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1 Gamma Function

$$\begin{aligned}\int_0^1 \log(x) dx &= \left[x \log(x) - 1 \cdot x \right]_0^1 = -1! \\ \int_0^1 \log^2(x) dx &= \left[x \log^2(x) - 2x \log(x) + 2 \cdot 1 \cdot x \right]_0^1 = 2! \\ \int_0^1 \log^3(x) dx &= \left[x \log^3(x) - 3x \log^2(x) - 2 \cdot 3x \log(x) - 3 \cdot 2 \cdot 1 \cdot x \right]_0^1 = -3!\end{aligned}$$

Observing the pattern, we can write,

$$\int_0^1 \log^n(x) dx = (-1)^n \cdot n!$$

Solving for $n!$ we get,

$$\begin{aligned}n! &= \frac{1}{(-1)^n} \int_0^1 \log^n(x) dx \\ n! &= \int_0^1 \left(\frac{\log(x)}{-1} \right)^n dx \\ n! &= \int_0^1 (-\log(x))^n dx \\ n! &= \int_0^1 \left(\log\left(\frac{1}{x}\right) \right)^n dx\end{aligned}$$

Let $u = \log\left(\frac{1}{x}\right)$. Using the property of logarithms $-u = \log(x)$. Raising to the power of e on both sides, we get $x = e^{-u}$ and thus, $dx = -e^{-u}du$. When $x = 0$, $u \rightarrow \infty$ and when $x = 1$, $u = 0$. Therefore, the integral becomes,

$$n! = \int_0^\infty u^n (e^{-u}) du$$

Now, by the definition of gamma function,

$$\begin{aligned}\Gamma(n+1) &= n! = \int_0^\infty u^n (e^{-u}) du \\ \Gamma(n) &= (n-1)! = \int_0^\infty u^{n-1} e^{-u} du\end{aligned}$$

Replacing n by x ,

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$$

1.1 $\Gamma(\frac{1}{2})$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty u^{-\frac{1}{2}} e^{-u} du$$

Let $t = u^{\frac{1}{2}}$ so that $dt = \frac{1}{2}u^{-\frac{1}{2}}du$. Making the substitution we get,

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^\infty e^{-t^2} dt$$

Since, e^{-t^2} is an even function we can write,

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^\infty e^{-t^2} dt$$

This is known as **Gaussian Integral**. Let's evaluate this Integral,

$$\begin{aligned} I &= \int_{-\infty}^\infty e^{-x^2} dx \\ I^2 &= \left(\int_{-\infty}^\infty e^{-x^2} dx \right)^2 \\ I^2 &= \left(\int_{-\infty}^\infty e^{-x^2} dx \right) \cdot \left(\int_{-\infty}^\infty e^{-y^2} dy \right) \\ I^2 &= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)} dx dy \end{aligned}$$

Now in polar coordinates, $dxdy = dA = r dr d\theta$ and when $x \rightarrow -\infty, r \rightarrow 0$ and $x \rightarrow \infty, r \rightarrow \infty$ when $y \rightarrow -\infty, \theta \rightarrow 0$ and $y \rightarrow \infty, \theta \rightarrow 2\pi$

Substituting, we get,

$$\begin{aligned} I^2 &= \int_0^{2\pi} \int_0^\infty r e^{-r^2} dr d\theta \\ I^2 &= \int_0^{2\pi} \frac{1}{2} d\theta \\ I^2 &= \frac{1}{2} [\theta]_0^{2\pi} \\ I^2 &= \frac{1}{2} 2\pi \\ I^2 &= \pi \\ I &= \sqrt{\pi} \\ \Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi} \end{aligned}$$

2 Leibniz Theorem

Let u and v be functions of x . This theorem gives the closed form for the n^{th} derivative of product of two functions. That will be $(uv)^{(n)}$ or $\frac{d^n(u \cdot v)}{dx^n}$ for functions u and v .

for small values of n,

$$\begin{aligned} n = 1, (uv)' &= u'v + uv' \\ n = 2, (uv)'' &= u''v + 2u'v' + uv'' \\ n = 3, (uv)''' &= u'''v + 3u''v' + 3u'v'' + uv''' \\ n = 4, (uv)^{(4)} &= u^{(4)}v + 4u^{(3)}v' + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + uv^{(4)} \end{aligned}$$

Looking at the pattern from above, coefficients are **binomial coefficients**. So, we can write general form as,

$$(uv)^{(n)} = \frac{d^n(uv)}{dx^n} = \sum_{r=0}^n C(n, r)u^{(n-r)}v^{(r)}$$

where $C(n, r)$ is the binomial coefficient and $C(n, r) = \frac{n!}{r!(n-r)!}$.

