UNIT-1

VECTOR SPACES :

1> $m \times n \rightarrow$ with real entries

Closure : $A + B \in R^{m \times n}$

Commutative : $A + B = B + A$

Associativity : $(A+B)+C = A+(B+C$

Zero vector : $A + 0 = A$

Additive Inverse $A + (-A) = 0$

Scalar multiplication :

Closure : $A \in R^{m \times n}$ $\quad CA \in R^{m \times n}$

distributivity over vector addition

$c(A+B) = CA + CB$

distributivity over scalar addition

$c(dA) = (cd)A$

scalar identity : $1A = A$

2> $f(-x) = -f(x)$

$(f+g)(-x) = -(f+g)(x)$

$f(-x) + g(-x) = -(f(x) + g(x))$

$= -(f+g)(x)$

$f+g \in V$

Let $f \in V$ & $C \in R$

$cf(-x) = -(cf)(x)$

$c \cdot f(-x) = -(c \cdot f(x)) = -(cf)(x)$

$cf \in V$

$z(x) = 0$ is odd

Set of all odd functions from R to R with field R, under usual addition & scalar multiplication is a vector spce.

3> $f, g \in C[a, b]$

$f+g \in C[a,b]$

$f \in C[a,b)$, $C \in R$

Cf is continuous on $[a,b]$

$cf \in C[a,b]$

$z(x) = 0$ is continuous

$C[a,b]$ satisfies all eight axioms & is vector space.

4> $V = R^2$

$(x_1, x_2) + (y_1, y_2) = (x_1+y_1, x_2+y_2)$

$\alpha(x_1, x_2) = (x_1, x_2)$

Scalar axiom

$1 \cdot V = V$

$1 \cdot (x_1, x_2) = (x_1, x_2)$

$(\alpha + \beta) \cdot V = \alpha V + \beta V$

Let $V = (x_1, x_2)$

LHS

$(\alpha+\beta) \cdot (x_1, x_2) = (x_1, x_2)$

RHS $\alpha(x_1, x_2) + \beta(x_1, x_2)$

$= (x_1 + x_1, x_2 + x_2) = (2x_1, 2x_2)$

LHS $\neq$ RHS

No, V is not a vector space with these operations.

5> $V = R$ with scalar multiplication

$a \cdot x = a \cdot x$ & ~~$x \oplus y$~~

$x \oplus (-x) = z$

$x \oplus y = max(x, y)$

$y \oplus x = max(y, x)$

$(x \oplus y) \oplus w = max(x, y, w)$

$x \oplus (y \oplus w) = max(x, y, w)$

$(\alpha+\beta) \cdot x = (\alpha+\beta)x$

$\alpha \cdot x \oplus \beta \cdot x = max(\alpha x, \beta x)$

These are not equal

No, R is not vector space with these operations.

6) $V = R^+$

$\alpha \oplus \beta = \alpha\beta$ $\quad c \cdot \alpha = \alpha^c$

vector addition

Closure : $\alpha\beta > 0$ $\quad \alpha, \beta \in R$

Commutative : $\alpha\beta = \beta\alpha$

Associative : $(\alpha\beta)\gamma = \alpha(\beta\gamma)$

Zero vector : $\alpha \oplus z = \alpha$

$\alpha z = \alpha$

Additive inverse : $\alpha \oplus (-\alpha) = z$

Scalar multiplication

Closure : $\alpha^c > 0$ for $\alpha \in R^+$

$c \in R$

Distributivity over vector addition

$c \cdot (\alpha \oplus \beta) = c(\alpha\beta) = (\alpha\beta)^c = \alpha^c\beta^c$

$c \cdot \alpha \oplus c \cdot \beta = \alpha^c \oplus \beta^c = \alpha^c\beta^c$

$(c+d) \cdot \alpha = \alpha^{(c+d)} = \alpha^c\alpha^d$

$c \cdot \alpha \oplus d \cdot \alpha = \alpha^c \oplus \alpha^d = \alpha^c\alpha^d$

$c \cdot (d \cdot \alpha) = c \cdot \alpha^d = (\alpha^d)^c = \alpha^{cd}$

Scalar identity : $1 \cdot \alpha = \alpha$

SUBSPACES, LINEAR COMBINATION & LINEAR SPAN

7) Let $p(x)$ & $q(x)$ be S

$p(0) = 0$ & $q(0) = 0$

$(p+q)(0) = p(0) + q(0) = 0$

set S is closed under addition

Let $p(x)$ be in S & c be scalar

$p(0) = 0$

$(cp)(0) = c \cdot p(0) = c \cdot 0 = 0$

set S is closed under scalar multiplication

S is nonempty & closed under both addition & scalar multiplication

S is subspace of P_n.

8) $W = s\begin{bmatrix}1\\1\\2\\0\end{bmatrix} + t\begin{bmatrix}3\\-1\\-1\\4\end{bmatrix} = sv_1 + tv_2$

$W = \text{span}\{v_1, v_2\}$

Yes, W is subspace of R^4 because can be expressed as span of vectors

$\begin{bmatrix}1\\1\\2\\0\end{bmatrix}$ & $\begin{bmatrix}3\\-1\\-1\\4\end{bmatrix}$

9) $H = \left\{ \begin{bmatrix} a & b \\ c & a^2 \end{bmatrix} \mid a, b, c \in R \right\}$

$a = 1, b = 0, c = 0$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$k = 2$

$k \cdot A = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$2 \neq 4$

10) a) Yes, it is a subspace.

