1 Gamma Function

$$\int_0^1 log(x)dx = \Big[xlog(x) - 1 \cdot x\Big]_0^1 = -1!$$

$$\int_0^1 log^2(x)dx = \Big[xlog^2(x) - 2xlog(x) + 2 \cdot 1 \cdot x\Big]_0^1 = 2!$$

$$\int_0^1 log^3(x)dx = \Big[xlog^3(x) - 3xlog^2(x) - 2 \cdot 3xlog(x) - 3 \cdot 2 \cdot 1 \cdot x\Big]_0^1 = -3!$$

Observing the pattern, we can write,

$$\int_0^1 log^n(x)dx = (-1)^n \cdot n!$$

Solving for $n!$ we get,

$$n! = \frac{1}{(-1)^n} \int_0^1 log^n(x)dx$$

$$n! = \int_0^1 \left(\frac{log(x)}{-1}\right)^n dx$$

$$n! = \int_0^1 \big(-log(x)\big)^n dx$$

$$n! = \int_0^1 \left(log\left(\frac{1}{x}\right)\right)^n dx$$

Let $u = log(\frac{1}{x})$. Using the property of logarithms $-u = log(x)$. Raising to the power of e on both sides, we get $x = e^{-u}$ and thus, $dx = -e^{-u}du$. When $x = 0$, $u \to \infty$ and when $x = 1$, $u = 0$. Therefore, the integral becomes,

$$n! = \int_0^\infty u^n(e^{-u})du$$

Now, by the definition of gamma function,

$$\Gamma(n+1) = n! = \int_0^\infty u^n(e^{-u})du$$

$$\Gamma(n) = (n-1)! = \int_0^\infty u^{n-1}e^{-u}du$$

Replacing u by x,

$$\Gamma(x) = \int_0^\infty x^{n-1}e^{-x}du$$