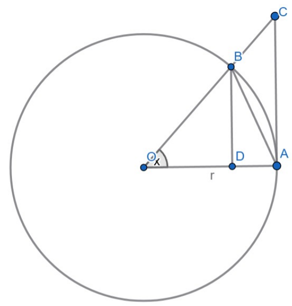
Contents

1 Basel Problem(Cauchy's Solution)



Relating the areas of $\triangle OAB$, sector OAB and $\triangle OAC$

$$\frac{r^2\sin(x)}{2} < \frac{r^2x}{2} < \frac{r^2\tan(x)}{2}$$

Simplifying the inequality, we obtain,

$$\sin(x) < x < \tan(x)$$
$$\cot^2(x) < \frac{1}{x^2} < \csc^2(x)$$
$$\cot^2(x) < \frac{1}{x^2} < 1 + \cot^2(x)$$

Substituting $x = \frac{n\pi}{2N+1}$ for $1 \leq n \leq N$,

$$\cot^2\left(\frac{n\pi}{2N+1}\right) < \frac{(2N+1)^2}{n^2\pi^2} < 1 + \cot^2\left(\frac{n\pi}{2N+1}\right)$$

Now, multiply each term by $\frac{\pi^2}{(2N+1)^2}$,

$$\frac{\pi^2}{(2N+1)^2}\cot^2\left(\frac{n\pi}{2N+1}\right) < \frac{1}{n^2} < \frac{\pi^2}{(2N+1)^2} + \frac{\pi^2}{(2N+1)^2}\cot^2\left(\frac{n\pi}{2N+1}\right)$$

Now, we sum up the terms from n to N,

$$\sum_{n=1}^{N}\frac{\pi^2}{(2N+1)^2}\cot^2\left(\frac{n\pi}{2N+1}\right) < \sum_{n=1}^{N}\frac{1}{n^2} < \sum_{n=1}^{N}\left(\frac{\pi^2}{(2N+1)^2}+\frac{\pi^2}{(2N+1)^2}\cot^2\left(\frac{n\pi}{2N+1}\right)\right)$$

Simplifying further,

$$\frac{\pi^2}{(2N+1)^2}\sum_{n=1}^{N}\cot^2\left(\frac{n\pi}{2N+1}\right) < \sum_{n=1}^{N}\frac{1}{n^2} < \frac{N\pi^2}{(2N+1)^2}+\frac{\pi^2}{(2N+1)^2}\sum_{n=1}^{N}\cot^2\left(\frac{n\pi}{2N+1}\right)$$

Now, we take the limit as $N \to \infty$

$$\lim_{N\to\infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right) < \sum_{n=1}^{\infty} \frac{1}{n^2} < \lim_{N\to\infty} \frac{N\pi^2}{(2N+1)^2} + \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right)$$

Evaluating the limit of the term $\frac{N\pi^2}{(2N+1)^2}$, i.e

$$\lim_{N\to\infty} \frac{N\pi^2}{(2N+1)^2} = 0$$

Substituting back into the inequality,

$$\lim_{N\to\infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right) < \sum_{n=1}^{\infty} \frac{1}{n^2} < \lim_{N\to\infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right)$$

Now, the goal is to evaluate the limit and use **Squeeze Theorem**,

$$\lim_{N\to\infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right)$$

Using De Moivre's Theorem,

$$(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$$

Dividing both sides by $\sin^n(x)$

$$(\cot(x) + i)^n = \frac{\cos(nx) + i\sin(nx)}{\sin^n(x)}$$

Using the Binomial Theorem on $(\cot(x) + i)^n$ we get,

$$(\cot(x)+i)^n = \binom{n}{0}\cot^n(x) + \binom{n}{1}\cot^{n-1}(x)\cdot i + ... + \binom{n}{n-1}\cot(x)\cdot i^{n-1} + \binom{n}{n}\cdot i^n$$

Equating the imaginary coefficients,

$$\frac{\sin(nx)}{\sin^n(x)} = \binom{n}{1}\cot^{n-1}(x) - \binom{n}{3}\cot^{n-3}(x) + ...$$

Now, substituting $x = \frac{n\pi}{2N+1}$ and $n = (2N+1)$, we obtain, $\sin(nx) = 0$
Thus,

$$0 = \binom{2N+1}{1}\cot^{2N}(x) - \binom{2N+1}{3}\cot^{2N-2}(x) + ...$$

Let $t = \cot^2(x)$,

$$P(t) = \binom{2N+1}{1} t^N - \binom{2N+1}{3} t^{N-1} + ...$$

$P(t)$ is polynomial.

Now, using Vieta's sum of roots of polynomial formula on $P(t)$,

$$\begin{aligned}
\sum_{n=1}^{N} t_n &= \frac{\binom{2N+1}{3}}{\binom{2N+1}{1}} \\
&= \frac{(2N+1)!}{(2N+1-3)! \cdot 3!} \cdot \frac{(2N+1-1)!1!}{(2N+1)!} \\
&= \frac{(2N)!}{(2N-2)! \cdot 6} \\
&= \frac{(2N)(2N-1)}{6}
\end{aligned}$$

But we know $t_n = \cot^2(\frac{n\pi}{2N+1})$ is the root of $P(t)$. So,

$$\sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right) = \frac{(2N)(2N-1)}{6}$$

Our goal was to evaluate the limit,

$$\begin{aligned}
\lim_{N\to\infty} \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2\left(\frac{n\pi}{2N+1}\right) &= \lim_{N\to\infty} \frac{\pi^2}{(2N+1)^2} \frac{(2N)(2N-1)}{6} \\
&= \frac{\pi^2}{6} \lim_{N\to\infty} \frac{4N^2 - 2N}{4N^2 + 4N + 1} \\
&= \frac{\pi^2}{6}
\end{aligned}$$

Now, substituting this value in the inequality, we obtain,

$$\frac{\pi^2}{6} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{\pi^2}{6}$$

From Squeeze Theorem,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

2 Bernoulli Differential Equation

Let general equation be,

$$\frac{dy}{dx} + Py = Qy^n$$

Dividing both sides by y^n we obtain,

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

Let $z = y^{1-n}$, $\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$. Multiplying above equation by $(1-n)$ and substituting we obtain,

$$(1-n)y^{-n}\frac{dy}{dx} + (1-n)Py^{1-n} = Q(1-n)$$

$$\frac{dz}{dx} + (1-n)Pz = Q(1-n)$$

Now, this equation can be solved by using **Integrating Factor Method**. Let $(1-n)P = P_1$ and $Q(1-n) = Q_1$, also let I.F be Integrating Factor, I.F $= e^{\int P_1 dx}$

Therefore,

$$\text{IF} \cdot z = \int Q_1 \cdot \text{IF} dx$$

We solve for z first and then use $z = y^{1-n}$ relation to solve for y.

3 Electro-chemistry and Buffer

3.0.1 Electro-Chemical cells

Electro-chemical cells are devices that either generate electrical energy from chemical reactions or use electrical energy to cause non-spontaneous chemical reactions.

*They convert **chemical energy** into **electrical energy** and vice-versa.*

Electrodes are solid conductors. They are of two types: **anode** and **cathode** ***(either reactive (Copper, Zinc and Nickel) or non-reactive (graphite, platinum or silicon))***

Anode: The electrode where **oxidation**(loss of electrons) occurs.
Cathode: The electrode where **reduction**(gain of electrons) occurs.

