UNIT-3

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TUTORIAL SHEET-1 DISCRETE RANDOM VARIABLES

1. Urn contains:
   8 white (lose ₹1 each)
   4 black (win ₹2 each)
   2 orange (no effect)

Let X = total winnings (₹)

Balls drawn	Winnings(X)
WW	-2
WB	1
WO	-1
BB	4
BO	2
OO	0

Total ways of selecting 2 balls $= {}^{14}C_2 = \frac{14\times13}{2} = 91$

$\frac{\cancel{13}\cancel{14}^{7}}{2}$

Probability $(X=-2)$

$P = \frac{{}^8C_2}{91} = \frac{28}{91}$

$P(X=1)$

$P = \frac{8\times4}{91} = \frac{32}{91}$

$P(X=-1)$

$P = \frac{8\times2}{91} = \frac{16}{91}$

$P(X=4)$

$P = \frac{{}^4C_2}{91} = \frac{6}{91}$

$P(X=2)$, $P(X=0)$

$P = \frac{4\times2}{91} = \frac{8}{91}$, $P = \frac{{}^2C_2}{91} = \frac{1}{91}$

2 i) $k+2k+2k+3k+3k^2+2k^2+7k^2+k=1$

$9k+12k^2=1 \Rightarrow 12k^2+9k-1=0$

$k=\frac{1}{12}$

ii) $P(X\geq5) = P(5)+P(6)+P(7)$

$= 3k^2+2k^2+(7k^2+k)$

$= 12k^2+k = \frac{12}{144}+\frac{1}{12} = 0.1667$

iii) $P(X<3) = P(0)+P(1)+P(2)$

$= 0+k+2k = 3k = \frac{1}{4}$

iv) $P(2<X\leq5) = P(3)+P(4)+P(5)$

$= 2k+3k+3k^2 = \frac{5}{12}+\frac{3}{144} = 0.4583$

v) Mean: $\mu = \Sigma x P(X=x)$

$= 0+1\times\frac{1}{12}+\frac{4}{12}+\frac{6}{12}+\frac{12}{12}+\frac{15}{144}+\frac{12}{144}+\frac{49}{144}$

$+\frac{7}{12} = 3.08$

vi) Variance $V(X) = E(X^2)-(E(X))^2$

$= 0+\frac{1}{12}+\frac{8}{12}+\frac{18}{12}+\frac{48}{12}+\frac{125}{144}+\frac{72}{144}+\frac{343}{144}+$

$\frac{49}{12} - 3.08^2$

$= 2.15$

vii) $V(-3X)$ $V(aX) = a^2V(X)$

$= 9\times2.15 = 19.35$

3. Total parts = 850, defective = 50,

Sample = 2

$P(X=0) = \frac{{}^{800}C_2}{{}^{850}C_2} = \frac{\frac{800\times799}{2}}{\frac{850\times849}{2}} = 0.8857$

$P(X=1) = \frac{{}^{800}C_1}{{}^{850}C_2} = \frac{800\times2}{850\times849} = 0.00221$

$P(X=2) = \frac{{}^{50}C_2}{{}^{850}C_2} = \frac{\frac{50\times49}{2}}{\frac{850\times849}{2}} = 0.0033$

x	$F(x)$
$x<0$	0
$0\leq x<1$	$P(0)$
$1\leq x<2$	$P(0)+P(1)$
$x\geq2$	1

4. Probability of particle $= 0.01$

Let X be no. of tested particles found

$P(X=x) = (0.99)^{x-1}(0.01)$

$P(X=2) = (0.99)^{1} \cdot 0.01$
$= 0.0099$

5. $F(x) = \begin{cases} 0 & x<1 \\ 0.7, & 1 \le x < 4 \\ 0.9 & 4 \le x < 7 \\ 1 & x \ge 7 \end{cases}$

$P(X=1) = 0.7$

$P(X=4) = 0.9 - 0.7 = 0.2$

$P(X=7) = 1 - 0.9 = 0.1$

a) $P(X \le 4) = 0.9$

b) $P(X > 7) = 0$

c) $P(X \le 5) = 0.9$

d) $P(X \le 2) = 0.7$

e) $P(X > 4) = 1 - 0.9 = 0.1$

f) $V(-5X) = 25V(X)$
$= 25(1.56) = 39$

TUTORIAL SHEET-2
CONT. RANDOM VARIABLES

1. $f(x) = \begin{cases} ax & 0 \le x \le 1 \\ a & 1 \le x \le 2 \\ -ax+3a & 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$

i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 -ax+3a\,dx + 0 = 1$

$= a\left[\frac{x^2}{2}\right]_0^1 + a[x]_1^2 + \left[-\frac{ax^2}{2} + 3ax\right]_2^3$

