UNIT - V

Inferential Statistics

In previous unit, we emphasized sampling properties of the sample mean and proportion. The purpose of these presentations is to build a foundation that allows us to draw conclusions about the population parameters from experimental data. For example, the Central Limit Theorem provides information about the distribution of the sample mean $\overline{X}$. The distribution involves the population mean μ. Thus, any conclusions concerning μ drawn from an observed sample average must depend on knowledge of this sampling distribution. Similar comments apply to S^2 and σ^2. Clearly, any conclusions we draw about the variance of a normal distribution will likely involve the sampling distribution of S^2.

Statistical inference may be divided into two major areas: estimation and tests of hypotheses.

t-distribution (Student's t-distribution)

Let X_1, X_2, ..., X_n be a random sample of size n from a normal distribution with mean μ and unknown variance σ^2. The random variable

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

has a t-distribution with degree of freedom $\nu = n - 1$, where $\overline{x} = \frac{1}{n}\sum x_i$ is the sample mean and $s^2 = \frac{1}{n-1}\sum(x_i - \overline{x})^2$ is the sample variance. The probability density function is

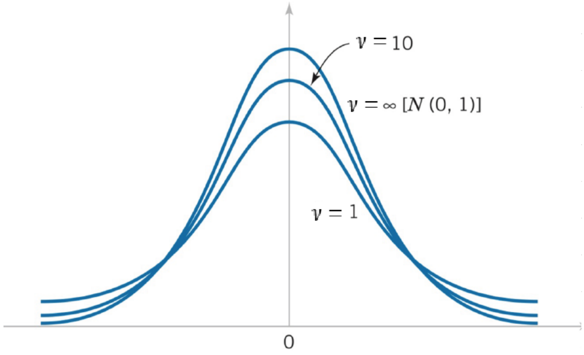
$$f(t) = \frac{\Gamma[(\nu+1)/2]}{\sqrt{\pi\nu}\,\Gamma(\nu/2)}(1 + t^2/\nu)^{-(\nu+1)/2}, \quad -\infty < t < \infty$$

where ν is the number of degree of freedom. The mean and variance of t-distribution are zero and $\nu/(\nu - 2)$ (for $\nu > 2$), respectively.

Properties of t-distribution

- It is symmetric and unimodal, maximum ordinate value is reached when $\mu = 0$.
- It is similar to the standard normal distribution. However, the t-distribution has heavier tails than the normal.
- As the number of degrees of freedom ν approaches ∞, the t-distribution approaches standard normal distribution.

Problem 1. *The following are the IQs of a randomly chosen sample of 10 boys: 70, 120, 110, 101, 88, 83, 95, 98, 107, 100. Find the value of student's t variate for the given sample taking the mean of population to be 100.*

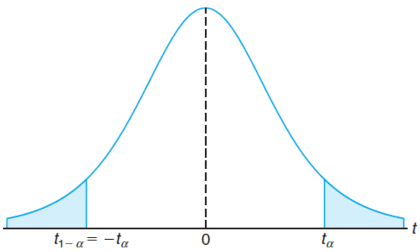


Probability density functions of several t-distributions

Solution:Given $\mu = 100$

$$\begin{aligned}
\overline{x} &= \frac{70+120+110+101+88+83+95+98+107+100}{10} = 97.2 \\
s^2 &= \frac{(-27.2)^2+22.8^2+12.8^2+3.8^2+(-9.2)^2+(-14.2)^2+(-2.2)^2+0.8^2+9.8^2+2.8^2}{10-1} \\
s^2 &= 203.73 \\
s &= \sqrt{203.73} = 14.27 \\
t &= \frac{\overline{x}-\mu}{s/\sqrt{n}} = \frac{97.2-100}{14.27/\sqrt{10}} = -0.62
\end{aligned}$$

Note: In t-distribution, $t_\alpha(k)$ or $t_{\alpha,k}$ indicates the t value in t-distribution curve corresponding to k degree of freedom such that the area under the curve and to the right of the point t is α.



Symmetry property (about 0) of the t-distribution and $t_{\alpha,k}$-value

Problem 2. *Find the value of $t_{\alpha,\nu}$ for $\alpha = 0.05$ and $\nu = 9$.*

Solution: $t_{\alpha,\nu} = t_{0.05,9} = 1.833$

χ^2 - distribution

Let X_1, X_2, ... X_n be a random sample from a normal distribution with mean μ and variance σ^2. Let S^2 be the random variable associated with sample variance. Then the random variable

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2}$$

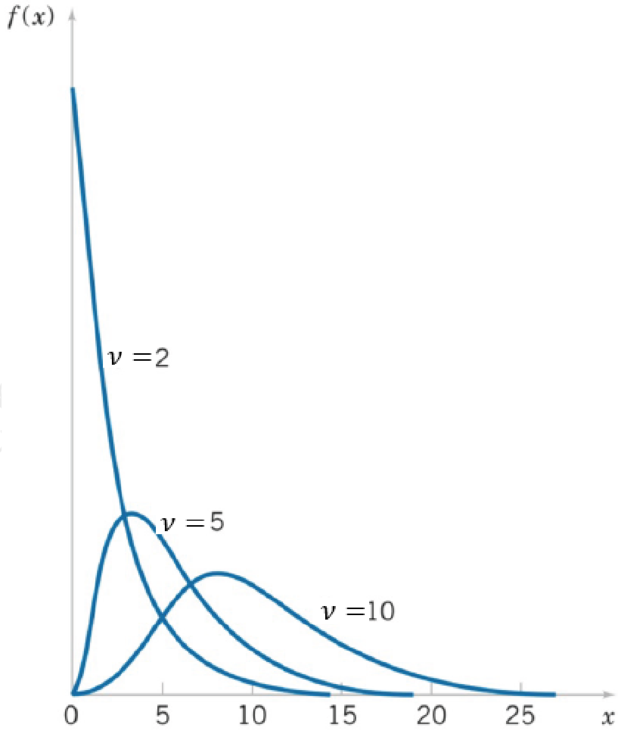
has a chi-square (χ^2) distribution with $\nu = n - 1$ degree of freedom. The probability density function of a χ^2 random variable is

$$f(x) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{(\nu/2)-1} e^{-x/2}, \quad x > 0$$

where ν is the number of degrees of freedom.

Properties of χ^2-distribution

- The mean of χ^2-distribution is $n - 1$.
- Variance of χ^2-distribution is $2(n - 1)$
- The random variable is non-negative.
- The probability distribution is skewed to the right.
- As n increases, the distribution becomes more symmetric.
- As n approaches ∞, the χ^2 distribution approaches normal distribution.



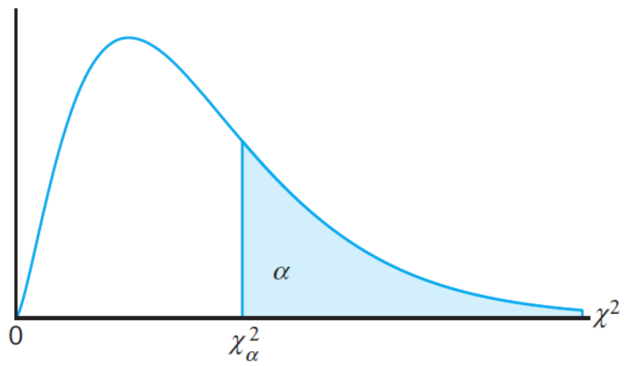
χ^2-distribution for different values of degrees of freedom $k = n - 1$

Problem 3. *A manufacturer of car batteries guarantees that the batteries will last, on average, 3 years with a standard deviation of 1 year. Assuming the battery lifetime follows a normal distribution, find χ^2 for the life time 1.9, 2.4, 3.0, 3.5 and 4.2 years of five of these batteries.*

Solution: Given $\mu = 3$ and $\sigma = 1$.

$$\begin{aligned}
\chi^2 &= \frac{((n-1)s^2)}{\sigma^2} \\
\overline{x} &= \frac{1.9+2.4+3.0+3.5+4.2}{5} = 3 \\
s^2 &= \frac{(1.9-3)^2+(2.4-3)^2+(3.0-3)^2+(3.5-3)^2+(4.2-3)^2}{5-1} = 0.815 \\
\chi^2 &= \frac{4\times 0.815}{1} = 3.26
\end{aligned}$$

Note: In χ^2-distribution, $\chi^2_\alpha(k)$ or $\chi^2_{\alpha,k}$ indicates the χ^2 value in χ^2-distribution curve corresponding to k degree of freedom such that the area under the curve and to the right of the point χ^2 is α.



$\chi^2_{\alpha,k}$-value

F-distribution

Let U and V be two independent random variables having chi-squared distributions with ν_1 and ν_2 degrees of freedom, respectively. Then the distribution of the random variable $F = \frac{U/\nu_1}{V/\nu_2}$ has the density function

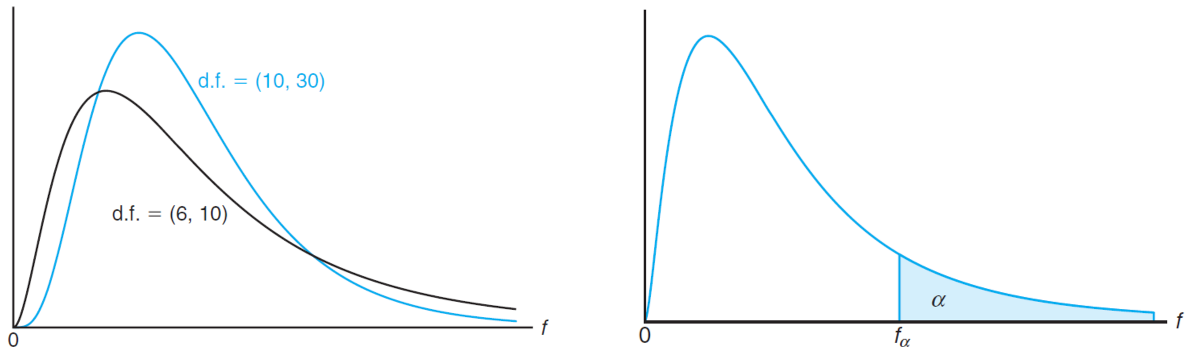
$$h(f) = \begin{cases} \frac{\Gamma((\nu_1+\nu_2)/2)}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)} \left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} \frac{f^{(\nu_1/2)-1}}{(1+\nu_1 f/\nu_2)^{(\nu_1+\nu_2)/2}}, & f > 0, \\ 0, & f \le 0. \end{cases}$$

This is known as the F**-distribution** with ν_1 and ν_2 degrees of freedom (d.f.). If S_1^2 and S_2^2 are the variances of independent random samples of size n_1 and n_2 taken from normal populations with variances σ_1^2 and σ_2^2, respectively, then

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2}$$

has an F-distribution with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom.

For any real $\alpha \in [0,1]$ and degrees of freedom ν_1 and ν_2, $f_\alpha(\nu_1,\nu_2)$ indicate the F value with degrees of freedom ν_1 and ν_2 such that area to the right side of $f_\alpha(\nu_1,\nu_2)$ is α.



F-distribution for different values of degrees of freedom and illustration of the f_α value

Properties of F-distribution

- The mean of F-distribution is $\frac{\nu_2}{\nu_2-2}$, for $\nu_2 > 2$.
- Variance of F-distribution is $\frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)}$, for $\nu_2 > 4$.
- The random variable is non-negative.
- The probability distribution is skewed to the right.
- $f_\alpha(\nu_1, \nu_2) = \frac{1}{f_{1-\alpha,\nu_2,\nu_1}}$

Problem 4. *The following are the weights (in grams) of apples produced by two different trees:*

Tree A:	*120,*	*130,*	*125,*	*135,*	*128,*	*132,*	*127*
Tree B:	*118,*	*122,*	*125,*	*121,*	*119,*	*124,*	*123*

Calculate the F value for the given data if the standard deviation in weight of apples of tree A and tree B are 5 and 2 grams respectively.

