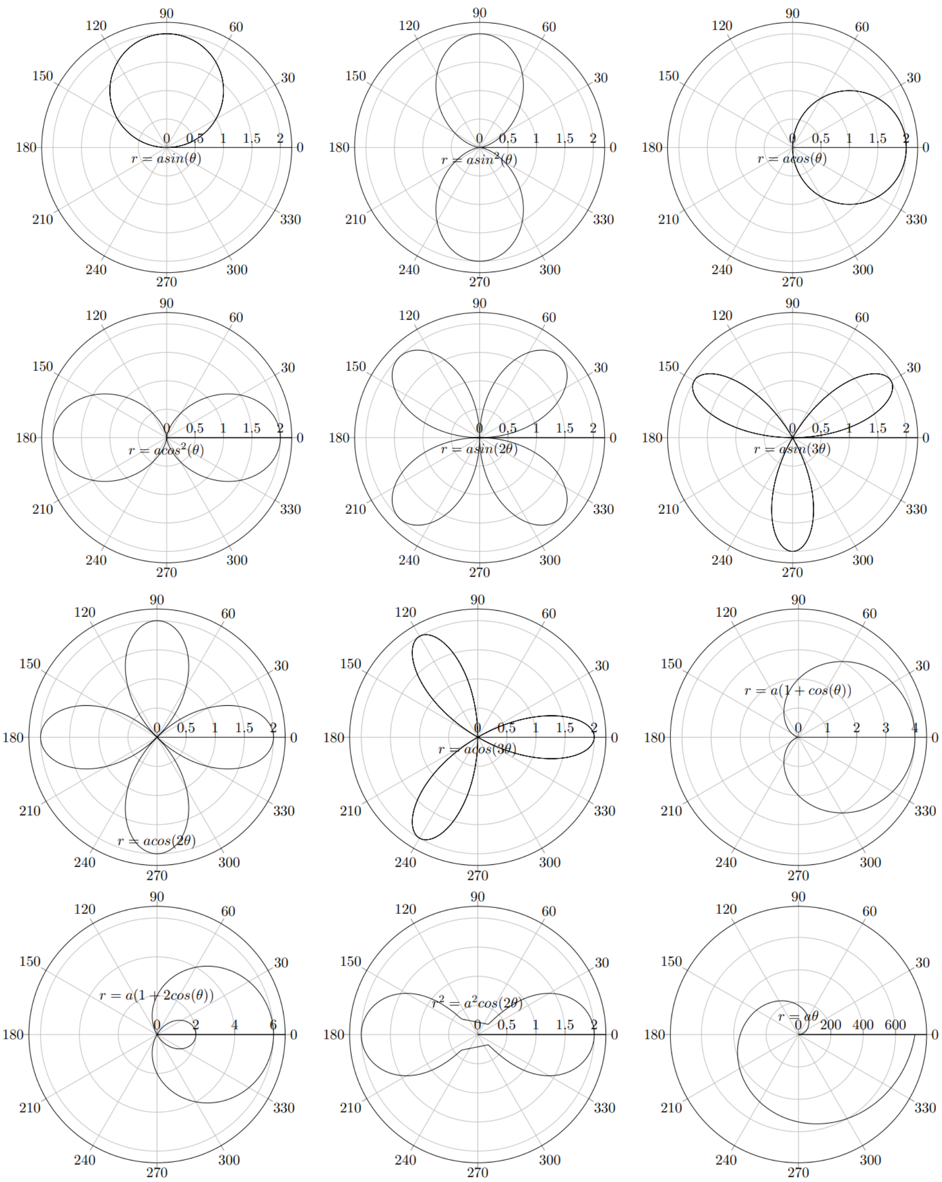
$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

$\sin(x)$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$
$\cos(x)$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$
$\tan(x)$	$x + \frac{x^3}{2} + \frac{2x^5}{15} - ...$
$\sinh(x)$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$
$\cosh(x)$	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} ...$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - ...$
$\tan^{-1}(x)$	$x - \frac{x^3}{3} + \frac{x^5}{5} + ...$

Now, the more general expansion gives the **Taylor Series** which is shifted by h.

$$f(x+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x)$$

5 Polar Curves



6 Infinite factors of $\sin(x)$

We know, the half angle formula for $\sin(x)$ is

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)cos\left(\frac{x}{2}\right) = 2\sin\left(\frac{x}{2}\right)sin\left(\frac{\pi}{2}+\frac{x}{2}\right)$$

