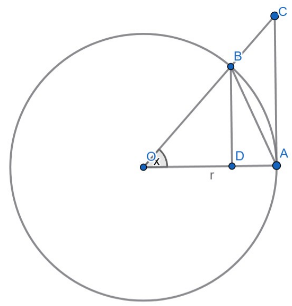
Contents

1 Basel Problem(Cauchy's Solution)



Relating the areas of $\triangle OAB$, sector OAB and $\triangle OAC$

$$\frac{r^2\sin(x)}{2} < \frac{r^2 x}{2} < \frac{r^2\tan(x)}{2}$$

Simplifying the inequality, we obtain,

$$\sin(x) < x < \tan(x)$$
$$\cot^2(x) < \frac{1}{x^2} < \csc^2(x)$$
$$\cot^2(x) < \frac{1}{x^2} < 1+\cot^2(x)$$

Substituting $x = \frac{n\pi}{2N+1}$ for $1 \le n \le N$,

$$\cot^2\left(\frac{n\pi}{2N+1}\right) < \frac{(2N+1)^2}{n^2\pi^2} < 1+\cot^2\left(\frac{n\pi}{2N+1}\right)$$

Now, multiply each term by $\frac{\pi^2}{(2N+1)^2}$,

$$\frac{\pi^2}{(2N+1)^2}\cot^2\left(\frac{n\pi}{2N+1}\right) < \frac{1}{n^2} < \frac{\pi^2}{(2N+1)^2} + \frac{\pi^2}{(2N+1)^2}\cot^2\left(\frac{n\pi}{2N+1}\right)$$

Now, we sum up the terms from n to N,

$$\sum_{n=1}^{N}\frac{\pi^2}{(2N+1)^2}\cot^2\left(\frac{n\pi}{2N+1}\right) < \sum_{n=1}^{N}\frac{1}{n^2} < \sum_{n=1}^{N}\left(\frac{\pi^2}{(2N+1)^2}+\frac{\pi^2}{(2N+1)^2}\cot^2\left(\frac{n\pi}{2N+1}\right)\right)$$

Simplifying further,

$$\frac{\pi^2}{(2N+1)^2}\sum_{n=1}^{N}\cot^2\left(\frac{n\pi}{2N+1}\right) < \sum_{n=1}^{N}\frac{1}{n^2} < \frac{N\pi^2}{(2N+1)^2}+\frac{\pi^2}{(2N+1)^2}\sum_{n=1}^{N}\cot^2\left(\frac{n\pi}{2N+1}\right)$$

Now, we take the limit as $N \to \infty$

$$\lim_{N\to\infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right) < \sum_{n=1}^{\infty} \frac{1}{n^2} < \lim_{N\to\infty} \frac{N\pi^2}{(2N+1)^2} + \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right)$$

Evaluating the limit of the term $\frac{N\pi^2}{(2N+1)^2}$, i.e

$$\lim_{N\to\infty} \frac{N\pi^2}{(2N+1)^2} = 0$$

Substituting back into the inequality,

$$\lim_{N\to\infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right) < \sum_{n=1}^{\infty} \frac{1}{n^2} < \lim_{N\to\infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right)$$

Now, the goal is to evaluate the limit and use **Squeeze Theorem**,

$$\lim_{N\to\infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right)$$

Using De Moivre's Theorem,

$$(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$$

Dividing both sides by $\sin^n(x)$

$$(\cot(x) + i)^n = \frac{\cos(nx) + i\sin(nx)}{\sin^n(x)}$$

Using the Binomial Theorem on $(\cot(x) + i)^n$ we get,

$$(\cot(x)+i)^n = \binom{n}{0} \cot^n(x) + \binom{n}{1} \cot^{n-1}(x) \cdot i + ... + \binom{n}{n-1} \cot(x) \cdot i^{n-1} + \binom{n}{n} \cdot i^n$$

Equating the imaginary coefficients,

$$\frac{\sin(nx)}{\sin^n(x)} = \binom{n}{1} \cot^{n-1}(x) - \binom{n}{3} \cot^{n-3}(x) + ...$$

Now, substituting $x = \frac{n\pi}{2N+1}$ and $n = (2N+1)$, we obtain, $\sin(nx) = 0$
Thus,

$$0 = \binom{2N+1}{1} \cot^{2N}(x) - \binom{2N+1}{3} \cot^{2N-2}(x) + ...$$

Let $t = \cot^2(x)$,

$$P(t) = \binom{2N+1}{1} t^N - \binom{2N+1}{3} t^{N-1} + ...$$

$P(t)$ is polynomial.

Now, using Vieta's sum of roots of polynomial formula on $P(t)$,

$$\begin{aligned}
\sum_{n=1}^{N} t_n &= \frac{\binom{2N+1}{3}}{\binom{2N+1}{1}} \\
&= \frac{(2N+1)!}{(2N+1-3)! \cdot 3!} \cdot \frac{(2N+1-1)!1!}{(2N+1)!} \\
&= \frac{(2N)!}{(2N-2)! \cdot 6} \\
&= \frac{(2N)(2N-1)}{6}
\end{aligned}$$

But we know $t_n = \cot^2(\frac{n\pi}{2N+1})$ is the root of $P(t)$. So,

$$\sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right) = \frac{(2N)(2N-1)}{6}$$

Our goal was to evaluate the limit,

$$\begin{aligned}
\lim_{N\to\infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right) &= \lim_{N\to\infty} \frac{\pi^2}{(2N+1)^2} \frac{(2N)(2N-1)}{6} \\
&= \frac{\pi^2}{6} \lim_{N\to\infty} \frac{4N^2-2N}{4N^2+4N+1} \\
&= \frac{\pi^2}{6}
\end{aligned}$$

Now, substituting this value in the inequality, we obtain,

$$\frac{\pi^2}{6} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{\pi^2}{6}$$

From Squeeze Theorem,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

2 Bernoulli Differential Equation

Let general equation be,

$$\frac{dy}{dx} + Py = Qy^n$$

Dividing both sides by y^n we obtain,

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

Let $z = y^{1-n}$, $\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$. Multiplying above equation by $(1-n)$ and substituting we obtain,

$$(1-n)y^{-n}\frac{dy}{dx} + (1-n)Py^{1-n} = Q(1-n)$$

$$\frac{dz}{dx} + (1-n)Pz = Q(1-n)$$

Now, this equation can be solved by using **Integrating Factor Method**. Let $(1-n)P = P_1$ and $Q(1-n) = Q_1$, also let I.F be Integrating Factor, I.F $= e^{\int P_1 dx}$

Therefore,

$$\text{IF} \cdot z = \int Q_1 \cdot \text{IF} dx$$

We solve for z first and then use $z = y^{1-n}$ relation to solve for y.

3 Cauchy-Schwarz Inequality

Let $\vec{x}$ and $\vec{y}$ be two non-zero vectors in $\Re^n$.

Let $p(t) = |t\vec{y} - \vec{x}|^2, p(t) \geq 0$

$$\begin{aligned}
p(t) &= (t\vec{y} - \vec{x}) \cdot (t\vec{y} - \vec{x}) \\
p(t) &= t\vec{y} \cdot t\vec{y} - \vec{x} \cdot t\vec{y} - t\vec{y} \cdot \vec{x} + \vec{x} \cdot \vec{x} \\
p(t) &= (\vec{y} \cdot \vec{y})t^2 - 2(\vec{x} \cdot \vec{y})t + \vec{x} \cdot \vec{x} \geq 0
\end{aligned}$$

Let $a = (\vec{y} \cdot \vec{y}), b = 2(\vec{x} \cdot \vec{y})$ and $c = \vec{x} \cdot \vec{x}$.

Evaluating $p(t)$ at $t = \frac{b}{2a}$,

$$\begin{aligned}
p\left(\frac{b}{2a}\right) &= a\frac{b^2}{4a^2} - b \cdot \ \frac{b}{2a} + c \geq 0 \\
&= \frac{b^2}{4a} - \frac{2b^2}{4a} + c \geq 0 \\
&= -\frac{b^2}{4a} + c \geq 0
\end{aligned}$$

Now, rearranging and substituting we obtain,

$$\begin{aligned}
c &\geq \frac{b^2}{4a} \\
4ac &\geq b^2 \\
4(\vec{y} \cdot \vec{y})(\vec{x} \cdot \vec{x}) &\geq 4(\vec{x} \cdot \vec{y})^2 \\
(\vec{y} \cdot \vec{y})(\vec{x} \cdot \vec{x}) &\geq (\vec{x} \cdot \vec{y})^2 \\
|\vec{x}|^2|\vec{y}|^2 &\geq (\vec{x} \cdot \vec{y})^2 \\
|\vec{x}||\vec{y}| &\geq \vec{x} \cdot \vec{y}
\end{aligned}$$

This inequality is known as Cauchy-Schwarz Inequality.

4 Electro-chemistry and Buffer

4.0.1 Electro-Chemical cells

Electro-chemical cells are devices that either generate electrical energy from chemical reactions or use electrical energy to cause non-spontaneous chemical reactions.

*They convert **chemical energy** into **electrical energy** and vice-versa.*

Electrodes are solid conductors. They are of two types: **anode** and **cathode** ***(either reactive (Copper, Zinc and Nickel) or non-reactive (graphite, platinum or silicon))***

Anode: The electrode where **oxidation**(loss of electrons) occurs.
Cathode: The electrode where **reduction**(gain of electrons) occurs.

These electrodes are immersed in an **electrolyte**, which is an **ion-conducting** solution. **(It means liquid itself behave as if it is copper wire carrying electrons but not exactly.)**

There are two types of electro-chemical cells **on the basis of direction of the energy conversion**.

1. Galvanic Cells(Voltaic Cells): Converts **chemical energy** into **electrical energy** via **spontaneous** redox reaction($\Delta G < 0$). These cells are **batteries**. **Anode** is **NEGATIVE** and **Cathode** is **POSITIVE**
2. Electrolytic Cells: Converts **electrical energy** into **chemical energy** via **non-spontaneous** redox reaction($\Delta G > 0$). These cells are used in **electroplating, purification of metals, and industrial production of chemicals (e.g., chlorine, sodium)**. **Anode** is **POSITIVE** and **Cathode** is **NEGATIVE**

4.1 Electrode Potential and Standard Electrode Potential

Electrode Potential is the potential difference that develops between the metal electrode and the electrolyte solution in which it is immersed.