These electrodes are immersed in an **electrolyte**, which is an **ion-conducting** solution. **(It means liquid itself behave as if it is copper wire carrying electrons but not exactly.)**

There are two types of electro-chemical cells **on the basis of direction of the energy conversion**.

1. Galvanic Cells(Voltaic Cells): Converts **chemical energy** into **electrical energy** via **spontaneous** redox reaction($\Delta G < 0$). These cells are **batteries**. **Anode** is **NEGATIVE** and **Cathode** is **POSITIVE**

2. Electrolytic Cells: Converts **electrical energy** into **chemical energy** via **non-spontaneous** redox reaction($\Delta G > 0$). These cells are used in **electroplating, purification of metals, and industrial production of chemicals (e.g., chlorine, sodium)**. **Anode** is **POSITIVE** and **Cathode** is **NEGATIVE**

3.1 Electrode Potential and Standard Electrode Potential

Electrode Potential is the potential difference that develops between the metal electrode and the electrolyte solution in which it is immersed.

When a metal electrode is placed in a solution containing its own ions, two opposing processes occur at the metal-solution interface, leading to an equilibrium:

Oxidation(Dissolution):

$$M(s) \rightarrow M^{n+}(aq) + ne^-$$

Reduction(Deposition):

$$M^{n+}(aq) + ne^- \rightarrow M(s)$$

At **equilibrium**, the rate of oxidation is equal to the rate of reduction. The **net charge** built up on the metal surface relative to the solution creates a **potential difference** known as electrode potential.

Absolute electrode potential cannot be measured. This is because any measurement requires a complete circuit with a second electrode which leads to second potential. Thus, it is always measured **relative to a reference electrode**(like the **Standard Hydrogen Electrode,** $E^0 = 0V$)

Standard Electrode Potential(E^o) is the electrode potential measured under standard conditions ($1M$ concentration, 1 bar pressure, $298K$ temperature).

3.1.1 Measurement of Electrode Potential

The universal reference for measuring standard electrode potentials is the **Standard Hydrogen Electrode(SHE)**.

SHE has following characteristics:

1. **Defined Potential**: The potential of the SHE is defined as exactly $0V$ at **all temperatures**
2. **Setup**: It consists of a **platinum electrode** coated with platinum black (to increase surface area) **immersed in an acidic solution** with a hydrogen ion concentration of 1 M, while pure hydrogen gas at a pressure of 1 bar is continuously bubbled over the electrode. YouTube Video
3. **Reaction:** The half-reaction occurring is:

$$2\mathrm{H}^+(\mathrm{aq}) + 2\mathrm{e}^- \rightleftharpoons \mathrm{H}_2(\mathrm{g})$$

4. **Standard Conditions:** When measured under standard conditions (1 M H^+, 1 bar H_2, and 298 K), the potential is the Standard Electrode Potential (**E**°).

The Measurement Process:
To measure the standard electrode potential (E°) of any unknown electrode (let's call it the **Test Half-Cell**), the following steps are taken:

1. **Construct the Half-Cell**: The test electrode (e.g., a zinc strip in a 1 M $ZnSO_4$ solution) is prepared **under standard conditions**.

2. **Form a Galvanic Cell:** The test half-cell is connected to the **SHE** to form a complete galvanic cell.
   - The two half-cells are connected externally by a wire **(to allow electron flow).**
   - The two electrolyte solutions are connected internally by a **salt bridge** (to complete the circuit and maintain electrical neutrality).

3. **Measure the Potential:** A **high-resistance voltmeter** is connected across the two electrodes to measure the **electromotive force (EMF)** of the complete cell.

4. **Determine E°:** Since the potential of the SHE (E°_{SHE}) is 0 V, the measured cell potential (E°_{cell}) is equal to the standard potential of the test half-cell (E°_{test}).

The relationship used is:

$$E^\circ_{\text{cell}} = E^\circ_{\text{cathode}} - E^\circ_{\text{anode}}$$

- If the test half-cell acts as the cathode (undergoes reduction), the measured E°_{cell} is positive, and $E^\circ_{\text{test}} = +E^\circ_{\text{cell}}$.
- If the test half-cell acts as the anode (undergoes oxidation), the measured E°_{cell} is negative, and $E^\circ_{\text{test}} = -E^\circ_{\text{cell}}$.

By convention, all standard electrode potentials are reported as Standard Reduction Potentials.

4 Epsilon Delta Definition of Limits

$\lim_{x \to a} f(x) = L$ means

$$\forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

5 Gamma Function

$$\int_0^1 log(x)dx = \Big[xlog(x) - 1 \cdot x\Big]_0^1 = -1!$$

$$\int_0^1 log^2(x)dx = \Big[xlog^2(x) - 2xlog(x) + 2 \cdot 1 \cdot x\Big]_0^1 = 2!$$

$$\int_0^1 log^3(x)dx = \Big[xlog^3(x) - 3xlog^2(x) - 2 \cdot 3xlog(x) - 3 \cdot 2 \cdot 1 \cdot x\Big]_0^1 = -3!$$

Observing the pattern, we can write,

$$\int_0^1 log^n(x)dx = (-1)^n \cdot n!$$

Solving for $n!$ we get,

$$n! = \frac{1}{(-1)^n} \int_0^1 log^n(x)dx$$

$$n! = \int_0^1 \left(\frac{log(x)}{-1}\right)^n dx$$

$$n! = \int_0^1 \big(-log(x)\big)^n dx$$

$$n! = \int_0^1 \left(log\left(\frac{1}{x}\right)\right)^n dx$$

Let $u = log\big(\frac{1}{x}\big)$. Using the property of logarithms $-u = log(x)$. Raising to the power of e on both sides, we get $x = e^{-u}$ and thus, $dx = -e^{-u}du$. When $x = 0$, $u \to \infty$ and when $x = 1$, $u = 0$. Therefore, the integral becomes,

$$n! = \int_0^\infty u^n(e^{-u})du$$

Now, by the definition of gamma function,

$$\Gamma(n+1) = n! = \int_0^\infty u^n(e^{-u})du$$

$$\Gamma(n) = (n-1)! = \int_0^\infty u^{n-1}e^{-u}du$$

Replacing n by x,

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du$$

5.1 $\Gamma(\frac{1}{2})$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} u^{-\frac{1}{2}} e^{-u} du$$

Let $t = u^{\frac{1}{2}}$ so that $dt = \frac{1}{2} u^{\frac{-1}{2}} du$. Making the substitution we get,

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^{\infty} e^{-t^2} dt$$

Since, e^{-t^2} is an even function we can write,

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^{\infty} e^{-t^2} dt$$

This is know as **Gaussian Integral**. Let's evaluate this Integral,

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2$$

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right) \cdot \left(\int_{-\infty}^{\infty} e^{-y^2} dy\right)$$

$$I^2 = \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy$$

Now in polar coordinates, $dxdy = dA = rdrd\theta$ and when $x \to -\infty, r \to 0$ and $x \to \infty, r \to \infty$ when $y \to -\infty, \theta \to 0$ and $y \to \infty, \theta \to 2\pi$