Now, we find $\sin(\frac{x}{2})$ and $\sin(\frac{\pi}{2}+\frac{x}{2})$ and substitute them back into the above identity.

$$\sin\left(\frac{x}{2}\right) = 2\sin\left(\frac{x}{2^2}\right)\sin\left(\frac{\pi}{2}+\frac{x}{2^2}\right) = 2\sin\left(\frac{x}{2^2}\right)sin\left(\frac{2\pi+x}{2^2}\right)$$

$$\sin\left(\frac{\pi}{2}+\frac{x}{2}\right) = 2\sin\left(\frac{\pi}{2^2}+\frac{x}{2^2}\right)\sin\left(\frac{\pi}{2}+\frac{\pi}{2}+\frac{x}{2^2}\right) = 2\sin\left(\frac{\pi}{2^2}+\frac{x}{2^2}\right)sin\left(\frac{3\pi+x}{2^2}\right)$$

After substitution,

$$\sin(x) = 2^3\sin\left(\frac{x}{2^2}\right)\sin\left(\frac{\pi+x}{2^2}\right)\sin\left(\frac{2\pi+x}{2^2}\right)\sin\left(\frac{3\pi+x}{2^2}\right)$$

Repeat for each sine function that we obtain,

$$\sin(x) = 2^7\sin\left(\frac{x}{2^3}\right)\sin\left(\frac{\pi+x}{2^3}\right)...\sin\left(\frac{7\pi+x}{2^3}\right)$$

from the pattern, for any $p =$ power of 2,

$$2^{p-1}\sin\left(\frac{x}{p}\right)\sin\left(\frac{\pi+x}{p}\right)...\sin\left(\frac{(p-1)\pi+x}{p}\right)$$

Last factor is,

$$\begin{aligned}\sin\left(\frac{(p-1)\pi+x}{p}\right) &= \sin\left(\pi - \frac{\pi-x}{p}\right)\\ &= \sin\left(\frac{\pi-x}{p}\right)\end{aligned}$$

The factor, one before the last factor,

$$\begin{aligned}\sin\left(\frac{(p-2)\pi+x}{p}\right) &= \sin\left(\pi - \frac{2\pi-x}{p}\right)\\ &= \sin\left(\frac{2\pi-x}{p}\right)\end{aligned}$$

Now, grouping the factors,

$$\sin(x) = 2^{p-1}\sin\left(\frac{x}{p}\right)\left[\sin\left(\frac{\pi+x}{p}\right)\sin\left(\frac{\pi-x}{p}\right)\right]\left[\sin\left(\frac{2\pi+x}{p}\right)\sin\left(\frac{2\pi-x}{p}\right)\right]...$$

The middle factor has been left out in the above grouping, which is,

$$\sin\left(\frac{\frac{p}{2}\pi+x}{p}\right) = \sin\left(\frac{\pi}{2}+\frac{x}{p}\right) = \cos(x)$$

Including this middle factor as well, we get,

$$\sin(x) = 2^{p-1}\sin\left(\frac{x}{p}\right)\left[\sin\left(\frac{\pi+x}{p}\right)\sin\left(\frac{\pi-x}{p}\right)\right]...\left[\sin\left(\frac{\frac{p}{2}\pi+x}{p}\right)\sin\left(\frac{\frac{p}{2}\pi-x}{p}\right)\right]\cos\left(\frac{x}{p}\right)$$

Using the formula, $\sin(a+b)\cdot\sin(a-b) = sin^2(a) - sin^2(b)$, which can be verified by substituting the expansions for $\sin(a+b)$ and $\sin(a-b)$.

Since,

$$\lim_{x\to 0}\left[\frac{\sin(x)}{\sin\left(\frac{x}{p}\right)}\right] = \lim_{x\to 0}\left[p\frac{\sin(x)}{x}\cdot\frac{x/p}{\sin(x/p)}\right] = p$$

Substituting for $\frac{\sin(x)}{\sin(x/p)}$, and taking the limit as x approaches zero, we get,

$$p = 2^{2p-1}\sin^2\left(\frac{\pi}{p}\right)\sin^2\left(\frac{2\pi}{p}\right)...\sin^2\left(\frac{(\frac{p}{2}-1)\pi}{p}\right)$$

