



HANDBOOK OF MATHEMATICS
FOR
SECOND YEAR B.E. PROGRAM



TRIGONOMETRY

Basic Functions

- $\sin \theta = \frac{\text{Opposite Side}}{\text{Hypotenuse}}$
- $\cos \theta = \frac{\text{Adjacent Side}}{\text{Hypotenuse}}$
- $\tan \theta = \frac{\text{Opposite Side}}{\text{Adjacent Side}} = \frac{\sin \theta}{\cos \theta}$
- $\sec \theta = \frac{\text{Hypotenuse}}{\text{Adjacent Side}} = \frac{1}{\cos \theta}$
- $\operatorname{cosec} \theta = \frac{\text{Hypotenuse}}{\text{Opposite Side}} = \frac{1}{\sin \theta}$
- $\cot \theta = \frac{\text{Adjacent Side}}{\text{Opposite Side}} = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}$

Identities

- $\sin(-x) = -\sin x$
- $\cos(-x) = \cos x$
- $\tan(-x) = -\tan x$
- $\sin(\pi - x) = \sin x$
- $\cos(\pi - x) = -\cos x$
- $\tan(\pi - x) = -\tan x$
- $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
- $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
- $\tan\left(\frac{\pi}{2} - x\right) = \cot x$
- $\sin(\pi + x) = -\sin x$
- $\cos(\pi + x) = -\cos x$
- $\tan(\pi + x) = \tan x$
- $\sin\left(\frac{\pi}{2} + x\right) = \cos x$
- $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$
- $\tan\left(\frac{\pi}{2} + x\right) = -\cot x$
- $\sin\left(\frac{3\pi}{2} - x\right) = -\cos x$
- $\cos\left(\frac{3\pi}{2} - x\right) = \sin x$
- $\tan\left(\frac{3\pi}{2} - x\right) = -\cot x$
- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x$
 $= 2 \cos^2 x - 1$
- $\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$
- $\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$
- $\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$
- $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\cos^2 x + \sin^2 x = 1$
- $\sec^2 x - \tan^2 x = 1$
- $\operatorname{cosec}^2 x - \cot^2 x = 1$
- $\sin 3x = 3 \sin x - 4 \sin^3 x$
- $\cos 3x = 4 \cos^3 x - 3 \cos x$
- $\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$
- $\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$



BASIC CALCULUS

Differentiation

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
a	0	a^x	$a^x \log_e a$
$x^n, n \neq -1$	nx^{n-1}	e^{ax}	ae^{ax}
\sqrt{x}	$\frac{1}{2\sqrt{x}}$	$\log_e x$	$\frac{1}{x}$
$\sin x$	$\cos x$	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos x$	$-\sin x$	$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan x$	$\sec^2 x$	$\tan^{-1} x$	$\frac{1}{1+x^2}$
cosec x	$-\cot x \operatorname{cosec} x$	$\operatorname{cosec}^{-1} x$	$-\frac{1}{ x \sqrt{x^2-1}}$
$\sec x$	$\tan x \sec x$	$\sec^{-1} x$	$\frac{1}{ x \sqrt{x^2-1}}$
$\cot x$	$-\operatorname{cosec}^2 x$	$\cot^{-1} x$	$-\frac{1}{1+x^2}$
$\sinh x = \frac{e^x - e^{-x}}{2}$	$\cosh x$	$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh x = \frac{e^x + e^{-x}}{2}$	$\sinh x$	$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh x$	$\operatorname{sech}^2 x$	$\tanh^{-1} x$	$\frac{1}{1-x^2}$
cosech x	$-\coth x \operatorname{cosech} x$	$\operatorname{cosech}^{-1} x$	$-\frac{1}{ x \sqrt{x^2+1}}$
$\operatorname{sech} x$	$-\tanh x \operatorname{sech} x$	$\operatorname{sech}^{-1} x$	$-\frac{1}{ x \sqrt{1-x^2}}$
$\coth x$	$-\operatorname{cosech}^2 x$	$\coth^{-1} x$	$\frac{1}{1-x^2}$

Rules of differentiation

- $\frac{d}{dx}(fg) = gf' + fg'$



- $\frac{d}{dx} \left(\frac{f}{g} \right) = \frac{gf' - fg'}{g^2}$
- $\frac{d}{dx} (f(t)) = \frac{d}{dt} (f(t)) \frac{dt}{dx}$

Integration

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1}$	$\frac{1}{x}$	$\log_e x$
e^{ax}	$\frac{e^{ax}}{a}$	$\log_e x$	$x(\log_e x - 1)$
a^x	$\frac{a^x}{\log_e a}$	cosec x	$\log_e (\cosec x - \cot x)$
$\sin x$	$-\cos x$	$\sec x$	$\log_e (\sec x + \tan x)$
$\cos x$	$\sin x$	$\cot x$	$\log_e \sin x$
$\tan x$	$\log_e \sec x$	$\sec^2 x$	$\tan x$
$\sinh x$	$\cosh x$	$\cosec^2 x$	$-\cot x$
$\cosh x$	$\sinh x$	$\tanh x$	$\log_e \cosh x$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1} \left(\frac{x}{a} \right)$	$\frac{1}{\sqrt{a^2 + x^2}}$	$\sinh^{-1} \left(\frac{x}{a} \right)$
$\frac{1}{a^2 + x^2}$	$\frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$	$\frac{1}{\sqrt{x^2 - a^2}}$	$\cosh^{-1} \left(\frac{x}{a} \right)$
$\frac{1}{a^2 - x^2}$	$\frac{1}{2a} \log_e \left(\frac{a+x}{a-x} \right)$	$\frac{1}{x^2 - a^2}$	$\frac{1}{2a} \log_e \left(\frac{x-a}{x+a} \right)$
$\sqrt{a^2 - x^2}$	$\frac{1}{2} \left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \left(\frac{x}{a} \right) \right]$	$e^{ax} \sin bx$	$\frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$
$u(x)v(x)$	$u \int v dx - \int \left[\frac{du}{dx} \left[\int v dx \right] dx \right]$	$e^{ax} \cos bx$	$\frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$



COMPLEX ANALYSIS

Algebra of complex numbers:

- If $z = x + iy$, then $|z| = \sqrt{x^2 + y^2}$ is non-negative real number.
- $z = x + iy$ is represented by a point $P(x, y)$ in the XY plane, $x - axis$ is real axis, $y - axis$ is imaginary axis, plane is complex plane.
- $z = re^{i\theta}$ is the polar form of complex number z where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} \left(\frac{y}{x} \right)$.
- $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ represents the distance between the points z_1 and z_2 in complex plane.
- $|z - z_0| = R$ represents complex equation of circle with centre z_0 and radius R .
- $|z - z_0| < R$ represents the region with in, but not on, a circle of radius R centred at the point z_0 , the point z_0 is said to be interior point.
- $|z - z_0| \leq R$ represents the region with in, and on, a circle of radius R centred at the point z_0 .
- $|z - z_0| > R$ represents the region outside the circle with centre z_0 and radius R .

Cauchy-Riemann (C-R) equations:

- In Cartesian form: If $f(z) = u(x, y) + i v(x, y)$ is differentiable at $z = x + iy$, then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
- In polar form: If $f(z) = u(r, \theta) + i v(r, \theta)$, then $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$

Harmonic Functions: A function ϕ is said to be a harmonic function if it satisfies Laplace equation $\nabla^2 \phi = 0$.

- In Cartesian form $\phi(x, y)$ is harmonic if $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$.
- In polar form $\phi(r, \theta)$ is harmonic if $\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0$.

Taylor theorem: Taylor series expansion for the function $f(z)$ about the point $z = a$ is

$$f(z) = f(a) + (z - a)f'(a) + (z - a)^2 \frac{f''(a)}{2!} + \dots$$

Maclaurin theorem: Maclaurin series for the function $f(z)$ about the point $z = 0$ is

$$f(z) = f(0) + zf'(0) + z^2 \frac{f''(0)}{2!} + \dots$$

Binomial Expansion: If $|x| < 1$, then

- $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \dots$
- $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots$
- $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$
- $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$



Laurent's theorem: Let c_1 and c_2 are concentric circles centered at " a ", then Laurent's series of $f(z)$ about the point $z = a$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - a)^n},$$

$$\text{where } a_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)^{n+1}} dw \quad b_n = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-a)^{-n+1}} dw \text{ for } n = 0, 1, 2 \dots$$

Determination of poles: If $f(z) = \frac{\phi(z)}{(z-a)^m}$ where $\phi(z)$ is analytic and not zero at the point ' a ', then ' a ' is a pole of order m of $f(z)$. The poles of $f(z)$ may be obtained by solving the equation $\frac{1}{f(z)} = 0$.

