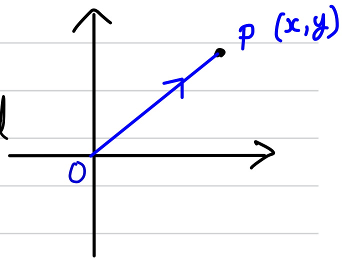
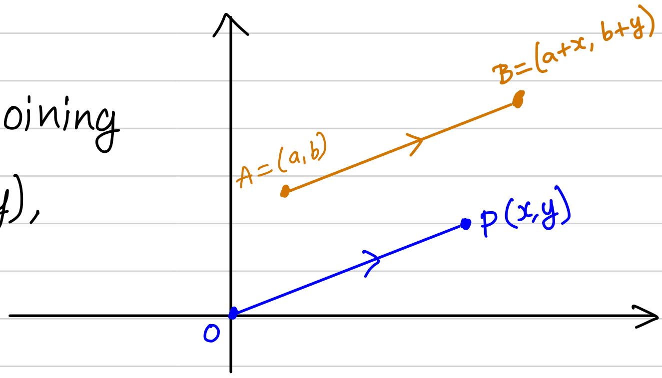
Definition [Vectors]

- Vector is a quantity which has both magnitude and direction.
- Let P be a point in 2-space as shown. The position vector of a point $P(x,y)$ is $\vec{OP}$ and is referred as $\begin{bmatrix} x \\ y \end{bmatrix}$.



- Any vector in 2-space can be represented by a position vector of a point in the space.
  For instance, if $\vec{AB}$ is a vector joining the points (a,b) and $(a+x, b+y)$, then $\vec{AB} = \vec{OP}$ as shown.



- An ordered pair (x, y) can be interpreted as a point, in which case x and y are the coordinates, or it can be interpreted as a vector, in which case x and y are the components. We prefer to refer vector as $\begin{bmatrix} x \\ y \end{bmatrix}$. The distinction is mathematically unimportant.
- Set of all vectors in 2-space is denoted by $\mathbb{R}^2$

  i.e $\quad \mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$

- In general, the ordered n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as a "generalized point" or a "generalized vector" in n-space.

- The set of a vectors in n-space is denoted by $\mathbb{R}^n$

i.e $\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \middle| \; x_i \in \mathbb{R} \right\}$

- Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ be any vectors in $\mathbb{R}^n$

  - The addition $X+Y$ is defined as

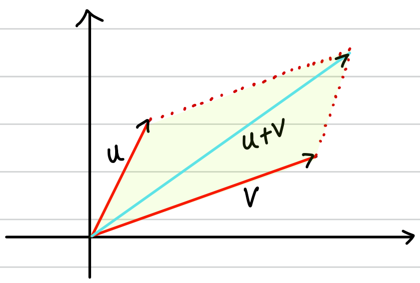
$$X + Y = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$$

  - Let α be a scalar (a real no.). Then the scalar multiple $\alpha \cdot X$ is defined by

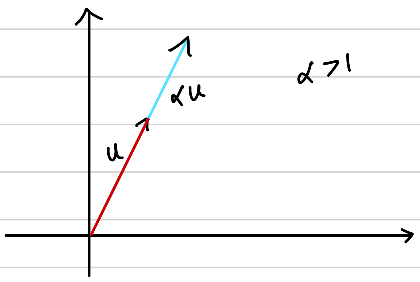
$$\alpha \cdot X = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

- Geometrically, suppose $u = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $v = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ are vectors in $\mathbb{R}^2$.

Addition



Scalar multiplication



  - Scalar multiplication of a vector is a vector in the same direction or a vector in the opposite direction.

- Addition of two vectors is a vector that is in the same plane as the original two vectors.

- These two operations of addition and scalar multiplication are called the standard operations (or usual operations) on $\mathbb{R}^n$.

The set $\mathbb{R}^n$ under the binary operations $+$ and $\cdot$ as defined above is called a vector space. It is usually referred as $\mathbb{R}^n$-vectorspace.