Now, if we take the ratio of $\sin(x)$ to p, we get,

$$\frac{\sin(x)}{p} = \sin\left(\frac{x}{p}\right)\left[1-\frac{\sin^2(x/p)}{\sin^2(\pi/p)}\right]...\left[1-\frac{\sin^2(x/p)}{\sin^2\left((\frac{p}{2}-1)\pi/p\right)}\right]\cos(x/p)$$

We want to express $\sin(x)$ as infinite product. So we let $p\to\infty$. Thus, as $p\to\infty$, $p\sin(x/p) = x$ and $\frac{\sin^2(x/p)}{sin^2(\pi/p)} = \frac{x^2}{\pi^2}$

Substituting, we get:

$$\sin(x) = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)...$$

$$\sin(x) = x\prod_{r=1}^{\infty}\left(1-\frac{x^2}{r^2\pi^2}\right)$$

6.1 Basel Problem (Euler's Solution)

Using the infinite product formula and the infinite sum formula for $\frac{\sin(x)}{x}$.

$$\frac{\sin(x)}{x} = \left(1 - \frac{x^2}{1^2\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)...$$

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - ...$$

Multiplying the infinite product formula to obtain the x^2 term and comparing it with the x^2 term of the infinite sum formula, we get:

$$\frac{-x^2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + ...\right) = \frac{-x^2}{3!}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}... = \frac{\pi^2}{6}$$

7 Stirling's Approximation

Using **Gamma Function**,

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx = n!$$

Substituting $x = nz$ and $dx = ndz$

$$\begin{aligned} n! &= \int_0^\infty (nz)^n e^{-nz} ndz \\ &= n^{n+1} \int_0^\infty z^n e^{-nz} dz \\ &= n^{n+1} \int_0^\infty e^{\ln(z^n)} e^{-nz} dz \\ &= n^{n+1} \int_0^\infty e^{n\ln(z)} e^{-nz} dz \\ &= n^{n+1} \int_0^\infty e^{n(\ln(z)-z)} dz \end{aligned}$$

Now, writing the **Taylor Series** expansion for $\ln(z) - z$ centered at $z = 1$,

$$f(z) = \ln(z) - z = f(1) + f'(1)(z-1) + \frac{1}{2!} f''(1)(z-1)^2 ...$$

For first three term approximation of $f(z)$, Substituting $f(1) = -1, f'(1) = 0, f''(1) = -1$,

$$f(z) = \ln(z) - z \approx -1 - \frac{1}{2!}(z-1)^2$$

Substituting this approximation in above expression,

$$
\begin{aligned}
n! &\approx n^{n+1}\int_0^\infty e^{n(-1-\frac{1}{2!}(z-1)^2)}dz \\
&= n^{n+1}e^{-n}\int_0^\infty e^{\frac{-n}{2}(z-1)^2}dx
\end{aligned}
$$

For large values of n,

$$
\begin{aligned}
&= n^{n+1}e^{-n}\sqrt{\frac{2}{n}\pi} \\
&= \frac{n^{n+1}}{n^{1/2}n^{1/2}}e^{-n}\sqrt{2\pi n} \\
&= n^n e^{-n}\sqrt{2\pi n} \\
&= \left(\frac{n}{e}\right)^n\sqrt{2\pi n}
\end{aligned}
$$

$$
n! \approx \left(\frac{n}{e}\right)^n\sqrt{2\pi n}
$$

8 Vieta's Formula for π

We take the formula for $\sin(x) = 2\sin(\frac{x}{2})\cos(\frac{x}{2})$ and apply this formula to itself repeatedly for the sine function as follows:

$$\begin{aligned}\sin(x) &= 2\sin(x/2)\cos(x/2)\\ &= 2\cdot 2\sin(x/4)\cos(x/4)\cos(x/2)\\ &= 2\cdot 2\cdot 2\sin(x/8)\cos(x/8)\cos(x/4)\cos(x/2)\end{aligned}$$

Repeating this process for n-times,

$$\sin(x) = 2^n \sin\left(\frac{x}{2^n}\right)\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)$$

Now, we take the limit of $n \to \infty$ and $\sin(x/2^n) \to x/2^n$ because as n gets larger, $x/2^n \to 0$

$$\lim_{n\to\infty}\sin(x) = \lim_{n\to\infty} 2^n \sin\left(\frac{x}{2^n}\right)\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)$$

$$\begin{aligned}\sin(x) &= \lim_{n\to\infty} 2^n \left(\frac{x}{2^n}\right)\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)\\ &= \lim_{n\to\infty} x\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)\end{aligned}$$