Residue: The coefficient of $\frac{1}{z-a}$ in the Laurent's expansion of $f(z)$ is the Residue of $f(z)$ at the pole $z = a$.

Determination of a residue: If ' a ' is a pole of order $m \geq 1$ of $f(z)$, then residue of $f(z)$ at ' a ' is $\frac{1}{(m-1)!} \left[\lim_{z \rightarrow a} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\} \right]$.

Cauchy's residue theorem: Let C be a simple closed curve and $f(z)$ be analytic within and on C except at a finite number of poles a_1, a_2, \dots, a_n which lie inside C , then

$$\int_C f(z) dz = 2\pi i(R_1 + R_2 + \dots + R_n)$$

where $R_1, R_2 \dots R_n$ are the residues of $f(z)$ at a_1, a_2, \dots, a_n respectively.

PARTIAL DIFFERENTIAL EQUATIONS

Lagrange's linear equation: The first order linear partial differential equation of the form $P \frac{\partial u}{\partial x} + Q \frac{\partial u}{\partial y} = R$, where P, Q and R are functions of x, y, z is known as Lagrange's Linear equation.

Subsidiary/Auxiliary equation: The equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ is known as the subsidiary/auxiliary equation of as Lagrange's Linear equation $P \frac{\partial u}{\partial x} + Q \frac{\partial u}{\partial y} = R$.

One-dimensional wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, where $c^2 = \frac{T}{\rho}$ the phase speed, T is the tension, and ρ density of the string.

One-dimensional heat equation: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, where $c^2 = \frac{\kappa}{s\rho}$ the thermal diffusivity, κ thermal conductivity, s specific heat and ρ density of the material of the body.

Two-dimensional Laplace equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

Laplace equation: $u_{xx} + u_{yy} = 0$



- Standard 5-point formula: $u_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}]$
- Diagonal 5-point formula: $u_{i,j} = \frac{1}{4}[u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}]$.

One-dimensional heat equation: $u_t = c^2 u_{xx}$

- Schmidt formula: $u_{i,j+1} = \alpha(u_{i-1,j} + u_{i+1,j}) + (1 - 2\alpha)u_{i,j}$ where $\alpha = \frac{kc^2}{h^2}$.
- Bendre-Schmidt relation: $u_{i,j+1} = \frac{1}{2}[u_{i-1,j} + u_{i+1,j}]$ when $\alpha = \frac{1}{2}$.

One-dimensional wave equation: $u_{tt} = c^2 u_{xx}$

- Explicit formula: $u_{i,j+1} = \beta^2(u_{i+1,j} + u_{i-1,j}) + 2(1 - \beta^2)u_{i,j} - u_{i,j-1}$, where $\beta^2 = \frac{c^2 k^2}{h^2}$.
- For $\beta = 1$ and $k = \frac{h}{c}$, $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$.

The above scheme is used with standard initial and boundary conditions.

CALCULUS OF VARIATION

Euler's equation: A necessary condition for the functional $I = \int_{x_1}^{x_2} f(x, y, y') dx$ to be an extremum is that $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$.

Alternate forms:

- $\frac{d}{dx}\left(f - y' \frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x}$
- $\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y' \frac{\partial f}{\partial y'}\right) = 0$
- $\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - \frac{\partial^2 f}{\partial y \partial y'} y' - \frac{\partial^2 f}{\partial (y')^2} y'' = 0$

Cases of Euler's Equation:

- If f is only the function of y' , then Euler's equation will be: $\frac{df}{dy'} = c$ where c is an arbitrary constant.
- If f is independent of y , then Euler's equation will be: $\frac{\partial f}{\partial y'} = c$ where c is an arbitrary constant.
- If f is independent of y' then Euler's equation will be: $\frac{\partial f}{\partial y} = 0$.
- If f is independent of x and y then Euler's equation will be: $y'' \frac{\partial^2 f}{\partial y'^2} = 0$.
- If f does not contain x then Euler's equation will be: $f - y' \frac{\partial f}{\partial y'} = c$ where c is an arbitrary constant.

Cartesian coordinate system: $u_1 = x, u_2 = y, u_3 = z$

Element of arc length $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$.

Cylindrical coordinate system: $u_1 = r, u_2 = \theta, u_3 = z$

Element of arc length $ds = \sqrt{(dr)^2 + r^2(d\theta)^2 + (dz)^2}$

Spherical coordinate system: $u_1 = r, u_2 = \theta$ and $u_3 = \phi$.



Element of arc length $ds = \sqrt{(dr)^2 + r^2(d\theta)^2 + r^2 \sin^2 \theta (d\phi)^2}$.

LAPLACE TRANSFORM

Gamma function:

- $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, (n > 0)$
- $\Gamma(1) = 1$
- $\Gamma(n) = 2 \int_0^\infty e^{-x^2} x^{2n-1} dx$
- $\Gamma(n+1) = \begin{cases} n\Gamma(n), & n > 0 \\ n!, & n \text{ positive integer} \end{cases}$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- $\Gamma(n) = \frac{\Gamma(n+1)}{n}, (n < 0, \neq -1, -2, \dots)$

Beta Function:

- $\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$
- $\beta(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$
- $\beta(m, n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt$
- $\beta(m, n) = \beta(n, m)$
- $\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Laplace transform of $f(t)$: $L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$

Transform of elementary functions:

- $L(e^{at}) = \frac{1}{s-a}, s > a$
- $L(\sinh at) = L\left(\frac{e^{at}-e^{-at}}{2}\right) = \frac{a}{s^2-a^2}, s > |a|$
- $L(\sin at) = \frac{a}{s^2+a^2}, s > 0$
- $L(\cosh at) = \frac{s}{s^2-a^2}, s > |a|$
- $L(\cos at) = \frac{s}{s^2+a^2}, s > 0$
- $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$
- $L[H(t-a)] = \frac{e^{-as}}{s}$, where H is Heaviside unit step function

Properties of Laplace transform:

- $L[a f(t) + b g(t)] = a L[f(t)] + b L[g(t)].$
- If $L[f(t)] = F(s)$, then $L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$, where a is a positive constant.
- Let a be any real constant then $L[e^{at}f(t)] = F(s-a)$
- If $L[f(t)] = F(s)$, then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s), n = 1, 2, 3, \dots$
- If $L[f(t)] = F(s)$, then $L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s) ds.$
- If $L[f(t)] = F(s)$, then $L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{n-1}(0)$
- If $L[f(t)] = F(s)$, then $L \int_0^t f(t) dt = \frac{1}{s} F(s)$
- If $f(t)$ is a periodic function of period T , then $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt.$



- If $L\{f(t)\} = F(s)$, then $L[f(t-a)H(t-a)] = e^{-as}F(s)$
- If $f(t)$ is a continuous function at $t = a$, then $\int_0^\infty f(t)\delta(t-a)dt = f(a)$, where $\delta(t-a)$ is unit impulse function.

Inverse Laplace transform of $F(s)$ using Convolution theorem: If $L^{-1}[F(s)] = f(t)$ and $L^{-1}[G(s)] = g(t)$, then $L^{-1}[F(s)G(s)] = \int_0^t f(u)g(t-u)du = f(t) * g(t)$.