2.1 Proof

Now, we prove this theorem by **induction**,

We assume for $n=k$, $(uv)^{(k)} = \frac{d^k(uv)}{dx^k} = \sum_{r=0}^k C(k, r)u^{(k-r)}v^{(r)}$ is true

And now for $n=k+1$, we prove,

$$(uv)^{(k+1)} = \frac{d^{k+1}(uv)}{dx^{k+1}} = \sum_{r=0}^{k+1} C(k+1, r)u^{(k+1-r)}v^{(r)}$$

By definition of successive differentiation,

$$\begin{aligned}
(uv)^{(k+1)} &= \frac{d^{k+1}(uv)}{dx^{k+1}} = \frac{d}{dx} \sum_{r=0}^k C(k, r) u^{(k-r)} v^{(r)} \\
&= \sum_{r=0}^k C(k, r) \left(u^{(k-r+1)} v^{(r)} + u^{(k-r)} v^{(r+1)} \right) \\
&= \sum_{r=0}^k C(k, r) u^{(k-r+1)} v^{(r)} + \sum_{r=0}^k C(k, r) u^{(k-r)} v^{(r+1)} \\
&= \sum_{r=0}^k C(k, r) u^{(k-r+1)} v^{(r)} + \sum_{r=1}^{k+1} C(k, r-1) u^{(k-r+1)} v^{(r)} \\
&= C(k, 0) u^{(k+1)} v^{(0)} + C(k, k) u^{(0)} v^{k+1} + \sum_{r=1}^k \left(C(k, r) + C(k, r-1) \right) u^{(k-r+1)} v^{(r)} \\
&= u^{(k+1)} v^{(0)} + u^{(0)} v^{k+1} + \sum_{r=1}^k C(k+1, r) u^{(k-r+1)} v^{(r)} \\
&= \sum_{r=0}^k C(k+1, r) u^{(k+1-r)} v^{(r)}
\end{aligned}$$

which is what we wanted to prove.

3 Maclaurin's Series

Let $f(x)$ be function of x . The goal is to write the function as an infinite polynomial.

$$f(x) = a + bx + cx^2 + dx^3 + ex^4 + \dots$$

when $x=0$, $f(0) = a$ In order to find other coefficients we differentiate the function with respect to x and put $x = 0$. $f'(0) = 1 \cdot b$ $f''(0) = 1 \cdot 2 \cdot b$ $f'''(0) = 1 \cdot 2 \cdot 3 \cdot c$ and so on. Looking at the pattern, the general form would be,