Solution: Given $\sigma_A = 5$ and $\sigma_B = 2$.

$$\begin{aligned}
\overline{x}_A &= \frac{120+130+125+135+128+132+127}{7} = 128.1429 \\
s_A^2 &= 23.8095 \\
\overline{x}_B &= \frac{118+122+125+121+119+124+123}{7} = 121.7142 \\
s_A^2 &= 6.5714 \\
f &= \frac{2^2 \times 23.8095}{5^2 \times 6.5714} = 0.5797
\end{aligned}$$

Confidence Interval

Let $X_1, X_2, ..., X_n$ be a random sample from a population $f(x)$ with mean μ and variance σ^2. Recall that a point estimate is a single value estimate for a population parameter. The most unbiased point estimate of the population mean μ is the sample mean $\overline{x}$. A point estimate for a given sample is not exactly equal to the population parameter. We determine an interval within which we would expect to find the value of the parameter. Such an interval is called interval estimate. The interval estimate for a population parameter is called a confidence interval. Below we construct confidence intervals on

- The mean of a normal distribution, using either the normal distribution or the t-distribution method.
- The variance and standard deviation of a normal distribution.
- A population proportion.

This confidence intervals play vital role in performing hypothesis testing.

Confidence Interval (CI) for the population mean μ

A confidence interval estimate for μ is an interval of the form $l \leq \mu \leq u$, where l and u are called lower and upper confidence limits, respectively. They are computed from the sample data. Because different samples with produce different l and u, these end-points are values of the random variable L and U, respectively. Suppose the probability $P(L \leq \mu \leq U) = 1 - \alpha$ is true, where $0 \leq \alpha \leq 1$. There is a Probability of $1 - \alpha$ of selecting a sample for which the CI will contain the true value of μ. The interval $l \leq \mu \leq u$ is called a $100(1-\alpha)\%$ CI on μ and $1 - \alpha$ is called the confidence coefficient or the degree of confidence.

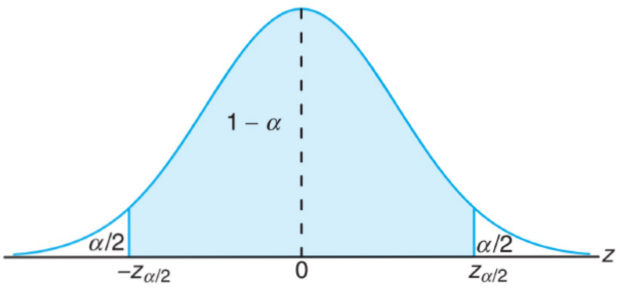
Confidence Interval for μ when variance σ^2 known

If $\overline{x}$ is the mean of a random sample of size n from a population $f(x)$ ($f(x)$ is normal or n is sufficiently large) with known variance σ^2, then a $100(1-\alpha)\%$ confidence interval for μ is given by

$$\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

where $z_{\alpha/2}$ is the z-value leaving an area of $\alpha/2$ to the right.

Let the sample mean $\overline{x}$ be the point estimate of μ. We find e such that $\mu \in (\overline{x} - e, \overline{x} + e)$. By Central Limit theorem, the sampling distribution of $\overline{X}$ is approximately normally distributed with mean $\mu_{\overline{x}} = \mu$ and variance $\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}}$. We may standardize $\overline{X}$ by $Z = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ The random variable Z has a standard normal distribution. We write $z_{\alpha/2}$ for the z-value above which (to the right side) we find an area of $\frac{\alpha}{2}$ under the normal curve.



We see from the figure that

$$P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$$

$$\Rightarrow P(-z_{\alpha/2} < \frac{\overline{x} - \mu}{\sigma/\sqrt{n}} < z_{\alpha/2}) = 1 - \alpha$$

$$\Rightarrow P(\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) = 1 - \alpha$$

$\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ is $100(1-\alpha)\%$ CI on μ.
Here lower confidence limit $l = \overline{x} - z_{\alpha/2}\sigma/\sqrt{n}$ and upper confidence limit $u = \overline{x} + z_{\alpha/2}\sigma/\sqrt{n}$. In particular, if $\alpha = 0.05$, we have 95% confidence interval, and when $\alpha = 0.01$, we have 99% confidence interval. From the cumulative distribution table $z_{0.05/2} = z_{0.025} = 1.96$ and $z_{0.01/2} = z_{0.005} = 2.575 \approx 2.58$

95% confidence interval : $\overline{x} - 1.96\sigma/\sqrt{n} < \mu < \overline{x} + 1.96\sigma/\sqrt{n}$

99% confidence interval : $\overline{x} - 2.58\sigma/\sqrt{n} < \mu < \overline{x} + 2.58\sigma/\sqrt{n}$

Maximum error of estimate e for μ when σ^2 known

Given a $100(1-\alpha)\%$ confidence interval, the error $|\overline{x} - \mu|$ will not exceed $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$. Therefore maximum error e is $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$. That is

$$e = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

Sample size for specified error for μ when σ^2 known

For a point estimate $\overline{x}$ we can be $100(1-\alpha)\%$ confident that the error will not exceed specified amount e when the sample size n is at least $\left(\frac{z_{\alpha/2}\sigma}{e}\right)^2$.

Interpreting a confidence interval

A CI is a random interval because the end points of the interval are random variables. If several samples are collected and a $100(1-\alpha)\%$ CI for μ is computed for each sample, then $100(1-\alpha)\%$ of these intervals contain the true value of μ.

Problem 5. *The average zinc concentration recovered from a sample of measurements taken in 36 different locations in a river is found to be 2.6 grams per milliliter. Find the* 95% *and* 99% *confidence intervals for the mean zinc concentration in the river. Assume that the population standard deviation is 0.3 gram per milliliter. How large a sample is required if we want to be* 95% *confident that our estimate of* μ *is off by less than 0.05?*

Solution: Given: sample size $n = 36$, population standard deviation $\sigma = 0.3$ gm/ml. The point estimate of μ is $\overline{x} = 2.6$ gm/ml. The 95% confidence interval is

$$\begin{aligned} \overline{x} - z_{0.05/2}\frac{\sigma}{\sqrt{n}} &< \mu < \overline{x} + z_{0.05/2}\frac{\sigma}{\sqrt{n}} \\ 2.6 - (1.96)\frac{0.3}{\sqrt{36}} &< \mu < 2.6 + (1.96)\frac{0.3}{\sqrt{36}} \\ 2.5 &< \mu < 2.7 \end{aligned}$$

The 99% confidence interval is

$$\begin{aligned} \overline{x} - z_{0.01/2}\frac{\sigma}{\sqrt{n}} &< \mu < \overline{x} + z_{0.01/2}\frac{\sigma}{\sqrt{n}} \\ 2.6 - (2.575)\frac{0.3}{\sqrt{36}} &< \mu < 2.6 + (2.575)\frac{0.3}{\sqrt{36}} \\ 2.47 &< \mu < 2.73 \end{aligned}$$

Given error $|\overline{x} - \mu| < 0.05$. Sample size, $n = \left(\frac{z_{0.05/2}\sigma}{e}\right)^2 = \left(\frac{(1.96)(0.3)}{0.05}\right)^2 \approx 139$

One-sided confidence bounds for μ when σ^2 known

Above what we discussed is a two-sided CI. It is also possible to obtain one-sided confidence bounds for μ by setting either the upper or lower limit to infinity

1. $100(1-\alpha)\%$ lower confidence bound for μ is $\mu \geq \overline{x} - z_\alpha \frac{\sigma}{\sqrt{n}}$
2. $100(1-\alpha)\%$ upper confidence bound for μ is $\mu \leq \overline{x} + z_\alpha \frac{\sigma}{\sqrt{n}}$

Problem 6. *The average zinc concentration recovered from a sample of measurements taken in 36 different locations in a river is found to be 2.6 grams per milliliter. Find the* 95% *lower confidence interval and* 99% *upper interval for the mean zinc concentration in the river. Assume that the population standard deviation is 0.3 gram per milliliter.*

Solution: Given: sample size $n = 36$, sample mean $\overline{x} = 2.6$, population standard deviation $\sigma = 0.3$ gm/ml. The point estimate of μ is $\overline{x} = 2.6$ gm/ml. The 95% lower confidence interval is

$$\begin{aligned}
\mu &\geq \overline{x} - z_{0.05}\frac{\sigma}{\sqrt{n}} \\
\mu &\geq 2.6 - 1.65\frac{0.3}{\sqrt{36}} \\
\mu &\geq 2.52
\end{aligned}$$

The 99% upper confidence interval is

$$\begin{aligned}
\mu &\leq \overline{x} + z_{0.01}\frac{\sigma}{\sqrt{n}} \\
\mu &\leq 2.6 + 2.33\frac{0.3}{\sqrt{36}} \\
\mu &\leq 2.7165
\end{aligned}$$

Large sample confidence interval for μ when σ^2 Unknown $(n \geq 30)$

When n is large, we can replace σ by the sample standard deviation S. Thus $Z = \frac{\overline{X}-\mu}{S/\sqrt{n}}$ has an approximate standard normal distribution. Consequently

$$\overline{x} - z_{\alpha/2}\frac{s}{\sqrt{n}} \leq \mu \leq \overline{x} + z_{\alpha/2}\frac{s}{\sqrt{n}}$$

is large sample $100(1-\alpha)\%$ confidence interval for μ.

Large sample one-sided Confidence bounds for μ when σ^2 unknown

A $100(1-\alpha)\%$ upper-confidence bound for μ is $\mu \leq \overline{x} + z_\alpha \frac{s}{\sqrt{n}}$.
A $100(1-\alpha)\%$ lower - Confidence bound for μ is $\mu \geq \overline{x} - z_\alpha \frac{s}{\sqrt{n}}$.

Maximum error of estimate e for μ when σ^2 unknown, n large

Given a $100(1-\alpha)\%$ confidence interval, the error of estimate $|\overline{x} - \mu|$ will not exceed $z_{\alpha/2}\frac{s}{\sqrt{n}}$. Therefore maximum error e is $z_{\alpha/2}\frac{s}{\sqrt{n}}$. That is

$$e = z_{\alpha/2}\frac{s}{\sqrt{n}}$$

Confidence interval for μ of a normal distribution when σ^2 unknown (small sample size: $n < 30$)

When n is small, a different distribution must be employed to construct the CI.

Confidence interval for μ when σ^2 unknown (n is small)

If $\overline{x}$ and s are the mean and standard deviation of a random sample from a normal distribution with unknown variance σ^2, then a $100(1-\alpha)\%$ confidence interval on μ is given by

$$\overline{x} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} \leq \mu \leq \overline{x} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}$$

where $t_{\alpha/2,n-1}$ is the upper $100\alpha/2$ percentage point of t distribution with $n-1$ degree of freedom.

One-sided confidence bounds

A $100(1-\alpha)\%$ upper-confidence bound for μ is $\mu \leq \overline{x} + t_{\alpha,n-1}\frac{s}{\sqrt{n}}$.
A $100(1-\alpha)\%$ lower - confidence bound for μ is $\overline{x} - t_{\alpha,n-1}\frac{s}{\sqrt{n}}$.

Problem 7. *The contents of seven similar containers of sulfuric acid are 9.8, 10.2, 10.4, 9.8, 10.0, 10.2, and 9.6 liters. Assuming population is approximately normal distribution, for the mean contents of all such containers find* 95% *(i) two sided (ii)lower and (iii) upper confidence intervals.*

Solution:
Given: Seven samples 9.8, 10.2, 10.4, 9.8, 10, 10.2, 9.6

$$\begin{aligned}
\overline{x} &= \frac{9.8+10.2+10.4+9.8+10+10.2+9.6}{7} = 10 \\
s^2 &= \frac{(-0.2)^2+0.2^2+0.4^2+(-0.2)^2+0+(0.2)^2+(-0.4)^2}{7-1} = 0.08 \\
s &= 0.283
\end{aligned}$$

Hence 95%. two-sided confidence interval of μ is

$$\begin{aligned}
\overline{x} - t_{0.025,6}\frac{s}{\sqrt{n}} &< \mu < \overline{x} + t_{0.025,6}\frac{s}{\sqrt{n}} \\
10 - 2.447\left(\frac{0.283}{\sqrt{7}}\right) &< \mu < 10 + 2.447\left(\frac{0.283}{\sqrt{7}}\right) \\
9.74 &< \mu < 10.26
\end{aligned}$$

The 95% lower confidence interval is

$$\begin{aligned}
\mu &\geq \overline{x} - t_{0.05,6}\frac{\sigma}{\sqrt{n}} \\
\mu &\geq 10 - 1.943\frac{0.283}{\sqrt{7}} \\
\mu &\geq 9.7922
\end{aligned}$$

The 95% upper confidence interval is

$$\begin{aligned}\mu &\leq \overline{x}+t_{0.05,6}\frac{\sigma}{\sqrt{n}}\\ \mu &\leq 10+1.943\frac{0.283}{\sqrt{7}}\\ \mu &\leq 10.2078\end{aligned}$$

Confidence interval for a population proportion p

Recall that the sampling distribution of the sample proportion $\hat{p}$ is approximately normal with mean p and Variance $\frac{p(1-p)}{n}$, if p is not close to 0 or 1 or n is relatively large. Typically, np and $n(1-p)$ are at least 5. Thus, if n is large, the distribution of

$$Z=\frac{\hat{P}-p}{\sqrt{\frac{p(1-p)}{n}}}$$

is approximately standard normal.