Substituting, we get,

$$I^2 = \int_0^{2\pi} \int_0^{\infty} re^{-r^2} drd\theta$$

$$I^2 = \int_0^{2\pi} \frac{1}{2} d\theta$$

$$I^2 = \frac{1}{2} \left[\theta\right]_0^{2\pi}$$

$$I^2 = \frac{1}{2} 2\pi$$

$$I^2 = \pi$$

$$I = \sqrt{\pi}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

6 Integrating Factor Method for Linear Differential Equations

Consider general linear first order differential equation,

$$\frac{dy}{dx} + Py = Q$$

Let IF be **Integrating Factor** defined as IF $= e^{\int P dx}$
Now, multiplying both sides with this integrating factor we obtain,

$$e^{\int P dx} \cdot \frac{dy}{dx} + Pe^{\int P dx} y = e^{\int P dx} \cdot Q$$

On the left hand side, we see the product rule of differentiation,

$$\frac{d}{dx}\left(ye^{\int P dx}\right) = Qe^{\int P dx}$$

Now, integrating with respect to x on both sides, we obtain,

$$ye^{\int P dx} = \int Qe^{\int P dx} dx$$

$$y \cdot \mathrm{IF} = \int Q \cdot \mathrm{IF} dx$$

7 Leibniz Theorem

Let u and v be functions of x. This theorem gives the closed form for the n^{th} derivative of product of two functions. That will be $(uv)^{(n)}$ or $\frac{d^n(u \cdot v)}{dx^n}$ for functions u and v.

for small values of n,

$$
\begin{aligned}
n = 1, (uv)' &= u'v + uv' \\
n = 2, (uv)'' &= u''v + 2u'v' + uv'' \\
n = 3, (uv)''' &= u'''v + 3u''v' + 3u'v'' + uv''' \\
n = 4, (uv)^{(4)} &= u^{(4)}v + 4u^{(3)}v' + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + uv^{(4)}
\end{aligned}
$$

Looking at the pattern from above, coefficients are **binomial coefficients**. So, we can write general form as,

$$(uv)^{(n)} = \frac{d^n(uv)}{dx^n} = \sum_{r=0}^{n} C(n,r)u^{(n-r)}v^{(r)}$$

where $C(n,r)$ is the binomial coefficient and $C(n,r) = \frac{n!}{r!(n-r)!}$.

7.1 Proof

Now, we prove this theorem by **induction**,
We assume for n=k, $(uv)^{(k)} = \frac{d^k(uv)}{dx^k} = \sum_{r=0}^{k} C(k,r)u^{(k-r)}v^{(r)}$ is true

And now for n=k+1, we prove,

$$(uv)^{(k+1)} = \frac{d^{k+1}(uv)}{dx^{k+1}} = \sum_{r=0}^{k+1} C(k+1,r)u^{(k+1-r)}v^{(r)}$$

By definition of successive differentiation,

$$
\begin{aligned}
(uv)^{(k+1)} = \frac{d^{k+1}(uv)}{dx^{k+1}} &= \frac{d}{dx}\sum_{r=0}^{k} C(k,r)u^{(k-r)}v^{(r)} \\
&= \sum_{r=0}^{k} C(k,r)\Big(u^{(k-r+1)}v^{(r)} + u^{(k-r)}v^{(r+1)}\Big) \\
&= \sum_{r=0}^{k} C(k,r)u^{(k-r+1)}v^{(r)} + \sum_{r=0}^{k} C(k,r)u^{(k-r)}v^{(r+1)} \\
&= \sum_{r=0}^{k} C(k,r)u^{(k-r+1)}v^{(r)} + \sum_{r=1}^{k+1} C(k,r-1)u^{(k-r+1)}v^{(r)}
\end{aligned}
$$

$$
\begin{aligned}
&= C(k,0)u^{(k+1)}v^{(0)} + C(k,k)u^{(0)}v^{k+1} + \sum_{r=1}^{k}\Big(C(k,r) + C(k,r-1)\Big)u^{(k-r+1)}v^{(r)} \\
&= u^{(k+1)}v^{(0)} + u^{(0)}v^{k+1} + \sum_{r=1}^{k} C(k+1,r)u^{(k-r+1)}v^{(r)} \\
&= \sum_{r=0}^{k} C(k+1,r)u^{(k+1-r)}v^{(r)}
\end{aligned}
$$

which is what we wanted to prove.

8 $\lim_{x\to\infty} \frac{x!}{x^x}$

Using the definition of factorial function and comparing it with the x^{x-1},

$$\begin{aligned} x! &= x(x-1)(x-2)...3\cdot 2\cdot 1 \\ &\leq x\cdot x\cdot x...\cdot 1 = x^{x-1} \end{aligned}$$

Factorial of positive integer is greater than or equal to 1. So,

$$1 \leq x! \leq x^{x-1}$$

Dividing all terms by x^x,

$$\frac{1}{x^x} \leq \frac{x!}{x^x} \leq \frac{x^{x-1}}{x^x}$$

$$\frac{1}{x^x} \leq \frac{x!}{x^x} \leq \frac{1}{x}$$

Taking limit as $x \to \infty$,

$$\lim_{x\to\infty} \frac{1}{x^x} \leq \lim_{x\to\infty} \frac{x!}{x^x} \leq \lim_{x\to\infty} \frac{1}{x}$$

$$0 \leq \lim_{x\to\infty} \frac{x!}{x^x} \leq 0$$

Now, from **Squeeze Theorem**,

$$\lim_{x\to\infty} \frac{x!}{x^x} = 0$$

9 Maclaurin's Series

Let $f(x)$ be function of x. The goal is to write the function as an infinite polynomial.

$$f(x) = a + bx + cx^2 + dx^3 + ex^4 + ...$$

when x=0, $f(0) = a$ In order to find other coefficients we differentiate the function with respect to x and put $x = 0$. $f'(0) = 1 \cdot b$ $f''(0) = 1 \cdot 2 \cdot b$ $f'''(0) = 1 \cdot 2 \cdot 3 \cdot c$ and so on. Looking at the pattern, the general form would be,

$$f^{(n)}(0) = n! \cdot (n^{th} coefficient)$$

Substituting in the above equation for coefficients, we get,

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$$

In Summation Notation,

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

$\sin(x)$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$
$\cos(x)$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$
$\tan(x)$	$x + \frac{x^3}{2} + \frac{2x^5}{15} - ...$
$\sinh(x)$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$
$\cosh(x)$	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} ...$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - ...$
$\tan^{-1}(x)$	$x - \frac{x^3}{3} + \frac{x^5}{5} + ...$

Now, the more general expansion gives the **Taylor Series** which is shifted by h.