When a metal electrode is placed in a solution containing its own ions, two opposing processes occur at the metal-solution interface, leading to an equilibrium:

Oxidation(Dissolution):

$$M(s) \to M^{n+}(aq) + ne^-$$

Reduction(Deposition):

$$M^{n+}(aq) + ne^- \to M(s)$$

At **equilibrium**, the rate of oxidation is equal to the rate of reduction. The **net charge** built up on the metal surface relative to the solution creates a **potential difference** known as electrode potential.

Absolute electrode potential cannot be measured. This is because any measurement requires a complete circuit with a second electrode which leads to second potential. Thus, it is always measured **relative to a reference electrode**(like the **Standard Hydrogen Electrode,** $E^0 = 0V$)

Standard Electrode Potential(E^o) is the electrode potential measured under standard conditions ($1M$ concentration, 1 bar pressure, $298K$ temperature).

4.1.1 Measurement of Electrode Potential

The universal reference for measuring standard electrode potentials is the **Standard Hydrogen Electrode(SHE)**.

SHE has following characteristics:

1. **Defined Potential**: The potential of the SHE is defined as exactly $0V$ at **all temperatures**

2. **Setup**: It consists of a **platinum electrode** coated with platinum black (to increase surface area) **immersed in an acidic solution** with a hydrogen ion concentration of 1 M, while pure hydrogen gas at a pressure of 1 bar is continuously bubbled over the electrode. YouTube Video

3. **Reaction:** The half-reaction occurring is:

$$2\mathrm{H}^+(\mathrm{aq}) + 2\mathrm{e}^- \rightleftharpoons \mathrm{H}_2(\mathrm{g})$$

4. **Standard Conditions:** When measured under standard conditions (1 M H^+, 1 bar H_2, and 298 K), the potential is the Standard Electrode Potential (**E**°).

The Measurement Process:
To measure the standard electrode potential (E°) of any unknown electrode (let's call it the **Test Half-Cell**), the following steps are taken:

1. **Construct the Half-Cell**: The test electrode (e.g., a zinc strip in a 1 M $ZnSO_4$ solution) is prepared **under standard conditions**.

2. **Form a Galvanic Cell:** The test half-cell is connected to the **SHE** to form a complete galvanic cell.

   - The two half-cells are connected externally by a wire **(to allow electron flow).**

   - The two electrolyte solutions are connected internally by a **salt bridge** (to complete the circuit and maintain electrical neutrality).

3. **Measure the Potential:** A **high-resistance voltmeter** is connected across the two electrodes to measure the **electromotive force (EMF)** of the complete cell.

4. **Determine E°:** Since the potential of the SHE (E°_{SHE}) is 0 V, the measured cell potential (E°_{cell}) is equal to the standard potential of the test half-cell (E°_{test}).

The relationship used is:

$$E^\circ_{\text{cell}} = E^\circ_{\text{cathode}} - E^\circ_{\text{anode}}$$

- If the test half-cell acts as the cathode (undergoes reduction), the measured E°_{cell} is positive, and $E^\circ_{\text{test}} = +E^\circ_{\text{cell}}$.
- If the test half-cell acts as the anode (undergoes oxidation), the measured E°_{cell} is negative, and $E^\circ_{\text{test}} = -E^\circ_{\text{cell}}$.

By convention, all standard electrode potentials are reported as Standard Reduction Potentials.

4.2 Gibbs Free Energy(G)

It is a **thermodynamic potential** that measures the amount of energy available in a system to do **useful work**(work other than associated with a change in volume) at constant temperature and pressure.

Importance of change in Gibbs Free Energy(ΔG) for a process predicts its **spontaneity**.

$$\Delta G = \Delta H - T\Delta S$$

where ΔH is **change in enthalpy**(heat content)(in KJ/mol or J/mol), T is the **absolute temperature**(in Kelvin, K) and ΔS is the **change in entropy(disorder)** of the system (in J/mol.K)

ΔG	Interpretation	Process
$\Delta G < 0$	Spontaneous (favorable) in the forward direction.	Exergonic
$\Delta G = 0$	The system is at equilibrium.	Equilibrium
$\Delta G > 0$	Non-spontaneous in the forward direction.	Endergonic

4.3 Nernst Equation

It is a fundamental relationship that allows us to calculate the **cell potential(E_{cell})** of an electrochemical cell (battery) under **non-standard conditions**.

Nernst equation corrects for deviations from these conditions, especially changes in reactant and product concentrations.

$$\mathbf{E_{cell}} = E^o_{cell} - \frac{RT}{nF}\ln(Q)$$

where, $\mathbf{E_{cell}}$ is **cell potential** under **non-standard conditions** (in Volts, V)
$\mathbf{E^o_{cell}}$ is **standard cell potential**(in Volts, V)
R is the **Universal Gas Constant** (128.314 J/mol $\cdot$ K)
T is the **absolute temperature** (in Kelvin, K)
n is the **number of moles of electrons** transferred in the balanced redox reaction.
F is the **Faraday Constant** ($1696, 485$ C/mol of e^-)
Q is the **reaction quotient** (the ratio of product concentrations to reactant concentrations raised to their stoichiometric coefficients).

4.3.1 Derivation:

The decrease in free energy equals the maximum useful work done by the system,

$$\Delta G = -W_{max}$$

The electrical work is the total charge transferred multiplied by the potential (voltage),

$$W_{\text{elec}} = (\text{charge}) \times (\text{potential})$$

$$\text{Charge Transferred} = nF$$

Combining these, we obtain,

$$\Delta G = -nFE_{cell}$$

Using **The Van't Hoff Isotherm**,

$$\Delta G = \Delta G^\circ + RT \ln Q$$

Substituting and Rearrangement,

$$\frac{-nFE_{\text{cell}}}{-nF} = \frac{-nFE^\circ_{\text{cell}}}{-nF} + \frac{RT \ln Q}{-nF}$$

$$\mathbf{E}_{\text{cell}} = \mathbf{E}^\circ_{\text{cell}} - \frac{\mathbf{RT}}{\mathbf{nF}} \ln \mathbf{Q}$$

4.4 Applications of Electrochemical Cells

- **Galvanic/Voltaic Cells**

  1. Batteries (Primary and Secondary)
     - **Alkaline Batteries:** Used in TV remotes, portable radios, etc (e.g. AA and AAA)
     - **Zinc-Carbon (Dry) Cells**: Older, cheaper batteries.
       (These are **non-rechargeable**.)
     - **Lithium-ion (Li-ion) Batteries:** Used in modern portable electronics like smartphones, laptops, and Electric Vehicles (EVs).
     - **Lead-Acid Batteries:** Used in automobiles for ignition and lighting.
       (These are **rechargeable**.)

5 Electromagnetism

5.1 Charges and Electric Fields

Take two long wires carrying equal currents placed at a separation of 1m attract each other with a force of $2\pi \times 10^{-7}$***N on each metre, each of the currents will be 1A. And the charge flowing in one second through any cross section of a wire carrying a current of 1A is call one coulomb(C) charge.***

Charge on proton($+e$) = **Charge of electron**($-e$) = $1.6 \times 10^{-9}C$.

5.1.1 Charge is Quantized and Conserved

Charges can only exist as **integral multiple of the fundamental charge(charge of electron or proton(e)).** This is known as **quantization of charge**. Quarks have charge which is non-integral multiple of **e**. *But quarks cannot be isolated.*

e is very small so we can use integral and differential calculus($dq \to 0$**) even though only discrete charges exist. It is approximation and not exact.**

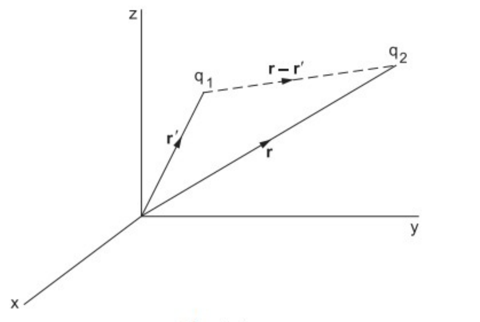
Net charge on an isolated system is never changed and total charge of the universe is constant. *We can produce a new positive or negative charge which never existed anywhere and also destroy a those positive or negative charge.* **But we have to create or destroy charges in pairs (each charge of a pair bearing an opposite sign to the other) so that we do not produce or destroy the net charge.**

For example, the positron is a particle with charge $+e$ (and mass equal to that of an electron). When it meets an electron, often the two destroy each other and gamma ray photons, having no charge, are created.

5.1.2 Force between Two Point Charges: Coulomb's Law

The force exerted by one point charge q_1 on another point charge q_2 placed at a separation r from it is,

$$F = \frac{q_1 q_1}{4\pi\epsilon_0 r^2}$$



In terms of position vectors,

$$F = \frac{q_1 q_2 (\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

where $\epsilon_0 = 8.85 \times 10^{-12} C^2/N \cdot m^2$ and $C^2/N \cdot m^2$ is same as F/m.