Large sample confidence interval on p

If $\hat{p}$ is the proportion of observation in a random sample of size n that belongs to a class of interest, then an approximate $100(1-\alpha)\%$ confidence interval on the proportion p of the population is

$$\hat{p}-z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\leq p\leq \hat{p}+z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Maximum error of estimate e on p

Given a $100(1-\alpha)\%$ confidence interval, the error $|\overline{x}-\mu|$ will not exceed $z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$. Therefore th maximum error e is $z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$. That is

$$e=z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Problem 8. *In a random sample of $n=500$ families owning television sets in the city of Hamilton, Canada, it is found that $x=340$ subscribe to HBO. Find a 95% confidence interval for the actual proportion of families with television sets in this city that subscribe to HBO.*

Solution: Given $n=500$, $\hat{p}=\frac{x}{n}=\frac{340}{500}=0.68$. The 95% confidence interval is given by

$$\begin{aligned}\hat{p}-Z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq \quad & p \quad \leq \hat{p}+Z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\\ 0.68-1.96\sqrt{\frac{0.68(1-0.68)}{500}} \leq \quad & p \quad \leq 0.68+1.96\sqrt{\frac{0.68(1-0.68)}{500}}\\ 0.6391 \leq \quad & p \quad \leq 0.7209.\end{aligned}$$

Confidence interval on the variance of a normal distribution

If s^2 is the variance of a random sample of size n from a normal distribution, then a $100(1-\alpha)\%$ confidence interval on σ^2 is

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}$$

where $\chi^2_{\alpha/2}$ and $\chi^2_{1-\alpha/2}$ are χ^2-values with $n-1$ degrees of freedom, leaving areas of $\frac{\alpha}{2}$ and $1-\frac{\alpha}{2}$ respectively, to the right. An approximate $100(1-\alpha)\%$ confidence interval for standard deviation is

$$\frac{(\sqrt{n-1})s}{\sqrt{\chi^2_{\frac{\alpha}{2},n-1}}} \leq \sigma \leq \frac{(\sqrt{n-1})s}{\sqrt{\chi^2_{1-\frac{\alpha}{2},n-1}}}$$

One-sided confidence bounds for the variance

The $100(1-\alpha)\%$ lower and upper confidence bounds for σ^2 are $\frac{(n-1)s^2}{\chi^2_{\alpha,n-1}} \leq \sigma^2$ and $\sigma^2 \geq \frac{(n-1)s^2}{\chi^2_{1-\alpha,n-1}}$ respectively.

Problem 9. *The following are the weights, in decagrams, of 10 packages of grass seed distributed by a certain company: 46.4, 46.1, 45.8, 47.0, 46.1, 45.9, 45.8, 46.9, 45.2, and 46.0. Find a 95% confidence interval for the variance of the weights of all such packages of grass seed distributed by this company, assuming a normal population.*

Solution: Sample is 46.4, 46.1, 45.8, 47, 46.1, 45.9, 45.8, 46.9, 45.2, 46.0 and $n = 10$.
$\overline{x} = \frac{\sum x_i}{n} = 46.12$ and $s^2 = \frac{\sum(x_i-\overline{x})^2}{n-1} = 0.286$. A 95% Confidence interval, $(\alpha = 0.05)$ is given by

$$\frac{9s^2}{\chi^2_{0.025,9}} \leq \sigma^2 \leq \frac{9s^2}{\chi^2_{1-0.025,9}}$$

Using table $\chi^2_{0.025,9} = 19.023$ and $\chi^2_{0.975,9} = 2.700$

$$0.135 \leq \sigma^2 \leq 0.953$$

Confidence interval for the ratios of two variance

Let σ_1^2 and σ_2^2 be the variances of two normal populations. Let s_1^2 and s_2^2 be the variances of independent samples of size n_1 and n_2, respectively, from normal populations. Then the estimate of the ratio $\frac{\sigma_1^2}{\sigma_2^2}$ is $\frac{s_1^2}{s_2^2}$. Hence the statistic $\frac{s_1^2}{s_2^2}$ is called an estimator of $\frac{\sigma_1^2}{\sigma_2^2}$. The interval estimate of $\frac{\sigma_1^2}{\sigma_2^2}$ is given by the statistic

$$F = \frac{\sigma_2^2 s_1^2}{\sigma_1^2 s_2^2}$$

The random variable F has F-distribution with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom. A $100(1-\alpha)\%$ confidence interval for $\frac{\sigma_1^2}{\sigma_2^2}$ is

$$\frac{s_1^2}{s_2^2} \times \frac{1}{f_{\alpha/2,\nu_1,\nu_2}} \leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{s_1^2}{s_2^2} \times \frac{1}{f_{1-\alpha/2,\nu_1,\nu_2}}$$

where $f_{\alpha/2,\nu_1,\nu_2}$ and $f_{1-\alpha/2,\nu_1,\nu_2}$ are f-values with $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ degrees of freedom, leaving areas $\frac{\alpha}{2}$ and $1 - \frac{\alpha}{2}$, respectively, to the right. Note that $f_{\alpha/2,\nu_2,\nu_1} = \frac{1}{f_{1-\alpha/2,\nu_1,\nu_2}}$.

Problem 10. *A researcher wants to compare the variability in exam scores between two different classes. A random sample of scores is taken from each class:*

Class A: *Sample size* $n_1 = 10$, *sample variance* $s_1^2 = 16$
Class B: *Sample size* $n_2 = 13$, *sample variance* $s_2^2 = 9$

Construct a 90% confidence interval for the ratio of the population variances $\frac{\sigma_1^2}{\sigma_2^2}$, *assuming the populations are normally distributed.*

Solution: Given: $n_1 = 10$, $s_1^2 = 16$, $n_2 = 13$, $s_2^2 = 9$, $\alpha = 0.05$.
The 90% confidence interval for the ratio of the population variance is

$$\begin{aligned}
\frac{s_1^2}{s_2^2} \times \frac{1}{f_{\alpha/2,n_1-1,n_2-1}} &\leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{s_1^2}{s_2^2} \times \frac{1}{f_{1-\alpha/2,n_1-1,n_2-1}} \\
\implies \frac{16}{9} \times \frac{1}{f_{0.05,9,12}} &\leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{16}{9} \times \frac{1}{f_{0.95,9,12}} \\
\implies \frac{16}{9} \times \frac{1}{f_{0.05,9,12}} &\leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{16}{9} \times f_{1-0.95,12,9} \\
\implies \frac{16}{9} \times \frac{1}{2.8} &\leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{16}{9} \times 3.07 \\
\implies 0.6349 &\leq \frac{\sigma_1^2}{\sigma_2^2} \leq 5.4578
\end{aligned}$$

Statistical Hypothesis

A statistical hypothesis is a statement concerning about the parameters of one or more population.

Examples:

1. H: $p = 0.1$ where p is population proportion of defective items.
2. H: $\mu = 1100$ hr, where μ is average life span of electric bulbs manufactured by a company.
3. Let μ_1 and μ_2 denote the true average tensile strengths of two different materials. A hypothesis might be assertion that $\mu_1 - \mu_2 \geq 2$ or another statement is $\mu_1 - \mu_2 = 0$.
4. Let Vaccine A be 25% effective on a certain viral infection after 2 years. To determine new and more expensive vaccine B is superior in providing protection against the same virus after 2 years. A hypothesis might be assertion that $P(B) > 0.25$, or $P(B) = 0.25$

Steps involved in hypothesis testing

1. Formulate the hypothesis to be tested.
2. Determine the appropriate test statistic and calculate it using the sample data
3. Compare test statistic to critical region to draw initial conclusion.
4. Calculate p-value
5. Conclusion, written in terms of the original problem.

Null and Alternate hypothesis

In any hypothesis-testing problem, there are always two competing hypothesis under consideration: The null hypothesis (H_0) and The alternate hypothesis (H_1). The **alternate hypothesis** H_1 usually represents the question to be answered or the theory to be tested. The **null hypothesis** H_0 nullifies or opposes H_1, and is often the logical complement to H_1. Basically, null hypothesis is tested for possible rejection under the assumption that it is true.

Example. If a person encounter a jury trial, then the null and alternate hypothesis are
H_0: defendant is innocent & H_1: defendant is guilty

Example. A factory has a machine that dispenses 100 ml of juice in a bottle. The employ believes that the average amount of fluid is not 100 ml. Then the null and alternate hypothesis are H_0: $\mu = 100$ ml and H_1: $\mu \neq 100$ ml.

Example. A company manufactures car batteries with an average life span of 2 or more years. An engineer believes this value to be less. Then the null and alternate hypothesis are H_0: $\mu \geq 2$ years and H_1: $\mu < 2$ years (we consider $H_0 : \mu = 2$ years).

Note: Null hypothesis H_0 will often be stated with equality sign.

The test statistic

A test statistic assesses how consistent your sample data with the hull hypothesis in a hypothesis test. The test statistic is a function of sample statistic, its value which calculated using the sample data tell us by how much sample diverges from the null hypothesis. As a test statistic value become more extreme, it indicates large differences between your sample data and the null hypothesis. If the difference is significant, the null hypothesis is rejected. If it is not, then the null hypothesis is not rejected.

Example. A factory has a machine that dispenses 80 ml of juice in a bottle. The employ believes that the average amount of fluid is not 80 ml. Here, null and alternate hypothesis are H_0: $\mu = 80\ ml$ and H_1 $\mu \neq 80\ ml$. Using 40 samples, he measures the average amount dispensed by the machine to be 78 ml with SD 2.5ml Given $\mu = 80\ ml$, $\overline{x} = 78\ ml$, $s = 2.5\ ml$ and $n = 40$. Since n is large ($n > 30$), the test statistic is Z-distribution and the value,

$$z = \frac{\overline{x} - \mu}{s/\sqrt{n}} = \frac{78 - 80}{2.5/\sqrt{40}} = -5.06$$

tells us by how much sample mean diverges from the null hypothesis.

Errors in Hypothesis Testing

When we perform hypothesis test, we make one of two decisions: a) reject the null hypothesis or b) fail to reject the null hypothesis. Because our decision is based on the sample, rather than the entire population, there is a possibility we may make one of the following errors.

- Type 1 error: The error occurs if we reject null hypothesis, when it is true.
- Type 2 error: This error occurs if we do not reject null hypothesis, when it is false.

