Contents

1 Gamma Function

$$\int_0^1 log(x)dx = \Big[xlog(x) - 1 \cdot x\Big]_0^1 = -1!$$

$$\int_0^1 log^2(x)dx = \Big[xlog^2(x) - 2xlog(x) + 2 \cdot 1 \cdot x\Big]_0^1 = 2!$$

$$\int_0^1 log^3(x)dx = \Big[xlog^3(x) - 3xlog^2(x) - 2 \cdot 3xlog(x) - 3 \cdot 2 \cdot 1 \cdot x\Big]_0^1 = -3!$$

Observing the pattern, we can write,

$$\int_0^1 log^n(x)dx = (-1)^n \cdot n!$$

Solving for $n!$ we get,

$$n! = \frac{1}{(-1)^n}\int_0^1 log^n(x)dx$$

$$n! = \int_0^1 \left(\frac{log(x)}{-1}\right)^n dx$$

$$n! = \int_0^1 \big(-log(x)\big)^n dx$$

$$n! = \int_0^1 \left(log\left(\frac{1}{x}\right)\right)^n dx$$

Let $u = log\big(\frac{1}{x}\big)$. Using the property of logarithms $-u = log(x)$. Raising to the power of e on both sides, we get $x = e^{-u}$ and thus, $dx = -e^{-u}du$. When $x = 0$, $u \to \infty$ and when $x = 1$, $u = 0$. Therefore, the integral becomes,

$$n! = \int_0^\infty u^n(e^{-u})du$$

Now, by the definition of gamma function,

$$\Gamma(n+1) = n! = \int_0^\infty u^n(e^{-u})du$$

$$\Gamma(n) = (n-1)! = \int_0^\infty u^{n-1}e^{-u}du$$

Replacing n by x,

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du$$

1.1 $\Gamma(\frac{1}{2})$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty u^{-\frac{1}{2}} e^{-u} du$$

Let $t = u^{\frac{1}{2}}$ so that $dt = \frac{1}{2}u^{\frac{-1}{2}}du$. Making the substitution we get,

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-t^2} dt$$

Since, e^{-t^2} is an even function we can write,

$$\Gamma\left(\frac{1}{2}\right) = \int_{-\infty}^\infty e^{-t^2} dt$$

This is know as **Gaussian Integral**. Let's evaluate this Integral,

$$I = \int_{-\infty}^\infty e^{-x^2} dx$$

$$I^2 = \left(\int_{-\infty}^\infty e^{-x^2} dx\right)^2$$

$$I^2 = \left(\int_{-\infty}^\infty e^{-x^2} dx\right) \cdot \left(\int_{-\infty}^\infty e^{-y^2} dy\right)$$

$$I^2 = \int_{-\infty}^\infty e^{-(x^2+y^2)} dxdy$$

Now in polar coordinates, $dxdy = dA = rdrd\theta$ and when $x \to -\infty, r \to 0$ and $x \to \infty, r \to \infty$ when $y \to -\infty, \theta \to 0$ and $y \to \infty, \theta \to 2\pi$

Substituting, we get,

$$I^2 = \int_0^{2\pi} \int_0^\infty re^{-r^2} drd\theta$$

$$I^2 = \int_0^{2\pi} \frac{1}{2} d\theta$$

$$I^2 = \frac{1}{2}\big[\theta\big]_0^{2\pi}$$

$$I^2 = \frac{1}{2}2\pi$$

$$I^2 = \pi$$

$$I = \sqrt{\pi}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

2 Maclaurin's Series

Let $f(x)$ be function of x. The goal is to write the function as an infinite polynomial.