6 Epsilon Delta Definition of Limits

$\lim_{x \to a} f(x) = L$ means

$$\forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

7 Gamma Function

$$\int_0^1 log(x)dx = \Big[xlog(x) - 1 \cdot x\Big]_0^1 = -1!$$

$$\int_0^1 log^2(x)dx = \Big[xlog^2(x) - 2xlog(x) + 2 \cdot 1 \cdot x\Big]_0^1 = 2!$$

$$\int_0^1 log^3(x)dx = \Big[xlog^3(x) - 3xlog^2(x) - 2 \cdot 3xlog(x) - 3 \cdot 2 \cdot 1 \cdot x\Big]_0^1 = -3!$$

Observing the pattern, we can write,

$$\int_0^1 log^n(x)dx = (-1)^n \cdot n!$$

Solving for $n!$ we get,

$$n! = \frac{1}{(-1)^n} \int_0^1 log^n(x)dx$$

$$n! = \int_0^1 \left(\frac{log(x)}{-1}\right)^n dx$$

$$n! = \int_0^1 \big(-log(x)\big)^n dx$$

$$n! = \int_0^1 \left(log\left(\frac{1}{x}\right)\right)^n dx$$

Let $u = log\big(\frac{1}{x}\big)$. Using the property of logarithms $-u = log(x)$. Raising to the power of e on both sides, we get $x = e^{-u}$ and thus, $dx = -e^{-u}du$. When $x = 0$, $u \to \infty$ and when $x = 1$, $u = 0$. Therefore, the integral becomes,

$$n! = \int_0^\infty u^n(e^{-u})du$$

Now, by the definition of gamma function,

$$\Gamma(n+1) = n! = \int_0^\infty u^n(e^{-u})du$$

$$\Gamma(n) = (n-1)! = \int_0^\infty u^{n-1}e^{-u}du$$

Replacing n by x,

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du$$

7.1 $\Gamma(\frac{1}{2})$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty u^{-\frac{1}{2}} e^{-u} du$$

Let $t = u^{\frac{1}{2}}$ so that $dt = \frac{1}{2}u^{\frac{-1}{2}}du$. Making the substitution we get,

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-t^2} dt$$

Since, e^{-t^2} is an even function we can write,

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^\infty e^{-t^2} dt$$

This is know as **Gaussian Integral**. Let's evaluate this Integral,

$$I = \int_{-\infty}^\infty e^{-x^2} dx$$

$$I^2 = \left(\int_{-\infty}^\infty e^{-x^2} dx\right)^2$$

$$I^2 = \left(\int_{-\infty}^\infty e^{-x^2} dx\right) \cdot \left(\int_{-\infty}^\infty e^{-y^2} dy\right)$$

$$I^2 = \int_{-\infty}^\infty e^{-(x^2+y^2)} dxdy$$

Now in polar coordinates, $dxdy = dA = rdrd\theta$ and when $x \to -\infty, r \to 0$ and $x \to \infty, r \to \infty$ when $y \to -\infty, \theta \to 0$ and $y \to \infty, \theta \to 2\pi$

Substituting, we get,

$$I^2 = \int_0^{2\pi} \int_0^\infty re^{-r^2} drd\theta$$

$$I^2 = \int_0^{2\pi} \frac{1}{2} d\theta$$

$$I^2 = \frac{1}{2}\left[\theta\right]_0^{2\pi}$$

$$I^2 = \frac{1}{2}2\pi$$

$$I^2 = \pi$$

$$I = \sqrt{\pi}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

8 How Batteries Work

The Engineering Mindset

8.1 What is a Battery?

Device used to store energy. Energy is stored as chemical energy and can be turned into electrical energy when needed.
Battery gives the pushing force to electrons that make them flow through any electrical devices.
But Batteries can push electrons for certain time only, depending on how much of the energy is stored in that battery and what is demanded by the load(Resistors, LEDs, DC motors, etc).
Alkaline Batteries are non-rechargeable.

8.2 What is inside the Battery?

Battery is wrapped inside PVC wrapper which insulates the battery and also displays product details such as (storage capacity, voltage and positive and negative ends).

Positive end has little bump and negative end is flat. These two terminals are electrically isolated from each other.

Inside the wrapper, we find **steel casing** which holds the internal components in place and also **prevents interactions with elements of the atmosphere**.

Inside of the steel casing, there is layer of mixture of **Manganese Oxide and Graphite** with $\mathbf{NH_4Cl + ZnCl_2}$. **Graphite improves the conductivity and increase the energy density as well.**

Inside these materials, there is a fibrous barrier which **prevents direct contact** between Anode and Cathode. This barrier allows the battery to last longer, when not in use. (**But microscopic holes allow ion atoms to pass through**)

Alkaline Electrolyte(KOH) is spread over this barrier

Inside of this barrier, there is anode which is mixture of (**Zn powder and gelling agent**). **Gelling agent** keeps the zinc suspended so it **doesn't accumulate in one spot** and zinc is in the powder form **to increase the surface area** which **increases electron flow** and reduces internal resistance.

Steel cap is sealed with **Nylon Plastic Cap** and **Brass pin** is inserted with steel cap.

Positive and Negative terminal are separated by plastic cap.

Batteries provides voltage difference which is like pressure difference which makes electrons flow. Conductors can make electrons flow and rubber cover keeps the wire same.

8.2.1 Chemical Reactions

1. Zinc gets oxidized and releases electrons. **Zinc is Anode**

$$Zn(s) \rightarrow Zn^{2+} + 2e^-$$

2. Electrons travel through the external conductor wire and enter the carbon rod again. **Carbon rod is cathode because it takes the electrons**.

3. Manganese gets reduced. But since the electrons are passed eventually through carbon rod, we call it cathode.

$$MnO_2 + NH_4^+ + e^- \rightarrow MnO(OH) + NH_3$$

9 Integrating Factor Method for Linear Differential Equations

Consider general linear first order differential equation,

$$\frac{dy}{dx} + Py = Q$$

Let IF be **Integrating Factor** defined as IF $= e^{\int Pdx}$
Now, multiplying both sides with this integrating factor we obtain,

$$e^{\int Pdx} \cdot \frac{dy}{dx} + Pe^{\int Pdx}y = e^{\int Pdx} \cdot Q$$

On the left hand side, we see the product rule of differentiation,

$$\frac{d}{dx}\left(ye^{\int Pdx}\right) = Qe^{\int Pdx}$$

Now, integrating with respect to x on both sides, we obtain,

$$ye^{\int Pdx} = \int Qe^{\int Pdx}dx$$

$$y \cdot \mathrm{IF} = \int Q \cdot \mathrm{IF}dx$$

10 Leibniz Theorem

Let u and v be functions of x. This theorem gives the closed form for the n^{th} derivative of product of two functions. That will be $(uv)^{(n)}$ or $\frac{d^n(u \cdot v)}{dx^n}$ for functions u and v.

for small values of n,

$$\begin{aligned}
n = 1, (uv)' &= u'v + uv' \\
n = 2, (uv)'' &= u''v + 2u'v' + uv'' \\
n = 3, (uv)''' &= u'''v + 3u''v' + 3u'v'' + uv''' \\
n = 4, (uv)^{(4)} &= u^{(4)}v + 4u^{(3)}v' + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + uv^{(4)}
\end{aligned}$$

Looking at the pattern from above, coefficients are **binomial coefficients**. So, we can write general form as,

$$(uv)^{(n)} = \frac{d^n(uv)}{dx^n} = \sum_{r=0}^{n} C(n,r)u^{(n-r)}v^{(r)}$$

where $C(n,r)$ is the binomial coefficient and $C(n,r) = \frac{n!}{r!(n-r)!}$.

10.1 Proof

Now, we prove this theorem by **induction**,
We assume for n=k, $(uv)^{(k)} = \frac{d^k(uv)}{dx^k} = \sum_{r=0}^{k} C(k,r)u^{(k-r)}v^{(r)}$ is true

And now for n=k+1, we prove,

$$(uv)^{(k+1)} = \frac{d^{k+1}(uv)}{dx^{k+1}} = \sum_{r=0}^{k+1} C(k+1,r)u^{(k+1-r)}v^{(r)}$$

By definition of successive differentiation,

$$\begin{aligned}
(uv)^{(k+1)} = \frac{d^{k+1}(uv)}{dx^{k+1}} &= \frac{d}{dx}\sum_{r=0}^{k} C(k,r)u^{(k-r)}v^{(r)} \\
&= \sum_{r=0}^{k} C(k,r)\Big(u^{(k-r+1)}v^{(r)} + u^{(k-r)}v^{(r+1)}\Big) \\
&= \sum_{r=0}^{k} C(k,r)u^{(k-r+1)}v^{(r)} + \sum_{r=0}^{k} C(k,r)u^{(k-r)}v^{(r+1)} \\
&= \sum_{r=0}^{k} C(k,r)u^{(k-r+1)}v^{(r)} + \sum_{r=1}^{k+1} C(k,r-1)u^{(k-r+1)}v^{(r)}
\end{aligned}$$

$$= C(k,0)u^{(k+1)}v^{(0)} + C(k,k)u^{(0)}v^{k+1} + \sum_{r=1}^{k}\Big(C(k,r) + C(k,r-1)\Big)u^{(k-r+1)}v^{(r)}$$

$$= u^{(k+1)}v^{(0)} + u^{(0)}v^{k+1} + \sum_{r=1}^{k} C(k+1,r)u^{(k-r+1)}v^{(r)}$$

$$= \sum_{r=0}^{k} C(k+1,r)u^{(k+1-r)}v^{(r)}$$

which is what we wanted to prove.

11 $\lim_{x\to\infty} \frac{x!}{x^x}$

Using the definition of factorial function and comparing it with the x^{x-1},

$$\begin{aligned} x! &= x(x-1)(x-2)...3 \cdot 2 \cdot 1 \\ &\leq x \cdot x \cdot x... \cdot 1 = x^{x-1} \end{aligned}$$

Factorial of positive integer is greater than or equal to 1. So,

$$1 \leq x! \leq x^{x-1}$$

Dividing all terms by x^x,

$$\frac{1}{x^x} \leq \frac{x!}{x^x} \leq \frac{x^{x-1}}{x^x}$$

$$\frac{1}{x^x} \leq \frac{x!}{x^x} \leq \frac{1}{x}$$

Taking limit as $x \to \infty$,

$$\lim_{x\to\infty} \frac{1}{x^x} \leq \lim_{x\to\infty} \frac{x!}{x^x} \leq \lim_{x\to\infty} \frac{1}{x}$$

$$0 \leq \lim_{x\to\infty} \frac{x!}{x^x} \leq 0$$

Now, from **Squeeze Theorem**,

$$\lim_{x\to\infty} \frac{x!}{x^x} = 0$$

12 Maclaurin's Series

Let $f(x)$ be function of x. The goal is to write the function as an infinite polynomial.