$$f(x+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x)$$

10 MIT Integration Bee

$$\int \tan(x)\sqrt{2+\sqrt{4+\cos(x)}}dx$$

$$= \int \frac{\sin(x)}{\cos(x)}\sqrt{2+\sqrt{4+\cos(x)}}dx$$

Let $u = 4+\cos(x)$, $du = -\sin(x)dx$

$$= \int \frac{-1}{u-4}\sqrt{2+\sqrt{u}}du$$

Let $v^2 = u$, $2vdv = du$

$$= \int \frac{-1}{v^2-4}\sqrt{2+v}\cdot 2vdv$$

$$= \int \frac{-2v\sqrt{2+v}}{(v+2)(v-2)}dv$$

$$= \int \frac{-2v}{\sqrt{(v+2)}(v-2)}dv$$

Let $w = \sqrt{v+2}$, $v = w^2-2$, $dv = 2wdw$

$$= \int \frac{-2(w^2-2)}{w(w^2-4)}\cdot 2wdw$$

$$= \int \frac{-4(w^2-2)}{w^2-4}dw$$

$$\int \frac{-4w^2+8}{w^2-4}dw$$

$$= \int \frac{-4(w^2-4)-8}{w^2-4}dw$$

$$= \int \left(-4-\frac{8}{w^2-4}\right)dw$$

$$= \int -4dw - \int \frac{8}{(w+2)(w-2)}dw$$

$$= -4w - \int \left(\frac{2}{w-2}-\frac{2}{w+2}\right)dw$$

$$= -4w - 2\ln|w-2| + 2\ln|w+2| + c$$

$$= -4w - 2\ln\left|\frac{w+2}{w-2}\right| + c$$

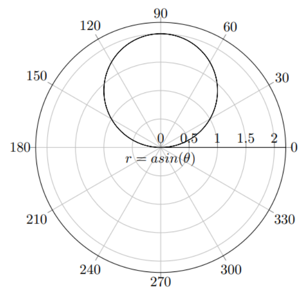
Substituting back,

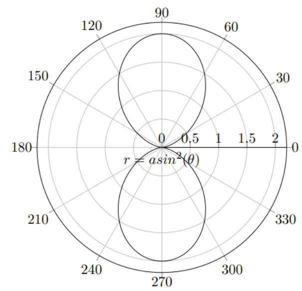
$$= -4\sqrt{v+2} - 2\ln\left|\frac{\sqrt{v+2}+2}{\sqrt{v+2}-2}\right| + c$$

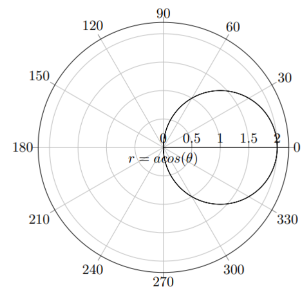
$$= -4\sqrt{\sqrt{u}+2} - 2\ln\left|\frac{\sqrt{\sqrt{u}+2}+2}{\sqrt{\sqrt{u}+2}-2}\right| + c$$

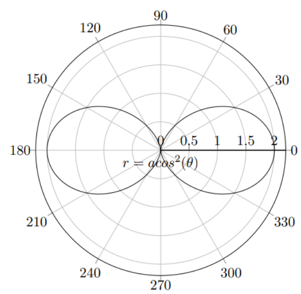
$$= -4\sqrt{\sqrt{4+\cos(x)}+2} - 2\ln\left|\frac{\sqrt{\sqrt{4+\cos(x)}+2}+2}{\sqrt{\sqrt{4+\cos(x)}+2}-2}\right| + c$$

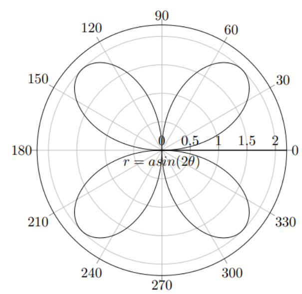
11 Polar Curves

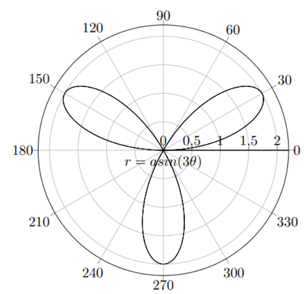


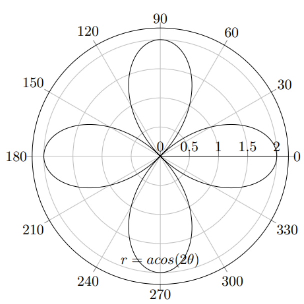


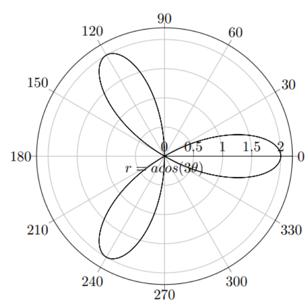


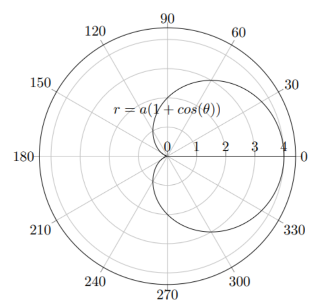


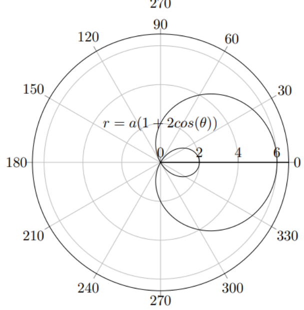


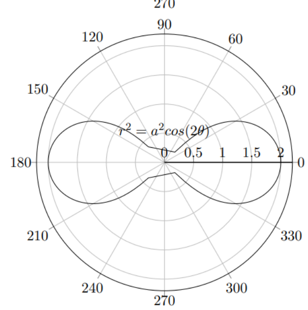


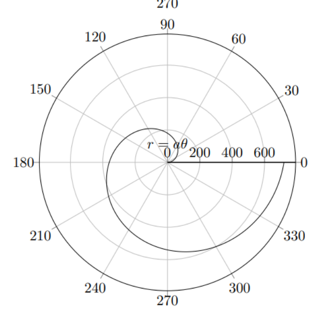












12 Relation Between Harmonic Numbers and Digamma Function

Firstly, we extend harmonic numbers and factorials into Reals($\Re$). Let $H(n) = \sum_{k=1}^{n} \frac{1}{k}$ be Harmonic Number for input n that belongs to natural numbers.

Recursive formula for harmonic numbers can be written as,

$$H(n) = H(n-1) + \frac{1}{n}$$

Replacing n by x, where x is real number we get,

$$H(x) = H(x-1) + \frac{1}{x}$$

Using this,

$$H(x+1) = H(x) + \frac{1}{x+1}$$

$$H(x+2) = H(x+1) + \frac{1}{x+2} = H(x) + \frac{1}{x+1} + \frac{1}{x+2}$$

General formula then becomes,

$$H(x+n) = H(x) + \sum_{k=1}^{n} \frac{1}{x+k}$$

For large value of N, $H(x+N) \approx H(x)$. Writing this as limit,

$$\lim_{N \to \infty} H(x+N) - H(N) = 0$$

Substituting above general formula and also substituting the definition of harmonic numbers,

$$\lim_{N \to \infty} H(x) + \sum_{k=1}^{n} \left(\frac{1}{x+k}\right) - \sum_{k=1}^{N} \frac{1}{k} = 0$$

Rearranging,

$$H(x) + \lim_{N \to \infty} \sum_{k=1}^{n} \left(\frac{1}{x+k} - \frac{1}{k}\right) = 0$$

$$H(x) = \lim_{N \to \infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{x+k}\right)$$

This is the extension of the harmonic numbers for all the real numbers.

Now, we do the same for factorials. Extension of factorials is know as **Gamma Function (**$\Gamma(x)$**).** $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ but in the following text we will derive Gamma Function in infinite product form which was how Euler did it for the first time.