$x_1 + x_2 = 0$ is linear homogeneous equation

b) No, it is not a subspace

$x_1 \cdot x_2 = 0$; $x_1 = 0$ or $x_2 = 0$

$\begin{bmatrix}1\\0\end{bmatrix}$ & $\begin{bmatrix}0\\1\end{bmatrix}$ are set

sum $= \begin{bmatrix}1\\1\end{bmatrix}$

c) Yes, it is a subspace.

$x_1 = 3x_2$ is linear homogeneous equation. Represents line through origin.

d) No, it is not a subspace

$x_1^2 = x_2^2 \Rightarrow x_1 = x_2$ or $x_1 = -x_2$

$\begin{pmatrix}1\\1\end{pmatrix}$ & $\begin{pmatrix}1\\-1\end{pmatrix}$ sum $= \begin{pmatrix}2\\0\end{pmatrix}$

11 a> No, it is not a subspace
$x_1+x_3=1$ is not homogeneous.
zero vector $\begin{pmatrix}0\\0\\0\end{pmatrix}$ does not satisfy the condition $(0+0\neq 1)$

b> Yes, it is a subspace.
$x_1=x_2=x_3$ are linear & homogeneous
Represent line through origin

c> Yes, it is a subspace
$x_3=x_1+x_2$ is linear homogeneous eq, represents plane through origin.

d> No, it is not a subspace
$x_3=1$ is not homogeneous

12> a> Yes it is a subspace.
Set of diagonal matrices is closed under addition & Scalar multiplication & contain zero matrix

b> Yes, it is subspace, set of upper triangular matrices is closed under addition & scalar multi & contains zero matrix.

c> Yes, it is a subspace - Set of lower triangular matrices is closed under addition & scalar multi & contains zero matrix

d> No, it is not a subspace.
$a_{12}=1$ is not homogeneous.
zero matrix does not satisfy condition $(0\neq 1)$

e> Yes, it is subspace, $b_{11}=0$ is linear & homogeneous.

f> Yes, it is subspace. Set of symmetric matrices $(A=A^T)$ is closed under addition $(A+B)^T=A^T+B^T=A+B$ & scalar multip. $(cA)^T=cA^T=cA$ & contains zero matrix

g> No, $n=2$.
$A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ $B=\begin{pmatrix}0&0\\0&1\end{pmatrix}$
$\det(A)=\det B=0$
sum $A+B=\begin{pmatrix}1&0\\0&1\end{pmatrix}$
set is not closed under addition

13> a) $A=\begin{pmatrix}2&1\\3&2\end{pmatrix}$
$\det(A)=4-3=1\neq 0$
$Ax=0 \quad x=0$
Null space is $\{0\}$ or $\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$

b) $B=\begin{pmatrix}1&1&-1&2\\2&2&-3&1\\-1&-1&0&5\end{pmatrix}$ $\begin{matrix}R_2\to R_2-2R_1\\R_3\to R_3+R_1\end{matrix}$

$\sim\begin{pmatrix}1&1&-1&2\\0&0&-1&-3\\0&0&-1&-3\end{pmatrix}$ $R_3\to R_3-R_2$

$\sim\begin{pmatrix}1&1&-1&2\\0&0&-1&-3\\0&0&0&0\end{pmatrix}$ $R_2\to -R_2$

$\sim\begin{pmatrix}1&1&-1&2\\0&0&1&3\\0&0&0&0\end{pmatrix}$ R_1+R_2 $\sim\begin{pmatrix}1&1&0&5\\0&0&1&3\\0&0&0&0\end{pmatrix}$

$x_1+x_2+5x_4=0 \Rightarrow x_1=-x_2-5x_4$
$x_3+3x_4=0 \Rightarrow x_3=-3x_4$
Let $x_2=s$, $x_4=t$

$x=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}-s-5t\\s\\-3t\\t\end{pmatrix}$

$=s\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}+t\begin{pmatrix}-5\\0\\-3\\1\end{pmatrix}$

c) $\begin{pmatrix} 1 & 3 & -4 \\ 2 & -1 & -1 \\ -1 & -3 & 4 \end{pmatrix}$ $R_2 \to R_2 - 2R_1$
$R_3 \to R_3 + R_1$

$\sim \begin{pmatrix} 1 & 3 & -4 \\ 0 & -7 & 7 \\ 0 & 0 & 0 \end{pmatrix}$ $R_2 \to -\frac{1}{7} R_2$ $\sim \begin{pmatrix} 1 & 3 & -4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $R_1 \to R_1 - 3R_2$

$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $x_1 - x_3 = 0 \Rightarrow x_1 = x_3$
$x_2 - x_3 = 0 \Rightarrow x_2 = x_3$

Let $x_3 = t$

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

14. a) yes det $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$\det(A) = 1 \cdot \det\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - 0 \cdot \det\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} +$
$1 \cdot \det\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = 1(1-0) - 0 + 1(0-0)$
$= 1 \neq 0$

Vectors are linearly independent

b) Set contains four vectors in R^3
First three vectors are linearly independent, which is already basis for R^3

c) $A = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ -2 & -2 & 0 \end{pmatrix}$

$\det A = 2(0+4) - 3(0+4) + 2(-2+4)$
$= 0 = 0$

Vector are linearly dependent & do not span $R3$

d) vectors are scalar multiples of each other & lie on same line through origin. They are collinear & linearly dependent.

e) Set contains 2 vectors in $R3$. set must contain atleast 3 vectors
Set is not spanning set for $R3$.