Arithmetic properties of addition and scalar multiplication of vectors in $\mathbb{R}^n$ are listed as follows:

If $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $W = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ are vectors in $\mathbb{R}^n$

and α and β are scalars, then

1) $u+v = v+u$

2) $u+(v+w) = (u+v)+w$

3) $u+0 = 0+u = 0$ (additive identity) ; $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

4) $u+(-u) = 0$ (additive inverse)

5) $\alpha\cdot(\beta u) = (\alpha\beta)\cdot u$; $-u = \begin{bmatrix} -u_1 \\ -u_2 \\ \vdots \\ -u_n \end{bmatrix}$

6) $\alpha\cdot(u+v) = \alpha\cdot u + \alpha\cdot v$

7) $(\alpha+\beta)\cdot u = \alpha\cdot u + \beta\cdot u$

8) $1\cdot u = u$, $1 \in \mathbb{R}$

Generalized Vector Space

The time has now come to generalize the concept of a vector.

We can think of a vector space in general, as a collection of objects that behave as vectors do in $\mathbb{R}^n$. The objects of such a set are called vectors.

Defn [Field]: A set $\mathbb{F}$ with operations $+$ and $\times$ is called a Field if

for any $\alpha, \beta, \gamma \in \mathbb{F}$

i) $\alpha+\beta \in \mathbb{F}$ (closed under addition)

ii) $\alpha+\beta = \beta+\alpha$

iii) $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$

iv) $\alpha+0 = 0+\alpha = \alpha$ (additive identity exist)

v) $\alpha+(-\alpha) = (-\alpha)+\alpha = 0$ (additive inverse)

vi) $\alpha \times \beta \in \mathbb{F}$

vii) $\alpha \times \beta = \beta \times \alpha$

viii) $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$

ix) $\alpha \times 1 = 1 \times \alpha = \alpha$ (Multiplicative identity)

x) $\alpha \times \alpha^{-1} = \alpha^{-1} \times \alpha = 1$ (Multiplicative inverse)

Ex 1: Under addition $+$ and multiplication $\times$,

i) $\mathbb{R}$ set of all real numbers is a Field

ii) $\mathbb{C}$ set of all complex no.s is a Field

iii) $\mathbb{Q}$ set of rational no.s is a Field

Ex 2: Set of integers $\mathbb{Z}$ is not a field.

Since every no. doesnot have multiplicative inverse.

Ex 3: Set of irrational no.s $\mathbb{Q}^c$ is not a field.

Since it is not closed under multiplication.

Axiomatic definition of a Vector Space

Any collection of mathematical objects W with binary operations $+$ on W and $\cdot$ on scalars (Fields, $\mathbb{F}$) with W such that

i) $w_1 + w_2 \in W$ for any $w_1, w_2 \in W$

ii) $\alpha \cdot w_1 \in W$ for any $w_1 \in W$ and $\alpha \in \mathbb{F}$

is a vector space over $\mathbb{F}$ if it satisfies the below eight axioms: For any $w_1, w_2, w_3 \in W$ and $\alpha, \beta \in \mathbb{F}$

VS1) $w_1 + w_2 = w_2 + w_1$ (Commutative)

VS2) $w_1 + (w_2 + w_3) = (w_1 + w_2) + w_3$ (Associative)

VS3) There exist an element $0 \in W$ such that

$w_1 + 0 = 0 + w_1 = w_1$ (Existence of additive identity)

VS4) For every $w_1 \in W$ there exist $-w_1 \in W$ such that

$w_1 + (-w_1) = (-w_1) + w_1 = 0$ (Existence of additive inverse)

VS5) $\alpha \cdot (\beta \cdot w_1) = (\alpha\beta) \cdot w_1$

VS6) $\alpha \cdot (w_1 + w_2) = \alpha \cdot w_1 + \alpha \cdot w_2$ (distributive law 1)

VS7) $(\alpha + \beta) \cdot w_1 = \alpha \cdot w_1 + \alpha \cdot w_2$ (distributive law 2)

VS8) $1 \cdot \alpha = \alpha$ (Multiplicative identity)

Notation:

A vector space W over $\mathbb{F}$ under $+$ and $\cdot$ is denoted by $(W(\mathbb{F}), +, \cdot)$

Ex 0: $(\mathbb{R}, +, \cdot)$ is a vector space with usual addition and scalar multiplication.

Ex 1: $(\mathbb{R}^n, + \cdot)$ is a vector space under standard addition and scalar multiplication.