FOURIER SERIES

Fourier series of $f(x)$ in the interval $(a, a + 2l)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right),$$

where Fourier coefficients a_0, a_n, b_n are given by

$$\begin{aligned} a_0 &= \frac{1}{l} \int_a^{a+2l} f(x) dx, & a_n &= \frac{1}{l} \int_a^{a+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx, & n &= 1, 2, 3, \dots \text{ and} \\ b_n &= \frac{1}{l} \int_a^{a+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx, & n &= 1, 2, 3, \dots \end{aligned}$$

Complex Fourier Series of $f(x)$ in the interval $(a, a + 2l)$:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}, \text{ where } c_n = \frac{1}{2l} \int_a^{a+2l} f(x) e^{\frac{-in\pi x}{l}} dx, n = 0, \pm 1, \pm 2, \pm 3, \dots$$

Relation between Fourier and complex Fourier coefficients:

$$a_0 = 2c_0, a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n})$$

Half-range Fourier series:

- Sine series of $f(x)$ in $(0, l)$: $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$, where $b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$.
- Cosine series of $f(x)$ in $(0, l)$: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$, where $a_0 = \frac{2}{l} \int_0^l f(x) dx$ and $a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$

Harmonic Analysis: Let the periodic function $y = f(x)$ takes values $y_0, y_1, y_2, \dots, y_m$ corresponding to a given set of equi-spaced values $x_0, x_1, x_2, \dots, x_m$ in $(a, a + 2l)$, Fourier series of $f(x)$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$, where a_0, a_n and b_n are computed using the formulae:

$$a_0 = 2 \frac{\sum y}{m}, \quad a_n = 2 \frac{\sum y \cos\left(\frac{n\pi x}{l}\right)}{m}, \quad b_n = 2 \frac{\sum y \sin\left(\frac{n\pi x}{l}\right)}{m}.$$

- Let the periodic function $y = f(x)$ takes values $y_0, y_1, y_2, \dots, y_m$ corresponding to a given set of equi spaced values $x_0, x_1, x_2, \dots, x_m$ in $(0, l)$, Half range cosine Fourier series and Half range sine Fourier series of $f(x)$ are respectively:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$



$$= \frac{1}{2} a_0 + a_1 \cos\left(\frac{\pi x}{l}\right) + a_2 \cos\left(\frac{2\pi x}{l}\right) + \dots$$

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \\ &= b_1 \sin\left(\frac{\pi x}{l}\right) + b_2 \sin\left(\frac{2\pi x}{l}\right) + \dots, \end{aligned}$$

where a_0 , a_n and b_n are computed from the table by using the formulae:

$$a_0 = 2 \frac{\Sigma y}{m}, \quad a_n = 2 \frac{\Sigma y \cos\left(\frac{n\pi x}{l}\right)}{m}, \quad b_n = 2 \frac{\Sigma y \sin\left(\frac{n\pi x}{l}\right)}{m}$$

FOURIER TRANSFORMS

Complex Fourier transform or Fourier transform of $f(x)$:

- Fourier transform of $f(x)$: $\hat{f}(\alpha) = F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$, provided the integral exists.
- Inverse Fourier transform of $\hat{f}(\alpha)$: $F^{-1}[\hat{f}(\alpha)] = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha x} d\alpha$.

Fourier sine and cosine transforms:

- Fourier sine transform of $f(x)$: $\hat{f}_s(\alpha) = F_s[f(x)] = \int_0^{\infty} f(x) \sin \alpha x dx$
- Inverse Fourier sine transform $\hat{f}_s(\alpha)$: $F_s^{-1}[\hat{f}_s(\alpha)] = \frac{2}{\pi} \int_{-\infty}^{\infty} \hat{f}_s(\alpha) \sin \alpha x d\alpha$
- Fourier cosine transform of $f(x)$: $\hat{f}_c(\alpha) = F_c[f(x)] = \int_0^{\infty} f(x) \cos \alpha x dx$
- Inverse Fourier cosine transform of $\hat{f}_c(\alpha)$: $F_c^{-1}[\hat{f}_c(\alpha)] = \frac{2}{\pi} \int_{-\infty}^{\infty} \hat{f}_c(\alpha) \cos \alpha x d\alpha$

Relation between Fourier sine and cosine transform:

- $F_s[xf(x)] = -\frac{d}{d\alpha} F_c[f(x)]$ and $F_c[xf(x)] = \frac{d}{d\alpha} F_s[f(x)]$

Properties of Fourier transforms:

- For any two functions $f(x)$ and $\phi(x)$ (whose Fourier transforms exist) and any two constants a and b , $F[af(x) + b\phi(x)] = aF[f(x)] + bF[\phi(x)]$
- If $F[f(x)] = \hat{f}(\alpha)$, then for any non-zero constant a , $F[f(ax)] = \frac{1}{|a|} \hat{f}\left(\frac{\alpha}{a}\right)$
- If $F[f(x)] = \hat{f}(\alpha)$, then for any non-zero constant a ,
 - $F[f(x-a)] = e^{i\alpha a} \hat{f}(\alpha)$
 - $F[e^{i\alpha x} f(x)] = \hat{f}(\alpha + a)$
- $F[f'(x)] = -isF[f(x)]$ and $F[f''(x)] = -s^2 F[f(x)]$
- If $F[f(x)] = \hat{f}(\alpha)$, then
 - $F[f(x) \cos ax] = \frac{1}{2} [\hat{f}(\alpha+a) + \hat{f}(\alpha-a)]$ and
 - $F[f(x) \sin ax] = \frac{1}{2} [\hat{f}(\alpha+a) - \hat{f}(\alpha-a)]$, where ' a ' is a real constant.
- If $\hat{f}_s(\alpha)$ and $\hat{f}_c(\alpha)$ are Fourier sine and cosine transforms of $f(x)$ respectively, then
 - $F_s[f(x) \cos ax] = \frac{1}{2} [\hat{f}_s(\alpha+a) + \hat{f}_s(\alpha-a)]$,
 - $F_c[f(x) \sin ax] = \frac{1}{2} [\hat{f}_s(\alpha+a) - \hat{f}_s(\alpha-a)]$,



$$(iii) F_s[f(x) \sin ax] = \frac{1}{2} [\widehat{f}_c(\alpha - a) + \widehat{f}_c(\alpha + a)].$$

Convolution theorem: If $F[f(x)] = \widehat{f}(\alpha)$ and $F[g(x)] = \widehat{g}(\alpha)$, then

$$F^{-1}[\widehat{f}(\alpha)\widehat{g}(\alpha)] = f(x) * g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du$$

Parseval's identity:

- If the Fourier transform of $f(x)$ and $g(x)$ are $F(\alpha)$ and $G(\alpha)$, respectively, then
 - (i) $\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha)\bar{G}(\alpha)d\alpha = \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx$
 - (ii) $\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha)\bar{G}(\alpha)d\alpha = \int_{-\infty}^{\infty} |f(x)|^2 dx$
- If Fourier cosine and sine transform of $f(x)$ and $g(x)$ are $F_c(\alpha)$, $G_c(\alpha)$ and $F_s(\alpha)$, $G_s(\alpha)$ respectively, then
 - (i) $\frac{2}{\pi} \int_0^{\infty} F_c(\alpha)G_c(\alpha)d\alpha = \int_0^{\infty} f(x)g(x)dx$
 - (ii) $\frac{2}{\pi} \int_0^{\infty} F_s(\alpha)G_s(\alpha)d\alpha = \int_0^{\infty} f(x)g(x)dx$
 - (iii) $\frac{2}{\pi} \int_0^{\infty} [F_c(\alpha)]^2 d\alpha = \int_0^{\infty} [f(x)]^2 dx$
 - (iv) $\frac{2}{\pi} \int_0^{\infty} [F_s(\alpha)]^2 d\alpha = \int_0^{\infty} [f(x)]^2 dx$

LINEAR ALGEBRA

Basic transformation matrices:

- Stretch matrix/dilation matrix: $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, where a is real constant.
- Rotation matrix: $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$
- Projection matrix: $\begin{bmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{bmatrix}$
- Reflection matrix: $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

Gram-Schmidt process: If $\{x_1, x_2, \dots, x_p\}$ is a basis for a subspace W of vector space R^n , then corresponding orthogonal basis $\{v_1, v_2, \dots, v_p\}$ for W , where

$$v_1 = x_1 \text{ and } v_i = x_i - \left(\frac{x_i \cdot v_1}{v_1 \cdot v_1} \right) v_1 - \left(\frac{x_i \cdot v_2}{v_2 \cdot v_2} \right) v_2 - \dots - \left(\frac{x_i \cdot v_{i-1}}{v_{i-1} \cdot v_{i-1}} \right) v_{i-1} \text{ for } i = 2, 3, 4, \dots, p.$$

STATISTICS

Moments for ungrouped data:

- The r^{th} moment about origin: $\mu'_r = \frac{1}{n} \sum_{i=1}^n x_i^r$, where $r = 1, 2, 3, \dots$, x_1, x_2, \dots, x_n are n observations
- The r^{th} central moment: $\mu_r = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^r$, where $r = 1, 2, 3, \dots$ and \bar{x} is mean

Moments for grouped data:



- The r^{th} moment about origin: $\mu'_r = \frac{1}{N} \sum_{i=1}^n f_i x_i^r$, $r = 1, 2, 3, \dots$, where observations x_1, x_2, \dots, x_n are the mid points of the class-intervals and f_1, f_2, \dots, f_n are their corresponding frequencies and $N = \sum_{i=1}^n f_i$
- The r^{th} central moment: $\mu_r = \frac{1}{N} \sum_{i=1}^n f_i (x_i - \bar{x})^r$, $r = 1, 2, 3 \dots$ & \bar{x} is mean
- The r^{th} moment about any point A: $\mu'_r = \frac{1}{N} \sum_{i=1}^n f_i (x_i - A)^r$, $r = 1, 2, 3 \dots$

Relation between raw moments (about origin or any point) and central moments:

- $\mu_r = \mu'_r - r C_1 \mu'_{r-1} \mu'_1 + r C_2 \mu'_{r-2} \mu'^2_1 + \dots + (-1)^r \mu'^r_1$, $r = 1, 2, 3 \dots$
- $\mu'_r = \mu_r + r C_1 \mu'_{r-1} \mu'_1 + r C_2 \mu'_{r-2} \mu'^2_1 + \dots + \mu'^r_1$

Measures of kurtosis: $\beta_2 = \frac{\mu_4}{\mu_2^2}$

Measures of Skewness (Karl Pearson's coefficient): $S_k = \frac{\sqrt{\beta_1} (\beta_2 + 3)}{2(5\beta_2 - 6\beta_1 - 9)}$, where $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$

Fitting of a straight line $y = a + bx$: The normal equations for estimating the values of a and b are

$$\sum y = na + b \sum x, \quad \sum xy = a \sum x + b \sum x^2$$

Fitting of a second-degree equation (quadratic) $y = a + bx + cx^2$: The normal equations for estimating the values of a, b, c are

$$\begin{aligned} \sum y &= na + b \sum x + c \sum x^2, \\ \sum xy &= a \sum x + b \sum x^2 + c \sum x^3, \\ \sum x^2y &= a \sum x^2 + b \sum x^3 + c \sum x^4. \end{aligned}$$

Correlation coefficient (Karl Pearson correlation coefficient):

- $r = \frac{\sum(x-\bar{x})(y-\bar{y})}{n\sigma_x\sigma_y}$, where $\sigma_x^2 = \frac{\sum(x-\bar{x})^2}{n}$ variance of the x series, $\sigma_y^2 = \frac{\sum(y-\bar{y})^2}{n}$ variance of the y series,
- $\bar{x} = \frac{\sum x}{n}$ → Mean of the x series $\bar{y} = \frac{\sum(y-\bar{y})^2}{n}$ → mean of the y series.
- $r = \frac{n \sum xy - (\sum x)(\sum y)}{\sqrt{n \sum x^2 - (\sum x)^2} \sqrt{n \sum y^2 - (\sum y)^2}}$.

Rank correlation coefficient r_s (Spearman's rank correlation coefficient):

- If x_1, x_2, \dots, x_n and y_1, y_2, \dots, y_n be the ranks of n individuals in characteristics A and B respectively, then $r_s = 1 - \frac{6 \sum_{i=1}^n d_i^2}{n(n^2-1)}$, where d_i is difference between ranks assigned in characteristics A and B. and n is number of pairs of data.
- Rank correlation coefficient for tied ranks: $r_s = 1 - \frac{6[\sum_{i=1}^n d_i^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \dots]}{n(n^2-1)}$, where m_1, m_2, \dots are number of repetitions of ranks.

Linear regression:

- Regression line of y on x : $y - \bar{y} = b_{yx}(x - \bar{x})$, where

$$b_{yx} = r \frac{\sigma_y}{\sigma_x} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2} = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2}$$
- Regression line of x on y : $x - \bar{x} = b_{xy}(y - \bar{y})$, where

$$b_{xy} = r \frac{\sigma_x}{\sigma_y} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(y-\bar{y})^2} = \frac{n \sum xy - \sum x \sum y}{n \sum y^2 - (\sum y)^2}$$



- Angle between two lines of regression: $\tan \theta = \frac{1-r^2}{r} \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$.

Multiple linear regression: Fitting of a multiple linear regression model: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_k x_k$ for n sets of data (y_i, x_{ij}) , where $i = 1, \dots, n$ and $j = 1, \dots, k$. The normal equations for estimating the values of $\beta_0, \beta_1, \beta_2, \dots, \beta_k$ are

$$\begin{aligned} n\beta_0 + \beta_1 \sum_{i=1}^n x_{i1} + \beta_2 \sum_{i=1}^n x_{i2} + \dots + \beta_k \sum_{i=1}^n x_{ik} &= \sum_{i=1}^n y_i, \\ \beta_0 \sum_{i=1}^n x_{i1} + \beta_1 \sum_{i=1}^n x_{i1}^2 + \beta_2 \sum_{i=1}^n x_{i1} x_{i2} + \dots + \beta_k \sum_{i=1}^n x_{i1} x_{ik} &= \sum_{i=1}^n x_{i1} y_i, \\ &\vdots \\ \beta_0 \sum_{i=1}^n x_{ik} + \beta_1 \sum_{i=1}^n x_{ik} x_{i1} + \beta_2 \sum_{i=1}^n x_{ik} x_{i2} + \dots + \beta_k \sum_{i=1}^n x_{ik}^2 &= \sum_{i=1}^n x_{ik} y_i. \end{aligned}$$

Multivariate data (Sample):

Dimensions	Sample	Sample Mean	Sample variance (unbiased) / Sample Covariance
1-dimensional	x_1, \dots, x_n	$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$	$s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$
p-dimensional	<p>p-random variable vector $X = \begin{bmatrix} X_1 \\ \vdots \\ X_p \end{bmatrix}$ where $x_1, x_2, \dots, x_n \in R^p$</p> <p>n-dimensional data matrix $X = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1p} \\ x_{21} & x_{22} & \dots & x_{2p} \\ \vdots & & & \\ x_{n1} & x_{n2} & \dots & x_{np} \end{bmatrix}_{n \times p}$</p>	<p>Sample Mean: $\bar{x}_j = \frac{1}{n} \sum_{i=1}^n x_{ij}, j = 1, 2, \dots, p.$ and Sample Mean Vector: $\bar{x} = \begin{bmatrix} \frac{1}{n} \sum_{i=1}^n x_{i1} \\ \vdots \\ \frac{1}{n} \sum_{i=1}^n x_{ip} \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_p \end{bmatrix}$</p>	<p>Sample Variance: $s_{jj} = s_j^2 = \frac{1}{n-1} \sum_{i=1}^n (x_{ij} - \bar{x}_j)^2, j = 1, 2, \dots, p$ and Sample Covariance: $s_{jk} = s_{kj} = \frac{1}{n-1} \sum_{i=1}^n (x_{ij} - \bar{x}_j)(x_{ik} - \bar{x}_k), 1 \leq k, j \leq p \text{ and } j \neq k.$</p> <p>Sample Covariance Matrix: $S = \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1p} \\ s_{21} & s_{22} & \dots & s_{2p} \\ \vdots & & & \\ s_{p1} & s_{p2} & \dots & s_{pp} \end{bmatrix}_{p \times p}$ $= \frac{1}{n-1} \sum_{i=1}^n \begin{bmatrix} (x_{i1} - \bar{x}_1)^2 & \dots & (x_{i1} - \bar{x}_1)(x_{ip} - \bar{x}_p) \\ \vdots & & \vdots \\ (x_{ip} - \bar{x}_p)(x_{i1} - \bar{x}_1) & \dots & (x_{ip} - \bar{x}_p)^2 \end{bmatrix}_{p \times p}$</p>

Multivariate data (Population):