15) $0 = 0 + 0 \in U + V$
Let $z_1, z_2 \in U + V$
$z_1 = u_1 + v_1 \quad z_2 = u_2 + v_2$
$u_1, u_2 \in U$ & $v_1, v_2 \in V$
$z_1 + z_2 = (u_1 + v_1) + (u_2 + v_2)$
$= (u_1 + u_2) + (v_1 + v_2)$
$u_1 + u_2 \in U$, $v_1 + v_2 \in V$
$z_1 + z_2 \in U + V$
$z \in U + V$ & c be scalar
$cz = c(u + v) = cu + cv$
Since U & V are subspaces, $cu \in U$ & $cv \in V$

16) a) $(-1, -1, 1) = -1(1, 1, -1)$
vectors are linearly dependent & span same 1-D space.

b) non zero vectors $(0, 1, 1)$ & $(1, 1, 0)$ are not scalar multiples & they are linearly independent

c) Rank of 2.
dim of col space = rank of matrix

d) set of all vectors with positive components is not subspace itself
subspace spanned is R^3

LINEAR INDEPENDENT & DEPENDENT

17 a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\det(A) = 1$
vectors are linearly independent.

b) $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

$\det(A) = 0$

vector are linearly dependent.

c) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

four vectors in R^3

any set of more than 3 vectors in R^3 must be linearly dependent.

18 a) $c_1 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $A_3 = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$

$A_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $A_1 = A_4$

$c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = 0$

$c_1 = c_2 = c_3 = c_4 = 0$

Vectors are linearly dependent

b) $B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$B_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B_4 = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$

$c_1B_1 + c_2B_2 + c_3B_3 + c_4B_4 = 0$

$c_1 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} +$

$c_4 \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$c_2 + 2c_4 = 0$ $\quad c_2 = -2c_4$

$c_1 + c_3 + 3c_4 = 0$ $\quad c_1 = 0$

$c_1 + c_2 + 2c_4 = 0$ $\quad c_2 = -2$

$c_3 = -3$

vectors are linearly dependent.

19. a) Linearly dependent.

set is $\{1, x^2, x^2 - 2\}$

$x^2 - 2 = 1 \cdot x^2 + (-2)$

one vector is linear combination of others & linearly dependent.

b) Linearly dependent

set is $\{2, x^2, x, 2x+3\}$

$2x + 3 = 2x + (1.5)2$

one set is linear combination of other

c) Linearly independent

set $\{x+2, x^2-1\}$

$c_1(x+2) + c_2(x^2-1) = 0$

$c_2x^2 + c_1x + 2c_1 - c_2 = 0$

$c_2 = 0, \; c_1 = 0$

20) null space $N(A)$ is set of all vectors x such that $Ax = 0$

Let $a_1, a_2 \ldots a_n$ be column vectors of A

$Ax = x_1a_1 + x_2a_2 + \ldots + x_na_n = 0$

$N(A) = \{0\}$

21. $3 + kt = c(2+t)$

$c = k$

$2c = 3$

$c = 3/2$

$k = 3/2$

22. $U = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} = a \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{E_{1}} + b \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{E_{2}}$

~~can~~ Standard basis of U is

$\left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

& its dimension is 2

23) a) 4 vectors are dependent because $R^3 = 3$ must be linearly dependent

b) two vectors v_1 & v_2 will be dependent if one is scalar multiple of other.

c) Vectors v_1 & $(0,0,0)$ are dependent because linear combination $c_1v_1 + c_2(0,0,0) = 0$

$c_1 = 0, c_2 = 1$

24) BASIS AND DIMENSION

24) a) $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$

$\det(A) = 6 - 4 = 2 \neq 0$

linearly independent

R^2 is 2-D vector space

R^2 spans space & forms basis

b) R^2 is 2

3 vectors (x_1, x_2, x_3) in 2D space

$\therefore$ must be linearly dependent

c) x_1 & x_2 already form basis of R^2. Vector x_3 can be expressed as linear combination of x_1 & x_2

Span of three vectors in R^2

is

25) a) $x, x-1, x^2+1$

$c_1x + c_2(x-1) + c_3(x^2+1) = 0$

$c_3x^2 + (c_1+c_2)x + (-c_2+c_3) = 0$

$c_3 = 0, c_1 + c_2 = 0, -c_2 + c_3 = 0$

$c_3 = 0, c_2 = 0, c_1 = 0$

vector are linearly independent

dim of subspace is 3.

