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DEPARTMENT OF MATHEMATICS
MAT231CT: LINEAR ALGEBRA AND PROBABILITY THEORY
UNIT 1: LINEAR ALGEBRA-1
TUTORIAL SHEET

Vector spaces:

1. Show that $\mathbb{R}^{m\times n}$, together with the usual addition and scalar multiplication of matrices, satisfies the eight axioms of a vector space.

2. Verify the set of all odd functions from $\mathbb{R}$ to $\mathbb{R}$ with field $\mathbb{R}$, under usual addition and scalar multiplication is a vector space.

3. Show that $C[a, b]$, together with the usual pointwise addition and scalar multiplication of functions, satisfies the eight axioms of a vector space. (where $C[a, b]$, is a set of all continuous real valued function on $[a, b]$,)

4. Let V be the set of all ordered pairs of real numbers with addition defined by
$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$
and scalar multiplication defined by
$$\alpha . (x_1, x_2) = (\alpha x_1, x_2)$$
Is V a vector space with these operations? Justify your answer.

5. Let R denote the set of real numbers. Define scalar multiplication by
$\alpha x = \alpha . x$ (the usual multiplication of real numbers)
and define addition, denoted $\oplus$, by
$x \oplus y = max(x, y)$ (the maximum of the two numbers)
Is R a vector space with these operations? Prove your answer.

6. Prove that the set of all positive real numbers $\mathbb{R}^+$, is a vector space over the real field, under the vector addition $\alpha + \beta = \alpha\beta\ \forall\ \alpha, \beta \in \mathbb{R}^+$ and scalar multiplication $c \cdot \alpha = \alpha^c\ \ \forall\ \alpha \in \mathbb{R}^+$ and $c \in \mathbb{R}$.

Subspaces, linear combination and linear span:

7. Let S be the set of all polynomials of degree $\le n$, with the property that $p(0) = 0$.
i.e. $S = \{p(x) \mid p(x)$ is a polynomial of degree $\le n$ and $p(0) = 0\}$.
The set S is nonempty since it contains the zero polynomial. Show that S is a subspace of P_n (where P_n denote the set of all polynomials of degree less than or equal to n.).

8. Verify $W = \left\{ \begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} \middle| s, t \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^4$.

9. Determine if the set H of all matrices of the form $\begin{bmatrix} a & b \\ c & a^2 \end{bmatrix}$ is a subspace of $M_{2\times 2}$.

10. Determine whether the following sets form subspaces of $\mathbb{R}^2$:

    (a) $\{(x_1, x_2)^T \mid x_1 + x_2 = 0\}$

    (b) $\{(x_1, x_2)^T \mid x_1 x_2 = 0\}$

    (c) $\{(x_1, x_2)^T \mid x_1 = 3x_2\}$

    (d) $\{(x_1, x_2)^T \mid x_1^2 = x_2^2\}$

11. Determine whether the following sets form subspaces of $\mathbb{R}^3$:

    (a) $\{(x_1, x_2, x_3)^T \mid x_1 + x_3 = 1\}$

    (b) $\{(x_1, x_2, x_3)^T \mid x_1 = x_2 = x_3\}$

    (c) $\{(x_1, x_2, x_3)^T \mid x_3 = x_1 + x_2\}$

    (d) $\{(x_1, x_2, x_3)^T \mid x_3 = x_1 \text{ or } x_3 = x_2\}$

    (e) $\{(x_1, x_2, x_3)^T \mid x_3 = 1\}$

12. Determine whether the following are subspaces of $\mathbb{R}^{n\times n}$:

    (a) The set of all $n \times n$ diagonal matrices

    (b) The set of all $n \times n$ upper triangular matrices

    (c) The set of all $n \times n$ lower triangular matrices

    (d) The set of all $n \times n$ matrices A such that $a_{12} = 1$

    (e) The set of all $n \times n$ matrices B such that $b_{11} = 0$

    (f) The set of all symmetric $n \times n$ matrices

    (g) The set of all singular $n \times n$ matrices

13. Determine the null space of each of the following matrices:

    (a) $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$

    (b) $\begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \\ -1 & -1 & 0 & -5 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 3 & -4 \\ 2 & -1 & -1 \\ -1 & -3 & 4 \end{pmatrix}$

14. Which of the sets that follow are spanning sets for $\mathbb{R}^3$? Justify your answers.

(a) $\{(1,0,0)^T, (0,1,1)^T, (1,0,1)^T\}$

(b) $\{(1,0,0)^T, (0,1,1)^T, (1,0,1)^T, (1,2,3)^T\}$

(c) $\{(2,1,-2)^T, (3,2,-2)^T, (2,2,0)^T\}$

(d) $\{(2,1,-2)^T, (-2,-1,2)^T, (4,2,-4)^T\}$

(e) $\{(1,1,3)^T, (0,2,1)^T$

15. Let U and V be subspaces of a vector space W. Define
$$U + V = \{\boldsymbol{z} \mid \boldsymbol{z} = \boldsymbol{u} + \boldsymbol{v} \text{ where } \boldsymbol{u} \in U \text{ and } \boldsymbol{v} \in V\}$$
Show that $U + V$ is a subspace of W.