$$f^{(n)}(0) = n! \cdot (n^{\text{th}} \text{coefficient})$$

Substituting in the above equation for coefficients, we get,

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

In Summation Notation,

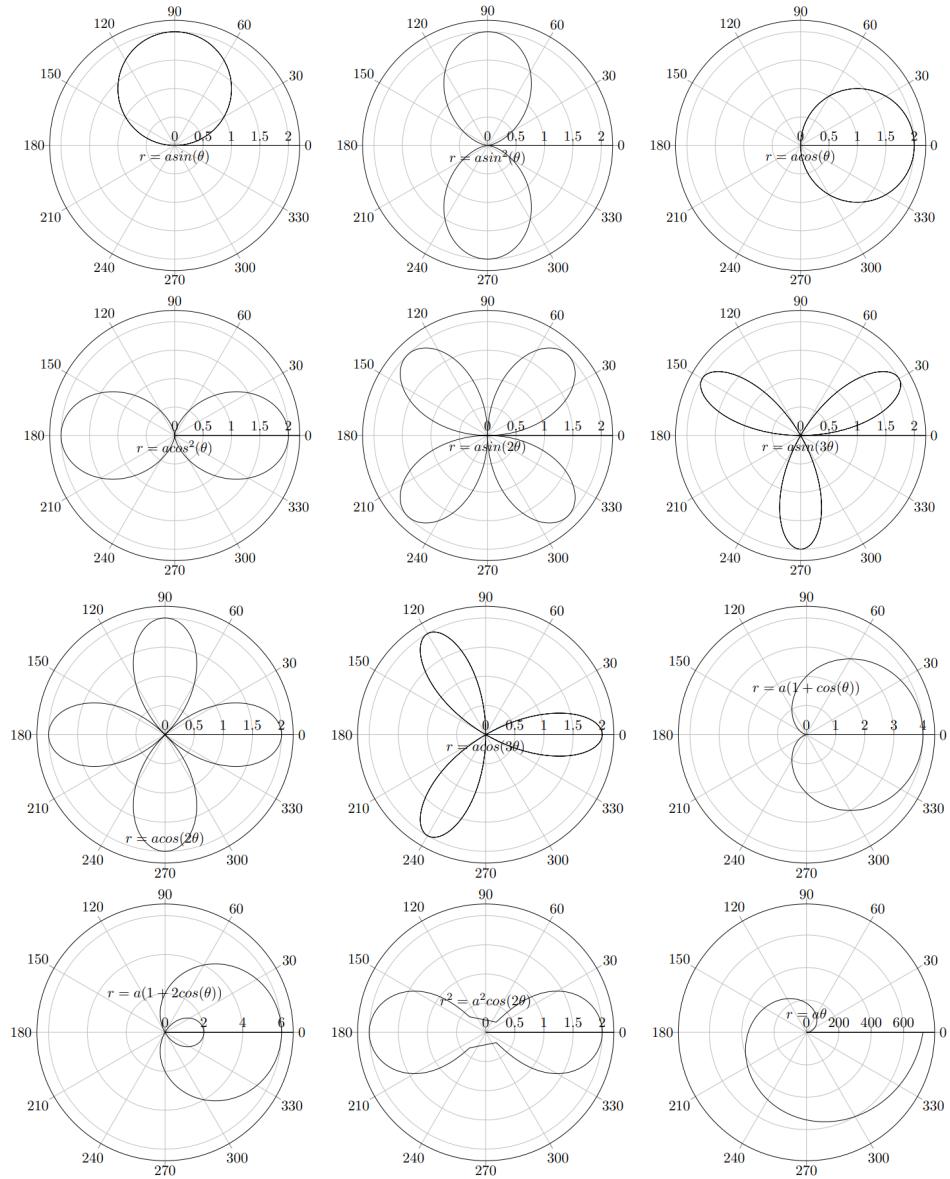
$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

$\sin(x)$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
$\cos(x)$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
$\tan(x)$	$x + \frac{x^3}{2} + \frac{2x^5}{15} - \dots$
$\sinh(x)$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$
$\cosh(x)$	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$
$\tan^{-1}(x)$	$x - \frac{x^3}{3} + \frac{x^5}{5} + \dots$

Now, the more general expansion gives the **Taylor Series** which is shifted by h .

$$f(x+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x)$$

4 Polar Curves



5 Infinite factors of $\sin(x)$

We know, the half angle formula for $\sin(x)$ is

$$\sin(x) = 2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) = 2 \sin\left(\frac{x}{2}\right) \sin\left(\frac{\pi}{2} + \frac{x}{2}\right)$$

Now, we find $\sin(\frac{x}{2})$ and $\sin(\frac{\pi}{2} + \frac{x}{2})$ and substitute back into the above identity.

$$\sin\left(\frac{x}{2}\right) = 2 \sin\left(\frac{x}{2^2}\right) \sin\left(\frac{\pi}{2} + \frac{x}{2^2}\right) = 2 \sin\left(\frac{x}{2^2}\right) \sin\left(\frac{2\pi + x}{2^2}\right)$$

$$\sin\left(\frac{\pi}{2} + \frac{x}{2}\right) = 2 \sin\left(\frac{\pi}{2^2} + \frac{x}{2^2}\right) \sin\left(\frac{\pi}{2} + \frac{\pi}{2} + \frac{x}{2^2}\right) = 2 \sin\left(\frac{\pi}{2^2} + \frac{x}{2^2}\right) \sin\left(\frac{3\pi + x}{2^2}\right)$$