$= \frac{a}{2} + a + \frac{a}{2} \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$

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ii) $P(X \le 1.5)$

$= \int_0^1 ax\,dx + \int_1^{1.5} a\,dx$

$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}[x]_1^{1.5} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

iii) Mean: $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$E(X) = 0$

Variance $V(X) = 2$

3. $p(x) = \begin{cases} 3x^{-4}, & x > 1 \\ 0, & \text{elsewhere} \end{cases}$

a) $\int_1^{\infty} 3x^{-4}\,dx = 3\left[\frac{x^{-3}}{-3}\right]_1^{\infty} = 1$

b) For $x > 1$
$F(x) = \int_1^{x} 3t^{-4}\,dt = 1 - \frac{1}{x^3}$

$F(x) = \begin{cases} 0, & x \le 1 \\ 1 - \frac{1}{x^3}, & x > 1 \end{cases}$

c) $P(X > 4) = 1 - F(4)$
$= 1 - \left(1 - \frac{1}{64}\right) = \frac{1}{64}$

4. $p(y) = \begin{cases} ky^4(1-y)^3, & 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$

a) $\int_0^1 ky^4(1-y)^3\,dy = 1$

$\int_0^1 y^4(1-y)^3\,dy = \frac{4!\,3!}{5!} = \frac{\cancel{4}\times\cancel{3}\times\cancel{2}\times 3\times 2}{5\times\cancel{4}\times\cancel{3}\times\cancel{2}} = \frac{6}{5}$

$k \cdot \frac{24 \cdot 6}{40320} = 1 \quad k = 280$

b) $P(Y \le 0.5)$

$= \int_0^{0.5} 280y^4(1-y)^3\,dy$

$= 0.1445$

c) $P(Y \ge 0.8) = 1 - F(0.8)$
$= 0.0579$

d) $V(2Y) = 4V(Y) \quad V(2Y) = \frac{80}{810}$

$V(Y) = \frac{5 \cdot 4}{9^2 \cdot 10} = \frac{20}{810}$

5. a) $F(x) = 1 - e^{-2x}$, $x > 0$

$f(x) = \frac{d}{dx} F(x) = 2e^{-2x}$, $x > 0$

b) $F(x) = \begin{cases} 0, & x < 0 \\ 0.2x, & 0 \le x < 4 \\ 0.04x + 0.64, & 4 \le x < 9 \\ 1, & x \ge 9 \end{cases}$

differentiateng piecewise

$f(x) = \begin{cases} 0.2 & , 0 \le x < 4 \\ 0.04 & , 4 \le x < 9 \\ 0 & , \text{elsewhere} \end{cases}$

TUTORIAL SHEET-3

JOINT DISCRETE RANDOM VARIABLES

1. $f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ -ax + 3a, & 2 \le x \le 3 \\ 0 & , \text{otherwise} \end{cases}$

High distortion = 0.01

moderate distortion = 0.04

low distortion = 0.95

Let X be no. of bits with high distortion

Y be no of bits with moderate distortion

a) $p(x,y) = \frac{3!}{x!\,y!\,(3-x-y)!} (0.01)^x (0.04)^y (0.95)^{3-x-y}$

b) $p(x) = {}^3C_x (0.01)^x (0.99)^{3-x}$

c) $E(X) = np = 3(0.01) = 0.03$

d) $p_{XY}(x,y) = p_X(x)\, p_Y(y)$

X & Y are NOT independent

e) $\text{cov}(X,Y) = -3(0.01)(0.04)$

$= -0.0012$

2. Box contains: 3 blue, 2 red, 3 green

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Let X be no of blue pens,

Y be no of red pens

a) ${}^8C_2 = \frac{8 \times 7}{2} = 28$

$p(x,y) = \frac{{}^3C_x \; {}^2C_y \; {}^3C_{2-x-y}}{28}$

Valid when $x + y \le 2$

b) $P(X+Y \le 1)$ cases: (0,0); (1,0); (0,1)