b) $x^2, x^2 - x - 1, x + 1$

$c_1x^2 + c_2(x^2 - x - 1) + c_3(x+1) = 0$

$(c_1 + c_2)x^2 + (-c_2 + c_3)x + (-c_2 + c_3) = 0$

$c_1 + c_2 = 0, -c_2 + c_3 = 0,$

$c_1 = -1, c_2 = 1, c_3 = 1$

Vector are linearly dependent.

dim of subspace is 2

c) $2x, x - 2$

$c_1 2x + c_2(x-2) = 0$

$(2c_1 + c_2)x + (-2c_2) = 0$

$2c_1 + c_2 = 0 \quad -2c_2 = 0$

$c_2 = 0, c_1 = 0$

Vectors are linearly independent

dimension of subspace is 2

26) $Ax = 0$

$Ax = x_1a_1 + x_2a_2 + \dots + x_na_n$

linear combination equals zero vecto

$N(A) = \{0\}$

27) $A = \begin{bmatrix} 1 & 2 & -3 \\ -2 & -2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - 2R_1$

$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & -3 \\ 0 & 0 & 12 \end{bmatrix}$ $R_3 \to \frac{R_3}{12}$ $\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

rank of matrix = 3

28) Basis = it is a subspace

dim = 2

29) a) $V = (x, y, z, w)$ in R^4

$x = y = z = w$

$V = (x, x, x, x)$

$V = x(1,1,1,1)$

basis $= \{(1,1,1,1)\}$.

dimension is 1

b) $V = (x, y, z, -x-y-z)$

$= (x,0,0,-x) + (0,y,0,-y) + (0,0,z,-z)$

$V = x(1,0,0,-1) + y(0,1,0,-1) + z(0,0,1,-1)$

They are linearly independent

dimension is 3

c) $V = (x, y, z, w)$ in R^4

$(x,y,z,w) \cdot (1,1,0,0) = 0$

$x + y = 0 \Rightarrow y = -x$

$(x,y,z,w)(1,0,1,0) = 0$

$x + \cancel{y} + z + w = 0 \Rightarrow w = -x-z$

$V = (x, -x, z, -x-z)$

$V = (x, -x, 0, -x) + (0, 0, z, -z)$

Basis $= \{(1,-1,0,-1), (0,0,1,-1)\}$

dimension $= 2$

d) $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}$

basis for colspace $\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$

$Ux = 0$

$x_1 + x_3 + x_5 = 0 \Rightarrow x_1 = -x_3 - x_5$

$x_2 + x_4 = 0 \Rightarrow x_2 = -x_4$

$x = (-x_3 - x_5, -x_4, x_3, x_4, x_5)$

Basis $\{(-1,0,1,0,0), (0,-1,0,1,0), (-1,0,0,0,1)\}$

ROW SPACE & COLUMN SPACE

30) a) $A|I = \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 1 & 0 \\ 4 & 7 & 8 & 0 & 0 & 1 \end{array}\right]$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 4R_1$

$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & -5 & 0 & -2 & 1 & 0 \\ 0 & -5 & 0 & -4 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 - R_2$

$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & -5 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & -1 & 1 \end{array}\right] \quad R_2 \to -\frac{1}{5} R_2$

$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2/5 & -1/5 & 0 \\ 0 & 0 & 0 & -2 & -1 & 1 \end{array}\right] \quad R_1 \to R_1 - 3R_2$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1/5 & 3/5 & 0 \\ 0 & 1 & 0 & 2/5 & -1/5 & 0 \\ 0 & 0 & 0 & -2 & -1 & 1 \end{array}\right]$

dim $\cancel{\text{of}}$ (RS) = 2

Basis for CS $= \left\{ \begin{bmatrix}1\\2\\4\end{bmatrix}, \begin{bmatrix}3\\1\\7\end{bmatrix} \right\}$

dim (CS) = 2

$R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 + 2x_3 = 0$

$x_3 = t$

$x = \begin{bmatrix} -2t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

b) Basis of NS: $\{n_1\} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\dim(NS) = 1$

Basis of LNS: $\{l_1\} = \{[-2 \ -1 \ 1]\}$

$\dim(LNS) = 1$

$\text{rank}(A) = \dim(CS) = \dim(RS) = 2$

$\text{nullity}(A) = \dim(NS) = 1$

$n = 3$

$2 + 1 = 3$

b) $A = \begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix}$ $R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{bmatrix}$ $R_2 \to R_2 + 3R_1$, $R_3 \to R_3 + 3R_1$

$\sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 14 & 1 & -4 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$

$\sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $R_2 \to \frac{1}{7}R_2$

$\sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $R_1 \to R_1 + R_3$

$\sim \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $R_1 \to R_1 - 2R_2$

$R = \begin{bmatrix} 1 & 0 & 0 & -10/7 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Basis of RS $= [1 \ 0 \ 0 \ \frac{-10}{7}]$, $[0 \ 1 \ 0 \ \frac{-2}{7}]$, $[0 \ 0 \ 1 \ 0]$