After substitution,

$$\sin(x) = 2^3 \sin\left(\frac{x}{2^2}\right) \sin\left(\frac{\pi + x}{2^2}\right) \sin\left(\frac{2\pi + x}{2^2}\right) \sin\left(\frac{3\pi + x}{2^2}\right)$$

Repeat for each sine function that we obtain,

$$\sin(x) = 2^7 \sin\left(\frac{x}{2^3}\right) \sin\left(\frac{\pi + x}{2^3}\right) \dots \sin\left(\frac{7\pi + x}{2^3}\right)$$

from the pattern, for any $p = \text{power of } 2$,

$$2^{p-1} \sin\left(\frac{x}{p}\right) \sin\left(\frac{\pi + x}{p}\right) \dots \sin\left(\frac{(p-1)\pi + x}{p}\right)$$

Last factor is,

$$\begin{aligned} \sin\left(\frac{(p-1)\pi + x}{p}\right) &= \sin\left(\pi - \frac{\pi - x}{p}\right) \\ &= \sin\left(\frac{\pi - x}{p}\right) \end{aligned}$$

The factor, one before the last factor,

$$\begin{aligned} \sin\left(\frac{(p-2)\pi + x}{p}\right) &= \sin\left(\pi - \frac{2\pi - x}{p}\right) \\ &= \sin\left(\frac{2\pi - x}{p}\right) \end{aligned}$$

Now, grouping the factors,

$$\sin(x) = 2^{p-1} \sin\left(\frac{x}{p}\right) \left[\sin\left(\frac{\pi+x}{p}\right) \sin\left(\frac{\pi-x}{p}\right) \right] \left[\sin\left(\frac{2\pi+x}{p}\right) \sin\left(\frac{2\pi-x}{p}\right) \right] \dots$$

The middle factor has been left out in the above grouping, which is,

$$\sin\left(\frac{\frac{p}{2}\pi+x}{p}\right) = \sin\left(\frac{\pi}{2} + \frac{x}{p}\right) = \cos(x)$$

Including this middle factor as well, we get,

$$\sin(x) = 2^{p-1} \sin\left(\frac{x}{p}\right) \left[\sin\left(\frac{\pi+x}{p}\right) \sin\left(\frac{\pi-x}{p}\right) \right] \dots \left[\sin\left(\frac{\frac{p}{2}\pi+x}{p}\right) \sin\left(\frac{\frac{p}{2}\pi-x}{p}\right) \right] \cos\left(\frac{x}{p}\right)$$

Using the formula, $\sin(a+b) \cdot \sin(a-b) = \sin^2(a) - \sin^2(b)$, which can be verified by substituting the expansions for $\sin(a+b)$ and $\sin(a-b)$.

Since,

$$\lim_{x \rightarrow 0} \left[\frac{\sin(x)}{\sin(\frac{x}{p})} \right] = \lim_{x \rightarrow 0} \left[p \frac{\sin(x)}{x} \cdot \frac{x/p}{\sin(x/p)} \right] = p$$

Substituting for $\frac{\sin(x)}{\sin(x/p)}$, and taking the limit as x approaches zero, we get,

$$p = 2^{2p-1} \sin^2\left(\frac{\pi}{p}\right) \sin^2\left(\frac{2\pi}{p}\right) \dots \sin^2\left(\frac{(\frac{p}{2}-1)\pi}{p}\right)$$

Now, if we take the ratio of $\sin(x)$ to p , we get,

$$\frac{\sin(x)}{p} = \sin\left(\frac{x}{p}\right) \left[1 - \frac{\sin^2(x/p)}{\sin^2(\pi/p)} \right] \dots \left[1 - \frac{\sin^2(x/p)}{\sin^2((\frac{p}{2}-1)\pi/p)} \right] \cos(x/p)$$