$P = \frac{{}^3C_0\,{}^2C_0\,{}^3C_2 + {}^3C_1\,{}^2C_0\,{}^3C_1 + {}^3C_0\,{}^2C_1\,{}^3C}{28}$

$= \frac{15}{28}$

3. $f(x,y) = \frac{x+y}{30}$ $x = 0, 1, 2, 3$ $y = 0, 1, 2$

a) $P(X \le 2, Y = 1)$

$f(0,1) + f(1,1) + f(2,1)$

$= \frac{1+2+3}{30} = \frac{6}{30}$

b) $P(X > 2, Y \le 1)$

Pairs : (3,0); (3,1)

$= \frac{3+4}{30} = \frac{7}{30}$

c) $P(X > Y) = \frac{19}{30}$

d) $P(X+Y=4)$ pairs: (2,2); (3,1)

$= \frac{4+4}{30} = \frac{8}{30}$

e) $E(2X + Y + 3)$

$= 2E(X) + E(Y) + 3$

$= 2 \times 1.8 + 1 + 3$

$= 7.6$

f) $E(2X + 3Y - 2)$

$= 2E(X) + 3E(Y) - 2$

$= 2 \times 1.8 + 3 - 2 = 6.6$

g) $COV(X,Y) = E(XY) - E(X)E(Y)$

$= \frac{72}{30} - (1.8 \times 1) = \cancel{0.4}$

$= 2.4 - 1.8$

$= 0.6$

h) $p_x(x) = \sum_y f(x,y)$

i) $P(Y>0 \mid X=2)$

$= \frac{f(2,1)+f(2,2)}{p_x(2)} = \frac{5}{7} = \frac{7/30}{9/30} = \frac{9}{30}$

j) $P(X>1 \mid Y=1) = \frac{5}{9}$

4. $P(Head) = 0.4$

Let Z = heads on first toss

W = total heads in two tosses

a)

Outcome	Z	W	Prob
HH	1	2	0.16
HT	1	1	0.24
TH	0	1	0.24
TT	0	0	0.36

b) $P(W=0) = 0.36$

$P(W=1) = 0.48$

$P(W=2) = 0.16$

c) $P(Z=1) = 0.4$

$P(Z=0) = 0.6$

d) $P(W \geq 1) = 1 - P(W=0)$

$= 1 - 0.36 = 0.64$

5.

Children	Probability
0	0.15
1	0.20
2	0.35
3	0.30

Let B = boys, G = girls

$p(b,g) = P(N=b+g)\ {}^{b+g}C_b \left(\frac{1}{2}\right)^{b+g}$

Ex: $p(2,1) = 0.30 \times {}^3C_2 \left(\frac{1}{2}\right)^3 = 0.1125$

TUTORIAL-4 JOINT CONT. RANDOM VARIABLES

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1. $f(x,y) = \begin{cases} 4xy, & 0<x<1,\ 0<y<1 \\ 0, & \text{elsewhere} \end{cases}$

a) $P(0 \leq X \leq \frac{1}{2};\ \frac{1}{4} \leq Y \leq \frac{1}{2})$

$P = \int_0^{1/2} \int_{1/4}^{1/2} 4xy\ dy\,dx$

$= 4x\left[\frac{y^2}{2}\right]_{1/4}^{1/2} = \int_0^{1/2} \frac{3x}{8}\,dx = \frac{3}{64}$

b) $P(X<Y)$

$P = \int_0^1 \int_0^y 4xy\ dx\,dy$

$= \int_0^1 4y\left[\frac{x^2}{2}\right]_0^y dy = \int_0^1 2y^3\,dy = \frac{1}{2}$

c) $f(x) = \int_0^1 4xy\ dy = 2$

$f_x(x) f_y(y) = 4xy = f(x,y)$

X and Y are independent

d) $E[5X - 3Y]$

$E(X) = \int_0^1 x(2x)\,dx = \frac{2}{3}$

$\cancel{E(Y)}\ E(5X-3Y) = 5E(X) - 3E(Y)$

$= \frac{4}{3}$

e) $V(2X+3Y)$; $V(X) = \frac{1}{18}$, $V(Y) = \frac{1}{18}$

$4V(X) + 9V(Y)$

$= \frac{4}{18} + \frac{9}{18} = \frac{13}{18}$

f) $COV(X,Y) = 0$

g) $\rho(X,Y) = \frac{0}{\sqrt{V(X)V(Y)}} = 0$

2. $f(x,y)=\begin{cases}\frac{1}{y}, & 0<x<y<1\\ 0, & \text{elsewhere}\end{cases}$