Ex 2: Let $\mathbb{C} = \{ a+ib \mid a, b \in \mathbb{R}, i = \sqrt{-1} \}$ with binary composition $+$ and $\cdot$ defined by

i) $(a+ib) + (c+id) = (a+c) + i(b+d)$

ii) $\alpha \cdot (a+ib) = \alpha a + i \alpha b$

$(\mathbb{C} + \cdot)$ is a vector space. (Verify)

Ex 3: Let $M_{2\times 2}(\mathbb{R})$ be set of all matrices with real enteries

i.e $M_{2\times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \in \mathbb{R} \right\}$ is a vector space with usual matrix addition and scalar multiplication defined by (it is usually denoted by $\mathbb{R}^{2\times 2}$)

i) $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$

ii) $\alpha \cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & \alpha d_1 \end{bmatrix}$

for every $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_2$ and $\alpha \in \mathbb{R}$.

Ex 4: The set of all $m \times n$ matrices with real enteries denoted by $M_{m\times n}(\mathbb{R})$ (or $\mathbb{R}^{m\times n}$) is also a vector space under usual addition and scalar multiplication.

Ex5: The set of all polynomials of degree $\leq n$ with Coefficients from the field $\mathbb{R}$, denoted by $P_n(\mathbb{R})$,

i.e $P_n(\mathbb{R}) = \{ a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid a_i \in \mathbb{R} \}$

is a vector space under addition and scalar multiplication defined as below:

For any $A = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, $B = b_0 + b_1 x + \cdots + b_r x^r$ $(r \leq n)$

in $P_n(\mathbb{R})$ and $\alpha \in \mathbb{R}$

i) $A + B = (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) + \cdots + (a_r + b_r) x^r + \cdots + a_n x^n$

ii) $\alpha \cdot A = \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \cdots + \alpha a_n x^n$

Here zero vector is $0 + 0x + \cdots + 0x^n = 0$

Ex6: The set of all polynomials with real co-efficients denoted by $P(\mathbb{R})$

i.e $P(\mathbb{R}^n) = \{ a_0 + a_1 x + a_2 x^2 + \cdots + a_r x^r \mid a_i \in \mathbb{R} \}$

is also a vector space under the operations defined as in Ex5.

Ex7: The set of all real valued functions defined on $[a, b]$

i.e $\mathcal{F} = \{ f \mid f : [a, b] \to \mathbb{R} \}$

is a vector space over $\mathbb{R}$ under pointwise addition and scalar multiplication defined as

for any $f_1, f_2 \in \mathcal{F}$ and $\alpha \in \mathbb{R}$

i) $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

ii) $(\alpha f_1)(x) = \alpha f_1(x)$

Clearly it is closed under addition and scalar multiplication

Since for $f_1, f_2 \in \mathcal{F}$

$(f_1+f_2)(x) = f_1(x)+f_2(x) \in \mathbb{R}$; $(\alpha f_1)(x) = \alpha f_1(x) \in \mathbb{R}$

$\Rightarrow f_1+f_2 \in \mathcal{F}$ $\qquad \Rightarrow \alpha f_1 \in \mathcal{F}$

VS1) $(f_1+f_2)(x) = f_1(x)+f_2(x) = f_2(x)+f_1(x) = (f_2+f_1)(x)$

$\Rightarrow f_1+f_2 = f_2+f_1$

VS2) $((f_1+f_2)+f_3)(x) = (f_1+f_2)(x)+f_3(x)$

$= (f_1(x)+f_2(x))+f_3(x)$

$= f_1(x)+(f_2(x)+f_3(x))$

$= (f_1+(f_2+f_3))(x)$

$\Rightarrow (f_1+f_2)+f_3 = f_1+(f_2+f_3)$

VS3) We define $\mathbf{0} \in \mathcal{F}$ as a zero function that sends each element in $[a,b]$ to 0 in $\mathbb{R}$

$(f+\mathbf{0})(x) = f(x)+\mathbf{0}(x) = f(x)+0 = f(x)$

$\Rightarrow f+\mathbf{0} = f$

Additive identity exist

VS4) $(f_1+(-1)f_1)(x) = f_1(x)+(-1)f_1(x) = f_1(x)-f_1(x) = 0 = \mathbf{0}(x)$

$\Rightarrow f_1+(-1)f_1 = \mathbf{0}$

$\therefore (-1)f_1 = -f_1$ is additive inverse of f_1

VS5) $(\alpha\cdot(\beta\cdot f_1))(x) = \alpha(\beta f_1(x)) = (\alpha\beta)f_1(x) = ((\alpha\beta)f_1)(x)$

