RV College of Engineering®
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DEPARTMENT OF MATHEMATICS
MAT241TA: LINEAR ALGEBRA AND PROBABILITY THEORY
Unit 1: Random Variables

TUTORIAL SHEET-1 : Discrete Random Variables

1. Two balls are chosen randomly from an urn containing 8 white, 4 black, and 2 orange balls. Suppose that we win \$2 for each black ball selected and we lose \$1 for each white ball selected. Let X denotes our winnings. What are the possible values of X, and what are the probabilities associated with each value?

2. Discrete random variable has the PMF as follows:

x	0	1	2	3	4	5	6	7
$p_X(x)$	0	k	$2k$	$2k$	$3k$	$3k^2$	$2k^2$	$7k^2+k$

Find (i) k; (ii) $P(X \geq 5)$; (iii) $P(X < 3)$; (iv) $P(2 < X \leq 5)$; (v) mean; (vi) Variance of X; (vii) $V(-3X)$.

3. Suppose that a day's production of 850 manufactured parts contains 50 parts that do not conform to customer requirements. Two parts are selected at random, without replacement, from the batch. Let the random variable X equal the number of nonconforming parts in the sample. What is the cumulative distribution function of X? The question can be answered by first finding the probability mass function of X.

4. Let the random variable X denote the number of semiconductor wafers that need to be analysed in order to detect a large particle of contamination. Assume that the probability that a wafer contains a large particle is 0.01 and that the wafers are independent. Determine the probability distribution of X.
(Hint : let p denote a wafer in which a large particle is present, let a denote a wafer in which it is absent, $\Omega = \{p, ap, aap, aaap, aaaap, aaaaap$, and so forth$\}$. $P(X = 2) = P(ap) = 0.0099$)

5. Errors in an experimental transmission channel are found when the transmission is checked by a certifier that detects missing pulses. The number of errors found in an eight-bit byte is a discrete random variable with the following cumulative distribution function:

$$F(x) = \begin{cases} 0, & x < 1 \\ 0.7, & 1 \leq x < 4 \\ 0.9, & 4 \leq x < 7 \\ 1, & 7 \leq x \end{cases}$$

Determine each of the following probabilities:
(a) $P(X \leq 4)$
(b) $P(X > 7)$
(c) $P(X \leq 5)$
(d) $P(X \leq 2)$
(e) $P(X > 4)$
(f) $V(-5X)$

TUTORIAL SHEET-2 : Continuous Random Variables

1. Let X be a continuous random variable with probability density function.

$$f(x) = \begin{cases} ax\,, & 0 \le x \le 1 \\ a\,, & 1 \le x \le 2 \\ -ax + 3a\,, & 2 \le x \le 3 \\ 0\,, & \text{otherwise} \end{cases}$$

(i) Determine the constant. (ii) Compute $P(X \le 1.5)$.

2. Find the mean and variance of the probability density function $f(x) = \frac{1}{2}e^{-|x|}$

3. An important factor in solid missile fuel is the particle size distribution. Significant problems occur if the particle sizes are too large. From production data in the past, it has been determined that the particle size (in micrometers) distribution is characterized by

$$p(x) = \begin{cases} 3x^{-4}, & x > 1, \\ 0, & elsewhere. \end{cases}$$

a) Verify that this is a valid density function.
b) Evaluate $F(x)$.
c) What is the probability that a random particle from the manufactured fuel exceeds 4 micrometers?

4. Suppose a certain type of small data processing firm is so specialized that some have difficulty making a profit in their first year of operation. The probability density function that characterizes the proportion Y that make a profit is given by

$$p(y) = \begin{cases} ky^4(1-y)^3, & 0 \le y \le 1, \\ 0, & elsewhere. \end{cases}$$

a) What is the value of k that renders the above a valid density function?
b) Find the probability that at most 50% of the firms make a profit in the first year
c) Find the probability that at least 80% of the firms make a profit in the first year
d) $V(2Y)$.