Thus,

$$\frac{\sin(x)}{x} = \prod_{k=1}^{\infty}\cos\left(\frac{x}{2^k}\right)$$

When we substitute $x = \frac{\pi}{2}$,

$$\frac{2}{\pi} = cos\left(\frac{\pi}{4}\right)cos\left(\frac{\pi}{8}\right)cos\left(\frac{\pi}{16}\right)...$$

we know the value for $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$. So, in order to obtain all the other values for which the angles are just half of the previous angle, we use the identity, $\cos(x/2) = 1/2\sqrt{2+2\cos(x)}$

$$\begin{aligned}
\cos(\pi/8) &= 1/2\sqrt{2 + 2\cos(\pi/4)} \\
&= 1/2\sqrt{2 + 2\sqrt{2}} \\
\cos(\pi/16) &= 1/2\sqrt{2 + 2\cos(\pi/8)} \\
&= 1/2\sqrt{2 + \sqrt{2 + 2\sqrt{2}}}
\end{aligned}$$

Using this pattern, we obtain,

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} ...$$

9 Vieta's formula for polynomials

Consider the quadratic equation,

$$x^2 + \frac{a_1}{a_2}x + \frac{a_0}{a_2} = (x - r_1)(x - r_2)$$

where r_1 and r_2 are the roots of the polynomial. Expanding the right hand side, we get,

$$x^2 + \frac{a_1}{a_2}x + \frac{a_0}{a_2} = x^2 - (r_1 + r_2)x + r_1 r_2$$

Now, comparing the coefficients of the like terms on both sides, we obtain,

$$r_1 + r_2 = \frac{-a_1}{a_2}$$

$$r_1 r_2 = \frac{a_0}{a_2}$$

Consider the cubic equation,

$$x^3 + \frac{a_2}{a_3}x^2 + \frac{a_1}{a_3}x + \frac{a_0}{a_3} = (x - r_1)(x - r_2)(x - r_3)$$

where r_1, r_2 and r_3 are the roots of the polynomial. Expanding the right hand side, we get,

$$x^3 + \frac{a_2}{a_3}x^2 + \frac{a_1}{a_3}x + \frac{a_0}{a_3} = x^3 - (r_1 + r_2 + r_3)x^2 + (r_1 r_2 + r_2 r_3 + r_3 r_1)x - r_1 r_2 r_3$$

Now, comparing the coefficients of the like terms on both sides, we obtain,

$$r_1 + r_2 + r_3 = \frac{-a_2}{a_3}$$

$$r_1 r_2 + r_2 r_3 + r_3 r_1 = \frac{a_1}{a_3}$$

$$r_1 r_2 r_3 = \frac{-a_0}{a_3}$$

Now, we consider the general polynomial,

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$$

Dividing both sides by a_n,

$$\frac{p(x)}{a_n} = x^n + \frac{a_{n-1}}{a_n}x^{n-1} + ... + \frac{a_2}{a_n}x^2 + \frac{a_1}{a_n}x + \frac{a_0}{a_n}$$

and consider $r_1, r_2, ..., r_n$ be the roots of this general polynomial. So,

$$p(x) = a_n(x - r_1)(x - r_2)...(x - r_n)$$

$$\frac{p(x)}{a_n} = (x - r_1)(x - r_2)...(x - r_n)$$

Expanding we obtain,

$$\frac{p(x)}{a_n} = x^n - (r_1 + r_2 + ... + r_n)x^{n-1} + (r_1r_2 + r_2r_3 + ... + r_{n-1}r_n)x^{n-2} - ... + (r_1r_2r_3...r_n)$$

These coefficients are **Elementary Symmetric Polynomials**

9.1 Elementary Symmetric Polynomials

These polynomials can be written in the form,

$$s_k = \sum_{1 \leq i_1 \leq i_2 \leq ... \leq i_k \leq n} r_{i_1} r_{i_2} r_{i_3} ... r_{i_k}$$

So, above expansion can be written as,

$$\frac{p(x)}{a_n} = x^n - s_1 x^{n-1} + s_2 x^{n-2} - ... + s_n$$

Now, comparing the coefficients of the like terms we obtain,

$$\frac{a_{n-1}}{a_n} = -s_1$$

$$\frac{a_{n-2}}{a_n} = s_2$$

$$\frac{a_{n-3}}{a_n} = -s_3$$

The general form is,

$$\frac{a_{n-k}}{a_n} = (-1)^k s_k$$

Substituting the definition of s_k,

$$\frac{a_{n-k}}{a_n} = (-1)^k \sum_{1 \leq i_1 \leq i_2 \leq ... \leq i_k \leq n} r_{i_1} r_{i_2} r_{i_3} ... r_{i_k}$$