$\Rightarrow \alpha(\beta f_1) = (\alpha\beta)f_1$

VS6) $((\alpha+\beta)\cdot f_1)(x) = (\alpha+\beta)f_1(x) = \alpha f_1(x)+\beta f_1(x) = (\alpha f_1+\beta f_1)(x)$

$\Rightarrow (\alpha+\beta).f_1 = \alpha f_1 + \beta f_1$

VS7) $(\alpha.(f_1+f_2))(x) = \alpha(f_1+f_2)(x) = \alpha(f_1(x)+f_2(x))$

$= (\alpha f_1(x) + \alpha f_2(x)) = (\alpha f_1 + \alpha f_2)(x)$

$\Rightarrow \alpha.(f_1+f_2) = \alpha f_1 + \alpha f_2$

VS8) $(1.f_1)x = f_1(x) \Rightarrow 1.f_1 = f_1$

Thus $\mathcal{F}$ is a vector space under pointwise addition and scalar multiplication.

Ex 8: [Counter example]

Let $\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; x, y \in \mathbb{R}^2 \right\}$. Define binary operations as below:

For any $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in $\mathbb{R}^n$ and $\alpha \in \mathbb{R}$

i) $u+v = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}$; ii) $\alpha.u = \begin{bmatrix} \alpha u_1 \\ 0 \end{bmatrix}$

Clearly $\mathbb{R}^2$ under above mentioned binary operations is not a vector space. Since $1 \cdot u = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \neq u$, doesnot satisfy VS8.

Ex 9: Let $S = \{(a_1,a_2) \mid a_1, a_2 \in \mathbb{R}\}$. For any $(a_1,a_2), (b_1,b_2) \in S$ and $c \in \mathbb{R}$, define $+$ and $\cdot$ as

i) $(a_1,a_2) + (b_1,b_2) = (a_1+b_1, a_2-b_2)$

ii) $c.(a_1,a_2) = (ca_1, ca_2)$. Is $(S,+,\cdot)$ a vector space?

Soln: No.

VS1) Let $(a_1,a_2), (b_1,b_2), (c_1,c_2) \in S$.

Consider $(a_1,a_2) + \underbrace{((b_1,b_2)+(c_1,c_2))}$

$= (a_1,a_2) + (b_1+c_1, b_2-c_2)$

$= (a_1+b_1+c_1,\ a_2-(b_2-c_2))$

Now, $((a_1,a_2)+(b_1,b_2))+(c_1,c_2)$

$= (a_1+b_1,\ a_2-b_2)+(c_1,c_2)$

$= (a_1+b_1,\ a_2-b_2-c_2)$

Thus, it doesnot satify associative law.

Even VS4 fails to hold.

Proposition: Let V be a vector space over $\mathbb{R}$, $u \in V$ and $\alpha \in \mathbb{R}$. Then

a) $0u = \mathbf{0}$ 0 is additive identity in $\mathbb{R}$ and $\mathbf{0}$ is additive identity in V

b) $\alpha \cdot \mathbf{0} = \mathbf{0}$

c) $(-\alpha)u = -(\alpha u) = \alpha(-u)$

d) If $\alpha u = 0 \Rightarrow \alpha = 0$ or $u = \mathbf{0}$

True or False

a) Every vector space contains a zero vector

Ans: True

b) Every vector space contain atleast two vectors.

Ans: False.

c) The rational numbers $\mathbb{Q}$ form a vector space over $\mathbb{Q}$.

Ans: True

d) Set of all functions $\mathcal{F} = \{f \mid f: \mathbb{Z} \to \mathbb{Z}\}$ form a vector space over $\mathbb{R}$ under pointwise addition and scalar multi.

Ans: False. Since not closed under scalar multiplication.

i.e for any $f \in F$ and $\alpha \in \mathbb{R}$, $(\alpha f)(x) = \alpha(f(x)) \notin \mathbb{Z}$

e) In any vector space $V(\mathbb{R})$, $\alpha \cdot u = \beta \cdot u \Rightarrow \alpha = \beta$ for any $u \in V$ $(u \neq 0)$ and $\alpha, \beta \in \mathbb{R}$

Ans: True (Verify)

f) A vector space may have more than one zero vector.