$$f(x) = a + bx + cx^2 + dx^3 + ex^4 + ...$$

when x=0, $f(0) = a$ In order to find other coefficients we differentiate the function with respect to x and put $x = 0$. $f'(0) = 1 \cdot b$ $f''(0) = 1 \cdot 2 \cdot b$ $f'''(0) = 1 \cdot 2 \cdot 3 \cdot c$ and so on. Looking at the pattern, the general form would be,

$$f^{(n)}(0) = n! \cdot (n^{th} coefficient)$$

Substituting in the above equation for coefficients, we get,

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$$

In Summation Notation,

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

$\sin(x)$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$
$\cos(x)$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$
$\tan(x)$	$x + \frac{x^3}{2} + \frac{2x^5}{15} - ...$
$\sinh(x)$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$
$\cosh(x)$	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} ...$
$\ln(1 + x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - ...$
$\tan^{-1}(x)$	$x - \frac{x^3}{3} + \frac{x^5}{5} + ...$

Now, the more general expansion gives the **Taylor Series** which is shifted by h.

$$f(x + h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x)$$

13 MIT Integration Bee

$$\int \tan(x)\sqrt{2+\sqrt{4+\cos(x)}}dx$$

$$= \int \frac{\sin(x)}{\cos(x)}\sqrt{2+\sqrt{4+\cos(x)}}dx$$

Let $u = 4+\cos(x)$, $du = -\sin(x)dx$

$$= \int \frac{-1}{u-4}\sqrt{2+\sqrt{u}}du$$

Let $v^2 = u$, $2vdv = du$

$$= \int \frac{-1}{v^2-4}\sqrt{2+v}\cdot 2vdv$$

$$= \int \frac{-2v\sqrt{2+v}}{(v+2)(v-2)}dv$$

$$= \int \frac{-2v}{\sqrt{(v+2)}(v-2)}dv$$

Let $w = \sqrt{v+2}$, $v = w^2-2$, $dv = 2wdw$

$$= \int \frac{-2(w^2-2)}{w(w^2-4)}\cdot 2wdw$$

$$= \int \frac{-4(w^2-2)}{w^2-4}dw$$

$$\int \frac{-4w^2+8}{w^2-4}dw$$

$$= \int \frac{-4(w^2-4)-8}{w^2-4}dw$$

$$= \int \left(-4-\frac{8}{w^2-4}\right)dw$$

$$= \int -4dw - \int \frac{8}{(w+2)(w-2)}dw$$

$$= -4w - \int \left(\frac{2}{w-2}-\frac{2}{w+2}\right)dw$$

$$= -4w - 2\ln|w-2| + 2\ln|w+2| + c$$

$$= -4w - 2\ln\left|\frac{w+2}{w-2}\right| + c$$

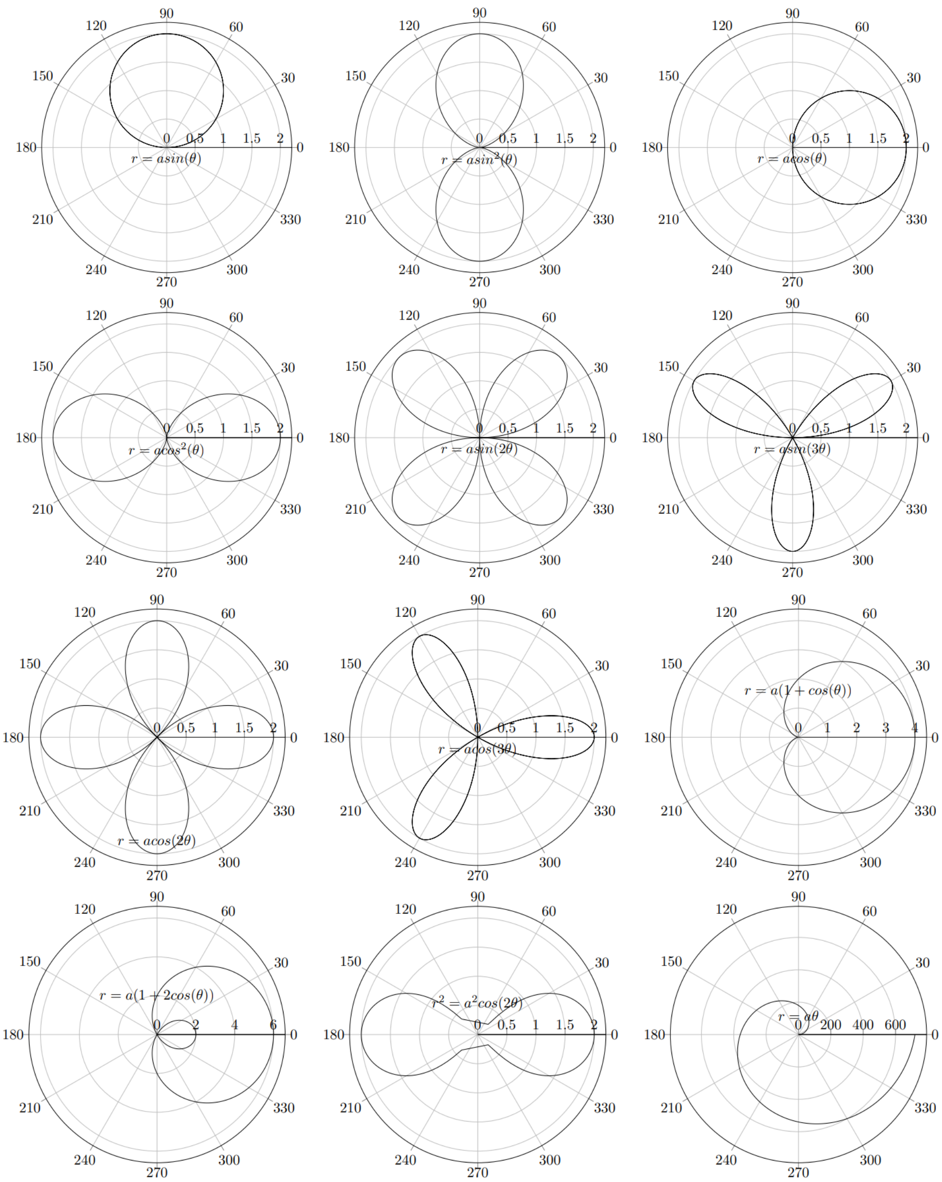
Substituting back,

$$= -4\sqrt{v+2} - 2\ln\left|\frac{\sqrt{v+2}+2}{\sqrt{v+2}-2}\right| + c$$

$$= -4\sqrt{\sqrt{u}+2} - 2\ln\left|\frac{\sqrt{\sqrt{u}+2}+2}{\sqrt{\sqrt{u}+2}-2}\right| + c$$

$$= -4\sqrt{\sqrt{4+\cos(x)}+2} - 2\ln\left|\frac{\sqrt{\sqrt{4+\cos(x)}+2}+2}{\sqrt{\sqrt{4+\cos(x)}+2}-2}\right| + c$$

14 Polar Curves



15 Relation Between Harmonic Numbers and Digamma Function

Firstly, we extend harmonic numbers and factorials into Reals($\Re$). Let $H(n) = \sum_{k=1}^{n} \frac{1}{k}$ be Harmonic Number for input n that belongs to natural numbers.

Recursive formula for harmonic numbers can be written as,

$$H(n) = H(n-1) + \frac{1}{n}$$

Replacing n by x, where x is real number we get,

$$H(x) = H(x-1) + \frac{1}{x}$$

Using this,

$$H(x+1) = H(x) + \frac{1}{x+1}$$

$$H(x+2) = H(x+1) + \frac{1}{x+2} = H(x) + \frac{1}{x+1} + \frac{1}{x+2}$$

General formula then becomes,

$$H(x+n) = H(x) + \sum_{k=1}^{n} \frac{1}{x+k}$$

For large value of N, $H(x+N) \approx H(x)$. Writing this as limit,

$$\lim_{N\to\infty} H(x+N) - H(N) = 0$$

Substituting above general formula and also substituting the definition of harmonic numbers,

$$\lim_{N\to\infty} H(x) + \sum_{k=1}^{n} \left(\frac{1}{x+k}\right) - \sum_{k=1}^{N} \frac{1}{k} = 0$$

Rearranging,

$$H(x) + \lim_{N\to\infty} \sum_{k=1}^{n} \left(\frac{1}{x+k} - \frac{1}{k}\right) = 0$$

$$H(x) = \lim_{N\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{x+k}\right)$$

This is the extension of the harmonic numbers for all the real numbers.

Now, we do the same for factorials. Extension of factorials is know as **Gamma Function** ($\Gamma(x)$). $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ but in the following text we will derive Gamma Function in infinite product form which was how Euler did it for the first time.

Factorial of any natural number (n) is defined as $n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot (n-2) \cdot (n-1) \cdot n$ The recursive formula then is,

$$n! = (n-1)! \cdot n$$

Replacing n by x where x is any real number,

$$x! = (x-1)! \cdot x$$

Using this,

$$(x+1)! = x! \cdot (x+1)$$

$$(x+2)! = (x+1)! \cdot (x) = x! \cdot (x+1) \cdot x$$

General formula then becomes,

$$(x+n)! = x! \cdot \prod_{k=1}^{n}(x+k)$$

Taking natural log on both sides to convert product into sum.

$$\ln((x+n)!) = \ln\left(x! \cdot \prod_{k=1}^{n}(x+k)\right)$$

$$\ln((x+n)!) = \ln(x!) + \ln\left(\prod_{k=1}^{n}(x+k)\right)$$

$$\ln((x+n)!) = \ln(x!) + \sum_{k=1}^{n}\ln(x+k)$$

Now, we can write, for very large N, $\ln(N) \approx \ln(N+k)$. Thus,

$$\lim_{N\to\infty} \ln(N) - \ln(N+k) = 0$$

$$\lim_{N\to\infty} \ln\left(\frac{N}{N+k}\right) = 0$$

Let $L(n) = \ln(n!)$. Now writing above equation in terms of L we obtain,

$$L(x+n) = L(x) + \sum_{k=1}^{n} \ln(x+k)$$

$$L(N+n) = L(N) + n\ln(N)$$

Replacing n by x,

$$L(N+x) = L(N) + x\ln(N)$$

here $L(N) = \ln(N!) = \sum_{k=1}^{N} \ln(k)$

Using above relation of $L(x+N) = \ln((x+N)!) = \ln(x!) + \sum_{k=1}^{N} \ln(x+k)$ and $L(N+x) = L(N) + x\ln(N)$

$$L(x) + \sum_{k=1}^{N} \ln(x+k) \approx \sum_{k=1}^{N} \ln(k) + x\ln(N)$$

$$L(x) \approx \sum_{k=1}^{N} (\ln(k) - \ln(x+k)) + x\ln(N)$$

$$L(x) \approx \sum_{k=1}^{N} (\ln(k) - \ln(x+k)) + x\ln(N)$$

$$L(x) \approx \sum_{k=1}^{N} \left(\ln\left(\frac{k}{x+k}\right)\right) + x\ln(N)$$

$$\ln(x!) = \lim_{N\to\infty} \sum_{k=1}^{N} \left(\ln\left(\frac{k}{x+k}\right)\right) + x\ln(N)$$