Dimensions	Population	Population Mean	Population variance / Population Covariance
1-dimensional	$x_1 \dots x_n$	$\mu = \sum_{i=1}^n x_i$	$\sigma^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \mu)^2$
p-dimensional	<p>p-random variable vector</p> $X = \begin{bmatrix} X_1 \\ \vdots \\ X_p \end{bmatrix}$ <p>where $x_1 \dots x_n \in R^p$</p> <p>n-dimensional data matrix</p> $X = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1p} \\ x_{21} & x_{22} & \dots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{np} \end{bmatrix}_{n \times p}$ <p>Here each column in the data matrix corresponds to a random variable X_j.</p>	<p>Population Mean:</p> $\mu_j = \frac{1}{n} \sum_{i=1}^n x_{ij}, \quad j = 1, 2, \dots, p.$ <p>and</p> <p>Population Mean Vector:</p> $\mu = \begin{bmatrix} \frac{1}{n} \sum_{i=1}^n x_{i1} \\ \vdots \\ \frac{1}{n} \sum_{i=1}^n x_{ip} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_p \end{bmatrix}$	<p>Population Variance:</p> $\sigma_{jj} = \sigma_j^2 = \frac{1}{n} \sum_{i=1}^n (x_{ij} - \mu_j)^2, \quad j = 1, 2, \dots, p \text{ and } i = j.$ <p>and</p> <p>Population Covariance:</p> $\sigma_{jk} = \sigma_{kj} = \frac{1}{n} \sum_{i=1}^n (x_{ij} - \mu_j)(x_{ik} - \mu_k), \quad 1 \leq k, j \leq p \text{ and } j \neq k.$ <p>Population Covariance Matrix:</p> $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \dots & \sigma_{pp} \end{bmatrix}_{p \times p}$ $= \frac{1}{n} \sum_{i=1}^n \begin{bmatrix} (x_{i1} - \mu_1)^2 & \dots & (x_{i1} - \mu_1)(x_{ip} - \mu_p) \\ \vdots & \ddots & \vdots \\ (x_{ip} - \mu_p)(x_{i1} - \mu_1) & \dots & (x_{ip} - \mu_p)^2 \end{bmatrix}_{p \times p}$

Rank Correlation, Partial Correlation, Multiple Correlation



In the case of tri variate data X_1, X_2 and X_3

i) Coefficient of partial correlation of X_1 and X_2 , keeping X_3 constant:

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{1 - r_{13}^2}\sqrt{1 - r_{23}^2}}$$

ii) Coefficient of partial correlation of X_1 and X_3 , keeping X_2 constant:

$$r_{13.2} = \frac{r_{13} - r_{12}r_{23}}{\sqrt{1 - r_{12}^2}\sqrt{1 - r_{23}^2}}$$

iii) Coefficient of partial correlation of X_2 and X_3 , keeping X_1 constant:

$$r_{23.1} = \frac{r_{23} - r_{12}r_{13}}{\sqrt{1 - r_{12}^2}\sqrt{1 - r_{13}^2}}$$

Note: In all the above three cases $r_{ij} = r_{ji}$ for all $i \neq j$.

In the case of tri variate data: X_1, X_2 and X_3

i. Multiple correlation coefficient of X_1 on X_2 and X_3 : $R_{1.23} = \sqrt{\frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2}}$

ii. Multiple correlation coefficient of X_2 on X_1 and X_3 : $R_{2.13} = \sqrt{\frac{r_{21}^2 + r_{23}^2 - 2r_{21}r_{23}r_{13}}{1 - r_{13}^2}}$

iii. Multiple correlation coefficient of X_3 on X_1 and X_2 : $R_{3.12} = \sqrt{\frac{r_{31}^2 + r_{32}^2 - 2r_{31}r_{32}r_{12}}{1 - r_{12}^2}}$

Given variables x, y and z , we define the multiple correlation coefficient.

$$R_{z.xy} = \sqrt{\frac{r_{zx}^2 + r_{zy}^2 - 2r_{zx}r_{zy}r_{xy}}{1 - r_{yx}^2}} \text{ and } r_{xy} = \frac{n\sum xy - \sum x\sum y}{\sqrt{\{n(\sum x^2) - (\sum x)^2\}\{n(\sum y^2) - (\sum y)^2\}}}$$

Note: In all the above three cases $r_{ij} = r_{ji}$ for all $i \neq j$.

Analysis of Variance (ANOVA)

One-way ANOVA Table

Sources of variation	Degrees of freedom	Sum of squares	Mean sum of squares	F_{cal}
Between classes	$k - 1$	SSB	MSSB	$F_{cal} = \frac{MSSB}{MSSW}$
Within classes	$N - k$	SSW	MSSW	
Total	$N - 1$	SST		

k : number of classes, $N = pq$ is number of rows \times number pf columns, n : number of entries in each X_i

T : Grand sum: $\sum X_i$, Correction factor (C.F.) = $\frac{T^2}{N}$

SSB: Sum of squares between classes = $\frac{(\sum X_i)^2}{n_i} - C.F.$

SST: Total sum of squares = $\sum X_i^2 - C.F.$



SSW: Sum of squares within classes = $SST - SSB$

MSSB: Mean sum of squares between classes = $\frac{SSB}{k-1}$

MSSW: Mean sum of squares within classes = $\frac{SSW}{N-k}$

Conclusion:

If $F_{cal} < F_{tab}$, there is NO significant difference, Null Hypothesis H_0 is accepted.

If $F_{cal} > F_{tab}$, there is a significant difference, Null Hypothesis H_0 is rejected and Alternative Hypothesis, H_1 is accepted.

Note: In one-way ANOVA, F_{cal} can also be computed using columns.

Two-way ANOVA table

Source of variation	Degrees of freedom	Sum of squares	Mean sum of squares	F_{cal}
Between rows	$r - 1$	SSR	MSSR	$F_{cal}(\text{rows}) = \frac{MSSR}{MSSE}$
Between columns	$c - 1$	SSC	MSSC	$F_{cal}(\text{columns}) = \frac{MSSC}{MSSE}$
Error (residual)	$(r - 1)(c - 1)$	SSE	MSSE	
Total	$rc - 1$	SST		

Number of levels of row factor = r , Number of levels of column factor = c

Total number of observations = rc = (rows \times columns)

Observations in ij^{th} cell of table = x_{ij} , Sum of c observations in i^{th} row = $T_{Ri} = \sum_j x_{ij}$

Sum of r observations in j^{th} column = $T_{Cj} = \sum_i x_{ij}$

Sum of all observations = $T = \sum_i T_{Ri} = \sum_j T_{Cj}$

Correction factor = $C.F. = \frac{T^2}{rc}$

Sum of squares between rows: $SSR = \sum \frac{T_{Ri}^2}{c} - C.F.$

Sum of squares between columns: $SSC = \sum \frac{T_{Cj}^2}{r} - C.F.$

Total sum of squares: $SST = \sum_i \sum_j x_{ij}^2 - C.F.$

Error (residual) sum of squares: $SSE = SST - SSR - SSC$

$MSSR$ = mean sum of squares between rows = $\frac{SSR}{r-1}$

$MSSC$ = mean sum of squares between columns = $\frac{SSC}{c-1}$



$$\text{MSSE} = \text{mean sum of squares of error} = \frac{\text{SSE}}{(r-1)(c-1)}$$

Conclusion:

If $F_{cal} < F_{\text{tab}}$, there is no significant difference, null hypothesis H_0 is accepted.

If $F_{cal} > F_{\text{tab}}$, there is a significant difference, null hypothesis H_0 is rejected and alternative hypothesis, H_1 is accepted.

PROBABILITY AND RANDOM VARIABLES

Probability of an event: If a trial results in n exhaustive, mutually exclusive and equally likely cases and m of them are favourable to the happening of an event E then the probability 'p' of happening of E is given by $p = P(E) = \frac{m}{n}$

Addition theorem:

For two events: If A and B are any two events with respective probabilities $P(A)$ and $P(B)$, then the probability of occurrence of at least one of the events is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

For three events: If A, B and C are any three events with respective probabilities $P(A), P(B)$ and $P(C)$, then the probability of occurrence of at least one of the events is given by

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Conditional probability: Conditional probability of B , given A , denoted by $P(B|A)$, is defined by $P(A|B) = \frac{P(A \cap B)}{P(A)}$, provided $P(A) > 0$.

Multiplication rule: Suppose A and B are events in a sample space S with $P(A) > 0$ multiplication rule is: $P(A \cap B) = P(A) P(B|A)$

Bayes' theorem (rule): If B_1, B_2, \dots, B_n are mutually disjoint events with $P(B_i) \neq 0$ ($i = 1, 2, \dots, n$) then for any arbitrary event A which is a subset of $\bigcup_{i=1}^n B_i$ such that $P(A) > 0$, then $P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{i=1}^n P(B_i)P(A|B_i)}$.

Discrete random variable: Let X be a discrete random variable. A function $p(x)$ is a probability mass function of the discrete random variable X if $p(x) \geq 0, \forall x \in X$ and $\sum_x p(x) = 1$.