Factorial of any natural number (n) is defined as $n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot (n-2) \cdot (n-1) \cdot n$ The recursive formula then is,

$$n! = (n-1)! \cdot n$$

Replacing n by x where x is any real number,

$$x! = (x-1)! \cdot x$$

Using this,

$$(x+1)! = x! \cdot (x+1)$$

$$(x+2)! = (x+1)! \cdot (x) = x! \cdot (x+1) \cdot x$$

General formula then becomes,

$$(x+n)! = x! \cdot \prod_{k=1}^{n}(x+k)$$

Taking natural log on both sides to convert product into sum.

$$\ln((x+n)!) = \ln\left(x! \cdot \prod_{k=1}^{n}(x+k)\right)$$

$$\ln((x+n)!) = \ln(x!) + \ln\left(\prod_{k=1}^{n}(x+k)\right)$$

$$\ln((x+n)!) = \ln(x!) + \sum_{k=1}^{n}\ln(x+k)$$

Now, we can write, for very large N, $\ln(N) \approx \ln(N+k)$. Thus,

$$\lim_{N\to\infty} \ln(N) - \ln(N+k) = 0$$

$$\lim_{N\to\infty} \ln\left(\frac{N}{N+k}\right) = 0$$

Let $L(n) = \ln(n!)$. Now writing above equation in terms of L we obtain,

$$L(x+n) = L(x) + \sum_{k=1}^{n} \ln(x+k)$$

$$L(N+n) = L(N) + n\ln(N)$$

Replacing n by x,

$$L(N+x) = L(N) + x\ln(N)$$

here $L(N) = \ln(N!) = \sum_{k=1}^{N} \ln(k)$

Using above relation of $L(x+N) = \ln((x+N)!) = \ln(x!) + \sum_{k=1}^{N} \ln(x+k)$ and $L(N+x) = L(N) + x\ln(N)$

$$L(x) + \sum_{k=1}^{N} \ln(x+k) \approx \sum_{k=1}^{N} \ln(k) + x\ln(N)$$

$$L(x) \approx \sum_{k=1}^{N} (\ln(k) - \ln(x+k)) + x\ln(N)$$

$$L(x) \approx \sum_{k=1}^{N} (\ln(k) - \ln(x+k)) + x\ln(N)$$

$$L(x) \approx \sum_{k=1}^{N} \left(\ln\left(\frac{k}{x+k}\right)\right) + x\ln(N)$$

$$\ln(x!) = \lim_{N\to\infty} \sum_{k=1}^{N} \left(\ln\left(\frac{k}{x+k}\right)\right) + x\ln(N)$$

Raising both side to power of e,

$$x! = \lim_{N\to\infty} e^{\sum_{k=1}^{N} \left(\ln\left(\frac{k}{x+k}\right)\right)} \cdot e^{\ln(N^x)}$$

$$x! = \lim_{N\to\infty} N^x \cdot \prod_{k=1}^{N} \left(\frac{k}{x+k}\right)$$

This is the infinite product formula for factorials.

$$x! = \lim_{N\to\infty} N^x \cdot \prod_{k=1}^{N} \left(\frac{k}{x+k}\right) = \Gamma(x+1)$$

In order to find the relation between factorials and harmonic numbers, we find $\frac{d}{dx}\ln(x!)$.

$$\begin{aligned}
\frac{d}{dx}\ln(x!) &= \frac{d}{dx}\left(\lim_{N\to\infty}\sum_{k=1}^{N}\ln\left(\frac{k}{x+k}\right) + x\ln(N)\right)\\
&= \lim_{N\to\infty}\sum_{k=1}^{N}\frac{d}{dx}\ln\left(\frac{k}{x+k}\right) + \frac{d}{dx}x\ln(N)\\
&= \lim_{N\to\infty}\sum_{k=1}^{N}\frac{d}{dx}(\ln(k) - \ln(x+k)) + \ln(N)\\
&= \lim_{N\to\infty}\sum_{k=1}^{N}\frac{-1}{x+k} + \ln(N)
\end{aligned}$$

Adding and Subtracting $\lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{k}$ we obtain,

$$\frac{d}{dx}\ln(x!) = \lim_{N\to\infty}\sum_{k=1}^{N}\frac{-1}{x+k} + \ln(N) + \lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{k} - \lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{k}$$

Simplifying,

$$\frac{d}{dx}\ln(x!) = \lim_{N\to\infty}\sum_{k=1}^{N}\left(\frac{1}{k} - \frac{1}{x+k}\right) + \ln(N) - \lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{k}$$

$$\frac{d}{dx}\ln(x!) = H_x - \lim_{N\to\infty}(H_n - \ln(N))$$

Let $\gamma = \lim_{N\to\infty}(H_n - \ln(N))$ is constant. $\gamma = 0.5722...$ which is known as **Euler-Mascheroni constant**. Thus,

$$\frac{d}{dx}\ln(x!) = H_x - \gamma$$

Since, $\Gamma(x+1) = x!$,

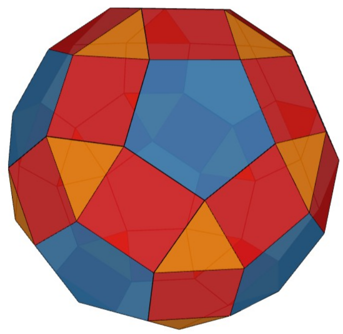
$$\frac{d}{dx}\ln(\Gamma(x+1)) = H_x - \gamma$$

$$\frac{d}{dx}\ln(\Gamma(x)) = H_{x-1} - \gamma$$

$\psi(x) = \frac{d}{dx}\Gamma(x)$ is known as **Digamma Function**. So,

$$\psi(x) = H_{x-1} - \gamma$$

13 Rhombicosidodecahedron



Vertices: 60 Edges: 120 Faces: 62 Vertex configuration: 3.4.5.4
Faces by type 20 triangles, 30 squares, 12 pentagons

13.1 Area and Volume

$$A = a^2(30 + 5\sqrt{3} + 3\sqrt{25 + 10\sqrt{5}}) \approx 59.306a^2$$

$$V = \frac{60 + 29\sqrt{5}}{3}a^3 \approx 41.625a^3$$

where a is edge length.

14 Second Order Differential Equations

Let's consider a general second order differential equation,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

It can also be written as,

$$ay'' + by' + cy = f(x)$$

14.1 Homogeneous Equations

If $f(x) = 0$, $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ is linear, constant coefficient, second order homogeneous differential equation.

Let u(x) and v(x) be two solutions of the equation, It satisfies above equation,

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = 0$$
$$a\frac{d^2v}{dx^2} + b\frac{dv}{dx} + cv = 0$$

Adding these two we get,

$$a\left(\frac{d^2u}{dx^2} + \frac{d^2v}{dx^2}\right) + b\left(\frac{du}{dx} + \frac{dv}{dx}\right) + c(u+v) = 0$$

$$a\frac{d^2(u+v)}{dx^2} + b\frac{d(u+v)}{dx} + c(u+v) = 0$$

Thus, $y = (u+v)$ is also solution.