Ans: No. Zero vector is unique. (prove it)

g) For any v in a vector space V has unique additive inverse.

Ans: Yes. (prove it)

Exercise:

1) Determine if the set $\mathbb{T} := \mathbb{R} \cup \{0\}$, with addition and scalar multiplication defined for all $v, w \in \mathbb{T}$ and $\alpha \in \mathbb{R}$ by

i) $v + w = \min(v, w)$

ii) $\alpha \cdot v = \alpha + v$

is a vector space over $\mathbb{R}$. If it is not, then list all of the defining axioms that fail to hold.

Subspace (a vector space inside a vector space)

Definition [Subspace]

Let $V(\mathbb{R})$ be a vector space over $\mathbb{R}$, let W be a non empty subset of V. A subset W is called a subspace of V if W is a vector space over $\mathbb{R}$ with the operations of addition and scalar multiplication defined on V.

It is usually denoted by $W \leq V$.

Ex 1: $P_n(\mathbb{R})$ is a subspace of $P(\mathbb{R})$.

Ex 2: Let $C[a,b]$ be set of all real valued continuous fns.

i.e $C[a,b] = \{ f \mid f : [a,b] \to \mathbb{R} \text{ and } f \text{ is continous} \}$

It is a vector space under pointwise addition and scalar multiplication and it is a subset of set of all real valued fn. $\mathcal{F}$ on $[a,b]$

$\therefore C[a,b] \leq \mathcal{F}$

Ex 3: Let $C^{(n)}[a,b]$ be set of all real valued functions on $[a,b]$ such that $f', f'', f''', \cdots f^{(n-1)}, f^{(n)}$ exist and are continuous. (Here n is any non negative integer).

i.e, $C^{(n)}[a,b] = \{ f : [a,b] \to \mathbb{R} \mid f', f'', f''', \cdots, f^{(n)} \text{ exist and continuous} \}$

It is a vector space over $\mathbb{R}$ under pointwise addition and scalar multiplication and $C^{(n)}[a,b] \subseteq C[a,b] \subseteq \mathcal{F}$

$\therefore C^{(n)}[a,b] \leq C[a,b] \leq \mathcal{F}$, where $\mathcal{F}$ is set of all real valued functions on $[a,b]$.

Below theorem can be used to verify which subset inherit the structure of a vector space.

Thm 1: Let $(V, +, \cdot)$ be a vector space. Let $W \subseteq V$ (non empty). Then $(W, +, \cdot)$ is said to be a subspace of $(V, +, \cdot)$ iff

i) $w_1 + w_2 \in W$ for all $w_1, w_2 \in W$

ii) $\alpha . w_1 \in W$ for all $\alpha \in \mathbb{R}$ and $w_1 \in W$.

} closure conditions

OR

linear combination of w_1 and w_2

$\alpha w_1 + \beta w_2 \in W$ for all $\alpha, \beta \in \mathbb{R}$ and $w_1, w_2 \in W$

pf: Let $(W, +, \cdot)$ be a subspace of $(V, +, \cdot)$.

Since W is a subspace, from defn W is by itself a vector space.

Thus i) and ii) hold.

Conversely, let $W \subseteq V$, and i) and ii) hold, we have to show that W is a subspace of V.

That is, we need to show that W by itself a vector space with same binary operations.

For all $w \in W$, $\alpha w \in W$ (from ii))

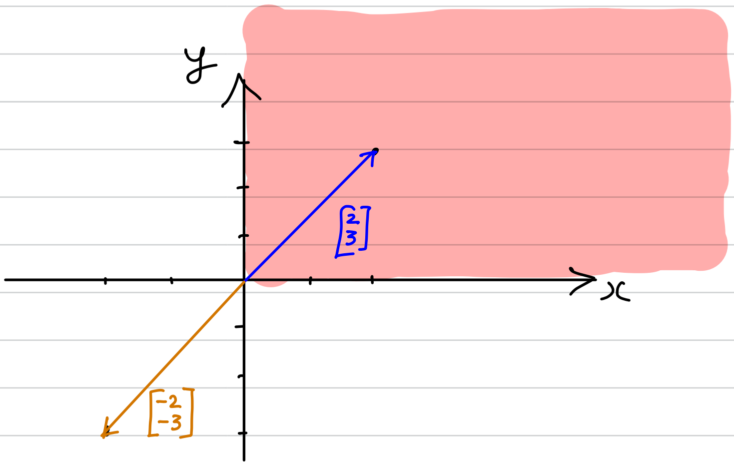
a) Let $\alpha = -1$, implies $-w \in W$ (Existence of inverse)

b) Let $\alpha = 0$, implies $0 . w = 0 \in W$ (Existence of additive identity)

All other 6 axioms are true for any element in V, thus it is true for elements in W as well.