- Expectation, $E(X) = \sum_x x p(x)$
- If $Y = g(X)$, then $E(Y) = \sum_x g(x)p(x)$
- Variance, $Var(X) = E[(X - E(X))^2] = E(X^2) - [E(X)]^2$
- Standard deviation, $\sigma_X = \sqrt{Var(X)}$
- The cumulative distribution function, $F(t) = P(X \leq t) = \sum_{x \leq t} p(x)$



Continuous random variable: Suppose X is a continuous random variable. A function $f(x)$ is called a probability density function of the continuous random variable X if $f(x) \geq 0 \forall x \in X$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

- Expectation, $E(X) = \int_{-\infty}^{\infty} x f(x)dx$
- If $Y = g(X)$, then $E(Y) = \int_{-\infty}^{\infty} g(x) f(x)dx$
- Variance, $Var(X) = E[(X - E(X))^2] = E(X^2) - [E(X)]^2$
- Standard deviation, $\sigma_X = \sqrt{Var(X)}$
- Cumulative distribution function, $F(t) = P(X \leq t) = \int_{-\infty}^t f(x)dx$
- $P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = P(a < X < b) = \int_a^b f(x)dx$

Markov's inequality: For any random variable X with finite $E[X]$ and any $k > 0$, the probability that X is at least k times its expected value is at-most $\frac{1}{k}$. That is, $P[X \geq kE[X]] \leq \frac{1}{k}$ or $P[X \geq k] \leq \frac{E[X]}{k}$.

Chebyshev's inequality: Any random variable X with expectation $\mu = E[X]$ and variance $\sigma^2 = Var[X]$ belongs to the interval $\mu \pm k = [\mu - k, \mu + k]$ with probability of at least $1 - \left(\frac{\sigma}{k}\right)^2$. That is, $P\{|X - \mu| \geq k\} \leq \left(\frac{\sigma}{k}\right)^2 = \frac{\sigma^2}{k^2}$

Joint probability mass function: Suppose X and Y are two discrete random variables. A function $p(x, y)$ is called a joint probability mass function of X and Y if $p(x, y) \geq 0, \forall x \in X, y \in Y$ and $\sum_x \sum_y p(x, y) = 1$.

- Let $Z = g(X, Y)$. Expectation, $E[Z] = \sum_x \sum_y g(x, y)p(x, y)$
- The marginal distributions of X alone and of Y alone are: $g(x) = \sum_y p(x, y)$ and $h(y) = \sum_x p(x, y)$
- Covariance, $Cov(X, Y) = E(XY) - E(X)E(Y)$
- Correlation of X and Y , $\rho(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$
- For discrete random variables X and Y with joint pmf $p(x, y)$ and x, y such that $g(x) > 0, h(y) > 0$, then the conditional probability mass functions are
 - (i) $P(X = x | Y = y) = \frac{p(x, y)}{h(y)}$, (ii) $P(Y = y | X = x) = \frac{p(x, y)}{g(x)}$.
- If X and Y are independent, then $E(XY) = E(X)E(Y)$

Joint probability density function: Suppose X and Y are two continuous random variables. A function $f(x, y)$ is called a joint probability density function of X and Y if $f(x, y) \geq 0, \forall x \in X, y \in Y$ and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$.

- Let $Z = g(X, Y)$. Expectation, $E[Z] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)f(x, y) dx dy$
- The marginal distributions of X alone and of Y alone are $g(x) = \int_{-\infty}^{\infty} f(x, y) dy$ and $h(y) = \int_{-\infty}^{\infty} f(x, y) dx$
- Covariance, $Cov(X, Y) = E(XY) - E(X)E(Y)$
- Correlation of X and Y , $\rho(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$
- If X and Y are independent, then $E(XY) = E(X)E(Y)$



- The conditional distribution of the random variable Y given that $X = x$ is $f(y|x) = \frac{f(x,y)}{g(x)}$, provided $g(x) > 0$
- The conditional distribution of the random variable X given that $Y = y$ is $f(x|y) = \frac{f(x,y)}{h(y)}$, provided $h(y) > 0$.

PROBABILITY DISTRIBUTIONS

Bernoulli distribution:

- The probability mass function is given by $f(x, p) = p^x(1 - p)^{1-x}$; $x \in (0,1)$
- Mean, $\mu = p$
- Variance, $\sigma^2 = pq$
- Standard deviation, $\sigma = \sqrt{pq}$

Binomial distribution:

- The probability function of the binomial distribution is given by $b(x; n, p) = n_{C_x} p^x q^{n-x}$, where $x = 1, 2, 3, \dots$, p is the probability of success and $q = 1 - p$ is the probability of failure.
- Mean, $\mu = np$
- Variance, $\sigma^2 = npq$
- Standard deviation, $\sigma = \sqrt{npq}$

Poisson distribution:

- The probability function of the Poisson distribution is given by $p(x; \lambda) = P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$, where λ is the parameter of the Poisson distribution.
- Mean, $\mu = \lambda$
- Variance, $\sigma^2 = \lambda$
- Standard deviation, $\sigma = \sqrt{\lambda}$

Geometric distribution:

- Probability mass function is given by $P(X = x) = (1 - p)^{x-1} p$, where $0 < p < 1$
- Mean, $\mu = 1/p$
- Variance, $\sigma^2 = \frac{1-p}{p^2}$
- Standard deviation $\sigma = \frac{\sqrt{1-p}}{p}$

Exponential distribution: A continuous random variable X assuming non-negative values is said to have an exponential distribution with parameter $\lambda > 0$, if its probability density function is given by

$$f(x; \lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

- Mean $\mu = \frac{1}{\lambda}$



- Variance $\sigma^2 = \left(\frac{1}{\lambda}\right)^2$
- Standard deviation $\sigma = \frac{1}{\lambda}$

Uniform distribution: The probability density function is $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$

- Mean $\mu = \frac{a+b}{2}$
- Variance $\sigma^2 = \frac{(b-a)^2}{12}$
- Standard deviation $\sigma = \sqrt{\frac{(b-a)^2}{12}}$

Normal distribution: A random variable X is said to have a normal distribution with parameters μ (called "mean") and σ^2 (called "variance") if its density function is given by the probability law:

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{x-\mu}{\sigma}\right)^2\right] \text{ for } -\infty < x < \infty, -\infty < \mu < \infty \text{ and } 0 < \sigma < \infty.$$

- Standard normal distribution or z – distribution is given by

$$f(z; 0, 1) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right], \quad -\infty < z < \infty,$$

- Cumulative standard normal distribution is

$$\phi(z) = P(Z \leq z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du.$$

Note: The value of the integral can be calculated by the Normal distribution table.

SAMPLING THEORY

Numerical summation of data: If the n observations in a sample are denoted by x_1, x_2, \dots, x_n , then

- Sample mean: $\bar{x} = \frac{\sum_{i=1}^n x_i}{n}$
- Sample variance: $s^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}$
- Standard deviation is the positive square root of the sample variance.
- If the population is finite with size N , then for the sampling distribution of \bar{x} :

$$\text{Mean: } \mu_{\bar{x}} = \mu$$

$$\text{Variance: } \sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} \times \frac{N-n}{N-1}, \text{ if the sampling is without replacement}$$

$$\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n}, \text{ if the sampling is with replacement or population is infinite.}$$

Sampling distributions: Let X be a random variable of a population with mean μ and variance σ^2 . If the random variables X_1, X_2, \dots, X_n are a random sample on size n , then sample mean $\bar{X} = \frac{x_1+x_2+\dots+x_n}{n}$

Sampling distribution of means

- Mean: $\mu_{\bar{x}} = \mu$
- Variance: $\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n}$



- If the population is normal or sample size is sufficiently large, then the distribution of $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ is approximately standard normal.

Sampling distribution of a difference in sample means

- Mean: $\mu_{(\bar{X}_1 - \bar{X}_2)} = \mu_{\bar{X}_1} - \mu_{\bar{X}_2} = \mu_1 - \mu_2$
- Variance: $\sigma_{(\bar{X}_1 - \bar{X}_2)}^2 = \sigma_{\bar{X}_1}^2 + \sigma_{\bar{X}_2}^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$
- If the populations are normal or sample sizes are sufficiently large, then the distribution of $Z = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$ is approximately standard normal.