If $a = 0$, This becomes first order differential equation with b and c as constant coefficients.

$$\begin{aligned} b\frac{dy}{dx} + cy &= 0 \\ b\frac{dy}{dx} &= -cy \\ \text{Let } \frac{c}{b} &= \text{k}, \\ \int\frac{dy}{y} &= \int -kdx \\ \ln|y| &= -kx + c \end{aligned}$$

$$
\begin{aligned}
y &= e^{-kx+c} \\
y &= e^{-kx} \cdot e^{c} \\
\text{Let } A = e^{c} & \\
y &= Ae^{-kx} \\
\text{Let } -k = m & \\
y &= Ae^{mx}
\end{aligned}
$$

Using this, we guess $y = Ae^{mx}$ is solution to the second order differential equation as well, $\frac{dy}{dx} = Ame^{mx}$ and $\frac{d^2y}{dx^2} = Am^2e^{mx}$. Substituting these derivatives,

$$
\begin{aligned}
aAm^2e^{mx} + bAme^{mx} + cAe^{mx} &= 0 \\
am^2 + bm + c &= 0
\end{aligned}
$$

This quadratic equation is know as **Auxiliary Equation**. Let m_1 and m_2 be two roots of the Auxiliary Equation. Thus, two solutions to the differential equation are, $y = Ae^{m_1x}$ and $y = Be^{m_2x}$ if these two are solutions, then sum of these two solutions is also solution.

$$y = Ae^{m_1x} + Be^{m_2x}$$

A and B are two necessary arbitrary constants for a second-order differential equations. So, there are no further solutions.

14.1.1 Types of Solution

Solution depends on roots of auxiliary equation,

$$am^2 + bm + c = 0$$

Real and Different Roots

If m_1 and m_2 are two **distinct real roots**,

$$y = Ae^{m_1x} + Be^{m_2x}$$

Real and Equal Roots

If $m_1 = m_2 = m$ are **real and equal roots**,

$$y = Ae^{mx} + Be^{mx}$$

$$y = e^{mx}(A + B)$$

But this has only one arbitrary constant $(A + B)$.
Thus, $Be^{mx}x$ can be checked and it is solution, So, solution is,

$$y = Ae^{mx} + Bxe^{mx}$$

Complex Roots

Let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$

$$\begin{aligned} y &= Ce^{(\alpha+i\beta)x} + De^{(\alpha-i\beta)x} \\ &= e^{\alpha x}(Ce^{i\beta x} + De^{-i\beta x}) \end{aligned}$$

Substituting $e^{i\beta x} = \cos(\beta x) + i\sin(\beta x)$ and $e^{-i\beta x} = \cos(\beta x) - i\sin(\beta x)$ and writing $A = C + D$ and $B = i(C - D)$ we obtain,

$$y = e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x))$$

14.1.2 Special Case

Let b = 0 and $\frac{c}{a} = \pm n^2$, we obtain,

$$\frac{d^2y}{dx^2} \pm n^2 y = 0$$

If constant coefficient of y is $+n^2$,

$$\frac{d^2y}{dx^2} + n^2 y = 0$$

Auxiliary Equation is,

$$m^2 + n^2 = 0$$

$$m^2 = -n^2$$

$$m = \pm ni$$

Thus, Solution is,

$$y = A\cos(nx) + B\sin(nx)$$

If constant coefficient of y is $-n^2$,

$$\frac{d^2y}{dx^2} - n^2y = 0$$

Auxiliary Equation is,

$$m^2 - n^2 = 0$$

$$m^2 = n^2$$

$$m = \pm n$$

Thus, Solution is,

$$y = Ce^{nx} + De^{-nx}$$

Using definitions of $\sinh(x)$ and $\cosh(x)$ we obtain, $\sinh(nx) = \frac{e^{nx}-e^{-nx}}{2}$ $\cosh(nx) = \frac{e^{nx}+e^{-nx}}{2}$.

Now, adding and subtracting we obtain following identities,

$$e^{nx} = \cosh(nx) + \sinh(nx)$$

$$e^{-nx} = \cosh(nx) - \sinh(nx)$$

Substituting in above obtained solution,

$$y = C\cosh(nx) + Ci\sinh(nx) + D\cosh(nx) - Di\sinh(nx)$$

$$y = (C + D)\cosh(nx) + i(C - D)\sinh(nx)$$

Writing $A = C + D$ and $B = i(C - D)$,

$$y = A\cosh(nx) + B\sinh(nx)$$

15 Infinite factors of $\sin(x)$

We know, the half angle formula for $\sin(x)$ is

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)cos\left(\frac{x}{2}\right) = 2\sin\left(\frac{x}{2}\right)sin\left(\frac{\pi}{2}+\frac{x}{2}\right)$$

Now, we find $\sin(\frac{x}{2})$ and $\sin(\frac{\pi}{2}+\frac{x}{2})$ and substitute them back into the above identity.

$$\sin\left(\frac{x}{2}\right) = 2\sin\left(\frac{x}{2^2}\right)\sin\left(\frac{\pi}{2}+\frac{x}{2^2}\right) = 2\sin\left(\frac{x}{2^2}\right)sin\left(\frac{2\pi+x}{2^2}\right)$$

$$\sin\left(\frac{\pi}{2}+\frac{x}{2}\right) = 2\sin\left(\frac{\pi}{2^2}+\frac{x}{2^2}\right)\sin\left(\frac{\pi}{2}+\frac{\pi}{2}+\frac{x}{2^2}\right) = 2\sin\left(\frac{\pi}{2^2}+\frac{x}{2^2}\right)sin\left(\frac{3\pi+x}{2^2}\right)$$

After substitution,

$$\sin(x) = 2^3\sin\left(\frac{x}{2^2}\right)\sin\left(\frac{\pi+x}{2^2}\right)\sin\left(\frac{2\pi+x}{2^2}\right)\sin\left(\frac{3\pi+x}{2^2}\right)$$

Repeat for each sine function that we obtain,

$$\sin(x) = 2^7\sin\left(\frac{x}{2^3}\right)\sin\left(\frac{\pi+x}{2^3}\right)...\sin\left(\frac{7\pi+x}{2^3}\right)$$

from the pattern, for any $p =$ power of 2,

$$2^{p-1}\sin\left(\frac{x}{p}\right)\sin\left(\frac{\pi+x}{p}\right)...\sin\left(\frac{(p-1)\pi+x}{p}\right)$$

Last factor is,

$$\begin{aligned}\sin\left(\frac{(p-1)\pi+x}{p}\right) &= \sin\left(\pi-\frac{\pi-x}{p}\right)\\ &= \sin\left(\frac{\pi-x}{p}\right)\end{aligned}$$

The factor, one before the last factor,

$$\begin{aligned}\sin\left(\frac{(p-2)\pi+x}{p}\right) &= \sin\left(\pi-\frac{2\pi-x}{p}\right)\\ &= \sin\left(\frac{2\pi-x}{p}\right)\end{aligned}$$

Now, grouping the factors,

$$\sin(x) = 2^{p-1} \sin\left(\frac{x}{p}\right)\left[\sin\left(\frac{\pi + x}{p}\right)\sin\left(\frac{\pi - x}{p}\right)\right]\left[\sin\left(\frac{2\pi + x}{p}\right)\sin\left(\frac{2\pi - x}{p}\right)\right]...$$

The middle factor has been left out in the above grouping, which is,

$$\sin\left(\frac{\frac{p}{2}\pi + x}{p}\right) = \sin\left(\frac{\pi}{2} + \frac{x}{p}\right) = \cos(x)$$

Including this middle factor as well, we get,

$$\sin(x) = 2^{p-1} \sin\left(\frac{x}{p}\right)\left[\sin\left(\frac{\pi + x}{p}\right)\sin\left(\frac{\pi - x}{p}\right)\right]...\left[\sin\left(\frac{\frac{p}{2}\pi + x}{p}\right)\sin\left(\frac{\frac{p}{2}\pi - x}{p}\right)\right]\cos\left(\frac{x}{p}\right)$$

Using the formula, $\sin(a+b) \cdot \sin(a-b) = sin^2(a) - sin^2(b)$, which can be verified by substituting the expansions for $\sin(a+b)$ and $\sin(a-b)$.