16. Describe the subspace of $\mathbb{R}^3$ (is it line or a plane or $\mathbb{R}^3$?) spanned by
a) The two vectors $(1,1,-1)$ and $(-1,-1,1)$.
b) The three vectors $(0,1,1)$, $(1,1,0)$ and $(0,0,0)$.
c) The columns of a 3 by 5 echelon matrix with 2 pivots
d) All vectors with positive components

Linear independence and dependence

17. Determine whether the following vectors are linearly independent in $\mathbb{R}^3$:

a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

18. Determine whether the following vectors are linearly independent in $\mathbb{R}^{2\times 2}$:

a) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

19. Determine whether the following vectors are linearly independent in P_3:
    a) $1,\ x^2,\ x^2 - 2$
    b) $2,\ x^2,\ x,\ 2x + 3$
    c) $x + 2,\ x^2 - 1$

20. Let A be an $m \times n$ matrix. Show that if A has linearly independent column vectors, then $N(A) = \{0\}$.

21. The value of k such that the polynomials $2 + t$ and $3 + kt$ are linearly dependent is ___.

22. Let $V = M_{2\times2}$, the vector space of 2×2 matrices and the set U consist of those matrices whose first row is zero. Then the standard basis of U is ____, and its dimension is ____.

23. Suppose v_1, v_2, v_3, v_4 are vectors in $\mathbb{R}^3$.
    a) These four vectors are dependent because ________.
    b) The two vectors v_1 and v_2 will be dependent if ________.
    c) The vectors v_1 and $(0, 0, 0)$ are dependent because _________.

Basis and Dimension

24. Consider the vectors

$$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

    a) Show that x_1 and x_2 form a basis for $\mathbb{R}^2$.
    b) Why must x_1, x_2, and x_3 be linearly dependent?
    c) What is the dimension of $\text{Span}(x_1, x_2, x_3)$.

25. In each of the following, find the dimension of the subspace of P_3 spanned by the given vectors:
    a) $x,\ x - 1,\ x^2 + 1$
    b) $x^2,\ x^2 - x - 1,\ x + 1$
    c) $2x,\ x - 2$

26. Let A ne am $m \times n$ matrix. Show that if A has linearly independent column vectors, then $N(A) = \{0\}$.

27. Determine the dimension of the subspace of $\mathbb{R}^3$ spanned by the vectors:

$$\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 3 \\ 6 \end{bmatrix}$$

28. Find the basis and dimension for all the subspaces in question 12.

29. Find the basis for each of these subspaces of $\mathbb{R}^4$:
    a) All vectors whose components are equal.
    b) All vectors whose components add to zero.
    c) All vectors that are perpendicular to $(1, 1, 0, 0)$ and $(1, 0, 1, 1)$.
    d) The column space (in $\mathbb{R}^2$) and nullspace (in $\mathbb{R}^5$) of $U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$.

Row Space and Column Space

30. For each of the following matrices, find a basis and the dimension for the row space, column space, null space and left null space. Hence verify rank-nullity theorem:

    a) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix}$

    b) $\begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix}$

    c) $\begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

    d) $A = \begin{bmatrix} 2 & -4 & 1 & 2 & -2 & -3 \\ -1 & 2 & 0 & 0 & 1 & -1 \\ 10 & -4 & -2 & 4 & -2 & 4 \end{bmatrix}$

31. Given a matrix:

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 2 & -3 & 3 & 3 & 5 \\ 4 & -6 & 9 & 5 & 9 \end{bmatrix}.$$

a) The $N(A)$ is a subspace of __________ vector space.

b) The $R(A)$ is a subspace of ___________ vector space.

c) The $C(A)$ is a subspace of __________ vector space.

d) The $N(A^T)$ is a subspace of __________ vector space.

e) Determine if $v = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ is in row space of A.

f) Determine if $v = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ is in column space of A.

g) Determine if $w = \begin{bmatrix} -5 \\ 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}$ is in null space of A.

h) Determine if $w = \begin{bmatrix} 4 \\ -2 \\ 9 \\ 5 \\ 5 \end{bmatrix}$ is in row space of A.

32. If A is a 9×7 matrix with three dimensional nullspace, the rank of A is ___ and the dimension of the left nullspace is ___ .

33. If A is any 4×7 matrix and if rank of A is 3, then the $dim\big(R(A)\big), \dim\big(N(A)\big)$ and $\dim\big(N(A^T)\big)$ are ____________.

34. Any solution X to $AX = b$ (if it exist) is always a sum of a vector in the ________ space of A plus a vector in the _______ space of A.

35. $AX = b$ is solvable if and only if b is orthogonal to every vector in the ______ space of A.

36. If X_1 and X_2 are both solutions of $AX = b$, then the vector $X_1 - X_2$ must be in the ________ space of A.

37. Let A be a 4×4 matrix with reduced row echelon form given by

$$U = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let a_i be i^{th} column of A. If

$$a_1 = \begin{bmatrix} -3 \\ 5 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad a_2 = \begin{bmatrix} 4 \\ -3 \\ 7 \\ -1 \end{bmatrix}$$

Find a_3 and a_4.

