DEPARTMENT OF MATHEMATICS
LINEAR ALGEBRA AND PROBABILITY THEORY (MA231TC)

UNIT 2: LINEAR ALGEBRA – II

1. Check whether the following functions defines an inner product on $\mathbb{R}^2$, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$.

   a) $\langle u, v\rangle = 3u_1v_1 + u_2v_2$
   b) $\langle u, v\rangle = 2u_1v_2 + u_2v_1 + u_1v_2 + 2u_2v_2$
   c) $\langle u, v\rangle = 3u_1v_2 - u_2v_1$
   [Ans: a) and b) are inner product and c) is not]

2. Show that the function defines an inner product on $\mathbb{R}^3$, where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$.

   a) $\langle u, v\rangle = 3u_1v_1 + 3u_2v_2 + u_3v_3$
   b) $\langle u, v\rangle = u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_2^2$
   [Ans: a) is inner product and b) is not]

3. Find $\langle u, v\rangle$, $\|u\|$, $\|v\|$ and distance between u and v, $\|u - v\|$ for the given inner product defined on $\mathbb{R}^n$.

   a) $u = (1,1,1)$ and $v = (2,5,2)$, $\langle u, v\rangle = u_1v_1 + 2u_2v_2 + u_3v_3$
   b) $u = (-1,2,0,1)$ and $v = (0,1,2,2)$, $\langle u, v\rangle = u \cdot v$

4. Find $\langle A, B\rangle$, $\|A\|$, $\|B\|$ and distance between A and B, $\|A - B\|$ for the matrices in $M_{2\times 2}$ using the inner product $\langle A, B\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$.

   a) $A = \begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$
   b) $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$

5. Use the functions f and g in $C[-1, 1]$ to find $\langle f, g\rangle$, $\|f\|$, $\|g\|$, and distance between f and g, $\|f - g\|$.

   a) $f(x) = \sin x,\ g(x) = \cos x,\quad \langle f, g\rangle = \int_0^{\frac{\pi}{4}} f(x)g(x)dx$
   b) $f(x) = x,\ g(x) = e^x,\quad \langle f, g\rangle = \int_0^1 f(x)g(x)dx$

6. Show that f and g are orthogonal in the inner product space $C[a, b]$ with the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$.

   a) $C[-1,1],\ f(x) = x,\ g(x) = \frac{1}{2}(3x^2 - 1)$
   b) $C\left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ f(x) = \cos x,\ g(x) = \sin x$

7. If $y = (3, 4)$ and $u = (1, 2)$, obtain the orthogonal projection of y onto u.

8. Find the coordinate matrix of w relative to the orthonormal basis B in $\mathbb{R}^n$.

a) $w = (2, -2, 1),\ B = \left\{\left(\frac{\sqrt{10}}{10}, 0, \frac{3\sqrt{10}}{10}\right), (0,1,0), \left(-\frac{3\sqrt{10}}{10}, 0, \frac{\sqrt{10}}{10}\right)\right\}$

b) $w = (5, 10, 15),\ B = \left\{\left(\frac{3}{5}, \frac{4}{5}, 0\right), \left(-\frac{4}{5}, \frac{3}{5}, 0\right), (0,0,1)\right\}$

c) $w = (2, -1, 4, 3),\ B = \left\{\left(\frac{5}{13}, 0, \frac{12}{13}, 0\right), (0,1,0,0), \left(\frac{12}{13}, 0, \frac{5}{13}, 0\right), (0,0,0,1)\right\}$

9. Find the projection of the vector v onto the subspace S. Also find the projection matrix onto S.

a) $S = span\left\{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\},\ v = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

b) $S = span\left\{\begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\},\ v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

c) $S = span\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\},\ v = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

10. Let

$$u_1 = \begin{bmatrix} 1/3\sqrt{2} \\ 1/3\sqrt{2} \\ -4/3\sqrt{2} \end{bmatrix}, \qquad u_2 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \qquad u_3 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

a) Show that $\{u_1, u_2, u_3\}$ is an orthonormal basis for $\mathbb{R}^3$

b) Find the projection matrix onto span$\{u_2, u_3\}$

c) Let $v = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Find the projection of v onto span$\{u_2, u_3\}$ and projection of v onto span$\{u_1, u_2, u_3\}$

[Hint : b) If $Q = [u_2\ u_3]$, then projection matrix is QQ^T, Ans : c) $\begin{pmatrix} \frac{23}{18} & \frac{41}{18} & \frac{8}{9} \end{pmatrix}^T$ and v itself.

11. Without finding the characteristic equation, verify whether $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is an eigenvector of the matrix $A = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$. If yes, then find the corresponding eigenvalue.

12. Given $A = \begin{bmatrix} 2 & 1 & 5 \\ -2 & -3 & -2 \\ 3 & 3 & 1 \end{bmatrix}$. Decompose the matrix A as $A = QR$, using the Gram-Schmidt process.

13. Factorize the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ as $A = PDP^T$.

14. Using the Gram-Schmidt process, orthonormalize the columns of the matrix

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

15. Obtain the Singular Value Decomposition of $A = \begin{bmatrix} 5 & 7 & 0 \\ 5 & 1 & 0 \end{bmatrix}$.

16. Obtain the third row of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ - & - & - \end{bmatrix}$, such that the rows are orthogonal.

17. Choose the second row of $A = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$ so that A has the eigenvalues 4 and 7.

18. Obtain the matrix P which diagonalizes the matrix $A = \begin{bmatrix} 7 & -4 & -2 \\ -4 & 1 & -4 \\ -2 & -4 & 7 \end{bmatrix}$.

19. Obtain the QR factorisation of the matrix A, by applying Gram-Schmidt process, where

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 1 \\ -2 & 0 & 4 \\ 1 & 0 & 2 \\ 2 & -2 & -1 \end{bmatrix}.$$

20. A matrix can be resolved as $U\Sigma V^T$, by singular value decomposition. Find the matrices U and Σ for the matrix $A = \begin{bmatrix} 4 & 2 \\ 4 & 2 \\ -2 & -1 \end{bmatrix}$.

21. If $y = (3, 4)$ and $u = (2, 2)$, obtain the orthogonal projection of y onto u.

22. Without finding the characteristic equation, verify whether $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector of the matrix $A = \begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$. If yes, then find the corresponding eigenvalue.

23. Given $A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \end{bmatrix}$, decompose the matrix A as $A = QR$, using the Gram-Schmidt process.

24. Factorize the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ as $A = PDP^T$.

25. Using the Gram-Schmidt process, orthonormalize the columns of the matrix

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -4 & 1 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

26. Obtain the Singular Value Decomposition of $A = \begin{bmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \end{bmatrix}$.

27. A matrix can be resolved as $U\Sigma V^T$, by singular value decomposition. Find the matrices U and Σ for the matrix $A = \begin{bmatrix} 6 & -2 \\ -3 & 1 \\ 6 & -2 \end{bmatrix}$

28. Obtain the QR factorisation of the matrix A, by applying Gram-Schmidt process, where

$$A = \begin{bmatrix} -6 & 1 & 0 \\ 1 & -3 & 2 \\ 4 & 2 & -2 \\ 0 & 1 & -5 \\ 5 & 2 & -1 \end{bmatrix}.$$

29. Convert the basis vectors $(3, 2, -2, 1, 3), (6, 0, 4, -1, 4), (6, -4, 4, 2, -1)$ to an orthonormal basis of a subspace of $\mathbb{R}^5$, using Gram-Schmidt orthogonalization.

30. Obtain the matrix P which diagonalizes the matrix $A = \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{bmatrix}$. Also find the matrices P^{-1} and D.