Since,

$$\lim_{x\to 0}\left[\frac{\sin(x)}{\sin\left(\frac{x}{p}\right)}\right] = \lim_{x\to 0}\left[p\frac{\sin(x)}{x} \cdot \frac{x/p}{\sin(x/p)}\right] = p$$

Substituting for $\frac{\sin(x)}{\sin(x/p)}$, and taking the limit as x approaches zero, we get,

$$p = 2^{2p-1}\sin^2\left(\frac{\pi}{p}\right)\sin^2\left(\frac{2\pi}{p}\right)...\sin^2\left(\frac{(\frac{p}{2}-1)\pi}{p}\right)$$

Now, if we take the ratio of $\sin(x)$ to p, we get,

$$\frac{\sin(x)}{p} = \sin\left(\frac{x}{p}\right)\left[1 - \frac{\sin^2(x/p)}{\sin^2(\pi/p)}\right]...\left[1 - \frac{\sin^2(x/p)}{\sin^2\left((\frac{p}{2}-1)\pi/p\right)}\right]\cos(x/p)$$

We want to express $\sin(x)$ as infinite product. So we let $p \to \infty$. Thus, as $p \to \infty$, $p\sin(x/p) = x$ and $\frac{\sin^2(x/p)}{sin^2(\pi/p)} = \frac{x^2}{\pi^2}$

Substituting, we get:

$$\sin(x) = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)...$$

$$\sin(x) = x\prod_{r=1}^{\infty}\left(1 - \frac{x^2}{r^2\pi^2}\right)$$

15.1 Basel Problem (Euler's Solution)

Using the infinite product formula and the infinite sum formula for $\frac{\sin(x)}{x}$.

$$\frac{\sin(x)}{x} = \left(1 - \frac{x^2}{1^2\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)...$$

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - ...$$

Multiplying the infinite product formula to obtain the x^2 term and comparing it with the x^2 term of the infinite sum formula, we get:

$$\frac{-x^2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + ...\right) = \frac{-x^2}{3!}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}... = \frac{\pi^2}{6}$$

16 Stirling's Approximation

Using **Gamma Function**,

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx = n!$$

Substituting $x = nz$ and $dx = ndz$

$$\begin{aligned} n! &= \int_0^\infty (nz)^n e^{-nz} ndz \\ &= n^{n+1} \int_0^\infty z^n e^{-nz} dz \\ &= n^{n+1} \int_0^\infty e^{\ln(z^n)} e^{-nz} dz \\ &= n^{n+1} \int_0^\infty e^{n\ln(z)} e^{-nz} dz \\ &= n^{n+1} \int_0^\infty e^{n(\ln(z)-z)} dz \end{aligned}$$

Now, writing the **Taylor Series** expansion for $\ln(z) - z$ centered at $z = 1$,

$$f(z) = \ln(z) - z = f(1) + f'(1)(z-1) + \frac{1}{2!} f''(1)(z-1)^2 ...$$

For first three term approximation of $f(z)$, Substituting $f(1) = -1, f'(1) = 0, f''(1) = -1$,

$$f(z) = \ln(z) - z \approx -1 - \frac{1}{2!}(z-1)^2$$

Substituting this approximation in above expression,

$$n! \approx n^{n+1} \int_0^{\infty} e^{n(-1-\frac{1}{2!}(z-1)^2)} dz$$

$$= n^{n+1} e^{-n} \int_0^{\infty} e^{\frac{-n}{2}(z-1)^2} dx$$

For large values of n,

$$= n^{n+1} e^{-n} \sqrt{\frac{2}{n}\pi}$$

$$= \frac{n^{n+1}}{n^{1/2} n^{1/2}} e^{-n} \sqrt{2\pi n}$$

$$= n^n e^{-n} \sqrt{2\pi n}$$

$$= \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

17 Vieta's Formula for π

We take the formula for $\sin(x) = 2\sin(\frac{x}{2})\cos(\frac{x}{2})$ and apply this formula to itself repeatedly for the sine function as follows:

$$\begin{aligned}\sin(x) &= 2\sin(x/2)\cos(x/2)\\ &= 2\cdot 2\sin(x/4)\cos(x/4)\cos(x/2)\\ &= 2\cdot 2\cdot 2\sin(x/8)\cos(x/8)\cos(x/4)\cos(x/2)\end{aligned}$$

Repeating this process for n-times,

$$\sin(x) = 2^n \sin\left(\frac{x}{2^n}\right)\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)$$

Now, we take the limit of $n \to \infty$ and $\sin(x/2^n) \to x/2^n$ because as n gets larger, $x/2^n \to 0$

$$\lim_{n\to\infty}\sin(x) = \lim_{n\to\infty} 2^n \sin\left(\frac{x}{2^n}\right)\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)$$

$$\begin{aligned}\sin(x) &= \lim_{n\to\infty} 2^n\left(\frac{x}{2^n}\right)\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)\\ &= \lim_{n\to\infty} x\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)\end{aligned}$$

Thus,

$$\frac{\sin(x)}{x} = \prod_{k=1}^{\infty}\cos\left(\frac{x}{2^k}\right)$$

When we substitute $x = \frac{\pi}{2}$,

$$\frac{2}{\pi} = cos\left(\frac{\pi}{4}\right)cos\left(\frac{\pi}{8}\right)cos\left(\frac{\pi}{16}\right)...$$

we know the value for $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$. So, in order to obtain all the other values for which the angles are just half of the previous angle, we use the identity, $\cos(x/2) = 1/2\sqrt{2 + 2\cos(x)}$

$$
\begin{aligned}
\cos(\pi/8) &= 1/2\sqrt{2+2\cos(\pi/4)} \\
&= 1/2\sqrt{2+2\sqrt{2}} \\
\cos(\pi/16) &= 1/2\sqrt{2+2\cos(\pi/8)} \\
&= 1/2\sqrt{2+\sqrt{2+2\sqrt{2}}}
\end{aligned}
$$