5. Determine the probability density function for each of the following cumulative distribution functions.

a) $F(x) = 1 - e^{-2x},\ \ x > 0$

b) $F(x) = \begin{cases} 0\,, & x < 0 \\ 0.2x\,, & 0 \le x < 4 \\ 0.04x + 0.64, & 4 \le x < 9 \\ 1, & 9 \le x \end{cases}$

TUTORIAL SHEET- 3 : Joint Discrete Random Variables

1. In the transmission of digital information, the probability that a bit has high, moderate, and low distortion is 0.01, 0.04, and 0.95, respectively. Suppose that three bits are transmitted and that the amount of distortion of each bit is assumed to be independent. Let X and Y denote the number of bits with high and moderate distortion out of the three, respectively. Determine:
   a) $p_{X,Y}(x,y)$; b) $p_X(x)$; c) $E(X)$; d) Are X and Y are independent? e) $Cov(X+Y)$,

2. Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find

a) The joint probability function $p(x,y)$
b) $P\big((X,Y) \in A\big)$, where A is the region $\{(x,y)|x+y \le 1\}$

3. If the joint probability distribution of X and Y is given by

$$f(x,y) = \frac{x\ +\ y}{30}, \qquad \text{for } x\ =\ 0,1,2,3;\ y\ =\ 0,1,2,$$

   find

a) $P(X \le 2, Y = 1)$
b) $P(X > 2, Y \le 1)$
c) $P(X > Y)$
d) $P(X + Y = 4)$
e) $E[2X + Y + 3]$
f) $V[2X + 3Y - 2]$
g) $Cov(X,Y)$
h) $\rho(X,Y)$
i) $P(Y > 0 \mid X = 2)$
j) $P(X > 1 \mid Y = 1)$

4. A coin is tossed twice. Let Z denote the number of heads on the first toss and W the total number of heads on the 2 tosses. If the coin is unbalanced and a head has a 40% chance of occurring, find

a) the joint probability distribution of W and Z;
b) the marginal distribution of W;
c) the marginal distribution of Z;
d) the probability that at least 1 head occurs

5. Suppose that 15 percent of the families in a certain community have no children, 20 percent have 1 child, 35 percent have 2 children, and 30 percent have 3. Suppose further that in each family each child is equally likely (independently) to be a boy or a girl. If a family is chosen at random from this community, then B, the number of boys, and G, the number of girls, in this family. Determine the joint probability mass function.

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TUTORIAL SHEET- 4: Joint Continuous Random Variables

1. Let X denote the reaction time, in seconds, to a certain stimulus and Y denote the temperature ($°F$) at which a certain reaction starts to take place. Suppose that two random variables X and Y have the joint density

$$f(x,y) = \begin{cases} 4xy, & 0 < x < 1, 0 < y < 1, \\ 0, & elsewhere \end{cases}$$

Find

a) $P\left(0 \leq X \leq \frac{1}{2} \; and \frac{1}{4} \leq Y \leq \frac{1}{2}\right)$;
b) $P(X < Y)$;
c) Are X and Y are independent?
d) $E[5X - 3Y]$
e) $V[2X + 3Y]$
f) $Cov(X, Y)$;
g) $\rho(X, Y)$;

2. Let X denote the diameter of an armored electric cable and Y denote the diameter of the ceramic mold that makes the cable. Both X and Y are scaled so that they range between 0 and 1. Suppose that X and Y have the joint density

$$f(x,y) = \begin{cases} \frac{1}{y}, & 0 < x < y < 1, \\ 0, & elsewhere. \end{cases}$$

Find $P\left(X + Y > \frac{1}{2}\right)$.

3. The amount of kerosene, in thousands of liters, in a tank at the beginning of any day is a random amount Y from which a random amount X is sold during that day. Suppose that the tank is not resupplied during the day so that $x \leq y$, and assume that the joint density function of these variables is

$$f(x,y) = \begin{cases} 2, & 0 < x \leq y < 1, \\ 0, & elsewhere. \end{cases}$$

a) Determine if X and Y are independent.
b) Find $P\left(\frac{1}{4} < X < \frac{1}{2} \mid Y = \frac{3}{4}\right)$.
c) $E[3X + 2Y]$

4. Given the joint density function

$$f(x,y) = \begin{cases} \frac{6 - x - y}{8}, & 0 < x < 2, \; 2 < y < 4, \\ 0, & elsewhere \end{cases}$$

Find $P(1 < Y < 3 \mid X = 1)$

5. Determine the value of c that makes the function $f(x,y) = c\,xy$ a joint probability density function over the range $0 < x < 3$ and $0 < y < x$.
Determine the following:

a) $P(X < 1, Y < 2)$
b) $P(1 < X < 2)$
c) $P(Y > 1)$
d) $P(X < 2, Y < 2)$
e) $E[X]$ and $E[Y]$
f) Marginal probability distribution of X
g) Conditional probability distribution of Y given $X = 1$
h) $P(Y > 2 \mid X = 1)$