$\dim(RS) = 3$

basis (CS) $= \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$

$\dim(CS) = 3$

$Rx = 0$

$x_1 - \frac{10}{7}x_4 = 0 \Rightarrow x_1 = \frac{10}{7}x_4$

$x_2 - \frac{2}{7}x_4 = 0 \Rightarrow x_2 = \frac{2}{7}x_4$

$x_3 = 0$, $x_4 = t$

$x = \begin{bmatrix} 10/7\,t \\ 2/7\,t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 10/7 \\ 2/7 \\ 0 \\ 1 \end{bmatrix}$

$x = t\begin{bmatrix} 10 \\ 2 \\ 0 \\ 7 \end{bmatrix}$

Basis ↑

$\dim(NS) = 1$

$A^T = \begin{bmatrix} -3 & 1 & -3 \\ 1 & 2 & 8 \\ 3 & -1 & 4 \\ 4 & -2 & 2 \end{bmatrix}$ rank $= 3$

Basis $= \{\ \}$ empty set

$\dim(LNS) = 0$ $\quad$ $\text{nullity}(A) = 1$

$n = 3 + 1 = 4$

c) $A = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$\sim \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & -5 & 7 & 0 \\ 0 & -5 & 11 & 3 \end{bmatrix}$ $R_3 \to R_3 - R_2$

$\sim \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & -5 & 7 & 0 \\ 0 & 0 & 4 & 3 \end{bmatrix}$ $R_2 \to -\frac{1}{5}R_2$, $R_3 \to \frac{1}{4}R_3$

$\sim \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & -7/5 & 0 \\ 0 & 0 & 1 & 3/4 \end{bmatrix}$ $R_2 \to R_2 + \frac{7}{5}R_3$, $R_1 \to R_1 + 2R_3$ 1 rank = 3

row space

$[1\ 3\ -2\ 1], [0\ -5\ 7\ 0], [0\ 0\ 4\ 3]$

dimension : 3

col space

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} \right\}$

null space = $\begin{bmatrix} 13 \\ -21 \\ -15 \\ 20 \end{bmatrix}$

dimension = 1

Rank - nullity = 3+1 = 4

d) $A = \begin{bmatrix} 2 & -4 & 1 & 2 & -2 & -3 \\ -1 & 2 & 0 & 0 & 1 & -1 \\ 10 & -4 & -2 & 4 & -2 & 4 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$\begin{bmatrix} -1 & 2 & 0 & 0 & 1 & -1 \\ 2 & -4 & 1 & 2 & -2 & -3 \\ 10 & -4 & -2 & 4 & -2 & 4 \end{bmatrix}$ $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 + 10R_1$

$\sim \begin{bmatrix} -1 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 & 0 & -5 \\ 0 & 16 & -2 & 4 & 8 & -6 \end{bmatrix}$ $R_2 \leftrightarrow R_3$

$\sim \begin{bmatrix} -1 & 2 & 0 & 0 & 1 & -1 \\ 0 & 16 & -2 & 4 & 8 & -6 \\ 0 & 0 & 1 & 2 & 0 & -5 \end{bmatrix}$ $R_2 \to \frac{1}{16} R_2$, $R_1 \to -R_1$

$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1/8 & 1/4 & 1/2 & -3/8 \\ 0 & 0 & 1 & 2 & 0 & -5 \end{bmatrix}$

$R_2 \to R_2 + \frac{1}{8} R_3$

$R_1 \to R_1 + 2R_2$

$R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1 \\ 0 & 0 & 1 & 2 & 0 & -5 \end{bmatrix}$

rank(A) = 3

null space $\Rightarrow x_1 + x_4 - x_6 = 0$

$x_1 = -x_4 + x_6$

$x_2 + \frac{1}{2}x_4 + \frac{1}{2}x_5 - x_6 = 0$

$x_2 = -\frac{1}{2}x_4 - \frac{1}{2}x_5 + x_6$

$x_3 + 2x_4 - 5x_6 = 0 \Rightarrow x_3 = -2x_4 + 5x_6$

$x = \begin{bmatrix} -x_4 + x_6 \\ -\frac{1}{2}x_4 - \frac{1}{2}x_5 + x_6 \\ -2x_4 + 5x_6 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ -1/2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -1/2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 1 \\ 1 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

dim(null space) = 3

Rank(A) + Nullity(A) = 3+3 = 6

3) a) R^5 A has 5 columns

b) $R^n = R^5$. as columns = 5

c) $R^m = R^3$ as rows = 3

d) $R^m = R^3$ rows = 3

e) $A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 2 & -3 & 3 & 3 & 5 \\ 4 & -6 & 9 & 5 & 9 \end{bmatrix}$

vector v has 3 components

A has 5 components

v cannot be in row space of A

f) $v = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$[A|v] = \left[\begin{array}{ccccc|c} 2 & -3 & 6 & 2 & 5 & 1 \\ 2 & -3 & 3 & 3 & 5 & 0 \\ 4 & -6 & 9 & 5 & 9 & 2 \end{array}\right]$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccccc|c} 2 & -3 & 6 & 2 & 5 & 1 \\ 0 & 0 & -3 & 1 & 0 & -1 \\ 0 & 0 & -3 & 1 & -1 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccccc|c} 2 & -3 & 6 & 2 & 5 & 1 \\ 0 & 0 & -3 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