Raising both side to power of e,

$$x! = \lim_{N\to\infty} e^{\sum_{k=1}^{N}\left(\ln\left(\frac{k}{x+k}\right)\right)} \cdot e^{\ln(N^x)}$$

$$x! = \lim_{N\to\infty} N^x \cdot \prod_{k=1}^{N} \left(\frac{k}{x+k}\right)$$

This is the infinite product formula for factorials.

$$x! = \lim_{N\to\infty} N^x \cdot \prod_{k=1}^{N} \left(\frac{k}{x+k}\right) = \Gamma(x+1)$$

In order to find the relation between factorials and harmonic numbers, we find $\frac{d}{dx}\ln(x!)$.

$$\begin{aligned}
\frac{d}{dx}\ln(x!) &= \frac{d}{dx}\left(\lim_{N\to\infty}\sum_{k=1}^{N}\ln\left(\frac{k}{x+k}\right) + x\ln(N)\right) \\
&= \lim_{N\to\infty}\sum_{k=1}^{N}\frac{d}{dx}\ln\left(\frac{k}{x+k}\right) + \frac{d}{dx}x\ln(N) \\
&= \lim_{N\to\infty}\sum_{k=1}^{N}\frac{d}{dx}(\ln(k) - \ln(x+k)) + \ln(N) \\
&= \lim_{N\to\infty}\sum_{k=1}^{N}\frac{-1}{x+k} + \ln(N)
\end{aligned}$$

Adding and Subtracting $\lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{k}$ we obtain,

$$\frac{d}{dx}\ln(x!) = \lim_{N\to\infty}\sum_{k=1}^{N}\frac{-1}{x+k} + \ln(N) + \lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{k} - \lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{k}$$

Simplifying,

$$\frac{d}{dx}\ln(x!) = \lim_{N\to\infty}\sum_{k=1}^{N}\left(\frac{1}{k} - \frac{1}{x+k}\right) + \ln(N) - \lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{k}$$

$$\frac{d}{dx}\ln(x!) = H_x - \lim_{N\to\infty}(H_n - \ln(N))$$

Let $\gamma = \lim_{N\to\infty}(H_n - \ln(N))$ is constant. $\gamma = 0.5722...$ which is known as **Euler-Mascheroni constant**. Thus,

$$\frac{d}{dx}\ln(x!) = H_x - \gamma$$

Since, $\Gamma(x+1) = x!$,

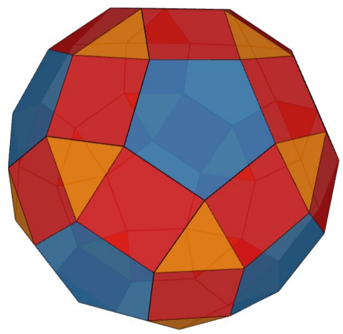
$$\frac{d}{dx}\ln(\Gamma(x+1)) = H_x - \gamma$$

$$\frac{d}{dx}\ln(\Gamma(x)) = H_{x-1} - \gamma$$

$\psi(x) = \frac{d}{dx}\Gamma(x)$ is known as **Digamma Function**. So,

$$\psi(x) = H_{x-1} - \gamma$$

16 Rhombicosidodecahedron



Vertices: 60 Edges: 120 Faces: 62 Vertex configuration: 3.4.5.4
Faces by type 20 triangles, 30 squares, 12 pentagons

16.1 Area and Volume

$$A = a^2(30 + 5\sqrt{3} + 3\sqrt{25 + 10\sqrt{5}}) \approx 59.306a^2$$

$$V = \frac{60 + 29\sqrt{5}}{3}a^3 \approx 41.625a^3$$

where a is edge length.

17 Second Order Differential Equations

Let's consider a general second order differential equation,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

It can also be written as,

$$ay'' + by' + cy = f(x)$$

17.1 Homogeneous Equations

If $f(x) = 0$, $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ is linear, constant coefficient, second order homogeneous differential equation.

Let u(x) and v(x) be two solutions of the equation, It satisfies above equation,

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = 0$$
$$a\frac{d^2v}{dx^2} + b\frac{dv}{dx} + cv = 0$$

Adding these two we get,

$$a\left(\frac{d^2u}{dx^2} + \frac{d^2v}{dx^2}\right) + b\left(\frac{du}{dx} + \frac{dv}{dx}\right) + c(u+v) = 0$$

$$a\frac{d^2(u+v)}{dx^2} + b\frac{d(u+v)}{dx} + c(u+v) = 0$$

Thus, $y = (u+v)$ is also solution.

If $a = 0$, This becomes first order differential equation with b and c as constant coefficients.

$$b\frac{dy}{dx} + cy = 0$$
$$b\frac{dy}{dx} = -cy$$
$$\text{Let } \frac{c}{b} = \text{k},$$
$$\int \frac{dy}{y} = \int -kdx$$
$$\ln|y| = -kx + c$$

$$
\begin{aligned}
y &= e^{-kx+c} \\
y &= e^{-kx} \cdot e^{c} \\
\text{Let } A = e^{c} & \\
y &= Ae^{-kx} \\
\text{Let } -k = m & \\
y &= Ae^{mx}
\end{aligned}
$$

Using this, we guess $y = Ae^{mx}$ is solution to the second order differential equation as well, $\frac{dy}{dx} = Ame^{mx}$ and $\frac{d^2y}{dx^2} = Am^2e^{mx}$. Substituting these derivatives,

$$
\begin{aligned}
aAm^2e^{mx} + bAme^{mx} + cAe^{mx} &= 0 \\
am^2 + bm + c &= 0
\end{aligned}
$$

This quadratic equation is know as **Auxiliary Equation**. Let m_1 and m_2 be two roots of the Auxiliary Equation. Thus, two solutions to the differential equation are, $y = Ae^{m_1x}$ and $y = Be^{m_2x}$ if these two are solutions, then sum of these two solutions is also solution.

$$y = Ae^{m_1x} + Be^{m_2x}$$

A and B are two necessary arbitrary constants for a second-order differential equations. So, there are no further solutions.

17.1.1 Types of Solution

Solution depends on roots of auxiliary equation,

$$am^2 + bm + c = 0$$

Real and Different Roots

If m_1 and m_2 are two **distinct real roots**,

$$y = Ae^{m_1x} + Be^{m_2x}$$

Real and Equal Roots

If $m_1 = m_2 = m$ are **real and equal roots**,

$$y = Ae^{mx} + Be^{mx}$$

$$y = e^{mx}(A + B)$$

But this has only one arbitrary constant $(A + B)$.
Thus, $Be^{mx}x$ can be checked and it is solution, So, solution is,

$$y = Ae^{mx} + Bxe^{mx}$$

Complex Roots

Let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$

$$\begin{aligned} y &= Ce^{(\alpha+i\beta)x} + De^{(\alpha-i\beta)x} \\ &= e^{\alpha x}(Ce^{i\beta x} + De^{-i\beta x}) \end{aligned}$$

Substituting $e^{i\beta x} = \cos(\beta x) + i\sin(\beta x)$ and $e^{-i\beta x} = \cos(\beta x) - i\sin(\beta x)$ and writing $A = C + D$ and $B = i(C - D)$ we obtain,

$$y = e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x))$$

17.1.2 Special Case

Let b = 0 and $\frac{c}{a} = \pm n^2$, we obtain,

$$\frac{d^2y}{dx^2} \pm n^2 y = 0$$

If constant coefficient of y is $+n^2$,

$$\frac{d^2y}{dx^2} + n^2 y = 0$$

Auxiliary Equation is,

$$m^2 + n^2 = 0$$

$$m^2 = -n^2$$

$$m = \pm ni$$

Thus, Solution is,

$$y = A\cos(nx) + B\sin(nx)$$

If constant coefficient of y is $-n^2$,

$$\frac{d^2y}{dx^2} - n^2y = 0$$

Auxiliary Equation is,

$$m^2 - n^2 = 0$$
$$m^2 = n^2$$
$$m = \pm n$$

Thus, Solution is,

$$y = Ce^{nx} + De^{-nx}$$

Using definitions of $\sinh(x)$ and $\cosh(x)$ we obtain, $\sinh(nx) = \frac{e^{nx}-e^{-nx}}{2}$ $\cosh(nx) = \frac{e^{nx}+e^{-nx}}{2}$.

Now, adding and subtracting we obtain following identities,

$$e^{nx} = \cosh(nx) + \sinh(nx)$$
$$e^{-nx} = \cosh(nx) - \sinh(nx)$$

Substituting in above obtained solution,

$$y = C\cosh(nx) + Ci\sinh(nx) + D\cosh(nx) - Di\sinh(nx)$$
$$y = (C + D)\cosh(nx) + i(C - D)\sinh(nx)$$

Writing $A = C + D$ and $B = i(C - D)$,

$$y = A\cosh(nx) + B\sinh(nx)$$

17.2 Inhomogeneous Equations

The equation of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ where $f(x) \neq 0$.

Then the solution is,

$$y = \text{CF} + \text{PI}$$

where CF is **Complementary Function**, which is the solution of the equation when $f(x) = 0$. This is just the solution of the **homogeneous part**.

Now, PI is known as **Particular Integral**. This is found by assuming the general form of $f(x)$. For example,

After assuming the general form for the $f(x)$, we compare the like terms and find the values of C,D,E,... and substitute back.

