

## 1 Gamma Function

$$\int_0^1 \log(x) dx = \left[ x \log(x) - 1 \cdot x \right]_0^1 = -1!$$

$$\int_0^1 \log^2(x) dx = \left[ x \log^2(x) - 2x \log(x) + 2 \cdot 1 \cdot x \right]_0^1 = 2!$$

$$\int_0^1 \log^3(x) dx = \left[ x \log^3(x) - 3x \log^2(x) - 2 \cdot 3x \log(x) - 3 \cdot 2 \cdot 1 \cdot x \right]_0^1 = -3!$$

Observing the pattern, we can write,

$$\int_0^1 \log^n(x) dx = (-1)^n \cdot n!$$

Solving for  $n!$  we get,

$$n! = \frac{1}{(-1)^n} \int_0^1 \log^n(x) dx$$

$$n! = \int_0^1 \left( \frac{\log(x)}{-1} \right)^n dx$$

$$n! = \int_0^1 (-\log(x))^n dx$$

$$n! = \int_0^1 \left( \log\left(\frac{1}{x}\right) \right)^n dx$$

Let  $u = \log\left(\frac{1}{x}\right)$ . Using the property of logarithms  $-u = \log(x)$ . Raising to the power of  $e$  on both sides, we get  $x = e^{-u}$  and thus,  $dx = -e^{-u} du$ . When  $x = 0$ ,  $u \rightarrow \infty$  and when  $x = 1$ ,  $u = 0$ . Therefore, the integral becomes,

$$n! = \int_0^\infty u^n (e^{-u}) du$$

Now, by the definition of gamma function,

$$\Gamma(n+1) = n! = \int_0^\infty u^n (e^{-u}) du$$

$$\Gamma(n) = (n-1)! = \int_0^\infty u^{n-1} e^{-u} du$$

Replacing  $n$  by  $x$ ,

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$$

### 1.1 $\Gamma(\frac{1}{2})$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty u^{-\frac{1}{2}} e^{-u} du$$

Let  $t = u^{\frac{1}{2}}$  so that  $dt = \frac{1}{2}u^{-\frac{1}{2}}du$ . Making the substitution we get,

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^\infty e^{-t^2} dt$$

Since,  $e^{-t^2}$  is an even function we can write,

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^\infty e^{-t^2} dt$$

This is known as **Gaussian Integral**. Let's evaluate this Integral,

$$\begin{aligned} I &= \int_{-\infty}^\infty e^{-x^2} dx \\ I^2 &= \left( \int_{-\infty}^\infty e^{-x^2} dx \right)^2 \\ I^2 &= \left( \int_{-\infty}^\infty e^{-x^2} dx \right) \cdot \left( \int_{-\infty}^\infty e^{-y^2} dy \right) \\ I^2 &= \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)} dx dy \end{aligned}$$

Now in polar coordinates,  $dxdy = dA = r dr d\theta$  and when  $x \rightarrow -\infty, r \rightarrow 0$  and  $x \rightarrow \infty, r \rightarrow \infty$  when  $y \rightarrow -\infty, \theta \rightarrow 0$  and  $y \rightarrow \infty, \theta \rightarrow 2\pi$

Substituting, we get,

$$\begin{aligned} I^2 &= \int_0^{2\pi} \int_0^\infty r e^{-r^2} dr d\theta \\ I^2 &= \int_0^{2\pi} \frac{1}{2} d\theta \\ I^2 &= \frac{1}{2} [\theta]_0^{2\pi} \\ I^2 &= \frac{1}{2} 2\pi \\ I^2 &= \pi \\ I &= \sqrt{\pi} \\ \Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi} \end{aligned}$$

## 2 Maclaurin's Series

Let  $f(x)$  be function of  $x$ . The goal is to write the function as an infinite polynomial.

$$f(x) = a + bx + cx^2 + dx^3 + ex^4 + \dots$$

when  $x=0$ ,  $f(0) = a$  In order to find other coefficients we differentiate the function with respect to  $x$  and put  $x = 0$ .  $f'(0) = 1 \cdot b$   $f''(0) = 1 \cdot 2 \cdot b$   $f'''(0) = 1 \cdot 2 \cdot 3 \cdot c$  and so on. Looking at the pattern, the general form would be,

$$f^{(n)}(0) = n! \cdot (n^{\text{th}} \text{coefficient})$$

Substituting in the above equation for coefficients, we get,

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

In Summation Notation,

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

$\sin(x)$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
$\cos(x)$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
$\tan(x)$	$x + \frac{x^3}{2} + \frac{2x^5}{15} - \dots$
$\sinh(x)$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$
$\cosh(x)$	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$
$e^x$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$
$\tan^{-1}(x)$	$x - \frac{x^3}{3} + \frac{x^5}{5} + \dots$

Now, the more general expansion gives the **Taylor Series** which is shifted by  $h$ .

$$f(x+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x)$$