10 Wallis Formula

Let's find reduction formulas for $\int \sin^n(x)dx$ and $\int \cos^n(x)dx$. Let $I_n = \int \sin^n(x)dx$

$$
\begin{aligned}
I_n = \int \sin^n(x) &= \int \sin^{n-1}(x)\sin(x)dx \\
&= \int \sin^{n-1} d(-\cos(x)) \\
&= \sin^{n-1}(x)(-\cos(x)) + (n-1)\int \cos(x)\cdot\sin^{n-2}(x)\cdot\cos(x)dx \\
&= -\sin^{n-1}(x)\cdot\cos(x) + (n-1)\int (1-\sin^2(x))\cdot\sin^{n-2} dx \\
&= -\sin^{n-1}(x)\cdot\cos(x) + (n-1)\left[\int \sin^{n-2} dx - \int \sin^n(x))dx\right] \\
I_n &= -\sin^{n-1}(x)\cdot\cos(x) + (n-1)[I_{n-2} - I_n] \\
n\cdot I_n &= -\sin^{n-1}(x)\cdot\cos(x) + (n-1)I_{n-2} \\
I_n &= -\frac{1}{n}\sin^{n-1}(x)\cdot\cos(x) + \frac{n-1}{n}I_{n-2}
\end{aligned}
$$

Using same method for evaluating $\int \cos^n(x)dx$ we get following results

$$\int \sin^n(x)dx = -\frac{1}{n}\sin^{n-1}(x)\cdot\cos(x) + \frac{n-1}{n}I_{n-2}$$

$$\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\cdot\sin(x) + \frac{n-1}{n}I_{n-2}$$

Now, evaluating these integrals from 0 to $\frac{\pi}{2}$ we get,

$$\int_0^{\frac{\pi}{2}} \cos^n(x)dx = \int_0^{\frac{\pi}{2}} \sin^n(x)dx = I_n = \frac{n-1}{n}I_{n-2}$$

10.1 Wallis Product formula for π

Taking the ratios of integrals,

$$\frac{I_n}{I_{n-2}} = \frac{n-1}{n}$$

for **even** numbers, $I_0 = \pi, I_2 = \frac{1}{2}\frac{\pi}{2}, I_4 = \frac{3}{4}\frac{1}{2}\frac{\pi}{2}$

$$I_{2n} = \frac{\pi}{2}\prod_{k=1}^{n}\frac{2k-1}{2k}$$

for **odd** numbers, $I_1 = 1, I_3 = \frac{2}{3}, I_5 = \frac{4}{5}\frac{2}{3}$

$$I_{2n+1} = 1\prod_{k=1}^{n}\frac{2k}{2k+1}$$

since, when $0 < x < \frac{\pi}{2}$, $0 < \sin(x) < 1$ which means, $\sin^{2n+1}(x) \leq \sin^{2n}(x) \leq \sin^{2n-1}(x)$. Thus, $I_{2n+1} \leq I_{2n} \leq I_{2n-1}$.

Dividing by I_{2n+1} we get,

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{I_{2n-1}}{I_{2n+1}}$$

Using the iterative relation from above,

$$\frac{I_{2n-1}}{I_{2n+1}} = \frac{2n}{2n+1}$$

Now, by substitution,

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{2n}{2n+1}$$

As $n \to \infty$

$$\lim_{n\to\infty} 1 \leq \lim_{n\to\infty}\frac{I_{2n}}{I_{2n+1}} \leq \lim_{n\to\infty}\frac{2n}{2n+1}$$

$$1 \leq \lim_{n\to\infty}\frac{I_{2n}}{I_{2n+1}} \leq 1$$

Thus, by squeeze theorem,

$$\lim_{n\to\infty}\frac{I_{2n}}{I_{2n+1}} = 1$$

$$\frac{\pi}{2}\prod_{k=1}^{\infty}\frac{2k-1}{2k}\cdot\frac{2k+1}{2k} = 1$$

$$\frac{\pi}{2} = \prod_{k=1}^{\infty}\frac{2k}{2k-1}\cdot\frac{2k}{2k+1}$$

This is known as **Wallis product**.

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6...}{1\cdot 3\cdot 3\cdot 5\cdot 5...}$$