$\mathbf{f(x)}$	$\mathbf{y}$
k	C
kx	$Cx + D$
kx^2	$Cx^2 + Dx + E$
$k\sin(x)$ or $k\cos(x)$	$C\cos(x) + D\sin(x)$
$k\sinh(x)$ or $k\cosh(x)$	$C\cosh(x) + D\sinh(x)$
e^{kx}	Ce^{kx}

18 Infinite factors of $\sin(x)$

We know, the half angle formula for $\sin(x)$ is

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)cos\left(\frac{x}{2}\right) = 2\sin\left(\frac{x}{2}\right)sin\left(\frac{\pi}{2}+\frac{x}{2}\right)$$

Now, we find $\sin(\frac{x}{2})$ and $\sin(\frac{\pi}{2}+\frac{x}{2})$ and substitute them back into the above identity.

$$\sin\left(\frac{x}{2}\right) = 2\sin\left(\frac{x}{2^2}\right)\sin\left(\frac{\pi}{2}+\frac{x}{2^2}\right) = 2\sin\left(\frac{x}{2^2}\right)sin\left(\frac{2\pi+x}{2^2}\right)$$

$$\sin\left(\frac{\pi}{2}+\frac{x}{2}\right) = 2\sin\left(\frac{\pi}{2^2}+\frac{x}{2^2}\right)\sin\left(\frac{\pi}{2}+\frac{\pi}{2}+\frac{x}{2^2}\right) = 2\sin\left(\frac{\pi}{2^2}+\frac{x}{2^2}\right)sin\left(\frac{3\pi+x}{2^2}\right)$$

After substitution,

$$\sin(x) = 2^3\sin\left(\frac{x}{2^2}\right)\sin\left(\frac{\pi+x}{2^2}\right)\sin\left(\frac{2\pi+x}{2^2}\right)\sin\left(\frac{3\pi+x}{2^2}\right)$$

Repeat for each sine function that we obtain,

$$\sin(x) = 2^7\sin\left(\frac{x}{2^3}\right)\sin\left(\frac{\pi+x}{2^3}\right)...\sin\left(\frac{7\pi+x}{2^3}\right)$$

from the pattern, for any $p =$ power of 2,

$$2^{p-1}\sin\left(\frac{x}{p}\right)\sin\left(\frac{\pi+x}{p}\right)...\sin\left(\frac{(p-1)\pi+x}{p}\right)$$

Last factor is,

$$\begin{aligned}\sin\left(\frac{(p-1)\pi+x}{p}\right) &= \sin\left(\pi-\frac{\pi-x}{p}\right)\\ &= \sin\left(\frac{\pi-x}{p}\right)\end{aligned}$$

The factor, one before the last factor,

$$\begin{aligned}\sin\left(\frac{(p-2)\pi+x}{p}\right) &= \sin\left(\pi-\frac{2\pi-x}{p}\right)\\ &= \sin\left(\frac{2\pi-x}{p}\right)\end{aligned}$$

Now, grouping the factors,

$$\sin(x) = 2^{p-1}\sin\left(\frac{x}{p}\right)\left[\sin\left(\frac{\pi+x}{p}\right)\sin\left(\frac{\pi-x}{p}\right)\right]\left[\sin\left(\frac{2\pi+x}{p}\right)\sin\left(\frac{2\pi-x}{p}\right)\right]...$$

The middle factor has been left out in the above grouping, which is,

$$\sin\left(\frac{\frac{p}{2}\pi+x}{p}\right) = \sin\left(\frac{\pi}{2}+\frac{x}{p}\right) = \cos(x)$$

Including this middle factor as well, we get,

$$\sin(x) = 2^{p-1}\sin\left(\frac{x}{p}\right)\left[\sin\left(\frac{\pi+x}{p}\right)\sin\left(\frac{\pi-x}{p}\right)\right]...\left[\sin\left(\frac{\frac{p}{2}\pi+x}{p}\right)\sin\left(\frac{\frac{p}{2}\pi-x}{p}\right)\right]\cos\left(\frac{x}{p}\right)$$

Using the formula, $\sin(a+b)\cdot\sin(a-b) = sin^2(a) - sin^2(b)$, which can be verified by substituting the expansions for $\sin(a+b)$ and $\sin(a-b)$.

Since,

$$\lim_{x\to 0}\left[\frac{\sin(x)}{\sin\left(\frac{x}{p}\right)}\right] = \lim_{x\to 0}\left[p\frac{\sin(x)}{x}\cdot\frac{x/p}{\sin(x/p)}\right] = p$$

Substituting for $\frac{\sin(x)}{\sin(x/p)}$, and taking the limit as x approaches zero, we get,

$$p = 2^{2p-1}\sin^2\left(\frac{\pi}{p}\right)\sin^2\left(\frac{2\pi}{p}\right)...\sin^2\left(\frac{(\frac{p}{2}-1)\pi}{p}\right)$$

Now, if we take the ratio of $\sin(x)$ to p, we get,

$$\frac{\sin(x)}{p} = \sin\left(\frac{x}{p}\right)\left[1-\frac{\sin^2(x/p)}{\sin^2(\pi/p)}\right]...\left[1-\frac{\sin^2(x/p)}{\sin^2\left((\frac{p}{2}-1)\pi/p\right)}\right]\cos(x/p)$$

We want to express $\sin(x)$ as infinite product. So we let $p\to\infty$. Thus, as $p\to\infty$, $p\sin(x/p) = x$ and $\frac{\sin^2(x/p)}{sin^2(\pi/p)} = \frac{x^2}{\pi^2}$

Substituting, we get:

$$\sin(x) = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)...$$

$$\sin(x) = x\prod_{r=1}^{\infty}\left(1-\frac{x^2}{r^2\pi^2}\right)$$

18.1 Basel Problem (Euler's Solution)

Using the infinite product formula and the infinite sum formula for $\frac{\sin(x)}{x}$.

$$\frac{\sin(x)}{x} = \left(1 - \frac{x^2}{1^2\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)...$$

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - ...$$

Multiplying the infinite product formula to obtain the x^2 term and comparing it with the x^2 term of the infinite sum formula, we get:

$$\frac{-x^2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + ...\right) = \frac{-x^2}{3!}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}... = \frac{\pi^2}{6}$$

19 Sophomore's Dream

$$\int_0^1 x^{-x}dx = \int_0^1 e^{\ln(x^{-x})}dx$$
$$= \int_0^1 e^{-x\ln(x)}dx$$

Now, using the **Taylor Series**, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{x^n(-\ln(x))^n}{n!}dx$$

Swapping the **integral and sum(absolute convergence)** and also, substituting $u = -\ln(x)$, so that, $x = e^{-u}$ and $dx = -e^{-u}du$,

$$= \sum_{n=0}^{\infty} \int_{\infty}^0 \frac{u^n e^{-nu}}{n!}(-e^{-u})du$$
$$= \sum_{n=0}^{\infty} \int_0^{\infty} \frac{u^n e^{-u(n+1)}}{n!}du$$

Substituting $v = u(n+1)$ so that $dv = (n+1)du$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} \frac{v^n}{(n+1)^n} \frac{e^{-v}}{n!} \frac{dv}{(n+1)}$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!(n+1)^{n+1}} \int_0^{\infty} v^n e^{-v}dv$$

$\Gamma(n+1) = \int_0^\infty v^n e^{-v} dv = n!$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{1}{n!(n+1)^{n+1}} n! \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)^{n+1}} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^n}
\end{aligned}$$

Therefore,

$$\int_0^1 x^{-x} dx = \sum_{n=1}^{\infty} n^{-n}$$

This result is known as Sophomore's Dream.

20 Stirling's Approximation

Using **Gamma Function**,

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx = n!$$

Substituting $x = nz$ and $dx = ndz$

$$\begin{aligned} n! &= \int_0^\infty (nz)^n e^{-nz} ndz \\ &= n^{n+1} \int_0^\infty z^n e^{-nz} dz \\ &= n^{n+1} \int_0^\infty e^{\ln(z^n)} e^{-nz} dz \\ &= n^{n+1} \int_0^\infty e^{n\ln(z)} e^{-nz} dz \\ &= n^{n+1} \int_0^\infty e^{n(\ln(z)-z)} dz \end{aligned}$$

Now, writing the **Taylor Series** expansion for $\ln(z) - z$ centered at $z = 1$,

$$f(z) = \ln(z) - z = f(1) + f'(1)(z-1) + \frac{1}{2!} f''(1)(z-1)^2 ...$$

For first three term approximation of $f(z)$, Substituting $f(1) = -1, f'(1) = 0, f''(1) = -1$,

$$f(z) = \ln(z) - z \approx -1 - \frac{1}{2!}(z-1)^2$$

Substituting this approximation in above expression,

$$
\begin{aligned}
n! &\approx n^{n+1} \int_0^\infty e^{n(-1-\frac{1}{2!}(z-1)^2)} dz \\
&= n^{n+1} e^{-n} \int_0^\infty e^{\frac{-n}{2}(z-1)^2} dx
\end{aligned}
$$

For large values of n,

$$
\begin{aligned}
&= n^{n+1} e^{-n} \sqrt{\frac{2}{n}\pi} \\
&= \frac{n^{n+1}}{n^{1/2} n^{1/2}} e^{-n} \sqrt{2\pi n} \\
&= n^n e^{-n} \sqrt{2\pi n} \\
&= \left(\frac{n}{e}\right)^n \sqrt{2\pi n}
\end{aligned}
$$

$$
n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}
$$

21 Vector Calculus

Textbook

21.1 Vectors in Two and Three Dimensions

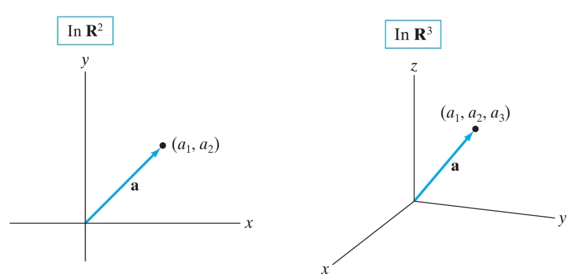
A **vector** in $\mathbb{R}^2$ is simply an **ordered pair** of real numbers and a **vector in** $\mathbb{R}^3$ is simply an **ordered triple** of real numbers, written as (a_1, a_2) and (a_1, a_2, a_3) resp.