$$f(x) = a + bx + cx^2 + dx^3 + ex^4 + ...$$

when x=0, $f(0) = a$ In order to find other coefficients we differentiate the function with respect to x and put $x = 0$. $f'(0) = 1 \cdot b$ $f''(0) = 1 \cdot 2 \cdot b$ $f'''(0) = 1 \cdot 2 \cdot 3 \cdot c$ and so on. Looking at the pattern, the general form would be,

$$f^{(n)}(0) = n! \cdot (n^{th} coefficient)$$

Substituting in the above equation for coefficients, we get,

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$$

In Summation Notation,

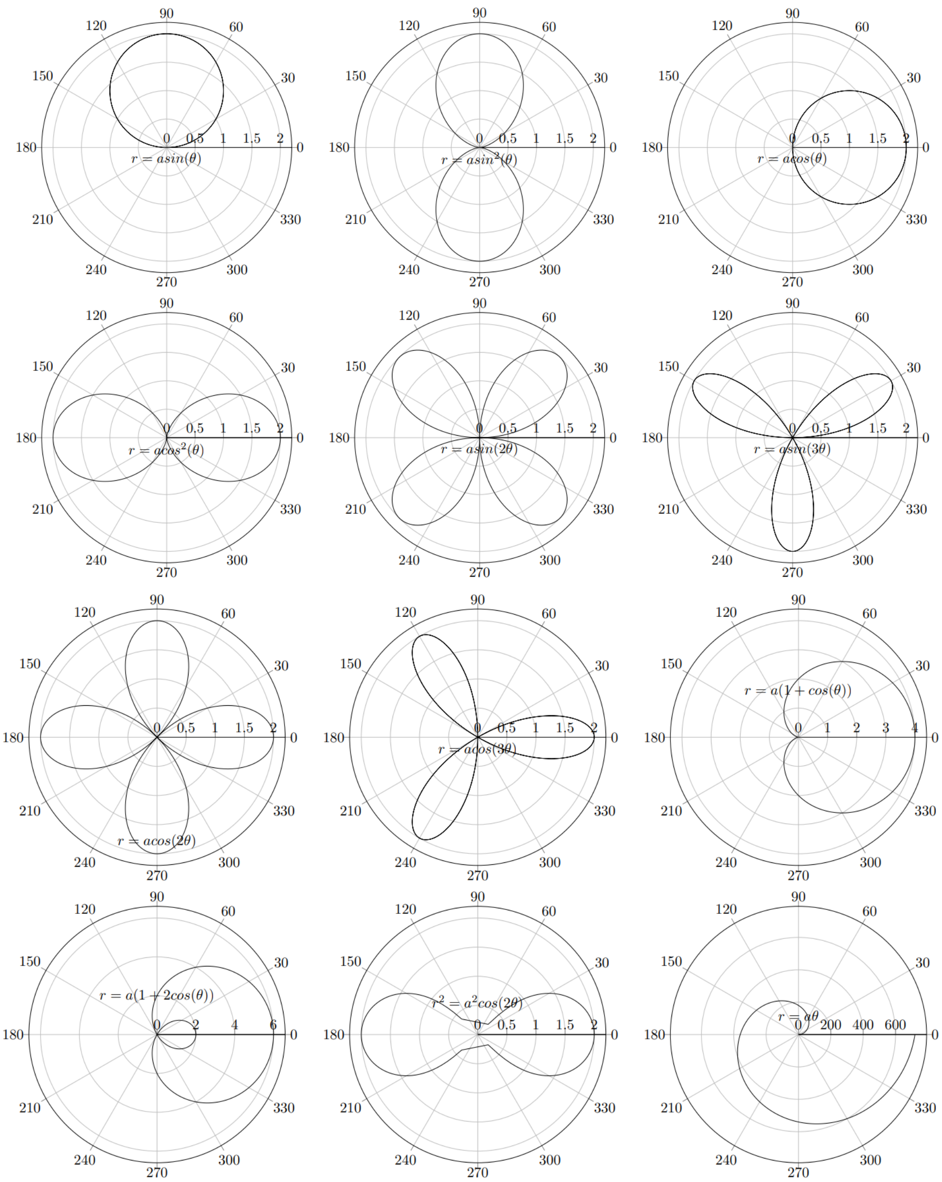
$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