Sampling distribution of proportions:

- Mean $\mu_{\hat{p}} = p$
- Standard deviation $\sigma_{\hat{p}} = \sqrt{\frac{pq}{n}} = \left[\frac{p(1-p)}{n} \right]^{\frac{1}{2}}$
- The distribution of $Z = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}}$ is approximately standard normal if n is large or $np \geq 5$ and $np(1 - p) \geq 5$.

Statistical decisions:

- Levels of significance for two-tailed test

Level of significance	Critical value	Acceptance region
0.05	$z_c = 1.96$	(-1.96, 1.96)
0.01	$z_c = 2.58$	(-2.58, 2.58)

- Levels of significance for one-tailed test

Level of significance	Critical value		Critical region	
	Right-tailed test	Left-tailed test	Right-tailed test	Left-tailed test
0.05	$z_c = 1.645$	$-z_c = -1.645$	$(1.645, \infty)$	$(-\infty, -1.645)$
0.01	$z_c = 2.33$	$-z_c = -2.33$	$(2.33, \infty)$	$(-\infty, -2.33)$

- For the standard normal variate Z , and the test value of Z is z_0 , then the P-value is determined as follows:
 - For left-tail test, the P-value is $P(Z \leq z_0)$
 - For right-tail test, the P-value is $P(Z \geq z_0)$
 - For two tail test, the P-value is $P(Z \leq -|z_0|) + P(Z \geq |z_0|) = 2(1 - \phi(|z_0|))$

Test of significance:

z-test:

- Test statistics:
 - $z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$, if σ is known.



(ii) $z = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$, if σ is unknown and n is large.

t-test:

- Test statistic:

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}, \text{ if } \sigma \text{ is unknown.}$$

Chi – square (χ^2) test:

- If s^2 is the variance of a random sample of size n taken from a normal population having variance σ^2 , then the statistic

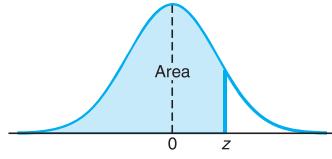
$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \sum_{i=1}^n \frac{(x_i - \bar{x})^2}{\sigma^2}$$

is a chi-squared distribution with $v = n - 1$ degrees of freedom.

- If $f_1, f_2, f_3, \dots, f_n$ are the observed frequencies and $e_1, e_2, e_3, \dots, e_n$ are the expected or theoretical frequencies. The statistic $\chi^2 = \sum_{i=1}^n \frac{(f_i - e_i)^2}{e_i}$

F- test:

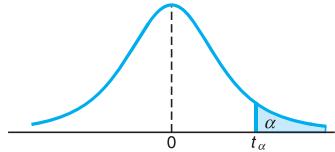
- Test statistic: $F_0 = \frac{s_1^2}{s_2^2}$



Areas under the Normal Curve

<i>z</i>	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

(continued) Areas under the Normal Curve

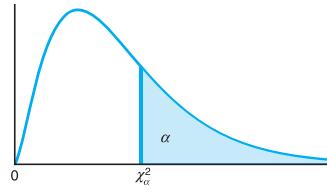


Critical Values of the t -Distribution

v	α						
	0.40	0.30	0.20	0.15	0.10	0.05	0.025
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306
9	0.261	0.543	0.883	1.100	1.383	1.833	2.262
10	0.260	0.542	0.879	1.093	1.372	1.812	2.228
11	0.260	0.540	0.876	1.088	1.363	1.796	2.201
12	0.259	0.539	0.873	1.083	1.356	1.782	2.179
13	0.259	0.538	0.870	1.079	1.350	1.771	2.160
14	0.258	0.537	0.868	1.076	1.345	1.761	2.145
15	0.258	0.536	0.866	1.074	1.341	1.753	2.131
16	0.258	0.535	0.865	1.071	1.337	1.746	2.120
17	0.257	0.534	0.863	1.069	1.333	1.740	2.110
18	0.257	0.534	0.862	1.067	1.330	1.734	2.101
19	0.257	0.533	0.861	1.066	1.328	1.729	2.093
20	0.257	0.533	0.860	1.064	1.325	1.725	2.086
21	0.257	0.532	0.859	1.063	1.323	1.721	2.080
22	0.256	0.532	0.858	1.061	1.321	1.717	2.074
23	0.256	0.532	0.858	1.060	1.319	1.714	2.069
24	0.256	0.531	0.857	1.059	1.318	1.711	2.064
25	0.256	0.531	0.856	1.058	1.316	1.708	2.060
26	0.256	0.531	0.856	1.058	1.315	1.706	2.056
27	0.256	0.531	0.855	1.057	1.314	1.703	2.052
28	0.256	0.530	0.855	1.056	1.313	1.701	2.048
29	0.256	0.530	0.854	1.055	1.311	1.699	2.045
30	0.256	0.530	0.854	1.055	1.310	1.697	2.042
40	0.255	0.529	0.851	1.050	1.303	1.684	2.021
60	0.254	0.527	0.848	1.045	1.296	1.671	2.000
120	0.254	0.526	0.845	1.041	1.289	1.658	1.980
∞	0.253	0.524	0.842	1.036	1.282	1.645	1.960

(continued) Critical Values of the *t*-Distribution

<i>v</i>	α						
	0.02	0.015	0.01	0.0075	0.005	0.0025	0.0005
1	15.894	21.205	31.821	42.433	63.656	127.321	636.578
2	4.849	5.643	6.965	8.073	9.925	14.089	31.600
3	3.482	3.896	4.541	5.047	5.841	7.453	12.924
4	2.999	3.298	3.747	4.088	4.604	5.598	8.610
5	2.757	3.003	3.365	3.634	4.032	4.773	6.869
6	2.612	2.829	3.143	3.372	3.707	4.317	5.959
7	2.517	2.715	2.998	3.203	3.499	4.029	5.408
8	2.449	2.634	2.896	3.085	3.355	3.833	5.041
9	2.398	2.574	2.821	2.998	3.250	3.690	4.781
10	2.359	2.527	2.764	2.932	3.169	3.581	4.587
11	2.328	2.491	2.718	2.879	3.106	3.497	4.437
12	2.303	2.461	2.681	2.836	3.055	3.428	4.318
13	2.282	2.436	2.650	2.801	3.012	3.372	4.221
14	2.264	2.415	2.624	2.771	2.977	3.326	4.140
15	2.249	2.397	2.602	2.746	2.947	3.286	4.073
16	2.235	2.382	2.583	2.724	2.921	3.252	4.015
17	2.224	2.368	2.567	2.706	2.898	3.222	3.965
18	2.214	2.356	2.552	2.689	2.878	3.197	3.922
19	2.205	2.346	2.539	2.674	2.861	3.174	3.883
20	2.197	2.336	2.528	2.661	2.845	3.153	3.850
21	2.189	2.328	2.518	2.649	2.831	3.135	3.819
22	2.183	2.320	2.508	2.639	2.819	3.119	3.792
23	2.177	2.313	2.500	2.629	2.807	3.104	3.768
24	2.172	2.307	2.492	2.620	2.797	3.091	3.745
25	2.167	2.301	2.485	2.612	2.787	3.078	3.725
26	2.162	2.296	2.479	2.605	2.779	3.067	3.707
27	2.158	2.291	2.473	2.598	2.771	3.057	3.689
28	2.154	2.286	2.467	2.592	2.763	3.047	3.674
29	2.150	2.282	2.462	2.586	2.756	3.038	3.660
30	2.147	2.278	2.457	2.581	2.750	3.030	3.646
40	2.123	2.250	2.423	2.542	2.704	2.971	3.551
60	2.099	2.223	2.390	2.504	2.660	2.915	3.460
120	2.076	2.196	2.358	2.468	2.617	2.860	3.373
∞	2.054	2.170	2.326	2.432	2.576	2.807	3.290