Two vectors $\vec{a}$ and $\vec{b}$, $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ in $\mathbb{R}^2$ are **equal** if their corresponding components are equal, $a_1 = b_1$ and $a_2 = b_2$. The same definition holds for vectors in $\mathbb{R}^3$: $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ are **equal** if $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$

Vectors are **added componentwise**, $\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$. Vector addition is **commutative**, $\vec{a} + \vec{b} = \vec{b} + \vec{a}$, **associative**, $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$ and $\vec{a} + \vec{0} = \vec{a}$ where $\vec{0}$ is **zero vector**

Vectors when multiplied by **scalar**, it only scales the **length(magnitude)** of the vector and direction remains **unchanged**. For $\vec{a} = (a_1, a_2, a_3)$ vector in $\mathbb{R}^3$, **scalar product** with k which is real number is $k\vec{a} = (ka_1, ka_2, ka_3)$. Scalar multiplication is **distributive**, $(k + l)\vec{a} = k\vec{a} + l\vec{a}$ and also $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$ and **associative**, $k(l\vec{a}) = (kl)\vec{a} = l(k\vec{a})$.

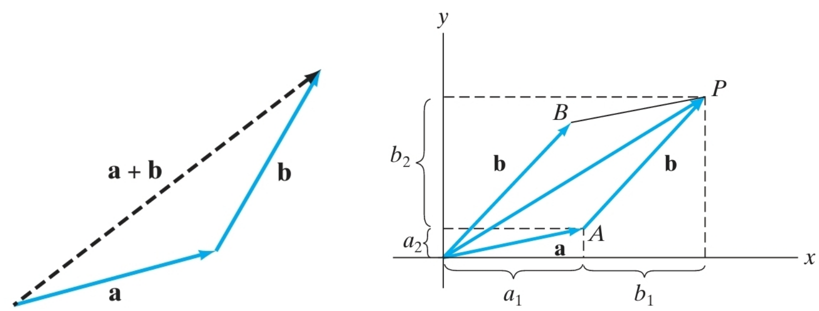
But these ideas of vector addition and scalar multiplication can be applied to any n-ordered tuple in $\mathbb{R}^n$***,*** $(a_1, a_2, ..., a_n)$ ***in analogous to what we did in*** $\mathbb{R}^2$ ***and*** $\mathbb{R}^3$



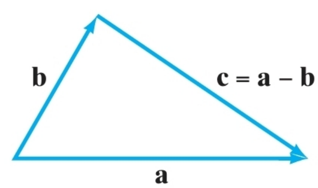
If the vector has **origin** as starting point and some **ordered pair** or **ordered triple** as ending point, it is known as **position vector** and the ending point can be thought of the point in **cartesian coordinate system**.

Zero Vector($\vec{0}$) sits at the origin, like a point, and has no magnitude and, therefore, an indeterminate direction.

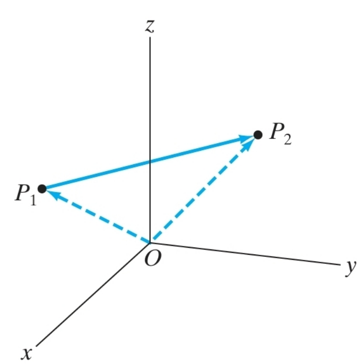
The geometry of adding two vectors is known as **Triangle Law** and **Parallelogram Law** of vector addition.



and this is geometry of vector subtraction,

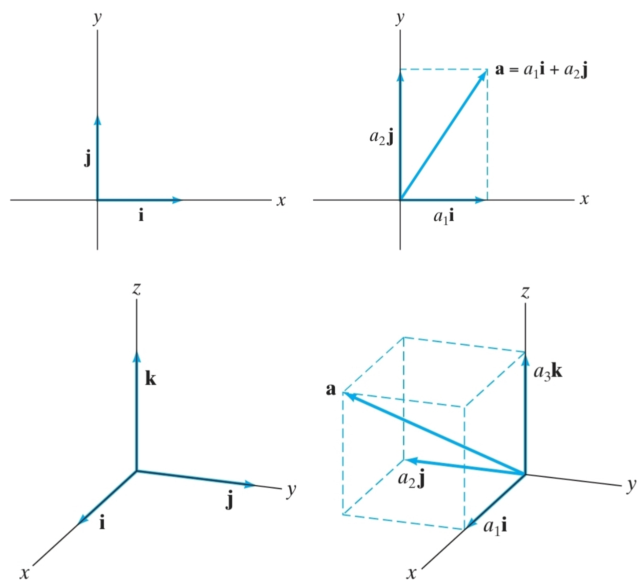


The vector pointing from one point $\mathbf{P_1}$ to another $\mathbf{P_2}$ is given by difference between position vectors i.e. $O\vec{P}_2 - O\vec{P}_1$. Thus,



$$\vec{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

21.1.1 The Standard Basis Vectors



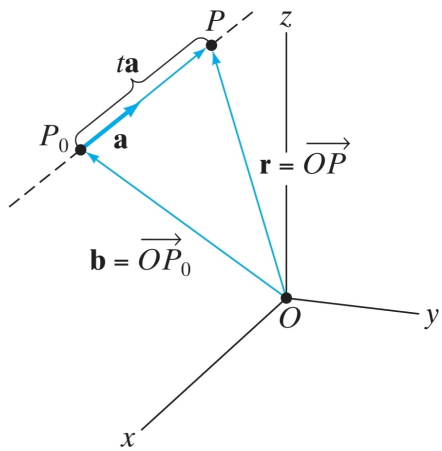
Here, $\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)$ and a_1, a_2, a_3 are components in the respective coordinates axes.

21.1.2 Parametric Equations of Lines

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t)$$

There are **two advantages** of representing a lin in parametric form, ***first*** is they offer a uniform way of describing curves in any number of dimensions and the ***second*** is that they allow we to get the dynamic sense of a curve if we consider the parameter variable **'t'** to represent the time and imagine that a particle is traveling along the curve with time according to these parametric equations.

A line in $\mathbb{R}^2$ and $\mathbb{R}^3$ is uniquely determined by two pieces of geometric information: (1) a vector whose **direction is parallel** to that of the line and (2) any **particular point** lying on the line.



We define $\vec{r} = \vec{OP}$ to be the position vector of point $\mathbf{P}$ whose tip traces out the required line.

Now, from the figure, we see that $\vec{OP}$ is the vector sum of the position vector $\vec{b}$ of the given point $\mathbf{P_0}$(i.e., $\vec{OP_0}$) and a vector parallel to $\vec{a}$. Any vector parallel to $\vec{a}$ must be scalar multiple of $\vec{a}$. Letting this scalar be the parameter variable $\mathbf{t}$, we have,

$$\vec{r} = \vec{OP_0} = \vec{OP_0} + t\vec{a} = \vec{b} + t\vec{a}$$

$$\vec{r}(t) = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} + t(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$$

$$= (a_1t + b_1)\hat{i} + (a_2t + b_2)\hat{j} + (a_3t + b_3)\hat{k}$$

We see from above that P has coordinates of $(a_1t + b_1, a_2t + b_2, a_3t + b_3)$

$$x = a_1t + b_1$$

$$y = a_2t + b_2$$

$$z = a_3t + b_3$$

where t is any real number.

Now if we consider $\mathbb{R}^n$, the parametric equations then will be, $x_1 = a_1t + b_1$, $x_2 = a_2t + b_2$, $x_3 = a_3t + b_3$ of the form,

$$x_n = a_nt + b_n$$

In general, given two *arbitrary* points,

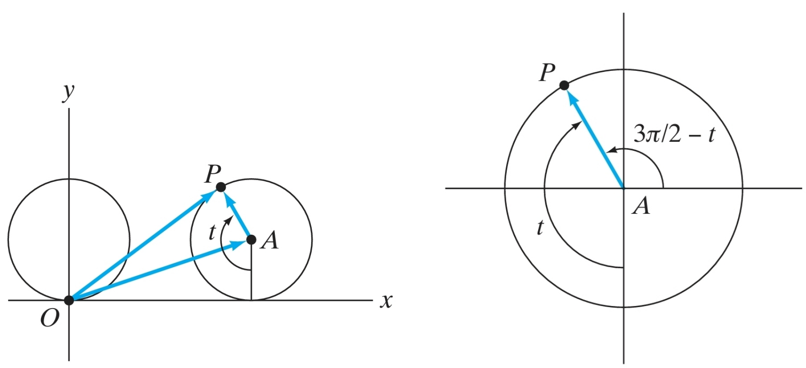
$$P_0(a_1, a_2, a_3) \text{ and } P_1(b_1, b_2, b_3)$$

finding $\mathbf{t}$ from three equations in $\mathbb{R}^3$ and equating we get,

$$\frac{x - b_1}{a_1} = \frac{y - b_2}{a_2} = \frac{z - b_3}{a_3}$$

This form is known as symmetric form of a line in $\mathbb{R}^3$

21.1.3 Parametric Equation of Cycloid



In the figure, let t(parameter) be the angle made by $\vec{AP}$ with the vertical as shown and the radius of the circle be r.

The curve traced out by the point P when the circle rolls without slipping is cycloid.