$\sin(x)$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$
$\cos(x)$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$
$\tan(x)$	$x + \frac{x^3}{2} + \frac{2x^5}{15} - ...$
$\sinh(x)$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$
$\cosh(x)$	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} ...$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - ...$
$\tan^{-1}(x)$	$x - \frac{x^3}{3} + \frac{x^5}{5} + ...$

Now, the more general expansion gives the **Taylor Series** which is shifted by h.

$$f(x+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x)$$

3 Polar Curves



4 Wallis Formula

Let's find reduction formulas for $\int \sin^n(x)dx$ and $\int \cos^n(x)dx$. Let $I_n = \int \sin^n(x)dx$

$$\begin{aligned}
I_n = \int \sin^n(x) &= \int \sin^{n-1}(x)\sin(x)dx \\
&= \int \sin^{n-1} d(-\cos(x)) \\
&= \sin^{n-1}(x)(-\cos(x)) + (n-1)\int \cos(x)\cdot\sin^{n-2}(x)\cdot\cos(x)dx \\
&= -\sin^{n-1}(x)\cdot\cos(x) + (n-1)\int (1-\sin^2(x))\cdot\sin^{n-2} dx \\
&= -\sin^{n-1}(x)\cdot\cos(x) + (n-1)\left[\int \sin^{n-2} dx - \int \sin^n(x))dx\right] \\
I_n &= -\sin^{n-1}(x)\cdot\cos(x) + (n-1)[I_{n-2} - I_n] \\
n\cdot I_n &= -\sin^{n-1}(x)\cdot\cos(x) + (n-1)I_{n-2} \\
I_n &= -\frac{1}{n}\sin^{n-1}(x)\cdot\cos(x) + \frac{n-1}{n}I_{n-2}
\end{aligned}$$

Using same method for evaluating $\int \cos^n(x)dx$ we get following results

$$\int \sin^n(x)dx = -\frac{1}{n}\sin^{n-1}(x)\cdot\cos(x) + \frac{n-1}{n}I_{n-2}$$

$$\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\cdot\sin(x) + \frac{n-1}{n}I_{n-2}$$

Now, evaluating these integrals from 0 to $\frac{\pi}{2}$ we get,

$$\int_0^{\frac{\pi}{2}} \cos^n(x)dx = \int_0^{\frac{\pi}{2}} \sin^n(x)dx = I_n = \frac{n-1}{n}I_{n-2}$$

4.1 Wallis Product formula for π

Taking the ratios of integrals,

$$\frac{I_n}{I_{n-2}} = \frac{n-1}{n}$$

for **even** numbers, $I_0 = \pi, I_2 = \frac{1}{2}\frac{\pi}{2}, I_4 = \frac{3}{4}\frac{1}{2}\frac{\pi}{2}$

$$I_{2n} = \frac{\pi}{2}\prod_{k=1}^{n}\frac{2k-1}{2k}$$

for **odd** numbers, $I_1 = 1, I_3 = \frac{2}{3}, I_5 = \frac{4}{5}\frac{2}{3}$

$$I_{2n+1} = 1\prod_{k=1}^{n}\frac{2k}{2k+1}$$

since, when $0 < x < \frac{\pi}{2}$, $0 < \sin(x) < 1$ which means, $\sin^{2n+1}(x) \leq \sin^{2n}(x) \leq \sin^{2n-1}(x)$. Thus, $I_{2n+1} \leq I_{2n} \leq I_{2n-1}$.