We want to express $\sin(x)$ as infinite product. So we let $p \rightarrow \infty$. Thus, as $p \rightarrow \infty$, $p \sin(x/p) = x$ and $\frac{\sin^2(x/p)}{\sin^2(\pi/p)} = \frac{x^2}{\pi^2}$

Substituting, we get:

$$\sin(x) = x \left(1 - \frac{x^2}{\pi^2} \right) \left(1 - \frac{x^2}{2^2\pi^2} \right) \left(1 - \frac{x^2}{3^2\pi^2} \right) \dots$$

$$\sin(x) = x \prod_{r=1}^{\infty} \left(1 - \frac{x^2}{r^2\pi^2} \right)$$

5.1 Basel Problem

Using the infinite product formula and the infinite sum formula for $\frac{\sin(x)}{x}$.

$$\frac{\sin(x)}{x} = \left(1 - \frac{x^2}{1^2\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\dots$$
$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots$$

Multiplying the infinite product formula to obtain the x^2 term and comparing it with the x^2 term of the infinite sum formula, we get:

$$\frac{-x^2}{\pi^2} \left(\frac{1}{1^2} + \frac{1}{2^2} + \dots \right) = \frac{-x^2}{3!}$$
$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \dots = \frac{\pi^2}{6}$$

6 Vieta's Formula for π

We take the formula for $\sin(x) = 2 \sin(\frac{x}{2}) \cos(\frac{x}{2})$ and apply this formula to itself repeatedly for the sine function as follows:

$$\begin{aligned}\sin(x) &= 2 \sin(x/2) \cos(x/2) \\ &= 2 \cdot 2 \sin(x/4) \cos(x/4) \cos(x/2) \\ &= 2 \cdot 2 \cdot 2 \sin(x/8) \cos(x/8) \cos(x/4) \cos(x/2)\end{aligned}$$

Repeating this process for n-times,

$$\sin(x) = 2^n \sin\left(\frac{x}{2^n}\right) \prod_{k=1}^n \cos\left(\frac{x}{2^k}\right)$$

Now, we take the limit of $n \rightarrow \infty$ and $\sin(x/2^n) \rightarrow x/2^n$ because as n gets larger, $x/2^n \rightarrow 0$

$$\begin{aligned}\lim_{n \rightarrow \infty} \sin(x) &= \lim_{n \rightarrow \infty} 2^n \sin\left(\frac{x}{2^n}\right) \prod_{k=1}^n \cos\left(\frac{x}{2^k}\right) \\ \sin(x) &= \lim_{n \rightarrow \infty} 2^n \left(\frac{x}{2^n}\right) \prod_{k=1}^n \cos\left(\frac{x}{2^k}\right) \\ &= \lim_{n \rightarrow \infty} x \prod_{k=1}^n \cos\left(\frac{x}{2^k}\right)\end{aligned}$$

Thus,

$$\frac{\sin(x)}{x} = \prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right)$$

When we substitute $x = \frac{\pi}{2}$,

$$\frac{2}{\pi} = \cos\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{8}\right) \cos\left(\frac{\pi}{16}\right) \dots$$

we know the value for $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$. So, in order to obtain all the other values for which the angles are just half of the previous angle, we use the identity, $\cos(x/2) = 1/2\sqrt{2 + 2\cos(x)}$

$$\begin{aligned}
\cos(\pi/8) &= 1/2\sqrt{2 + 2\cos(\pi/4)} \\
&= 1/2\sqrt{2 + 2\sqrt{2}} \\
\cos(\pi/16) &= 1/2\sqrt{2 + 2\cos(\pi/8)} \\
&= 1/2\sqrt{2 + \sqrt{2 + 2\sqrt{2}}}
\end{aligned}$$

Using this pattern, we obtain,

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2 + \sqrt{2}}}{2} \cdot \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdot \frac{\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}}{2} \cdots$$