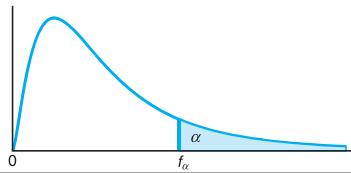


Critical Values of the Chi-Squared Distribution

v	α									
	0.995	0.99	0.98	0.975	0.95	0.90	0.80	0.75	0.70	0.50
1	0.04393	0.03157	0.03628	0.03982	0.00393	0.0158	0.0642	0.102	0.148	0.455
2	0.0100	0.0201	0.0404	0.0506	0.103	0.211	0.446	0.575	0.713	1.386
3	0.0717	0.115	0.185	0.216	0.352	0.584	1.005	1.213	1.424	2.366
4	0.207	0.297	0.429	0.484	0.711	1.064	1.649	1.923	2.195	3.357
5	0.412	0.554	0.752	0.831	1.145	1.610	2.343	2.675	3.000	4.351
6	0.676	0.872	1.134	1.237	1.635	2.204	3.070	3.455	3.828	5.348
7	0.989	1.239	1.564	1.690	2.167	2.833	3.822	4.255	4.671	6.346
8	1.344	1.647	2.032	2.180	2.733	3.490	4.594	5.071	5.527	7.344
9	1.735	2.088	2.532	2.700	3.325	4.168	5.380	5.899	6.393	8.343
10	2.156	2.558	3.059	3.247	3.940	4.865	6.179	6.737	7.267	9.342
11	2.603	3.053	3.609	3.816	4.575	5.578	6.989	7.584	8.148	10.341
12	3.074	3.571	4.178	4.404	5.226	6.304	7.807	8.438	9.034	11.340
13	3.565	4.107	4.765	5.009	5.892	7.041	8.634	9.299	9.926	12.340
14	4.075	4.660	5.368	5.629	6.571	7.790	9.467	10.165	10.821	13.339
15	4.601	5.229	5.985	6.262	7.261	8.547	10.307	11.037	11.721	14.339
16	5.142	5.812	6.614	6.908	7.962	9.312	11.152	11.912	12.624	15.338
17	5.697	6.408	7.255	7.564	8.672	10.085	12.002	12.792	13.531	16.338
18	6.265	7.015	7.906	8.231	9.390	10.865	12.857	13.675	14.440	17.338
19	6.844	7.633	8.567	8.907	10.117	11.651	13.716	14.562	15.352	18.338
20	7.434	8.260	9.237	9.591	10.851	12.443	14.578	15.452	16.266	19.337
21	8.034	8.897	9.915	10.283	11.591	13.240	15.445	16.344	17.182	20.337
22	8.643	9.542	10.600	10.982	12.338	14.041	16.314	17.240	18.101	21.337
23	9.260	10.196	11.293	11.689	13.091	14.848	17.187	18.137	19.021	22.337
24	9.886	10.856	11.992	12.401	13.848	15.659	18.062	19.037	19.943	23.337
25	10.520	11.524	12.697	13.120	14.611	16.473	18.940	19.939	20.867	24.337
26	11.160	12.198	13.409	13.844	15.379	17.292	19.820	20.843	21.792	25.336
27	11.808	12.878	14.125	14.573	16.151	18.114	20.703	21.749	22.719	26.336
28	12.461	13.565	14.847	15.308	16.928	18.939	21.588	22.657	23.647	27.336
29	13.121	14.256	15.574	16.047	17.708	19.768	22.475	23.567	24.577	28.336
30	13.787	14.953	16.306	16.791	18.493	20.599	23.364	24.478	25.508	29.336
40	20.707	22.164	23.838	24.433	26.509	29.051	32.345	33.66	34.872	39.335
50	27.991	29.707	31.664	32.357	34.764	37.689	41.449	42.942	44.313	49.335
60	35.534	37.485	39.699	40.482	43.188	46.459	50.641	52.294	53.809	59.335

(continued) Critical Values of the Chi-Squared Distribution

v	α									
	0.30	0.25	0.20	0.10	0.05	0.025	0.02	0.01	0.005	0.001
1	1.074	1.323	1.642	2.706	3.841	5.024	5.412	6.635	7.879	10.827
2	2.408	2.773	3.219	4.605	5.991	7.378	7.824	9.210	10.597	13.815
3	3.665	4.108	4.642	6.251	7.815	9.348	9.837	11.345	12.838	16.266
4	4.878	5.385	5.989	7.779	9.488	11.143	11.668	13.277	14.860	18.466
5	6.064	6.626	7.289	9.236	11.070	12.832	13.388	15.086	16.750	20.515
6	7.231	7.841	8.558	10.645	12.592	14.449	15.033	16.812	18.548	22.457
7	8.383	9.037	9.803	12.017	14.067	16.013	16.622	18.475	20.278	24.321
8	9.524	10.219	11.030	13.362	15.507	17.535	18.168	20.090	21.955	26.124
9	10.656	11.389	12.242	14.684	16.919	19.023	19.679	21.666	23.589	27.877
10	11.781	12.549	13.442	15.987	18.307	20.483	21.161	23.209	25.188	29.588
11	12.899	13.701	14.631	17.275	19.675	21.920	22.618	24.725	26.757	31.264
12	14.011	14.845	15.812	18.549	21.026	23.337	24.054	26.217	28.300	32.909
13	15.119	15.984	16.985	19.812	22.362	24.736	25.471	27.688	29.819	34.527
14	16.222	17.117	18.151	21.064	23.685	26.119	26.873	29.141	31.319	36.124
15	17.322	18.245	19.311	22.307	24.996	27.488	28.259	30.578	32.801	37.698
16	18.418	19.369	20.465	23.542	26.296	28.845	29.633	32.000	34.267	39.252
17	19.511	20.489	21.615	24.769	27.587	30.191	30.995	33.409	35.718	40.791
18	20.601	21.605	22.760	25.989	28.869	31.526	32.346	34.805	37.156	42.312
19	21.689	22.718	23.900	27.204	30.144	32.852	33.687	36.191	38.582	43.819
20	22.775	23.828	25.038	28.412	31.410	34.170	35.020	37.566	39.997	45.314
21	23.858	24.935	26.171	29.615	32.671	35.479	36.343	38.932	41.401	46.796
22	24.939	26.039	27.301	30.813	33.924	36.781	37.659	40.289	42.796	48.268
23	26.018	27.141	28.429	32.007	35.172	38.076	38.968	41.638	44.181	49.728
24	27.096	28.241	29.553	33.196	36.415	39.364	40.270	42.980	45.558	51.179
25	28.172	29.339	30.675	34.382	37.652	40.646	41.566	44.314	46.928	52.619
26	29.246	30.435	31.795	35.563	38.885	41.923	42.856	45.642	48.290	54.051
27	30.319	31.528	32.912	36.741	40.113	43.195	44.140	46.963	49.645	55.475
28	31.391	32.620	34.027	37.916	41.337	44.461	45.419	48.278	50.994	56.892
29	32.461	33.711	35.139	39.087	42.557	45.722	46.693	49.588	52.335	58.301
30	33.530	34.800	36.250	40.256	43.773	46.979	47.962	50.892	53.672	59.702
40	44.165	45.616	47.269	51.805	55.758	59.342	60.436	63.691	66.766	73.403
50	54.723	56.334	58.164	63.167	67.505	71.420	72.613	76.154	79.490	86.660
60	65.226	66.981	68.972	74.397	79.082	83.298	84.58	88.379	91.952	99.608



Critical Values of the *F*-Distribution

v_2	$f_{0.05}(v_1, v_2)$								
	1	2	3	4	5	6	7	8	9
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88

(continued) Critical Values of the *F*-Distribution

v_2	$f_{0.05}(v_1, v_2)$									
	v_1									
10	12	15	20	24	30	40	60	120	∞	
1	241.88	243.91	245.95	248.01	249.05	250.10	251.14	252.20	253.25	254.31
2	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

(continued) Critical Values of the *F*-Distribution

		$f_{0.01}(v_1, v_2)$								
		v_1								
v_2		1	2	3	4	5	6	7	8	9
1	4052.18	4999.50	5403.35	5624.58	5763.65	5858.99	5928.36	5981.07	6022.47	
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	

(continued) Critical Values of the *F*-Distribution

v_2	$f_{0.01}(v_1, v_2)$									
	v_1									
10	12	15	20	24	30	40	60	120	∞	
1	6055.85	6106.32	6157.28	6208.73	6234.63	6260.65	6286.78	6313.03	6339.39	6365.86
2	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48	99.49	99.50
3	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32	26.22	26.13
4	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46
5	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
8	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
15	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
16	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
17	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
18	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
20	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
21	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36	2.27	2.17
26	3.09	2.96	2.81	2.66	2.58	2.50	2.42	2.33	2.23	2.13
27	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29	2.20	2.10
28	3.03	2.90	2.75	2.60	2.52	2.44	2.35	2.26	2.17	2.06
29	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23	2.14	2.03
30	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
60	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
∞	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00