$-x_5 = 1 \to x_5 = -1$

vector v is column space of A

g) $w = \begin{bmatrix} -5 \\ 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}$ $\quad Aw = 0$

$$Aw = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 2 & -3 & 3 & 3 & 5 \\ 4 & -6 & 9 & 5 & 9 \end{bmatrix} \begin{bmatrix} -5 \\ 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}$$

$2 \cdot -5 + (-3 \cdot 2) + 6 \cdot 1 + 2 \cdot 3 + 5 \cdot 0 = -4$

$2 \cdot -5 + (-3 \cdot 2) + 3 \cdot 1 + 3 \cdot 3 + 5 \cdot 0 = -4$

$4 \cdot -5 + (-6 \cdot 2) + 9 \cdot 1 + 5 \cdot 3 + 9 \cdot 0 = -8$

$\begin{bmatrix} -4 \\ -4 \\ -8 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ w is not in null space of A

h) $w = \begin{bmatrix} 4 \\ -2 \\ 9 \\ 5 \\ 5 \end{bmatrix}$

$A^T x = w$ $\left[\begin{array}{ccc|c} 2 & 2 & 4 & 4 \\ -3 & -3 & -6 & -2 \\ 6 & 3 & 9 & 9 \\ 2 & 3 & 5 & 5 \\ 5 & 6 & 9 & 5 \end{array}\right]$

$R_2 \to R_2 + 3R_1$, $R_3 \to R_3 - 6R_1$

$R_4 \to R_4 - 2R_1$, $R_5 \to R_5 - 5R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & -3 & -3 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -5 \end{array}\right]$$

$R_3 \to -\frac{1}{3} R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -5 \end{array}\right]$$

$R_4 \to R_4 - R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -5 \end{array}\right]$$

$0x_1 + 0x_2 + 0x_3 = 4$

w is not row space of A

32. $\text{rank}(A) + \dim(N(A)) = n$

$\text{rank}(A) = 7 - 3 = 4$

Left null space

$m - \text{rank}(A)$

$\dim(N(A^T)) = 9 - 4 = 5$

33. $\dim(R(A)) = \text{rank}(A) = 3$

$\dim(N(A)) = n - \text{rank}(A) = 7 - 3 = 4$

$\dim(N(A^T)) = m - \text{rank}(A) = 4 - 3 = 1$

34. row, null

35. left null

36. $Ax_1 = b$ & $Ax_2 = b$

$A(x_1 - x_2) = b - b = 0$

$x_1 - x_2$ must be null space of A.

37. $u_3 = 2u_1 + 1u_2$

$u_4 = u_1 + 4u_2$

$a_3 = 2a_1 + 1a_2$

$a_4 = a_1 + 4a_2$

$$a_3 = 2\begin{bmatrix}-3\\5\\2\\1\end{bmatrix} + 1\begin{bmatrix}4\\-3\\7\\-1\end{bmatrix}$$

$$a_3 = \begin{bmatrix}-2\\7\\11\\1\end{bmatrix}$$

$$a_4 = 1\begin{bmatrix}-3\\5\\2\\1\end{bmatrix} + 4\begin{bmatrix}4\\-3\\7\\-1\end{bmatrix} = \begin{bmatrix}13\\-7\\30\\-3\end{bmatrix}$$

38. nullspace of A^TA = nullspace of A

left nullspace of A^TA = left nullspace of A

col space of A^TA = rowspace of A

row space of A^TA = row space of A

LINEAR TRANSFORMATION

39) a) let $u = \begin{bmatrix}a_1\\b_1\end{bmatrix}$ $v = \begin{bmatrix}a_2\\b_2\end{bmatrix}$

$$T(u+v) = T\begin{bmatrix}a_1+a_2\\b_1+b_2\end{bmatrix} = \begin{bmatrix}2(a_1+a_2)\\3(b_1+b_2)\end{bmatrix}$$

$$T(u)+T(v) = \begin{bmatrix}2a_1\\3b_1\end{bmatrix} + \begin{bmatrix}2a_2\\3b_2\end{bmatrix} = \begin{bmatrix}2a_1+2a_2\\3b_1+3b_2\end{bmatrix}$$

$T(u+v) = T(u) + T(v)$

$$T(cu) = T\begin{bmatrix}ca_1\\cb_1\end{bmatrix} = \begin{bmatrix}2(ca_1)\\3(cb_1)\end{bmatrix} = \begin{bmatrix}c(2a_1)\\c(3b_1)\end{bmatrix}$$

$$cT(u) = c\begin{bmatrix}2a_1\\3b_1\end{bmatrix}$$

$T(cu) = cT(u)$, scalar homogeneity condition

$\therefore$ linear transformation

b) $T\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}a+c & 0\\0 & c-d\end{bmatrix}$