Dividing by I_{2n+1} we get,

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{I_{2n-1}}{I_{2n+1}}$$

Using the iterative relation from above,

$$\frac{I_{2n-1}}{I_{2n+1}} = \frac{2n}{2n+1}$$

Now, by substitution,

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{2n}{2n+1}$$

As $n \to \infty$

$$\lim_{n\to\infty} 1 \leq \lim_{n\to\infty}\frac{I_{2n}}{I_{2n+1}} \leq \lim_{n\to\infty}\frac{2n}{2n+1}$$

$$1 \leq \lim_{n\to\infty}\frac{I_{2n}}{I_{2n+1}} \leq 1$$

Thus, by squeeze theorem,

$$\lim_{n\to\infty}\frac{I_{2n}}{I_{2n+1}} = 1$$

$$\frac{\pi}{2}\prod_{k=1}^{\infty}\frac{2k-1}{2k}\cdot\frac{2k+1}{2k} = 1$$

$$\frac{\pi}{2} = \prod_{k=1}^{\infty}\frac{2k}{2k-1}\cdot\frac{2k}{2k+1}$$

This is known as **Wallis product**.

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6...}{1\cdot 3\cdot 3\cdot 5\cdot 5...}$$

5 Leibniz Theorem

Let u and v be functions of x. This theorem gives the closed form for the n^{th} derivative of product of two functions. That will be $(uv)^{(n)}$ or $\frac{d^n(u \cdot v)}{dx^n}$ for functions u and v.

for small values of n,

$$\begin{aligned} n = 1, (uv)' &= u'v + uv' \\ n = 2, (uv)'' &= u''v + 2u'v' + uv'' \\ n = 3, (uv)''' &= u'''v + 3u''v' + 3u'v'' + uv''' \\ n = 4, (uv)^{(4)} &= u^{(4)}v + 4u^{(3)}v' + 6u^{(2)}v^{(2)} + 4u^{(1)}v^{(3)} + uv^{(4)} \end{aligned}$$

Looking at the pattern from above, coefficients are **binomial coefficients**. So, we can write general form as,

$$(uv)^{(n)} = \frac{d^n(uv)}{dx^n} = \sum_{r=0}^{n} C(n,r)u^{(n-r)}v^{(r)}$$

where $C(n,r)$ is the binomial coefficient and $C(n,r) = \frac{n!}{r!(n-r)!}$.

5.1 Proof

Now, we prove this theorem by **induction**,
We assume for n=k, $(uv)^{(k)} = \frac{d^k(uv)}{dx^k} = \sum_{r=0}^{k} C(k,r)u^{(k-r)}v^{(r)}$ is true

And now for n=k+1, we prove,

$$(uv)^{(k+1)} = \frac{d^{k+1}(uv)}{dx^{k+1}} = \sum_{r=0}^{k+1} C(k+1,r)u^{(k+1-r)}v^{(r)}$$

By definition of successive differentiation,

$$
\begin{aligned}
(uv)^{(k+1)} = \frac{d^{k+1}(uv)}{dx^{k+1}} &= \frac{d}{dx}\sum_{r=0}^{k} C(k,r)u^{(k-r)}v^{(r)} \\
&= \sum_{r=0}^{k} C(k,r)\Big(u^{(k-r+1)}v^{(r)} + u^{(k-r)}v^{(r+1)}\Big) \\
&= \sum_{r=0}^{k} C(k,r)u^{(k-r+1)}v^{(r)} + \sum_{r=0}^{k} C(k,r)u^{(k-r)}v^{(r+1)} \\
&= \sum_{r=0}^{k} C(k,r)u^{(k-r+1)}v^{(r)} + \sum_{r=1}^{k+1} C(k,r-1)u^{(k-r+1)}v^{(r)}
\end{aligned}
$$

$$
= C(k,0)u^{(k+1)}v^{(0)} + C(k,k)u^{(0)}v^{k+1} + \sum_{r=1}^{k}\Big(C(k,r) + C(k,r-1)\Big)u^{(k-r+1)}v^{(r)}
$$

$$
= u^{(k+1)}v^{(0)} + u^{(0)}v^{k+1} + \sum_{r=1}^{k} C(k+1,r)u^{(k-r+1)}v^{(r)}
$$

$$
= \sum_{r=0}^{k} C(k+1,r)u^{(k+1-r)}v^{(r)}
$$

which is what we wanted to prove.