Thus, in testing any statistical hypothesis, four different situations determine whether the final decision is correct or error listed in the following table:

Decision	H_0 **is true**	H_0 **is false**
Fail to reject H_0	Correct decision	type II error
Reject H_0	type I error	Correct decision

Level of Significance or α-error and β-error

The probability of type I error is called the level of significance or α-error.

$$\alpha = P(\text{type I error}) = P(\text{reject } H_0 \text{ when } H_0 \text{ is true})$$

The probability of type II error is called β-error.

$$\beta = P(\text{type II error}) = P(\text{fail to reject } H_0 \text{ when } H_0 \text{ false})$$

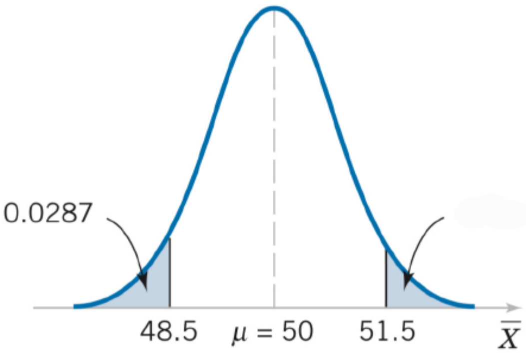
Example 11. Suppose $f(x)$ is a normal population with mean 50 and Standard deviation 2.5. Test the hypothesis that $\mu = 50$ against the alternative that $\mu \neq 50$ if the size of the random sample is 10. Given: Null hypothesis, H_0: $\mu = 50$; Alternate hypothesis $H_1 : \mu \neq 50$; $\sigma = 2.5$ and $n = 10$.
The sample mean can take on many different values. Suppose that if $48.5 \leq \overline{x} \leq 51.5$, we will not reject $H_0 : \mu = 50$, and if $\overline{x} < 48.5$ or $\overline{x} > 51.5$ we will reject H_0 in favor of H_1: $\mu \neq 50$. This is illustrated as below:

Reject H_0	Fail to reject H_0	Reject H_0
$\mu \neq 50$	$\mu = 50$	$\mu \neq 50$
48.5	50	51.5
	$\overline{x}$	

The level of significance or α-level

$$\begin{aligned}
\alpha &= P(\text{reject } H_0 \text{ when } \mu = 50) \\
&= P(\overline{x} < 48.5 \text{ or } \overline{x} > 51.5 \text{ when } \mu = 50) \\
&= P(\overline{x} < 48.5 \text{ when } \mu = 50) + P(\overline{x} > 51.5 \text{ when } \mu = 50
\end{aligned}$$

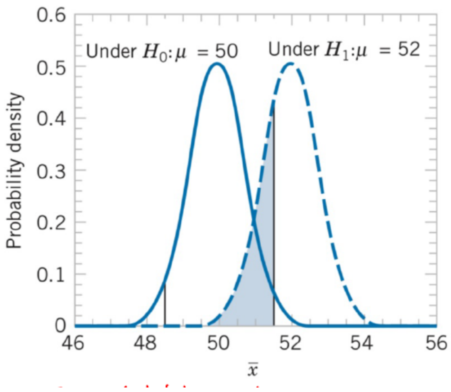
Z-values are $z_1 = \frac{48.5-\mu}{\sigma/\sqrt{n}} = \frac{48.5-50}{2.5/\sqrt{10}} = -1.9$ and $z_2 = \frac{51.5-\mu}{\sigma/\sqrt{n}} = \frac{51.5-50}{2.5/\sqrt{10}} = 1.9$.
$\alpha = P(Z < -1.9) + P(Z > 1.9) = 0.0287 + 0.0287 = 0.0574$



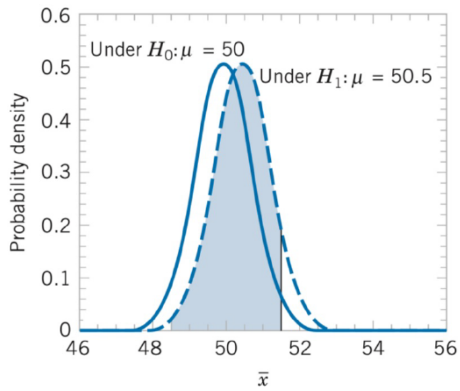
Note that decreasing with increase in n.
We Can't calculate β-level unless we have a specific alternate hypothesis that is, we must have a particular value of μ. Suppose that if $\mu = 52$ and $48.5 \leq \overline{x} \leq 51.5$, then we do not reject H_0. This leads to type II error.

$$\begin{aligned}
\beta &= P(\text{fail to reject } H_0 \text{ when } \ \mu = 52) \\
&= P(48.5 \leq \overline{x} \leq 51.5 \text{ when } \mu = 52)
\end{aligned}$$

The Z-values are $z_1 = \frac{48.5-\mu}{\sigma/\sqrt{n}} = \frac{48.5-52}{2.5/\sqrt{10}} = -4.43$ and $z_2 = \frac{5/.5-52}{2.5/\sqrt{10}} = -0.63$. Therefore $\beta = P(-4.43 \leq z \leq -0.63) = \phi(-0.63) - \phi(-4.43) = 0.2643$ By symmetry, if the true value of the mean is $\mu = 48$, the value of β will also be 0.2643.

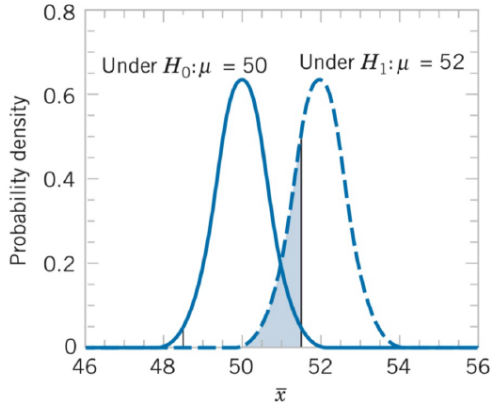


The probability of type-II error when $\mu = 52$ and $n = 10$



The probability of type-II error when $\mu = 50.5$ and $n = 10$

The probability of making type II error increases rapidly as the true value of μ approaches the hypothesized value. For example: $\beta = P(48.5 \leq \overline{x} \leq 51.5 \text{ when } \mu = 50.5) = 0.8923$.



The probability of type-II error when $\mu = 52$ and $n = 16$

Note:

1. Type I and type II errors are related. A decrease in the probability of one generally results in an increase in the probability of other.

2. An increase in sample size reduces β, provided that α is held constant.

Critical region and Critical value.

A **critical region**, also known as the **rejection region**, is a set of values for the test statistic for which the null hypothesis is rejected. That is, if the observed test statistic is in the critical region, then we reject the null hypothesis and accept the alternative hypothesis. **Acceptance region** is a set of values for the test statistic for which the null hypothesis is not reject. The value of test statistic which separates the critical region and the acceptance region is called the *critical value*.
In the Example 11, the acceptance region is $48.5 \leq \overline{x} \leq 51.5$ and the corresponding Z-values are $-1.9 \leq z \leq 1.9$. The critical region is $\overline{x} > 51.5$ or $\overline{x} < 48.5$ and the Z-values are $z > 1.9$ Or $z < -1.9$. The critical values are -1.9 and 1.9.

One and Two-Tailed Tests

A test of any statistical hypothesis where the alternate is one sided, such as H_0: $\theta = \theta_0$ and H_1: $\theta > \theta_0$ or perhaps H_0: $\theta = \theta_0$ H_1: $\theta < \theta_0$ is called a *one-tailed test*. A test of any statistical hypothesis where the alternative is two-sided, such as H_0: $\theta = \theta_0$ H_1: $\theta \neq \theta_0$ is called a *two-tailed test*.

Note:

1. The critical region for the alternative hypothesis $\theta > \theta_0$ lies in the right tail of the distribution of the test statistic.

2. The critical region for the alternative hypothesis $\theta < \theta_0$ lies in the left tail

3. The critical region for the alternative hypothesis $\theta \neq \theta_0$ lies in both tails.

Problem 12. A large manufacturing firm is being charged with discrimination in its hiring practices. (a) What hypothesis is being tested if a jury commits a type I error by finding the firm guilty? (b) What hypothesis is being tested if a jury commits a type II error by finding the firm guilty?

Solution: a) Null hypothesis, H_0: The firm is not guilty. Alternate hypothesis, H_1: The firm is guilty H_0 is tested. b) Null hypothesis, H_0: The firm is guilty, Alternate hypothesis, H_1: The firm is not guilty. H_0 is tested.

Problem 13. *A fabric manufacturer believes that the proportion of orders for raw material arriving late is $p = 0.6$. If a random sample of 10 orders shows that 3 or fewer arrived late, the hypothesis that $p = 0.6$ should be rejected in favor of the alternative $p < 0.6$ Use the binomial distribution. (a) Find the probability of committing a type-I error if the true proportion is $p = 0.6$. (b) Find the probability of committing a type-II error for the alternatives $p = 0.3$.*

Solution: Null hypothesis, $p = 0.6$
Alternate hypothesis, $p < 0.6$
Let X be the number of orders arrived late. Sample size $n = 10$. We reject the null hypothesis when $X \leq 3$.

$$\begin{aligned}
\alpha &= P(\text{Type-I error}) \\
&= P(\text{reject } H_0 \text{ when } p = 0.6) \\
&= P(X \leq 3 \text{ when } p = 0.6) = \sum_{x=0}^{3} b(x; n, p) \\
&= b(0; 10, 0.6) + b(1; 10, 0.6) + b(2; 10, 0.6) + b(3; 10, 0.6) \\
&= (0.4)^{10} + 10(0.6)(0.4)^9 + \binom{10}{2}(0.6)^2(0.4)^8 + \binom{10}{3}(0.6)^3(0.4)^7 \\
&= 0.05476
\end{aligned}$$

$$\begin{aligned}
\beta &= P(\text{Type-II error}) \\
&= P(\text{fail to reject } H_0, \text{ when } p = 0.3) \\
&= P(x > 3, = \text{when } p = 0.3) = \sum_{x=4}^{10} b(x; n, p) \\
&= 1 - \sum_{x=0}^{3} b(x; 10, 0.3) \\
&= 1 - b(0; 10, 0.3) - b(1; 10, 0.3) - b(2; 10, 0.3) - b(3; 10, 0.3) \\
&= 1 - \binom{10}{0}(0.7)^{10} - \binom{10}{1}(0.3)(0.7)^9 - \binom{10}{2}(0.3)^2(0.7)^8 - \binom{10}{3}(0.3)^3(0.7)^7 \\
&= 1 - 0.6496 \\
&= 0.3504
\end{aligned}$$

Problem 14. *The lifetime of a certain type of battery be normally distributed with standard deviation $\sigma = 10$ hours. A battery manufacturer claims that the mean lifetime is $\mu = 100$ hours. To test the null hypothesis:*

$$H_0 : \mu = 100 \quad vs \quad H_1 : \mu < 100,$$

a sample of size $n = 25$ is considered, and reject H_0 if the sample mean $\bar{x} < 96.9$.