Q: Is $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \ x \geq 0 \text{ and } y \geq 0 \right\}$ a subspace of $\mathbb{R}^2$ under standard addition and scalar multiplicatn.

Ans: No



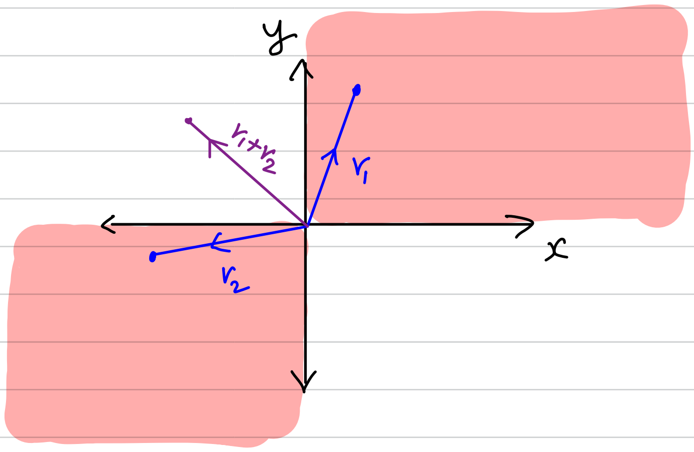
$\because$ if we consider $\alpha = -1$,

then $\alpha \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \notin V$.

Not closed under scalar multiplication.

Q: Is $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \ x \geq 0 \text{ and } y \geq 0 \text{ or } x \leq 0 \text{ and } y \leq 0 \right\}$ a subspace of $\mathbb{R}^2$?

Ans: No



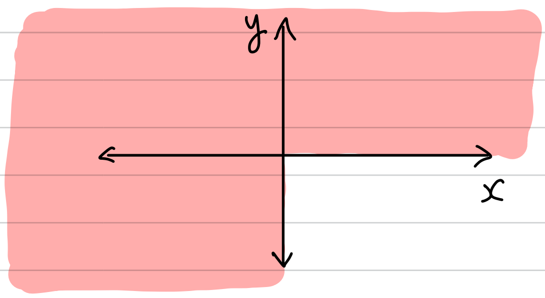
See fig v_1 and $v_2 \in V$, but not $v_1 + v_2$.

Consider $v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ -2 \end{bmatrix} \in V$.

$v_1 + v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \notin V$

Q: Is $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \ x \geq 0 \text{ and } y \geq 0 \text{ or } x \leq 0 \text{ and } y \leq 0 \text{ or } x \leq 0 \text{ and } y \geq 0 \right\}$ a subspace of $\mathbb{R}^2$?

Ans: No (Why?)



Ex4: Obtain all possible subspaces of the vector space $\mathbb{R}^2$ under standard $+$ and $\cdot$

Soln: i) $\mathbb{R}^2$ itself (we call improper subspace)

ii) The singleton set, $W = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ (It is called a trivial subspace)

iii) All lines through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Since linear combination of any two vectors on a line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a vector on the same line.

Ex5: Obtain all possible subspaces of $\mathbb{R}^3$ under standard addition and scalar multi.

Soln: i) $\mathbb{R}^3$ itself, improper subspace

ii) $W = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$, trivial subspace

iii) All lines through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

iv) All planes through $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Remarks:

i) Every subspace of a vector space is a vector space in its own right

ii) Any vector space V automatically contains two subspaces

a) The set $\{0\}$, set consisting of only zero vector, is called trivial subspace.

b) V itself is called improper subspace.

Ex6 : Check whether the following are subspace of $M_{2\times 2}(\mathbb{R})$ under usual matrix addition and scalar multiplication.

i) $M_1 = \left\{ \begin{pmatrix} a & b \\ c & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$

ii) $M_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = 0 \right\}$

Ans i) : No. Since $\begin{bmatrix} a_1 & b_1 \\ c_1 & 1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & 1 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & 2 \end{bmatrix} \notin M.$

Ans ii) : No.