$A = \begin{bmatrix}a_1 & b_1\\c_1 & d_1\end{bmatrix}$ $B = \begin{bmatrix}a_2 & b_2\\c_2 & d_2\end{bmatrix}$

$$T(A+B) = T\begin{bmatrix}a_1+a_2 & b_1+b_2\\c_1+c_2 & d_1+d_2\end{bmatrix}$$

$$= \begin{bmatrix}(a_1+a_2)+(c_1+c_2) & 0\\0 & (c_1+c_2)-(d_1+d_2)\end{bmatrix}$$

$$T(A)+T(B) = \begin{bmatrix}a_1+c_1 & 0\\0 & c_1-d_1\end{bmatrix} + \begin{bmatrix}a_2-c_2 & 0\\0 & c_2-d_2\end{bmatrix}$$

$$= \begin{bmatrix}(a_1+c_1)+(a_2+c_2) & 0\\0 & (c_1-d_1)+(c_2-d_2)\end{bmatrix}$$

$$T(kA) = \begin{bmatrix}ka_1 & kb_1\\kc_1 & kd_1\end{bmatrix} = \begin{bmatrix}ka_1+kc_1 & 0\\0 & kb_1+kd_1\end{bmatrix}$$

$$= \cancel{kT(A)} = k\begin{bmatrix}a_1+c_1 & 0\\0 & b_1+d_1\end{bmatrix}$$

$= kT(A)$

c) $T\begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}c & a\\d & b\end{bmatrix}$

Yes, this is linear transformation

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d> Yes, this is linear transformation

$T(f(t)+g(t)) = t f(t)+tg(t)$

$= T(f(t)) + T(g(t))$

$T(cf(t)) = t(cf(t)) = cT(f(t))$

e> No, not linear transformation

Let $f(t) = t$ & $g(t) = t$

$f(t)+g(t) = 2t$

$T(f(t)+g(t)) = T(2t) = (2t)^2$
$= 4t^2$

$T(f(t))+T(g(t)) = T(t)+T(t)$
$= t^2+t^2 = 2t^2$

f> No, this is not linear transformation.

$$T(x+y) = \begin{bmatrix} x+y \\ (x+y)^2 \\ (x+y)^3 \end{bmatrix} = \begin{bmatrix} x+y \\ x^2+2xy+y^2 \\ x^3+3x^2y+3xy^2+y^3 \end{bmatrix}$$

$$T(x)+T(y) = \begin{bmatrix} x \\ x^2 \\ x^3 \end{bmatrix} + \begin{bmatrix} y \\ y^2 \\ y^3 \end{bmatrix} = \begin{bmatrix} x+y \\ x^2+y^2 \\ x^3+y^3 \end{bmatrix}$$

$T(x+y) \neq T(x)+T(y)$

$$T(cx) = \begin{bmatrix} cx \\ (cx)^2 \\ (cx)^3 \end{bmatrix} = \begin{bmatrix} cx \\ c^2x^2 \\ c^3x^3 \end{bmatrix}$$

$$cT(x) = c\begin{bmatrix} x \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} cx \\ cx^2 \\ cx^3 \end{bmatrix}$$

40. a) $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+z \\ y+z \\ x+2y+2z \end{pmatrix}$

$v = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$$\begin{cases} x+z = 1 \\ y+z = -1 \\ x+2y+2z = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 2 & 0 \end{array}\right) R_3 \to R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 1 & -1 \end{array}\right] R_3 \to R_3 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

$-z = 1 \Rightarrow z = -1$

$y+z = -1$

$y = 0$

$x+z = 1$

$x = 2$

$x = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

b) $L(x) = Ax$

$$A = \left[\begin{array}{ccc|c} -1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 3 \\ 2 & -1 & 1 & 2 \end{array}\right]$$

$R_2 \to R_2 + R_1$
$R_3 \to R_3 + 2R_1$

$$\sim \left[\begin{array}{ccc|c} -1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 4 \\ 0 & 3 & 1 & 4 \end{array}\right] R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} -1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

System is consistent & has multiple solutions.

41. Let $A = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}$

$u = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $b = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$ $c = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$

a) $T(u) = Au = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$T(u) = \begin{pmatrix} 5 \\ 1 \\ -9 \end{pmatrix}$

b) $Ax = b$

$\begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$

$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right]$ $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 + R_1$

$\sim \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{array}\right]$ $R_2 \to \frac{R_2}{14}$, $R_3 \to R_3 - 4R_2$

$\sim \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{array}\right]$

$x_2 = -1/2$

$x_1 - 3x_2 = 3$

$x_1 = 3/2$

$x = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$

c) $Ax = b$ has unique solution

No, there is only 1 unique x whose image is b

d) $T(x) = c$

$Ax = c$

$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right]$ $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 + R_1$

$\sim \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{array}\right]$ $R_2 \to \frac{R_2}{14}$

$\sim \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 4 & 8 \end{array}\right]$ $R_3 \to R_3 - 4R_2$

$\sim \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 10 \end{array}\right]$

$\cancel{x_2 = -\frac{1}{2}}$ $0x_1 + 0x_2 = 10$

no solution.