(a) *Calculate the probability of a Type-I error.*

(b) *Calculate the probability of a Type-II error if the true mean is $\mu = 95$.*

Solution: Given: $\sigma = 10$, $n = 25$, $\mu_0 = 100$, Rejection region: $\bar{x} < 96.9$
(a) Probability of Type-I error: This is the probability of rejecting H_0 when it is true, i.e.,

$$\begin{aligned}
\alpha &= P(\bar{X} < 96.9 \mid \mu = 100) \\
&= P\left(Z < \frac{96.9 - 100}{10/\sqrt{25}}\right) \\
&= P(Z < -1.55) = \phi(-1.55) \\
&= 0.0606
\end{aligned}$$

(b) Probability Type-II error: This is the probability of not rejecting H_0 when $\mu = 95$, i.e.,

$$\begin{aligned}
\beta &= P(\bar{X} \geq 96.9 \mid \mu = 95) \\
&= P\left(Z \geq \frac{96.9 - 95}{10/\sqrt{25}}\right) \\
&= P(Z \geq 0.95) = 1 - \phi(0.95) \\
&= 1 - 0.8289 = 0.1711
\end{aligned}$$

Problem 15. *A machine is designed to fill bottles with 500 ml of a drink. The amount filled follows a normal distribution with known standard deviation $\sigma = 4$ ml. To test:*

$$H_0 : \mu = 500 \quad vs \quad H_1 : \mu \neq 500,$$

a sample of size $n = 36$ is taken, and the rejection region is

$$|\bar{x} - 500| > 1.96 \cdot \frac{4}{\sqrt{36}}.$$

(a) *What is the probability of a Type-I error?*

(b) *What is the probability of a Type-II error if the true mean is $\mu = 498.5$?*

Solution: Given: $\sigma = 4$, $n = 36$, $\mu_0 = 500$ Significance level: $\alpha = 0.05$, Two-tailed test. Rejection region: $\bar{x} < 498.6933$ or $\bar{x} > 501.3067$

(a) Probability of Type-I error: This is the probability of rejecting H_0 when it is true, i.e.,

$$\begin{aligned}
\alpha &= P(\bar{X} < 498.6933 \mid \mu = 500) + P(\bar{X} > 501.3067 \mid \mu = 500) \\
&= P\left(Z < \frac{498.6933 - 500}{4/\sqrt{36}}\right) + P\left(Z > \frac{501.3067 - 500}{4/\sqrt{36}}\right) \\
&= P(Z < -1.96) + P(Z > 1.96) \\
&= 2 \times P(Z < -1.96) \\
&= 2 \times 0.025 = 0.05
\end{aligned}$$

(b) Probability of Type-II error: This is the probability of not rejecting H_0 when $\mu = 498.5$, i.e.,

$$\begin{aligned}
\beta &= P(498.6933 \leq \bar{X} \leq 501.3067 \mid \mu = 498.5) \\
&= P\left(\frac{498.6933 - 498.5}{4/\sqrt{36}} \leq Z \leq \frac{501.3067 - 498.5}{4/\sqrt{36}}\right) \\
&= P(0.29 \leq Z \leq 2.10) \\
&= \phi(2.10) - \phi(0.29) \\
&= 0.9821 - 0.6141 = 0.3680
\end{aligned}$$

Problem 16. *A quality control department is testing whether the proportion of defective items in a production process is equal to 0.6. A sample of 100 items is taken, and the sample proportion of defectives $\hat{p}$ is used for the hypothesis test. The null and alternative hypotheses are:*

$$H_0 : p = 0.6 \quad vs. \quad H_1 : p < 0.6$$

The test uses a rejection region of $\hat{p} < 0.54$.

(a) Compute the probability of a Type-I error (i.e., the probability of rejecting H_0 when it is true).

(b) Compute the probability of a Type-II error if the true proportion is actually $p = 0.5$.

Use the normal approximation for the sampling distribution of $\hat{p}$, and assume the sample size is sufficiently large for the approximation to be valid.

Solution: **Given:** $p_0 = 0.6$, $n = 100$, Rejection region: $\hat{p} < 0.54$
(a) Probability of Type-I error: This is the probability of rejecting H_0 when it is true, i.e.,

$$\begin{aligned}
\alpha &= P(\hat{P} < 0.54 \mid p = 0.6) \\
&= P\left(Z < \frac{0.54 - 0.6}{\sqrt{\frac{0.6(1-0.6)}{100}}}\right) \\
&= P(Z < -1.54) = \phi(-1.54) \\
&= 0.0618
\end{aligned}$$

(b) Probability of Type-II error: This is the probability of not rejecting H_0 when $p = 0.5$, i.e.,

$$\begin{aligned}\beta &= P(\hat{P} \geq 0.54 \mid p = 0.5) \\ &= P\left(Z \geq \frac{0.54 - 0.5}{\sqrt{\frac{0.5(1-0.5)}{100}}}\right) \\ &= P(Z \geq 0.8) = 1 - \phi(0.8) \\ &= 1 - 0.7881 = 0.2119\end{aligned}$$

Preselection of a significance level and Critical region.

We can control the maximum risk of making type I error by preselecting a significance level. Let us fix the significance level as α, we say α level of significance. Then $100(1-\alpha)\%$ confidence interval for the parameter θ. Constitute the acceptance region for the null hypothesis $\theta = \theta_0$.That is, let $\hat{\theta}$ be the value of a test statistic in the hypothesis testing. Then:

1. The value $\hat{\theta}$ is said to be in the acceptance region for the α level of significance if it falls within the $100(1-\alpha)\%$ confidence interval.

2. The value $\hat{\theta}$ is said to be in the critical region for the α level of significance if it falls outside the 100 (1-α)% the confidence interval.

Note: The widely used procedure in hypothesis testing is to use Significance level $\alpha = 0.05$. This value has evolved through experience, and may not be appropriate for all situations.

P-values in hypotheses tests

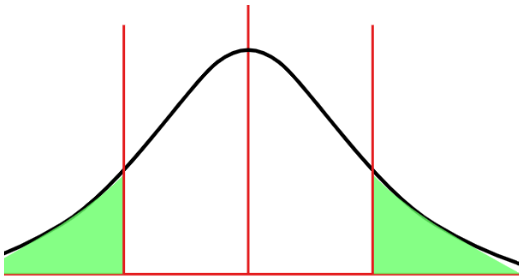
The fixed significance level is nice, but it is inadequate to conclude whether the computed value of the test statistic is just barely in the rejection region or whether is very far into this region. This approach may be unsatisfactory because some decision makers might be uncomfortable with the risk implied by $\alpha = 0.05$. To avoid this p-value approach is adopted.
The *p-value* is the smallest level of significance that would lead to rejection of the null hypothesis H_0 with the given data. The p-value is the probability that the test statistic take on a value that is at least as extreme as the observed value of the statistic when H_0 is true.

Example. Suppose H_0: $\mu = 50$, $H_1 : \mu \neq 50$, $\overline{x} = 48.5$, $n = 16$, $\sigma = 2.5$. Test statistic $z = \frac{48.5-50}{2.5/\sqrt{16}} = -2.40$. Then P-value $= 2\ P(Z \leq -2.40) = 0.0164$

Testing hypotheses on the mean when σ known (z-test)

Let us consider that the population is normal or sample size n is sufficiently large ($n \geq 30$).

$$\text{Null hypothesis: } H_0 : \mu = \mu_0$$
$$\text{Test Statistic: } Z_0 = \frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}$$



p-value (area shaded)

Let $\overline{x}$ be the value of the sample mean. The corresponding z-value is $z_0 = \frac{\overline{x}-\mu_0}{\sigma/\sqrt{n}}$. For an α level of significance, the Table 1 gives the details of z-test.

Alternate hypothesis	$H_1 : \mu \neq \mu_0$	$H_1 : \mu < \mu_0$	$H_1 : \mu > \mu_0$
Tests	Two-tailed test	lower-tailed test (left tailed test)	upper-tailed test (right tailed test)
Critical region	$z_0 > z_{\alpha/2}$, $z_0 < -z_{\alpha/2}$	$z_0 < -z_\alpha$	$z_0 > z_\alpha$
Acceptance region	$-z_{\alpha/2} \leq z_0 \leq z_{\alpha/2}$	$z_0 \geq -z_\alpha$	$z_0 \leq z_\alpha$
P**-value**	Probability above $\lvert z_0\rvert$ and below $-\lvert z_0\rvert$, $P = 2(1-\phi(\lvert z_0\rvert))$	Probability below z_0, $P = \phi(z_0)$	Probability above z_0, $P = 1-\phi(z_0)$

Table 1: z-test

Testing hypotheses on the mean when σ unknown (z-test)

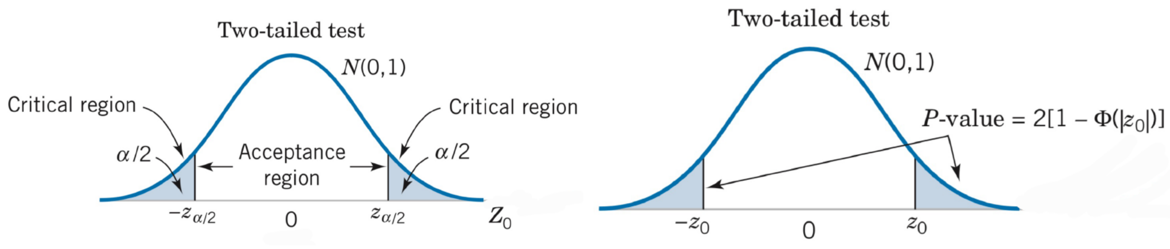
Let us consider that the sample size n is large ($n \geq 30$)

$$\text{Null hypothesis : } H_0\text{: } \mu = \mu_0$$
$$\text{Test Statistic: } Z_0 = \frac{\overline{X}-\mu_0}{S/\sqrt{n}}$$

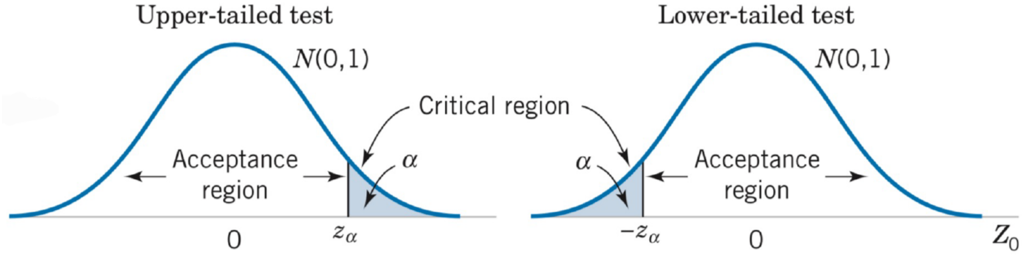
where S is sample standard deviation. Rest of the steps are same as given in the case of σ known (Table 1).

Level of significance	**Critical value**	**Acceptance region**
0.05	$z_c = 1.96$	$(-1.96, 1.96)$
0.01	$z_c = 2.58$	$(-2.58, 2.58)$

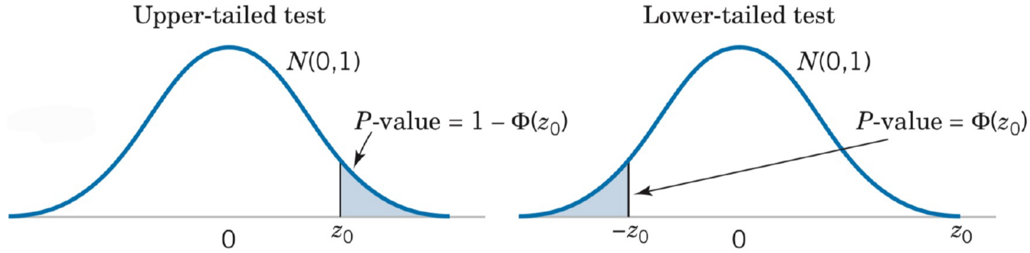
Critical value and region for given level of significance in two-tailed test



Critical region and p-value for two-tailed z-test.



Critical region for one-tailed z-test.



p-value for one-tailed z-test.

Level of significance	**Critical value**		**Critical region**	
	Right-tailed test	Left-tailed test	Right-tailed test	Left-tailed test
0.05	$z_c = 1.645$	$-z_c = -1.645$	$(1.645, \infty)$	$(-\infty, -1.645)$
0.01	$z_c = 2.33$	$-z_c = -2.33$	$(2.33, \infty)$	$(-\infty, -2.33)$

Critical value and critical region for given level of significance in one-tailed test

Testing hypotheses on the mean, σ unknown, $n < 30$ (t-test/Student's t-test)

Let us consider that the population is at least approximately normal.

$$\text{Null hypothesis: } H_0 : \mu = \mu_0$$
$$\text{Test Statistic: } T_0 = \frac{\overline{X}-\mu_0}{S/\sqrt{n}}$$

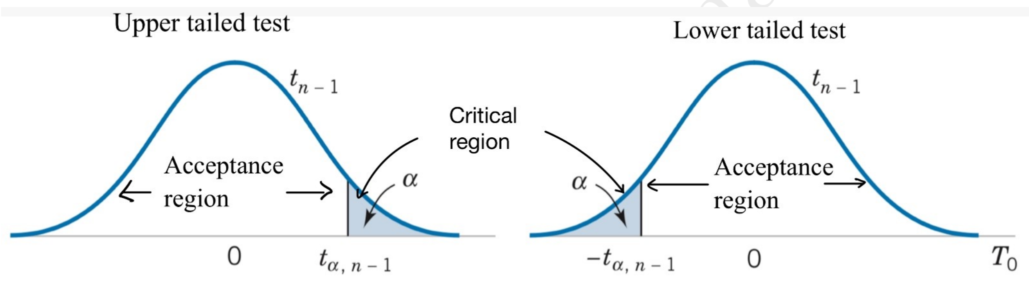
where S is sample standard deviation. Let $\overline{x}$ be the value of the sample mean, let s be its variance. The corresponding t-value is

$$t_0 = \frac{\overline{x} - \mu_0}{s/\sqrt{n}}$$

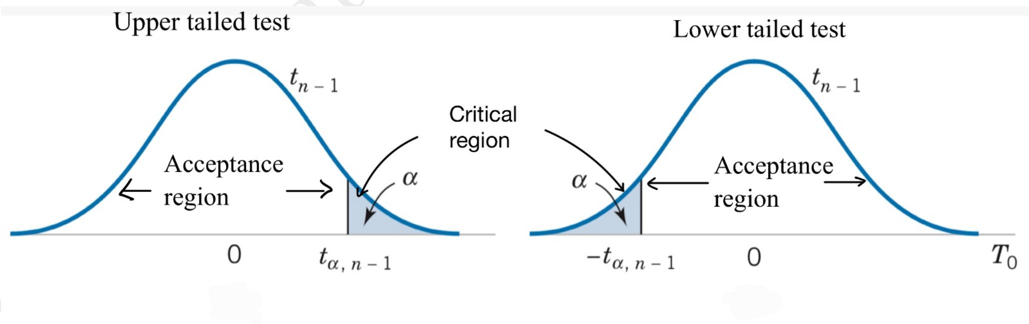
For a α level of significance. The Table 2 gives the details for t-test.