Using this pattern, we obtain,

$$
\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \ldots
$$

18 Vieta's Formula For Polynomials

Consider the quadratic equation,

$$x^2 + \frac{a_1}{a_2}x + \frac{a_0}{a_2} = (x - r_1)(x - r_2)$$

where r_1 and r_2 are the roots of the polynomial. Expanding the right hand side, we get,

$$x^2 + \frac{a_1}{a_2}x + \frac{a_0}{a_2} = x^2 - (r_1 + r_2)x + r_1 r_2$$

Now, comparing the coefficients of the like terms on both sides, we obtain,

$$r_1 + r_2 = \frac{-a_1}{a_2}$$

$$r_1 r_2 = \frac{a_0}{a_2}$$

Consider the cubic equation,

$$x^3 + \frac{a_2}{a_3}x^2 + \frac{a_1}{a_3}x + \frac{a_0}{a_3} = (x - r_1)(x - r_2)(x - r_3)$$

where r_1, r_2 and r_3 are the roots of the polynomial. Expanding the right hand side, we get,

$$x^3 + \frac{a_2}{a_3}x^2 + \frac{a_1}{a_3}x + \frac{a_0}{a_3} = x^3 - (r_1 + r_2 + r_3)x^2 + (r_1 r_2 + r_2 r_3 + r_3 r_1)x - r_1 r_2 r_3$$

Now, comparing the coefficients of the like terms on both sides, we obtain,

$$r_1 + r_2 + r_3 = \frac{-a_2}{a_3}$$

$$r_1 r_2 + r_2 r_3 + r_3 r_1 = \frac{a_1}{a_3}$$

$$r_1 r_2 r_3 = \frac{-a_0}{a_3}$$

Now, we consider the general polynomial,

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$$

Dividing both sides by a_n,

$$\frac{p(x)}{a_n} = x^n + \frac{a_{n-1}}{a_n}x^{n-1} + ... + \frac{a_2}{a_n}x^2 + \frac{a_1}{a_n}x + \frac{a_0}{a_n}$$

and consider $r_1, r_2, ..., r_n$ be the roots of this general polynomial. So,

$$p(x) = a_n(x - r_1)(x - r_2)...(x - r_n)$$

$$\frac{p(x)}{a_n} = (x - r_1)(x - r_2)...(x - r_n)$$

Expanding we obtain,

$$\frac{p(x)}{a_n} = x^n - (r_1 + r_2 + ... + r_n)x^{n-1} + (r_1r_2 + r_2r_3 + ... + r_{n-1}r_n)x^{n-2} - ... + (r_1r_2r_3...r_n)$$

These coefficients are **Elementary Symmetric Polynomials**

18.1 Elementary Symmetric Polynomials

These polynomials can be written in the form,

$$s_k = \sum_{1 \le i_1 \le i_2 \le ... \le i_k \le n} r_{i_1} r_{i_2} r_{i_3} ... r_{i_k}$$

So, above expansion can be written as,

$$\frac{p(x)}{a_n} = x^n - s_1x^{n-1} + s_2x^{n-2} - ... + s_n$$

Now, comparing the coefficients of the like terms we obtain,

$$\frac{a_{n-1}}{a_n} = -s_1$$

$$\frac{a_{n-2}}{a_n} = s_2$$

$$\frac{a_{n-3}}{a_n} = -s_3$$

The general form is,

$$\frac{a_{n-k}}{a_n} = (-1)^k s_k$$

Substituting the definition of s_k,

$$\frac{a_{n-k}}{a_n} = (-1)^k \sum_{1 \le i_1 \le i_2 \le ... \le i_k \le n} r_{i_1} r_{i_2} r_{i_3} ... r_{i_k}$$

19 Wallis Formula

Let's find reduction formulas for $\int \sin^n(x)dx$ and $\int \cos^n(x)dx$. Let $I_n = \int \sin^n(x)dx$

$$\begin{aligned}
I_n = \int \sin^n(x) &= \int \sin^{n-1}(x)\sin(x)dx \\
&= \int \sin^{n-1} d(-\cos(x)) \\
&= \sin^{n-1}(x)(-\cos(x)) + (n-1)\int \cos(x)\cdot\sin^{n-2}(x)\cdot\cos(x)dx \\
&= -\sin^{n-1}(x)\cdot\cos(x) + (n-1)\int (1-\sin^2(x))\cdot\sin^{n-2}dx \\
&= -\sin^{n-1}(x)\cdot\cos(x) + (n-1)\left[\int \sin^{n-2}dx - \int \sin^n(x))dx\right] \\
I_n &= -\sin^{n-1}(x)\cdot\cos(x) + (n-1)[I_{n-2} - I_n] \\
n\cdot I_n &= -\sin^{n-1}(x)\cdot\cos(x) + (n-1)I_{n-2} \\
I_n &= -\frac{1}{n}\sin^{n-1}(x)\cdot\cos(x) + \frac{n-1}{n}I_{n-2}
\end{aligned}$$

Using same method for evaluating $\int \cos^n(x)dx$ we get following results

$$\int \sin^n(x)dx = -\frac{1}{n}\sin^{n-1}(x)\cdot\cos(x) + \frac{n-1}{n}I_{n-2}$$

$$\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\cdot\sin(x) + \frac{n-1}{n}I_{n-2}$$

Now, evaluating these integrals from 0 to $\frac{\pi}{2}$ we get,

$$\int_0^{\frac{\pi}{2}} \cos^n(x)dx = \int_0^{\frac{\pi}{2}} \sin^n(x)dx = I_n = \frac{n-1}{n}I_{n-2}$$

19.1 Wallis Product Formula For π

Taking the ratios of integrals,

$$\frac{I_n}{I_{n-2}} = \frac{n-1}{n}$$

for **even** numbers, $I_0 = \pi, I_2 = \frac{1}{2}\frac{\pi}{2}, I_4 = \frac{3}{4}\frac{1}{2}\frac{\pi}{2}$

$$I_{2n} = \frac{\pi}{2}\prod_{k=1}^{n}\frac{2k-1}{2k}$$

for **odd** numbers, $I_1 = 1, I_3 = \frac{2}{3}, I_5 = \frac{4}{5}\frac{2}{3}$

$$I_{2n+1} = 1\prod_{k=1}^{n}\frac{2k}{2k+1}$$

since, when $0 < x < \frac{\pi}{2}$, $0 < \sin(x) < 1$ which means, $\sin^{2n+1}(x) \leq \sin^{2n}(x) \leq \sin^{2n-1}(x)$. Thus, $I_{2n+1} \leq I_{2n} \leq I_{2n-1}$.

Dividing by I_{2n+1} we get,

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{I_{2n-1}}{I_{2n+1}}$$

Using the iterative relation from above,

$$\frac{I_{2n-1}}{I_{2n+1}} = \frac{2n}{2n+1}$$

Now, by substitution,

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{2n}{2n+1}$$

As $n \to \infty$

$$\lim_{n\to\infty} 1 \leq \lim_{n\to\infty}\frac{I_{2n}}{I_{2n+1}} \leq \lim_{n\to\infty}\frac{2n}{2n+1}$$

$$1 \leq \lim_{n\to\infty}\frac{I_{2n}}{I_{2n+1}} \leq 1$$

Thus, by squeeze theorem,

$$\lim_{n\to\infty}\frac{I_{2n}}{I_{2n+1}} = 1$$

$$\frac{\pi}{2}\prod_{k=1}^{\infty}\frac{2k-1}{2k}\cdot\frac{2k+1}{2k} = 1$$

$$\frac{\pi}{2} = \prod_{k=1}^{\infty}\frac{2k}{2k-1}\cdot\frac{2k}{2k+1}$$

This is known as **Wallis product**.

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6...}{1\cdot 3\cdot 3\cdot 5\cdot 5...}$$