

1 Gamma Function

$$\int_0^1 \log(x) dx = \left[x \log(x) - 1 \cdot x \right]_0^1 = -1!$$

$$\int_0^1 \log^2(x) dx = \left[x \log^2(x) - 2x \log(x) + 2 \cdot 1 \cdot x \right]_0^1 = 2!$$

$$\int_0^1 \log^3(x) dx = \left[x \log^3(x) - 3x \log^2(x) - 2 \cdot 3x \log(x) - 3 \cdot 2 \cdot 1 \cdot x \right]_0^1 = -3!$$

Observing the pattern, we can write,

$$\int_0^1 \log^n(x) dx = (-1)^n \cdot n!$$

Solving for $n!$ we get,

$$n! = \frac{1}{(-1)^n} \int_0^1 \log^n(x) dx$$

$$n! = \int_0^1 \left(\frac{\log(x)}{-1} \right)^n dx$$

$$n! = \int_0^1 (-\log(x))^n dx$$

$$n! = \int_0^1 \left(\log\left(\frac{1}{x}\right) \right)^n dx$$

Let $u = \log\left(\frac{1}{x}\right)$. Using the property of logarithms $-u = \log(x)$. Raising to the power of e on both sides, we get $x = e^{-u}$ and thus, $dx = -e^{-u} du$. When $x = 0$, $u \rightarrow \infty$ and when $x = 1$, $u = 0$. Therefore, the integral becomes,

$$n! = \int_0^\infty u^n (e^{-u}) du$$

Now, by the definition of gamma function,

$$\Gamma(n+1) = n! = \int_0^\infty u^n (e^{-u}) du$$

$$\Gamma(n) = (n-1)! = \int_0^\infty u^{n-1} e^{-u} du$$

Replacing n by x ,

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$$

1.1 $\Gamma(\frac{1}{2})$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} u^{-\frac{1}{2}} e^{-u} du$$

Let $t = u^{\frac{1}{2}}$ so that $dt = \frac{1}{2} u^{-\frac{1}{2}} du$. Making the substitution we get,

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^{\infty} e^{-t^2} dt$$

Since, e^{-t^2} is an even function we can write,

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^{\infty} e^{-t^2} dt$$

This is known as **Gaussian Integral**. Let's evaluate this Integral,

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2$$

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) \cdot \left(\int_{-\infty}^{\infty} e^{-y^2} dy \right)$$

$$I^2 = \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy$$

Now in polar coordinates, $dx dy = dA = r dr d\theta$ and when $x \rightarrow -\infty, r \rightarrow 0$ and $x \rightarrow \infty, r \rightarrow \infty$ when $y \rightarrow -\infty, \theta \rightarrow 0$ and $y \rightarrow \infty, \theta \rightarrow 2\pi$

Substituting, we get,

$$I^2 = \int_0^{2\pi} \int_0^{\infty} r e^{-r^2} dr d\theta$$

$$I^2 = \int_0^{2\pi} \frac{1}{2} d\theta$$

$$I^2 = \frac{1}{2} [\theta]_0^{2\pi}$$

$$I^2 = \frac{1}{2} 2\pi$$

$$I^2 = \pi$$

$$I = \sqrt{\pi}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

2 Maclaurin's Series

Let $f(x)$ be function of x . The goal is to write the function as an infinite polynomial.

$$f(x) = a + bx + cx^2 + dx^3 + ex^4 + \dots$$

when $x=0$, $f(0) = a$ In order to find other coefficients we differentiate the function with respect to x and put $x = 0$. $f'(0) = 1 \cdot b$ $f''(0) = 1 \cdot 2 \cdot b$ $f'''(0) = 1 \cdot 2 \cdot 3 \cdot c$ and so on. Looking at the pattern, the general form would be,

$$f^{(n)}(0) = n! \cdot (n^{th} coefficient)$$

Substituting in the above equation for coefficients, we get,

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

In Summation Notation,

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

$\sin(x)$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
$\cos(x)$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
$\tan(x)$	$x + \frac{x^3}{2} + \frac{2x^5}{15} - \dots$
$\sinh(x)$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$
$\cosh(x)$	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$
$\tan^{-1}(x)$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

Now, the more general expansion gives the **Taylor Series** which is shifted by h .

$$f(x+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x)$$