Alternate hypothesis	$H_1 : \mu \neq \mu_0$	$H_1 : \mu < \mu_0$	$H_1 : \mu > \mu_0$
Tests	Two-tailed test	Lower-tailed test	Upper-tailed test
Critical region	$t_0 > t_{\alpha/2,n-1}$, $t_0 < -t_{\alpha/2,n-1}$	$t_0 < -t_{\alpha,n-1}$	$t_0 > t_{\alpha,n-1}$
Acceptance region	$-t_{\alpha/2,n-1} \leq t_0 \leq t_{\alpha/2,n-1}$	$t_0 \geq -t_{\alpha,n-1}$	$t_0 \leq t_{\alpha,n-1}$
P**-value**	Probability above $\lvert t_0 \rvert$ and below $-\lvert t_0 \rvert$	Probability below t_0	Probability above t_0

Table 2: t-test



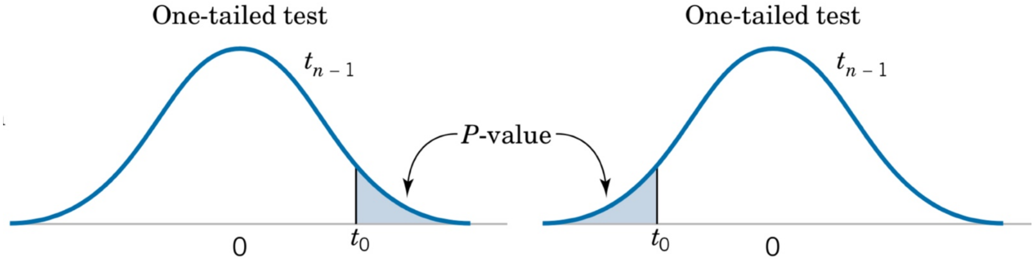
Critical region for two-tailed t-test.



Critical region for one-tailed t-test.

Testing hypotheses on a proportion, large sample size

Null hypothesis: $H_0 : p = p_0$



P-value for one-tailed t-test.

Let X be number of observations in a random sample of size n that belongs to a class of interest. Then sample proportion $\hat{P} = \frac{X}{n}$, its distribution is approximately normal as n is large. Therefore

$$\text{Test statistic: } z_0 = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

For α level of significance, the Table 3 gives the details of z-test for proportion.

Alternate Hypothesis	$H_1 : p \neq p_0$	$H_1 : p < p_0$	$H_1 : p > p_0$
Tests	Two-tailed test	Lower-tailed test	Upper-tailed test
Critical Region	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$	$z_0 \leq -z_\alpha$	$z_0 > z_\alpha$
Acceptance Region	$-z_{\alpha/2} \leq z_0 \leq z_{\alpha/2}$	$z_0 > -z_\alpha$	$z_0 \leq z_\alpha$
***P*-value**	$P = 2(1 - \Phi(\lvert z_0 \rvert))$	$P = \Phi(z_0)$	$P = 1 - \Phi(z_0)$

Table 3: z-test for population proportion

Note: Tests on a proportion when the sample size n is small ($n < 30$) are based on the binomial distribution, not the normal approximation to the binomial.

Problem 17. *A manufacturer of sports equipment has developed a new synthetic fishing line that the company claims has a mean breaking strength of 8 kilograms with a standard deviation of 0.5 kilogram. Test the hypothesis that $\mu = 8$ kilograms against the alternative that $\mu \neq 8$ kilograms if a random sample of 50 lines is tested and found to have a mean breaking strength of 7.8 kilograms. Use a 0.01 level of significance.*

Solution: Given: $\sigma = 0.5$ kg, $n = 50$, $\bar{x} = 7.8$ kg.

$$\begin{aligned} \text{Null hypothesis:} \quad & H_0 : \mu = 8 \\ \text{Alternate hypothesis:} \quad & H_1 : \mu \neq 8 \end{aligned}$$

Level of significance: $\alpha = 0.01$.
Test: Two-tailed z-test (since n is large)
Test statistic: $z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} = \frac{7.8-8}{0.5/\sqrt{50}} = -2.83$

Critical region: $z < -z_{\alpha/2}$ and $z > z_{\alpha/2}$, i.e. $z < -2.575$ and $z > 2.575$
Decision: Since $z = -2.83 < -2.575$, the value is in the critical region. We reject H_0 and conclude that the average breaking strength is not equal to 8 but in fact less than 8.
P**-value:** $= P(|z| > 2.83) = 2 \cdot P(z < -2.83)$
$= 2 \cdot (1 - \Phi(|-2.83|)) = 0.0046$

Which allows us to reject $H_0 : \mu = 8$ kg at a level of significance smaller than 0.01.

Critical region and P-value for the data in Problem 17

Problem 18. *A random sample of 100 recorded deaths in the United States during the past year showed an average life span of 71.8 years. Assuming a population standard deviation of 8.9 years, does this seem to indicate that the mean life span today is greater than 70 years? Use a 0.05 level of significance.*

Solution: Given: $\sigma = 8.9$ kg, $n = 100$, $\bar{x} = 71.8$.

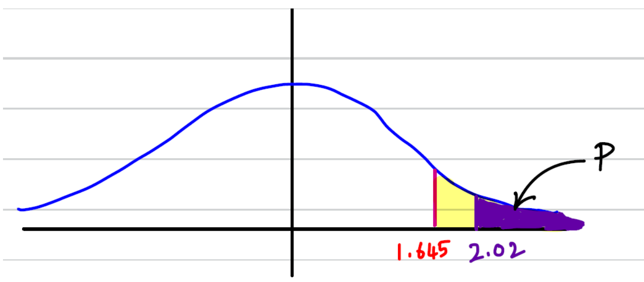
$$\text{Null hypothesis:} \quad H_0 : \mu = 70$$
$$\text{Alternate hypothesis:} \quad H_1 : \mu > 70$$

Level of significance: $\alpha = 0.05$
Test: Upper-tailed z-test (Since n is large)
Test statistic: $z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{71.8 - 70}{8.9/\sqrt{100}} = 2.02$
Critical region: $z > z_{0.05}$, i.e. $z > 1.645$

Decision: Since the test value $z = 2.02 > 1.645$, we reject null hypothesis H_0 and conclude that the mean life span today is greater than 70 years.

P**-value:** $= P(z > 2.02) = 1 - \Phi(2.02) = 0.0217$

This implies that the evidence is in favor of H_1 is even stronger than suggested by a 0.05 level of significance.



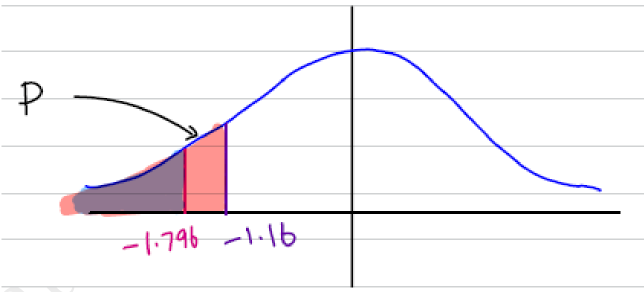
Critical region and P-value for the data in Problem 18

Problem 19. *The Edison Electric Institute has published figures on the number of kilowatt hours used annually by various home appliances. It is claimed that a vacuum cleaner uses an average of 46 kilowatt hours per year. If a random sample of 12 homes included in a planned study indicates that vacuum cleaners use an average of 42 kilowatt hours per year with a standard deviation of 11.9 kilowatt hours, does this suggest at the 0.05 level of significance that vacuum cleaners use, on average, less than 46 kilowatt hours annually? Assume the population of kilowatt hours to be normal.*

Solution: Given: $n = 12$, $\bar{x} = 42$, $s = 11.9$

$$\begin{aligned} \text{Null hypothesis:} \quad & H_0 : \mu = 46 \\ \text{Alternate hypothesis:} \quad & H_1 : \mu < 46 \end{aligned}$$

Level of significance: $\alpha = 0.05$
Test: Lower-tailed t-test (since σ is unknown, population is normal with $n < 30$)
Test statistic: $t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$
Computation: $t = \frac{42-46}{11.9/\sqrt{12}} = \frac{-4}{11.9/3.464} = \frac{-4}{3.436} \approx -1.16$
Critical region: $t < -t_{\alpha,n-1}$ i.e., $t < -t_{0.05,11} \iff t < -1.796$
Decision: Since the value $t = -1.16 > -1.796$, we do not reject null hypothesis H_0 and conclude that the average no. of kilowatt hours used annually by home vacuum cleaner is not significantly less than 46.



Critical region and P-value for the data in Problem 19

Problem 20. *A commonly prescribed drug for relieving nervous tension is believed to be only 60% effective. Experimental results with a new drug administered to a random sample of 100 adults who were suffering from nervous tension show that 70 received relief. Is this sufficient evidence to conclude that the new drug is superior to the one commonly prescribed? Use a 0.05 level of significance.*

Solution: Given: $n = 100$, $x = 70$.Sample proportion, $\hat{p} = \frac{x}{n} = \frac{70}{100} = 0.7$

$$\begin{aligned} \text{Null hypothesis:} \quad & H_0 : p = 0.6 \\ \text{Alternate hypothesis:} \quad & H_1 : p > 0.6 \end{aligned}$$

Level of significance: $\alpha = 0.05$
Test: Upper-tailed z-test (since n is large)

Test statistic: $z = \frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}}$
Computation: $z = \frac{0.7-0.6}{\sqrt{\frac{0.6(1-0.6)}{100}}} = \frac{0.1}{\sqrt{\frac{0.6\times0.4}{100}}} = \frac{0.1}{\sqrt{\frac{0.24}{100}}} = \frac{0.1}{\sqrt{0.0024}} = \frac{0.1}{0.049} \approx 2.04$
Critical region: $z > z_{0.05}$ i.e., $z > 1.645$
P-value: $= P(Z > 2.04) \approx 0.0207$
Decision: As the value $z = 2.04 > 1.645$, we reject H_0 and conclude that the new drug is superior. P-value gives stronger evidence in favor of H_1.