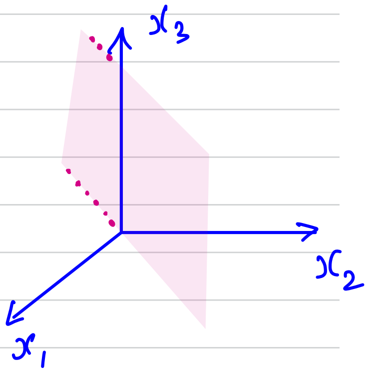
Because, $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in M$, and $A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin M$

$(\det(A+B) \neq 0)$

Ex7 : Let $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; x_1 = x_2 \;\; \forall x_i \in \mathbb{R} \right\}$. S.T S is a subspace of $\mathbb{R}^3$.

Soln: We need to verify that the two closure properties hold:

Let $\begin{bmatrix} a \\ a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ c \\ d \end{bmatrix}$ be any vectors in S.

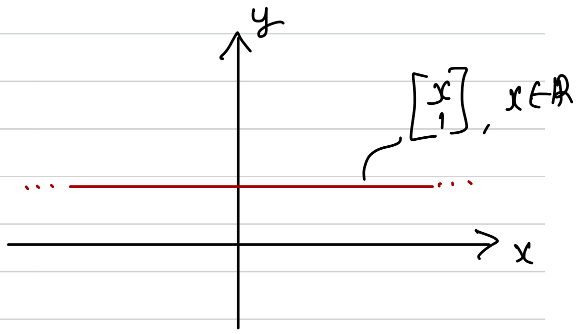


i) $\begin{bmatrix} a \\ a \\ b \end{bmatrix} + \begin{bmatrix} c \\ c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ a+c \\ b+d \end{bmatrix} \in S.$

ii) $\alpha . \begin{bmatrix} a \\ a \\ b \end{bmatrix} = \begin{bmatrix} \alpha a \\ \alpha a \\ \alpha b \end{bmatrix} \in S,$ for any $\alpha \in \mathbb{R}$.

$\because$ S is non-empty and satisfies two closure conditions, it follows that S is a subspace of $\mathbb{R}^3$.

Ex 8: Let $S = \left\{ \begin{bmatrix} x \\ 1 \end{bmatrix} \;\middle|\; x \in \mathbb{R} \right\}$. Check whether S is a subspace of $\mathbb{R}^2$ or not.



Soln: For any $\begin{bmatrix} x_1 \\ 1 \end{bmatrix}, \begin{bmatrix} x_2 \\ 1 \end{bmatrix}$ in S,

$$\begin{bmatrix} x_1 \\ 1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2 \end{bmatrix} \notin S$$

Since it fails to satisfy closure conditions, S is not a subspace of $\mathbb{R}^2$.

or

Since $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin S$, S is not a subspace of $\mathbb{R}^2$.

Ex 9: Let $S = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \;\middle|\; \begin{array}{l} a_{12} = -a_{21} \\ a_{ij} \in \mathbb{R} \end{array} \right\}$.

Is this a subspace of $\mathbb{R}^{2\times 2}$. ($\mathbb{R}^{2\times 2}$ is set of matrices of order 2)

Soln: S is non-empty because $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$.

We verify two closure conditions:

Let $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ be any two elements in S.

i) $$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} \in S$$

$$\left(\begin{array}{l} \because \; a_{21} = -a_{12} \text{ and } b_{21} = -b_{12} \\ \Rightarrow a_{21} + b_{21} = -(a_{12} + b_{12}) \end{array} \right)$$

ii) $\alpha \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{pmatrix} \in S$ $\left(\text{because } \alpha a_{21} = -(\alpha a_{12})\right)$

$\therefore$ S is a subspace of $\mathbb{R}^{2\times 2}$.

The Null Space of a Matrix

Defn: Let A be $m\times n$ matrix. The null space of A, denoted by $N(A)$ is the set of all solutions of the homogenous system $Ax = 0$

i.e $N(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$

Clearly $N(A)$ is a subset of $\mathbb{R}^n$.

Ex: If A is $m\times n$ matrix, then $N(A)$ is a subspace of $\mathbb{R}^n$.