7 Wallis Formula

Let's find reduction formulas for $\int \sin^n(x)dx$ and $\int \cos^n(x)dx$. Let $I_n = \int \sin^n(x)dx$

$$\begin{aligned}
I_n &= \int \sin^n(x) = \int \sin^{n-1}(x) \sin(x) dx \\
&= \int \sin^{n-1} d(-\cos(x)) \\
&= \sin^{n-1}(x)(-\cos(x)) + (n-1) \int \cos(x) \cdot \sin^{n-2}(x) \cdot \cos(x) dx \\
&= -\sin^{n-1}(x) \cdot \cos(x) + (n-1) \int (1 - \sin^2(x)) \cdot \sin^{n-2} dx \\
&= -\sin^{n-1}(x) \cdot \cos(x) + (n-1) \left[\int \sin^{n-2} dx - \int \sin^n(x) dx \right] \\
I_n &= -\sin^{n-1}(x) \cdot \cos(x) + (n-1)[I_{n-2} - I_n] \\
n \cdot I_n &= -\sin^{n-1}(x) \cdot \cos(x) + (n-1)I_{n-2} \\
I_n &= -\frac{1}{n} \sin^{n-1}(x) \cdot \cos(x) + \frac{n-1}{n} I_{n-2}
\end{aligned}$$

Using same method for evaluating $\int \cos^n(x)dx$ we get following results

$$\begin{aligned}
\int \sin^n(x)dx &= -\frac{1}{n} \sin^{n-1}(x) \cdot \cos(x) + \frac{n-1}{n} I_{n-2} \\
\int \cos^n(x)dx &= \frac{1}{n} \cos^{n-1}(x) \cdot \sin(x) + \frac{n-1}{n} I_{n-2}
\end{aligned}$$

Now, evaluating these integrals from 0 to $\frac{\pi}{2}$ we get,

$$\int_0^{\frac{\pi}{2}} \cos^n(x)dx = \int_0^{\frac{\pi}{2}} \sin^n(x)dx = I_n = \frac{n-1}{n} I_{n-2}$$

7.1 Wallis Product formula for π

Taking the ratios of integrals,

$$\frac{I_n}{I_{n-2}} = \frac{n-1}{n}$$

for **even** numbers, $I_0 = \pi, I_2 = \frac{1}{2} \frac{\pi}{2}, I_4 = \frac{3}{4} \frac{1}{2} \frac{\pi}{2}$

$$I_{2n} = \frac{\pi}{2} \prod_{k=1}^n \frac{2k-1}{2k}$$

for **odd** numbers, $I_1 = 1, I_3 = \frac{2}{3}, I_5 = \frac{4}{5} \frac{2}{3}$

$$I_{2n+1} = 1 \prod_{k=1}^n \frac{2k}{2k+1}$$

since, when $0 < x < \frac{\pi}{2}$, $0 < \sin(x) < 1$ which means, $\sin^{2n+1}(x) \leq \sin^{2n}(x) \leq \sin^{2n-1}(x)$. Thus, $I_{2n+1} \leq I_{2n} \leq I_{2n-1}$.

Dividing by I_{2n+1} we get,

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{I_{2n-1}}{I_{2n+1}}$$

Using the iterative relation from above,

$$\frac{I_{2n-1}}{I_{2n+1}} = \frac{2n}{2n+1}$$

Now, by substitution,

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{2n}{2n+1}$$

As $n \rightarrow \infty$

$$\begin{aligned} \lim_{n \rightarrow \infty} 1 &\leq \lim_{n \rightarrow \infty} \frac{I_{2n}}{I_{2n+1}} \leq \lim_{n \rightarrow \infty} \frac{2n}{2n+1} \\ 1 &\leq \lim_{n \rightarrow \infty} \frac{I_{2n}}{I_{2n+1}} \leq 1 \end{aligned}$$

Thus, by squeeze theorem,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{I_{2n}}{I_{2n+1}} &= 1 \\ \frac{\pi}{2} \prod_{k=1}^{\infty} \frac{2k-1}{2k} \cdot \frac{2k+1}{2k} &= 1 \\ \frac{\pi}{2} &= \prod_{k=1}^{\infty} \frac{2k}{2k-1} \cdot \frac{2k}{2k+1} \end{aligned}$$

This is known as **Wallis product**.

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \dots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \dots}$$