Testing hypotheses on the variance (χ^2-test)

We wish to test the hypothesis that the variance of a normal population σ^2 equals a specified value, say σ_0^2 or equivalently $\sigma = \sigma_0$.

$$\text{Null hypothesis: } H_0 : \sigma^2 = \sigma_0^2$$
$$\text{Test statistic: } \chi_0^2 = \frac{(n-1)S^2}{\sigma_0^2}$$

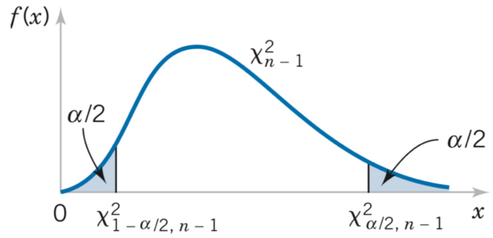
Let s be the value of the sample variance. Then the χ^2-value is

$$\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

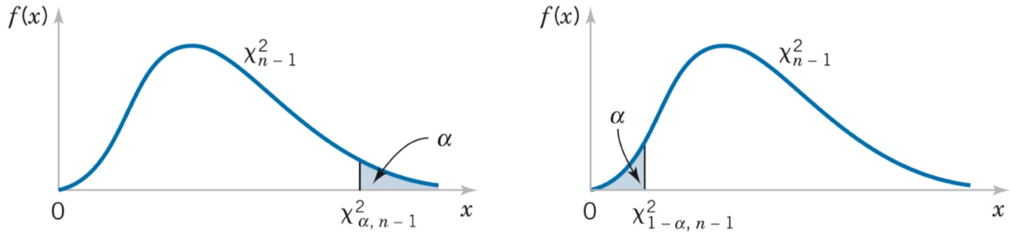
For α level of significance, the Table 4 gives the details of χ^2-test

Alternate hypothesis	$H_1 : \sigma^2 \neq \sigma_0^2$	$H_1 : \sigma^2 < \sigma_0^2$	$H_1 : \sigma^2 > \sigma_0^2$
Tests	Two-tailed test	Lower-tailed test	Upper-tailed test
Critical region	$\chi_0^2 > \chi^2_{\alpha/2,n-1}$, $\chi_0^2 < \chi^2_{1-\alpha/2,n-1}$	$\chi_0^2 < \chi^2_{1-\alpha,n-1}$	$\chi_0^2 > \chi^2_{\alpha,n-1}$
Acceptance region	$\chi^2_{1-\alpha/2,n-1} \leq \chi_0^2 \leq \chi^2_{\alpha/2,n-1}$	$\chi_0^2 > \chi^2_{1-\alpha,n-1}$	$\chi_0^2 < \chi^2_{\alpha,n-1}$

Table 4: χ^2-test



Critical region for two-tailed χ^2-test.



Critical region for one-tailed (upper and lower tailed) χ^2-test.

Problem 21. *An automated filling machine is used to fill bottles with liquid detergent. A random sample of 20 bottles results in a sample variance of fill volume of $s^2 = 0.0153$ (fluid ounces)^2. If the variance of fill volume exceeds 0.01 (fluid ounces)^2, an unacceptable proportion of bottles will be underfilled or overfilled. Is there evidence in the sample data to suggest that the manufacturer has a problem with underfilled or overfilled bottles? Use $\alpha = 0.05$, and assume that fill volume has a normal distribution.*

Solution: Given: $n = 20$, $s^2 = 0.0153$

$$\begin{aligned} \text{Null hypothesis: } & H_0 : \sigma^2 = 0.01 \\ \text{Alternate hypothesis: } & H_1 : \sigma^2 > 0.01 \end{aligned}$$

Test: Upper-tailed χ^2-test

Test statistic: $\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$

Computation: $\chi^2 = \frac{(20-1)(0.0153)}{(0.01)^2} = \frac{19 \times 0.0153}{0.0001} = \frac{0.2907}{0.0001} = 29.07$

Level of significance: $\alpha = 0.05$

Critical region: $\chi^2 > \chi^2_{\alpha,n-1} \implies \chi^2 > \chi^2_{0.05,19} \implies \chi^2 > 30.14$ (from table)

Decision: Since $\chi^2 = 29.07 < \chi^2_{0.05,19} = 30.14$, we conclude that there is no strong evidence to reject $H_0 : \sigma^2 = 0.01$. So there is no strong evidence of a problem with incorrectly filled bottles.
A manufacturer of car batteries claims that the life of the company's batteries is approximately normally distributed with a standard deviation equal to 0.9 year. If a random sample of 10 of these batteries has a standard deviation of 1.2 years, do you think that $\sigma > 0.9$ year? Use a 0.05 level of significance.
Solution: Given: $n = 10$, $s = 1.2 \implies s^2 = (1.2)^2 = 1.44$, $\alpha = 0.05$

$$\begin{aligned} \text{Null hypothesis: } & H_0 : \sigma^2 = 0.9^2 = 0.81 \\ \text{Alternate hypothesis: } & H_1 : \sigma^2 > 0.9^2 = 0.81 \end{aligned}$$

Test: Upper-tailed test (χ^2-test)
Test statistic: $\chi^2 = \frac{(n-1)s^2}{\sigma_0^2}$
Computation: $\chi^2 = \frac{(10-1)(1.44)}{(0.9)^2} = \frac{9 \times 1.44}{0.81} = \frac{12.96}{0.81} = 16$
Critical region: $\chi^2 > \chi^2_{0.05,n-1} \implies \chi^2 > \chi^2_{0.05,9} \implies \chi^2 > 16.919$
Decision: Since $\chi^2 = 16 < 16.919$, we have no sufficient evidence that $\sigma > 0.9$.

Testing hypotheses on the equality of variances of two populations (F-test)

We Now let us consider the problem of testing the equality of the variances σ_1^2 and σ_2^2 of two populations. That is, we shall test the null hypothesis H_0 that $\sigma_1^2 = \sigma_2^2$ against one of the usual alternatives.
Let S_1^2 and S_2^2 be sample variances of the two populations with corresponding sample size n_1 and n_2

$$\text{Null hypothesis:} \quad H_0 : \sigma_1^2 = \sigma_2^2$$

$$\text{Test statistic:} \quad F_0 = \frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2}$$

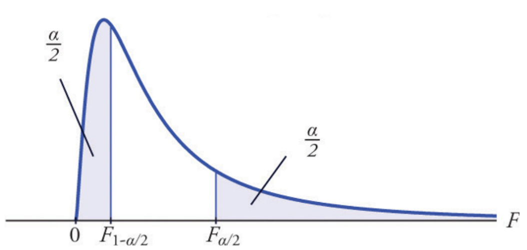
Since we are testing the hypothesis $\sigma_1^2 = \sigma_2^2$, the test statistic will reduces to $F_0 = \frac{S_1^2}{S_2^2}$. Let s_1^2 and s_2^2 be the value of the sample variance. Then the F-value is

$$f_0 = \frac{s_1^2}{s_2^2}$$

Here the degree of freedom $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$. For α level of significance, the Table 5 gives the details of F-test.

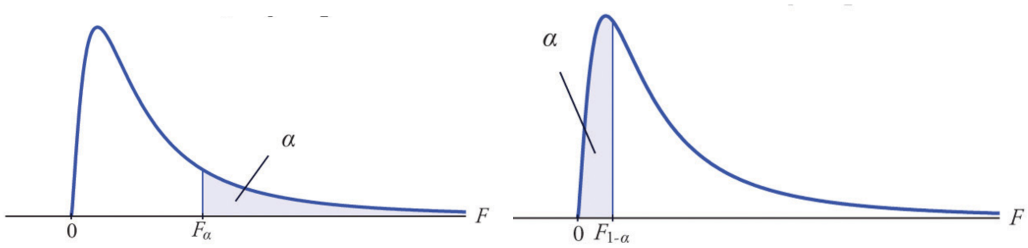
Alternate Hypothesis	$H_1 : \sigma_1^2 \neq \sigma_2^2$	$H_1 : \sigma_1^2 < \sigma_2^2$	$H_1 : \sigma_1^2 > \sigma_2^2$
Tests	Two-tailed test	Lower-tailed test (Left)	Upper-tailed test (Right)
Critical Region	$f_0 > f_{\alpha/2,\nu_1,\nu_2}$, $f_0 < f_{1-\alpha/2,\nu_1,\nu_2}$	$f_0 < f_{1-\alpha,\nu_1,\nu_2}$	$f_0 > f_{\alpha,\nu_1,\nu_2}$
Acceptance Region	$f_{1-\alpha/2,\nu_1,\nu_2} \leq f_0 \leq f_{\alpha/2,\nu_1,\nu_2}$	$f_0 \geq f_{1-\alpha,\nu_1,\nu_2}$	$f_0 \leq f_{\alpha,\nu_1,\nu_2}$

Table 5: F-test for comparing two variances



Critical region for two-tailed F-test.

Problem 22. *The household net expenditure on health care in south and north India, in two sample of household, expressed as percentage of total income in the following table.*



Critical region for one-tailed (upper and lower tailed) F-test.

South:	*15*	*8*	*3.8*	*6.4*	*27.4*	*19*	*35.3*	*13.6*
North:	*18.8*	*23.1*	*10.3*	*8*	*18*	*10.2*	*15.2*	*19*

Test the equality of variances of household's net expenditure in south and north India at 10% level of significance.

Solution:

Null hypothesis: $H_0 : \sigma_1^2 = \sigma_2^2$ (The population variances of net expenditure in South and North India are equal.)

Alternate hypothesis: $H_1 : \sigma_1^2 \neq \sigma_2^2$ (The population variances of net expenditure in South and North India are not equal.)

Test: Two-tailed F-test

Test statistic: $F = \frac{S_1^2}{S_2^2}$

Computation:South (Sample 1):

- $\bar{x}_1 = \frac{15+8+3.8+6.4+27.4+19+35.3+13.6}{8} = \frac{128.5}{8} = 16.0625$
- $s_1^2 = \frac{\sum(x_{1i}-\bar{x}_1)^2}{n_1-1}$

$$\begin{aligned}\sum(x_{1i}-\bar{x}_1)^2 &= (15-16.0625)^2 + (8-16.0625)^2 + (3.8-16.0625)^2 + (6.4-16.0625)^2 \\ &\quad + (27.4-16.0625)^2 + (19-16.0625)^2 + (35.3-16.0625)^2 + (13.6-16.0625)^2 \\ &= (-1.0625)^2 + (-8.0625)^2 + (-12.2625)^2 + (-9.6625)^2 \\ &\quad + (11.3375)^2 + (2.9375)^2 + (19.2375)^2 + (-2.4625)^2 \\ &= 1.1289 + 65.0039 + 150.3689 + 93.3639 + 128.5391 + 8.6291 + 360.0889 + 6.06 \\ &= 813.1866\end{aligned}$$

$s_1^2 = \frac{813.1866}{8-1} = \frac{813.1866}{7} \approx 116.1695$

North (Sample 2):

- $\bar{x}_2 = \frac{18.8+23.1+10.3+8+18+10.2+15.2+19}{8} = \frac{122.6}{8} = 15.325$

- $s_2^2 = \frac{\sum(x_{2i}-\bar{x}_2)^2}{n_2-1}$

$$\begin{aligned}\sum(x_{2i}-\bar{x}_2)^2 &= (18.8-15.325)^2+(23.1-15.325)^2+(10.3-15.325)^2+(8-15.325)^2\\ &\quad+(18-15.325)^2+(10.2-15.325)^2+(15.2-15.325)^2+(19-15.325)^2\\ &=(3.475)^2+(7.775)^2+(-5.025)^2+(-7.325)^2\\ &\quad+(2.675)^2+(-5.125)^2+(-0.125)^2+(3.675)^2\\ &=12.0756+60.4506+25.2506+53.6556+7.1556+26.2656+0.0156+13.5056\\ &=198.375\end{aligned}$$

$s_2^2 = \frac{198.375}{8-1} = \frac{198.375}{7} \approx 28.3393$

Test statistic: $f_0 = \frac{s_1^2}{s_2^2} = \frac{116.1695}{28.3393} \approx 4.099$

Level of significance: $\alpha = 0.1$

Critical region: $f_0 > f_{\alpha/2,\nu_1,\nu_2}$, $f_0 < f_{1-\alpha/2,\nu_1,\nu_2}$

- Numerator degrees of freedom, $\nu_1 = n_1 - 1 = 8 - 1 = 7$
- Denominator degrees of freedom, $\nu_2 = n_2 - 1 = 8 - 1 = 7$

From table, $f_{\alpha/2,\nu_1,\nu_2} = f_{0.05,7,7} = 3.79$ and $f_{1-\alpha/2,\nu_1,\nu_2} = f_{0.95,7,7} = \frac{1}{f_{0.05,7,7}} = 1/3.79 = 0.2639$. Therefore critical region is $f_0 > 3.79$, $f_0 < 0.2639$

Decision: Since the calculated F-statistic (4.099) is in the critical region, we reject the null hypothesis (H_0). That is, the population variances of net expenditure in South and North India are not equal.