Soln: Clearly, $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in N(A)$, so $N(A)$ is nonempty.

Let x_1 and x_2 be any vectors in $N(A)$.

i) $A(x_1 + x_2) = Ax_1 + Ax_2 = 0 + 0 = 0.$

Thus $x_1 + x_2 \in N(A)$.

ii) For any scalar α,

$A(\alpha x_1) = \alpha A(x_1) = \alpha . 0 = 0.$

implies that $\alpha x_1 \in N(A)$

It follows that $N(A)$ is a subspace of $\mathbb{R}^n$.

Ex: Determine $N(A)$ if

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

Soln: Consider Homogeneous System

$Ax = 0$, where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

Consider, Augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

$R_1 \to R_1 + R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

$R_2 \to -R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{array}\right]$$ is in rref.

Equivalent system

$$x_1 - x_3 + x_4 = 0$$
$$x_2 + 2x_3 + x_4 = 0$$

No. of free variable $= n - r$ (r is rank of A)

$= 4 - 2 = 2$

free variables are x_3 and x_4. Let $x_3 = k_1$ and $x_4 = k_2$

$\Rightarrow x_1 = k_1 - k_2$ and $x_2 = -2k_1 - k_2$

Thus, general soln

$$X = \begin{bmatrix} k_1 - k_2 \\ -2k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad (k_1, k_2 \text{ are scalars})$$

Null space of A,

$$N(A) = \left\{ k_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; \text{for all } k_1, k_2 \in \mathbb{R} \right\}$$

Here $N(A)$ is a set of all linear combinations of $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

Thm 2: The intersection of two subspaces of a vector space V, is again a subspace.

or

If W_1 and W_2 are subspaces of a vector space V, then $W_1 \cap W_2$ is also a subspace V.

pf: Let V_1 and V_2 be any vectors in $W_1 \cap W_2$

Since $V_1, V_2 \in W_1 \cap W_2$

$\Rightarrow \quad V_1, V_2 \in W_1 \quad$ and $\quad V_1, V_2 \in W_2$

$\Rightarrow \quad V_1 + V_2 \in W_1 \quad$ and $\quad V_1 + V_2 \in W_2 \quad$ ($\because$ W_1 and W_2 are subspaces)

Also,

$\alpha V_1 \in W_1 \quad$ and $\quad \alpha V_1 \in W_2 \quad$ (")

(for any scalar α)

$\Rightarrow \quad V_1 + V_2 \in W_1 \cap W_2 \quad$ and $\quad \alpha V_1 \in W_1 \cap W_2$

Thus, $W_1 \cap W_2$ is a subspace of V.

Note: Union of two subspaces of a vector space V need not be a subspace of V.

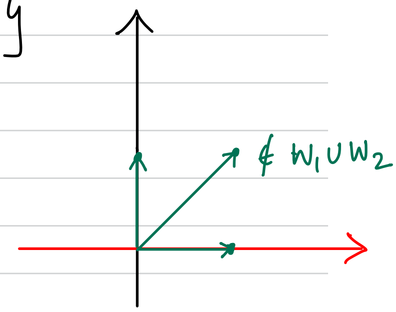
Counter example:

Consider $W_1 = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$, $W_2 = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} : y \in \mathbb{R} \right\}$

are subspaces of $\mathbb{R}^2$

But $W_1 \cup W_2$ is not a subspace of $\mathbb{R}^2$

$\left(\because \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \notin W_1 \cup W_2 \right)$



Thm 3: Let W_1 and W_2 be subspaces of a vector space V. Then $W_1 \cup W_2$ is a subspace of V iff $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Exercise:

True or false?

(a) If V is a vector space and W is a subset of V that is also a vector space, then W is a subspace of V.

(b) The empty set is a subspace of every vector space.

(c) If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$.

(d) The intersection of any two subsets of V is a subspace of V.

(e) Any union of subspaces of a vector space V is a subspace of V.

2) Which of the following subset are subspace of $\mathbb{R}^3$.

i) $S_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \; x_1 + x_2 - x_3 = 0 \right\}$

ii) $S_2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; x_1 x_2 - x_3 = 0 \right\}$

iii) $S_3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;\middle|\; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

iv) $S_4 = \left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \;\middle|\; \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

where c_1 and c_2 are scalars.