$x+y>\frac{1}{2}$, $0<x<y<1$

$P=\int_{1/4}^{1}\int_{1/2-y}^{y}\frac{1}{y}\,dx\,dy$

$=\int_{1/4}^{1}\frac{2y-\frac{1}{2}}{y}dy=\frac{3}{4}$

3. $f(x,y)=\begin{cases}2, & 0<x\le y<1\\ 0, & \text{elsewhere}\end{cases}$

a) Joint $\neq$ product of marginals
Not independent

b) $P\left(\frac{1}{4}\le X\le\frac{1}{2}\,\middle|\,Y=\frac{3}{4}\right)$

$f\left(\frac{x}{y}\right)=\frac{f(x,y)}{f_Y(y)}$

$f_Y(y)=\int_0^y 2\,dx=2y$

$P=\int_{1/4}^{1/2}\frac{1}{3/4}\,dx=\frac{1}{3}$

c) $E(3X+2Y)$ $\quad E(X)=\frac{1}{3}$

$\frac{3}{3}+\frac{4}{3}=\frac{7}{3}$ $\quad E(Y)=\frac{2}{3}$

4. $f(x,y)=\frac{6-x-y}{8}$ $\quad 0<x<2$, $2<y<4$

$P(1<Y<3\mid X=1)$

$f_X(1)=\int_2^4\frac{5-y}{8}\,dy=\frac{1}{4}$

$f.(y|1)=\frac{5-y}{2}$

$P=\int_2^3\frac{5-y}{2}\,dy=\frac{3}{8}$

5. $f(x,y)=cxy$ $\quad 0<x<3$, $0<y<x$

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$\int_0^3\int_0^x cxy\,dy\,dx=1$

$c\int_0^3 x\frac{x^2}{2}\,dx=1\Rightarrow c=\frac{8}{81}$

a) $P(X<1,Y<2)=\cancel{\frac{2}{27}}\int_0^1\int_0^x\frac{8}{81}xy\,dy\,dx$

$=\frac{8}{162}\cdot\frac{1}{4}=\frac{2}{27}$

b) $P(1<X<2)=\cancel{\ }$

$\int_1^2\int_0^x\frac{8}{81}xy\,dy\,dx$

$=\frac{8}{81}\int_1^2\frac{x^3}{2}dx=\frac{4}{81}\left[\frac{x^4}{4}\right]_1^2=\frac{4}{81}(16-1)$

$=\frac{20}{81}$

c) $P(Y>1)$ $\quad P=\int_1^3\int_1^x\frac{8}{81}xy\,dy\,dx$

$\frac{8}{81}\int_1^3\frac{x(x^2-1)}{2}dx=\frac{4}{81}\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_1^3$

$=\frac{16}{27}$

d) $P(X<2,Y<2)$

$P=\int_0^2\int_0^x\frac{8}{81}xy\,dy\,dx$

$=\frac{8}{81}\int_0^2\frac{x^3}{2}dx=\frac{4}{81}\cdot 4=\frac{20}{27}$

e) $E(X)=\int_0^3\int_0^x x\cdot\frac{8}{81}xy\,dy\,dx$

$=\frac{8}{81}\int_0^3\frac{x^4}{2}dx=\frac{4}{81}\int_0^3 x^4dx=2$

$E(Y)=\int_0^3\int_0^x y\cdot\frac{8}{81}xy\,dy\,dx$

$=\frac{8}{81}\int_0^3\frac{x^4}{3}dx=\frac{4}{3}$

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f) $f_X(x) = \int_0^x \frac{8}{81} xy \, dy$

$= \frac{8x}{81} \int_0^x y \, dy = \frac{8x}{81} \cdot \left[\frac{y^2}{2}\right]_0^x$

$= \frac{8x^3}{81 \times 2} = \frac{4x^3}{81} \quad 0 < x < 3$

$f_X(x) = \begin{cases} 4x^3/81, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$

g) $f_{Y|X}(y|1) = \frac{f(1,y)}{f_X(1)}$

$= \frac{\frac{8}{81} y}{\frac{8}{81}} = y \quad 0 < y < 1$

h) $P(Y > 2 \mid X = 1)$

$0 < y < 1$, when $x = 1$

$P(Y > 2 \mid X = 1) = 0$