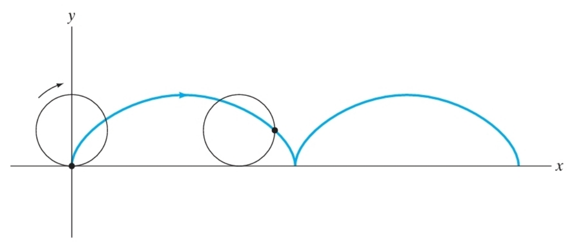
From the above figure,

$$\begin{aligned}
\vec{OP} &= \vec{OA} + \vec{AP} \\
\vec{OA} &= at\hat{i} + a\hat{j} + (-a\sin(t))\hat{i} + (-a\cos(t))\hat{j} \\
\vec{OA} &= (at\hat{i} - a\sin(t))\hat{i} + (a - a\cos(t))\hat{j} \\
\vec{OA} &= a(t - \sin(t))\hat{i} + a(1 - \cos(t))\hat{j}
\end{aligned}$$

Thus, **parametric equations of cycloid are**,

$$x = a(t - \sin(t))$$

$$x = a(1 - \cos(t))$$



22 Vieta's Formula for π

We take the formula for $\sin(x) = 2\sin(\frac{x}{2})\cos(\frac{x}{2})$ and apply this formula to itself repeatedly for the sine function as follows:

$$\begin{aligned}\sin(x) &= 2\sin(x/2)\cos(x/2)\\ &= 2\cdot 2\sin(x/4)\cos(x/4)\cos(x/2)\\ &= 2\cdot 2\cdot 2\sin(x/8)\cos(x/8)\cos(x/4)\cos(x/2)\end{aligned}$$

Repeating this process for n-times,

$$\sin(x) = 2^n \sin\left(\frac{x}{2^n}\right)\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)$$

Now, we take the limit of $n \to \infty$ and $\sin(x/2^n) \to x/2^n$ because as n gets larger, $x/2^n \to 0$

$$\lim_{n\to\infty}\sin(x) = \lim_{n\to\infty} 2^n \sin\left(\frac{x}{2^n}\right)\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)$$

$$\begin{aligned}\sin(x) &= \lim_{n\to\infty} 2^n\left(\frac{x}{2^n}\right)\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)\\ &= \lim_{n\to\infty} x\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)\end{aligned}$$

Thus,

$$\frac{\sin(x)}{x} = \prod_{k=1}^{\infty}\cos\left(\frac{x}{2^k}\right)$$

When we substitute $x = \frac{\pi}{2}$,

$$\frac{2}{\pi} = cos\left(\frac{\pi}{4}\right)cos\left(\frac{\pi}{8}\right)cos\left(\frac{\pi}{16}\right)...$$

we know the value for $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$. So, in order to obtain all the other values for which the angles are just half of the previous angle, we use the identity, $\cos(x/2) = 1/2\sqrt{2+2\cos(x)}$

$$\begin{aligned}
\cos(\pi/8) &= 1/2\sqrt{2 + 2\cos(\pi/4)} \\
&= 1/2\sqrt{2 + 2\sqrt{2}} \\
\cos(\pi/16) &= 1/2\sqrt{2 + 2\cos(\pi/8)} \\
&= 1/2\sqrt{2 + \sqrt{2 + 2\sqrt{2}}}
\end{aligned}$$

Using this pattern, we obtain,

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \ldots$$

23 Vieta's Formula For Polynomials

Consider the quadratic equation,

$$x^2 + \frac{a_1}{a_2}x + \frac{a_0}{a_2} = (x - r_1)(x - r_2)$$

where r_1 and r_2 are the roots of the polynomial. Expanding the right hand side, we get,

$$x^2 + \frac{a_1}{a_2}x + \frac{a_0}{a_2} = x^2 - (r_1 + r_2)x + r_1 r_2$$

Now, comparing the coefficients of the like terms on both sides, we obtain,

$$r_1 + r_2 = \frac{-a_1}{a_2}$$

$$r_1 r_2 = \frac{a_0}{a_2}$$

Consider the cubic equation,

$$x^3 + \frac{a_2}{a_3}x^2 + \frac{a_1}{a_3}x + \frac{a_0}{a_3} = (x - r_1)(x - r_2)(x - r_3)$$

where r_1, r_2 and r_3 are the roots of the polynomial. Expanding the right hand side, we get,

$$x^3 + \frac{a_2}{a_3}x^2 + \frac{a_1}{a_3}x + \frac{a_0}{a_3} = x^3 - (r_1 + r_2 + r_3)x^2 + (r_1 r_2 + r_2 r_3 + r_3 r_1)x - r_1 r_2 r_3$$

Now, comparing the coefficients of the like terms on both sides, we obtain,

$$r_1 + r_2 + r_3 = \frac{-a_2}{a_3}$$

$$r_1 r_2 + r_2 r_3 + r_3 r_1 = \frac{a_1}{a_3}$$

$$r_1 r_2 r_3 = \frac{-a_0}{a_3}$$

Now, we consider the general polynomial,

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$$

Dividing both sides by a_n,

$$\frac{p(x)}{a_n} = x^n + \frac{a_{n-1}}{a_n} x^{n-1} + ... + \frac{a_2}{a_n} x^2 + \frac{a_1}{a_n} x + \frac{a_0}{a_n}$$

and consider $r_1, r_2, ..., r_n$ be the roots of this general polynomial. So,

$$p(x) = a_n(x - r_1)(x - r_2)...(x - r_n)$$

$$\frac{p(x)}{a_n} = (x - r_1)(x - r_2)...(x - r_n)$$

Expanding we obtain,

$$\frac{p(x)}{a_n} = x^n - (r_1 + r_2 + ... + r_n)x^{n-1} + (r_1r_2 + r_2r_3 + ... + r_{n-1}r_n)x^{n-2} - ... + (r_1r_2r_3...r_n)$$

These coefficients are **Elementary Symmetric Polynomials**

23.1 Elementary Symmetric Polynomials

These polynomials can be written in the form,

$$s_k = \sum_{1 \le i_1 \le i_2 \le ... \le i_k \le n} r_{i_1} r_{i_2} r_{i_3} ... r_{i_k}$$

So, above expansion can be written as,

$$\frac{p(x)}{a_n} = x^n - s_1x^{n-1} + s_2x^{n-2} - ... + s_n$$

Now, comparing the coefficients of the like terms we obtain,

$$\frac{a_{n-1}}{a_n} = -s_1$$

$$\frac{a_{n-2}}{a_n} = s_2$$

$$\frac{a_{n-3}}{a_n} = -s_3$$

The general form is,

$$\frac{a_{n-k}}{a_n} = (-1)^k s_k$$

Substituting the definition of s_k,

$$\frac{a_{n-k}}{a_n} = (-1)^k \sum_{1 \le i_1 \le i_2 \le ... \le i_k \le n} r_{i_1} r_{i_2} r_{i_3} ... r_{i_k}$$

24 Wallis Formula

Let's find reduction formulas for $\int \sin^n(x)dx$ and $\int \cos^n(x)dx$. Let $I_n = \int \sin^n(x)dx$

$$\begin{aligned}
I_n = \int \sin^n(x) &= \int \sin^{n-1}(x)\sin(x)dx \\
&= \int \sin^{n-1} d(-\cos(x)) \\
&= \sin^{n-1}(x)(-\cos(x)) + (n-1)\int \cos(x)\cdot\sin^{n-2}(x)\cdot\cos(x)dx \\
&= -\sin^{n-1}(x)\cdot\cos(x) + (n-1)\int (1-\sin^2(x))\cdot\sin^{n-2} dx \\
&= -\sin^{n-1}(x)\cdot\cos(x) + (n-1)\left[\int \sin^{n-2} dx - \int \sin^n(x))dx\right] \\
I_n &= -\sin^{n-1}(x)\cdot\cos(x) + (n-1)[I_{n-2} - I_n] \\
n\cdot I_n &= -\sin^{n-1}(x)\cdot\cos(x) + (n-1)I_{n-2} \\
I_n &= -\frac{1}{n}\sin^{n-1}(x)\cdot\cos(x) + \frac{n-1}{n}I_{n-2}
\end{aligned}$$

Using same method for evaluating $\int \cos^n(x)dx$ we get following results

$$\int \sin^n(x)dx = -\frac{1}{n}\sin^{n-1}(x)\cdot\cos(x) + \frac{n-1}{n}I_{n-2}$$

$$\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\cdot\sin(x) + \frac{n-1}{n}I_{n-2}$$

Now, evaluating these integrals from 0 to $\frac{\pi}{2}$ we get,

$$\int_0^{\frac{\pi}{2}} \cos^n(x)dx = \int_0^{\frac{\pi}{2}} \sin^n(x)dx = I_n = \frac{n-1}{n}I_{n-2}$$

24.1 Wallis Product Formula For π

Taking the ratios of integrals,

$$\frac{I_n}{I_{n-2}} = \frac{n-1}{n}$$

for **even** numbers, $I_0 = \pi, I_2 = \frac{1}{2}\frac{\pi}{2}, I_4 = \frac{3}{4}\frac{1}{2}\frac{\pi}{2}$

$$I_{2n} = \frac{\pi}{2}\prod_{k=1}^{n}\frac{2k-1}{2k}$$

for **odd** numbers, $I_1 = 1, I_3 = \frac{2}{3}, I_5 = \frac{4}{5}\frac{2}{3}$

$$I_{2n+1} = 1\prod_{k=1}^{n}\frac{2k}{2k+1}$$

since, when $0 < x < \frac{\pi}{2}$, $0 < \sin(x) < 1$ which means, $\sin^{2n+1}(x) \leq \sin^{2n}(x) \leq \sin^{2n-1}(x)$. Thus, $I_{2n+1} \leq I_{2n} \leq I_{2n-1}$.

Dividing by I_{2n+1} we get,

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{I_{2n-1}}{I_{2n+1}}$$

Using the iterative relation from above,

$$\frac{I_{2n-1}}{I_{2n+1}} = \frac{2n}{2n+1}$$

Now, by substitution,

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{2n}{2n+1}$$

As $n \to \infty$

$$\lim_{n\to\infty} 1 \leq \lim_{n\to\infty}\frac{I_{2n}}{I_{2n+1}} \leq \lim_{n\to\infty}\frac{2n}{2n+1}$$

$$1 \leq \lim_{n\to\infty}\frac{I_{2n}}{I_{2n+1}} \leq 1$$

Thus, by squeeze theorem,

$$\lim_{n\to\infty}\frac{I_{2n}}{I_{2n+1}} = 1$$

$$\frac{\pi}{2}\prod_{k=1}^{\infty}\frac{2k-1}{2k}\cdot\frac{2k+1}{2k} = 1$$

$$\frac{\pi}{2} = \prod_{k=1}^{\infty}\frac{2k}{2k-1}\cdot\frac{2k}{2k+1}$$

This is known as **Wallis product**.

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6...}{1\cdot 3\cdot 3\cdot 5\cdot 5...}$$