38. Relate the four fundamental subspaces of $A^T A$ to the four fundamental subspaces of a real matrix A: nullspace of $A^T A =$ ________ space of A, left nullspace of $A^T A =$ ______ space of A, column space of $A^T A =$ ________ space of A, row space of $A^T A =$ ______ space of A.

Linear Transformation

39. Check whether the following are linear transformation:

a) $T: \mathbb{R}^2 \to \mathbb{R}^2$; $\quad T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 2a \\ 3b \end{bmatrix}$.

b) $T: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$; $\quad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+c & 0 \\ 0 & c-d \end{bmatrix}$.

c) $T: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$; $\quad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$.

d) $T: P_2(t) \to P_3(t)$; $\; T\big(f(t)\big) = tf(t)$.

e) $T: P_2(t) \to P_4(t)$; $\; T\big(f(t)\big) = f(t^2)$.

f) $T: \mathbb{R} \to \mathbb{R}^3$; $\quad T(x) = \begin{bmatrix} x \\ x^2 \\ x^3 \end{bmatrix}$.

40. Determine whether the vector v is in the range of the linear transformation L.

a) $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+z \\ y+z \\ x+2y+2z \end{bmatrix}$, $\; u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

b) $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\; u = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

41. Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ and $c = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$. Define a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(X) = AX,$ for all $X \in \mathbb{R}^2$.

a) Find $T(u)$, the image under the transformation T.

b) Find an $X \in \mathbb{R}^2$ whose image is b.

c) Is there more than one $X \in \mathbb{R}^2$ whose image is b.

d) Determine if c is in the range of the transformation.

42. Obtain the Linear Transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1\\-1\\5\end{bmatrix}$, $T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}5\\-2\\-2\end{bmatrix}$.

43. The vector $\begin{bmatrix}2\\1\end{bmatrix}$ is reflected along the line $y = x$. Then the reflection matrix is ____ and the resultant vector is ____.

44. Find the range space and kernel of the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\1\end{bmatrix}$.

45. Show that each of the following are linear operators on $\mathbb{R}^2$. Describe geometrically what each linear transformation accomplishes.

    a) $L(X) = \begin{bmatrix}-x_1\\x_2\end{bmatrix}$,

    b) $L(X) = -X$

    c) $L(X) = x_2 e_2$

46. Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator. If

$$L\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix} \quad \text{and} \quad L\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}5\\2\end{bmatrix}$$

find the value of $L\left(\begin{bmatrix}7\\5\end{bmatrix}\right)$.

47. Determine whether the following are linear transformations from $\mathbb{R}^2$ into $\mathbb{R}^3$:

    a) $L(X) = \begin{bmatrix}x_1\\x_2\\1\end{bmatrix}$

    b) $L(X) = \begin{bmatrix}x_1\\x_2\\x_1 + 2x_2\end{bmatrix}$

    c) $L(X) = \begin{bmatrix}x_1\\0\\0\end{bmatrix}$

    d) $L(X) = \begin{bmatrix}x_1\\x_2\\x_1^2 + x_2^2\end{bmatrix}$

48. Determine the kernel and range of each of the following linear operators on $\mathbb{R}^3$:

a) $L(X) = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}$

b) $L(X) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$

c) $L(X) = \begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix}$

49. For each of the following linear transformations L mapping $\mathbb{R}^3$ into $\mathbb{R}^2$, find a matrix A such that $L(X) = AX$ for every X in $\mathbb{R}^3$:

a) $L(X) = \begin{bmatrix} x_1 + x_2 \\ 0 \end{bmatrix}$

b) $L(X) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

c) $L(X) = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

50. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$L(X) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 3x_1 + 2x_2 \end{bmatrix}$$

Find a matrix A such that $LX = AX$ for each X in $\mathbb{R}^2$.

51. Find the matrix of the linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + 2y - 3z \\ 4x - 5y - 6z \\ 7x + 8y + 9z \end{bmatrix}$$

with respect to the standard basis.

52. Find the matrix of the linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y - z \\ 4y + 5z \end{bmatrix}$$

with respect to the standard basis.

53. Discuss the following maps on $\mathbb{R}^2$ and represent them graphically:
    a) Reflection through $x -$ axis
    b) Reflection through $y -$ xis
    c) Reflection through $y = x$.
    d) Reflection through $y = -x$.
    e) Reflection through origin.
    f) Rotation
    g) Horizontal contraction or expansion
    h) Vertical contraction or expansion
    i) Dilation
    j) Horizontal Shear
    k) Vertical Shear

54. Determine the matrix that describes a reflection about $x -$ axis, followed by rotation through $\frac{\pi}{2}$, followed by a dilation of factor 3. Find the image of the point $(4, 1)$ under this sequence of mappings.

55. Derive the matrix of the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, which rotates a vector $v \in \mathbb{R}^2$ by an angle θ in an anticlockwise direction. Find if there exist:
    a) A preimage of $(1, -3)$
    b) An image of $(3, -1)$ when $\theta = \frac{\pi}{2}$.