Problem 23. *A company manufactures two types of light bulbs, Type A and Type B. The consistency of the manufacturing process is crucial, and the company wants to compare the variability in the lifespan of these two bulb types. A random sample of 12 Type A bulbs and 10 Type B bulbs were tested, and their lifespans (in hours) were recorded as follows:*

Type A	1120	1080	1150	1090	1110	1130	1070	1100	1140	1060	1125
Type B	1010	980	1030	1000	1020	990	1040	970	1050	1005	

Test whether the variance in the lifespan of Type A bulbs is significantly less than that of Type B bulbs at a 5% level of significance. Assume that the lifespans are normally distributed.

Solution:

Null hypothesis:	$H_0 : \sigma_A^2 = \sigma_B^2$	(The population variances of lifespans for Type A and Type B bulbs are equal.)
Alternate hypothesis:	$H_0 : \sigma_A^2 < \sigma_B^2$	(The population variance of lifespans for Type A bulbs is less than that for Type B bulbs.)

Test: One-tailed (Left-tailed) F-test

Test statistic: $F = \frac{S_A^2}{S_B^2}$

Computation: Type A (Sample 1):

- $\bar{x}_A = \frac{1120+1080+1150+1090+1110+1130+1070+1100+1140+1060+1125}{11} = \frac{12075}{11} \approx 1097.7273$
- $\sum(x_{Ai} - \bar{x}_A)^2 \approx 9622.72$
- $s_A^2 = \frac{\sum(x_{Ai}-\bar{x}_A)^2}{n_A-1} = \frac{9622.72}{11-1} = \frac{9622.72}{10} = 962.272$

Type B (Sample 2):

- $\bar{x}_B = \frac{1010+980+1030+1000+1020+990+1040+970+1050+1005}{10} = \frac{10105}{10} = 1010.5$
- $\sum(x_{Bi} - \bar{x}_B)^2 = 532.5$
- $s_B^2 = \frac{\sum(x_{Bi}-\bar{x}_B)^2}{n_B-1} = \frac{532.5}{10-1} = \frac{532.5}{9} \approx 59.1667$

Test statistic: $f_0 = \frac{s_A^2}{s_B^2} == \frac{962.272}{59.1667} \approx 16.264$

Level of significance: $\alpha = 0.05$

Critical region: $f_0 < f_{1-\alpha,\nu_1,\nu_2} = \frac{1}{f_{\alpha,\nu_2,\nu_1}} = \frac{1}{f_{0.05,9,10}} = \frac{1}{3.02} \approx 0.331$

Decision: Since the calculated F-statistic (16.264) is not less than the critical F-value (0.345), it is not in the critical region. Therefore, we do not reject the null hypothesis (H_0). At the 5% level of significance, there is no sufficient evidence to conclude that the variance in the lifespan of Type A bulbs is significantly less than that of Type B bulbs. The sample data actually indicates that the variance of Type A bulbs is considerably larger than that of Type B bulbs.

Testing for goodness-of-fit

Till now, hypothesis-testing are designed for problems in which the population or probability distribution is known and the hypothesis involve the parameters of the distribution. We now encounter another kind of hypothesis: We don't know the underlying distribution of the population, and we wish to test the hypothesis that a particular distribution will be satisfactory as a population model.

Example. 1. *We wish to test the hypothesis that the population is normal.*

2. *We use **goodness-of-fit** procedure based on the Chi-square distribution.*

Steps involved in the procedure:

1. Consider a random sample of size n.
2. These n observations are arranged in a frequency histogram, having k bins or class intervals.

3. Let O_i be the observed frequency in the i^{th} class-interval.

4. From the hypothesized probability distribution, we compute the expected frequency E_i, in the i-th class interval.

Note that each of the expected frequency is at least equal to 5. The test statistic is

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

The distribution of the random variable χ^2-is approximately Chi-squared distribution with $v = k - 1$ degree of freedom. We reject the null hypothesis if the value of χ^2 lies in the critical region, i.e., when $\chi_0^2 > \chi_\alpha^2(k-1)$ for α level of significance.

Problem 24. *The number of defects in printed circuit boards is hypothesized to follow a Poisson distribution. A random sample of $n = 60$ printed boards has been collected, and the following number of defects observed. Use $\alpha = 0.05$.*

Number of Defects	*0*	*1*	*2*	*3*
Observed Frequency	*32*	*15*	*9*	*4*

Solution: Given $n = 60$
We estimate the mean of the assumed Poisson distribution from the sample data:

$$\lambda = \mu = \frac{0(32) + 1(15) + 2(9) + 3(4)}{60} = \frac{45}{60} = 0.75$$

We compute the theoretical, hypothesized probabilities p_i as follows:

$$\begin{aligned}
P(X = 0) &= \frac{e^{-0.75}(0.75)^0}{0!} = e^{-0.75} = 0.472 \\
P(X = 1) &= \frac{e^{-0.75}(0.75)^1}{1!} = 0.354 \\
P(X = 2) &= \frac{e^{-0.75}(0.75)^2}{2!} = 0.133 \\
P(X \geq 3) &= 1 - (p_1 + p_2 + p_3) = 1 - (0.472 + 0.354 + 0.133) = 0.041
\end{aligned}$$

The expected frequencies $E_i = nP(X = x_i)$:

Number of Defects	0	1	2	3 or more
Probability	0.472	0.354	0.133	0.041
Expected Frequency	28.32	21.24	7.98	2.46

Since the expected frequency in the last cell is < 5, we combine the last two cells:

Number of Defects	0	1	2 (or more)
Observed Frequency	32	15	13
Expected Frequency	28.32	21.24	10.44

Parameter of interest : The distribution of defects in printed circuit board.

Null hypothesis, H_0 : The form of distribution of defects is Poisson.

Alternate hypothesis, H_1 : The form of distribution of defects is not Poisson.

Test statistic : $\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$

Level of significance : $\alpha = 0.05$, reject H_0 if P-value is less than 0.05.

Computation : $\chi^2 = \frac{(32-28.32)^2}{28.32} + \frac{(15-21.24)^2}{21.24} + \frac{(13-10.44)^2}{10.44} = 2.94$

Critical region:$\chi^2 > \chi^2_{0.05,2}$, i. e. $\chi^2 > 5.99$

Conclusion : From table $\chi^2_{0.05,2} = 5.99$ and $\chi^2 = 2.94 < 5.99$. Thus P-value is greater that 0.05. We fail to reject H_0. The distribution of defects is Poisson.

Problem 25. *Fit a binomial distribution to the following data:*

x_i	0	1	2	3	4	5
f_i	2	14	20	34	22	8

Find the corresponding theoretical estimates for f. Test the hypothesis that the given data follows binomial distribution at 0.05 level of significance.

Solution: Given the data: $N = \sum f_i = 2 + 14 + 20 + 34 + 22 + 8 = 100$. We need to estimate the parameter p for the binomial distribution $B(n, p)$. Here, $n = 5$. The mean of the given distribution is

$$\begin{aligned}
\mu &= \frac{\sum x_i f_i}{\sum f_i} \\
&= \frac{(0 \times 2) + (1 \times 14) + (2 \times 20) + (3 \times 34) + (4 \times 22) + (5 \times 8)}{100} \\
&= \frac{284}{100} = 2.84
\end{aligned}$$

For a binomial distribution, the mean is np. So, $np = 2.84$

$$\begin{aligned}
5p &= 2.84 \\
p &= \frac{2.84}{5} = 0.568 \\
\text{Then } q &= 1 - p = 1 - 0.568 = 0.432
\end{aligned}$$

The probability mass function of a binomial distribution is $P(X = x) = \binom{n}{x} p^x q^{n-x}$. Here, $P(X = x) = \binom{5}{x} (0.568)^x (0.432)^{5-x}$.

We compute the theoretical, hypothesized probabilities p_i as follows:

$$
\begin{aligned}
P(X=0) &= \binom{5}{0}(0.568)^0(0.432)^5 = 1 \times 1 \times (0.432)^5 \approx 0.015 \\
P(X=1) &= \binom{5}{1}(0.568)^1(0.432)^4 = 0.0989 \\
P(X=2) &= \binom{5}{2}(0.568)^2(0.432)^3 = 0.2601 \\
P(X=3) &= \binom{5}{3}(0.568)^3(0.432)^2 = 0.342 \\
P(X=4) &= \binom{5}{4}(0.568)^4(0.432)^1 = 0.2248 \\
P(X=5 &= \binom{5}{5}(0.568)^5(0.432)^0 = 0.0592
\end{aligned}
$$

The expected frequencies $E_i = NP(X = x_i)$:

Number of defects (x_i)	0	1	2	3	4	5
Observed frequency (O_i)	2	14	20	34	22	8
Expected frequency (E_i)	1.5	9.89	26.01	34.2	22.48	5.91

Since some expected frequencies are less than 5, we combine adjacent cells (combining $x = 0$ and $x = 1$: $O_i = 2 + 14 = 16$). The minimum expected frequency should be 5. Also sum of the expected frequency must be equal to N. So, we have the revised observed and expected frequencies.

Number of defects (x_i)	0 or 1	2	3	4	5
Observed frequency (O_i)	16	20	34	22	8
Expected frequency (E_i)	11.4	26.01	34.2	22.48	5.91

Null hypothesis, H_0 : The given data follows a binomial distribution.

Alternate hypothesis, H_1 : The given data does not follow a binomial distribution.

Test statistic : $\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 3.9954$

Level of significance : $\alpha = 0.05$

Degrees of freedom: 4

Degrees of freedom: $\chi^2 \approx 2.131 + 1.248 + 0.001 + 0.094 \approx 3.474$
Critical region: $\chi^2 > \chi^2_{0.05,4}$, i. e. $\chi^2 > 9.488$
Conclusion: Since the calculated $\chi^2 = 3.9954$ is less than the critical value $\chi^2_{0.05,4} = 9.488$, we fail to reject the null hypothesis. Therefore, we conclude that the given data follows a binomial distribution at a 0.05 level of significance.

Problem 26. *In 200 tosses of a coin, 118 heads and 82 tails were observed. Test the hypothesis that the coin is unbiased at a 5% level of significance.*

Solution: Given the total number of tosses, $N = 200$. Observed number of heads $(O_1) = 118$. Observed number of tails $(O_2) = 82$. If the coin is unbiased, the probability of getting a head (p) is 0.5, and the probability of getting a tail (q) is also 0.5.

Expected Frequencies (E_i): Expected number of heads $(E_1) = N \times p = 200 \times 0.5 = 100$
Expected number of tails $(E_2) = N \times q = 200 \times 0.5 = 100$

Observed and Expected Frequencies:

Outcome	Heads	Tails
Observed frequency (O_i)	118	82
Expected frequency (E_i)	100	100

Null hypothesis, H_0 : The coin is unbiased (i.e., $p = 0.5$).

Alternate hypothesis, H_1 : The coin is biased (i.e., $p \neq 0.5$).

Test statistic : $\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 6.48$

Level of significance : $\alpha = 0.05$

Degrees of freedom: 1

Critical region: $\chi^2 > \chi^2_{0.05,1}$, i. e. $\chi^2 > 3.841$

Conclusion: Since the calculated $\chi^2 = 6.48$ is greater than the critical value $\chi^2_{0.05,1} = 3.841$, we reject the null hypothesis. Therefore, we conclude that the coin is biased at a 5% level of significance.

Video Lecture

1.	Confidence Interval	:	Lecture 1; Lecture 2; Lecture 3
2.	Statistical Hypothesis	:	Lecture 1
3.	Errors in hypothesis testing	:	Lecture 1
4.	Critical Region and P-value	:	Lecture 1
5.	Examples of z and t-test	:	Lecture 1; Lecture 2
6.	Testing hypothesis on variance (Chi-square test)	:	Lecture 1; Lecture 2 (Goodness of fit)
7.	Testing hypotheses on the equality of variances of two populations (F-test)	:	Lecture 1