42. $V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$e_1 = \frac{1}{3} V_1 - \frac{2}{3} V_2$

$e_2 = \frac{1}{3} V_1 + \frac{1}{3} V_2$

$T(e_1) = T\left(\frac{1}{3} V_1 - \frac{2}{3} V_2\right)$

$= \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 5 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix}$

$T(e_2) = T\left(\frac{1}{3}V_1 + \frac{1}{3}V_2\right)$

$= \frac{1}{3}\begin{bmatrix}1\\-1\\5\end{bmatrix} + \frac{1}{3}\begin{bmatrix}5\\-2\\-2\end{bmatrix} = \begin{bmatrix}2\\-1\\1\end{bmatrix}$

$A = \begin{bmatrix}-3 & 2\\1 & -1\\3 & 1\end{bmatrix}$

$T\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-3 & 2\\1 & -1\\3 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

$= \begin{bmatrix}-3x_1 + 2x_2\\x_1 - x_2\\3x_1 + x_2\end{bmatrix}$

43) $e_1 = \begin{bmatrix}1\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\end{bmatrix}$

$R = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

$R\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$

44) $V_1 = \begin{bmatrix}1\\1\end{bmatrix}, V_2 = \begin{bmatrix}1\\-1\end{bmatrix}$

$e_1 = \begin{bmatrix}1\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\end{bmatrix}$

$e_1 = \frac{1}{2}V_1 + \frac{1}{2}V_2$

$e_2 = \frac{1}{2}V_1 - \frac{1}{2}V_2$

$T(e_1) = \frac{1}{2}T(V_1) + \frac{1}{2}T(V_2)$

$= \frac{1}{2}\begin{bmatrix}1\\2\\3\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\2\end{bmatrix}$

$T(e_2) = \frac{1}{2}T(V_1) - \frac{1}{2}T(V_2)$

$A = \begin{bmatrix}1 & 0\\1 & 1\\2 & 1\end{bmatrix}$

$R(T) = \text{span}\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$

$n = c_1 \times c_2 = \begin{bmatrix}-1\\-1\\1\end{bmatrix}$

$-a - b + c = 0$

$a + b - c = 0$

$\begin{bmatrix}1 & 0\\1 & 1\\2 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$x = 0$

$x + y = 0$

$2x + y = 0$

45) a) Let $u = \begin{bmatrix}u_1\\u_2\end{bmatrix}$ $v = \begin{bmatrix}v_1\\v_2\end{bmatrix}$

$L(cu+v) = L\begin{bmatrix}cu_1+v_1\\cu_2+v_2\end{bmatrix} = \begin{bmatrix}-cu_1-v_1\\cu_2+v_2\end{bmatrix}$

$cL(u) + L(v) = c\begin{bmatrix}-u_1\\u_2\end{bmatrix} + \begin{bmatrix}-v_1\\v_2\end{bmatrix}$

$= \begin{bmatrix}-cu_1\\cu_2\end{bmatrix} + \begin{bmatrix}-v_1\\v_2\end{bmatrix}$

$= \begin{bmatrix}-cu_1 - v_1\\cu_2 + v_2\end{bmatrix}$

$L(cu+v) = cL(u) + L(v)$

L is a linear operator.

b) $L(x) = -x$

$L(u+v) = -(u+v) = -u - v$

$= L(u) + L(v)$

$L(cu) = -(cu) = -cu$

$= c(-u) = cL(u)$

Transformation is linear operator

c) $L(x) = x_2 e_2$

$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad L(x) = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$

Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$L(u+v) = \begin{bmatrix} 0 \\ u_2+v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ v_2 \end{bmatrix}$

$= L(u) + L(v)$

$L(cu) = \begin{bmatrix} 0 \\ cu_2 \end{bmatrix} = c\begin{bmatrix} 0 \\ u_2 \end{bmatrix} = cL(u)$

Transformation is linear operator

46) $\begin{bmatrix} 7 \\ 5 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$c_1 + c_2 = 7$

$2c_1 - c_2 = 5$

$(c_1+c_2) + (2c_1 - c_2) = 7+5$

$3c_1 = 12$

$c_1 = 4$

$4 + c_2 = 7$

$c_2 = 3$

$\begin{bmatrix} 7 \\ 5 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$L\begin{bmatrix} 7 \\ 5 \end{bmatrix} = L\left(4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

$= 4L\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3L\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$L\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$L\begin{bmatrix} 7 \\ 5 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 23 \\ 18 \end{bmatrix}$

47) a) $L(x) = [x_1, x_2, 1]$

$L\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ Not linear transformation

b) $L(x) = [x_1, x_2, x_1 + 2x_2]$

$L(x) = Ax$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$

$\therefore$ A linear transformation

$L(u+v) = L(u) + L(v)$

& $L(cu) = cL(u)$

c) $L(x) = [x_1, 0, 0]$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \leftarrow L(x) = Ax$

$\therefore$ Matrix transformation are linear

d) $L(x) = [x_1, x_2, x_1^2 + x_2^2]$

Let $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad L(x) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$2x = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad L(2x) = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$

$2L(x) = 2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

$L(2x) \neq 2L(x) \therefore$ not linear

48) a) $L(x) = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}$

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow L(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = 0, x_2 = 0, x_3 = 0$

kernel is $\{0\}$

range $= R^3$

b) $L(x) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$

$x_1 = 0, x_2 = 0$

x_3 can be any real no.

$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \Rightarrow$ kernel form $= \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$

range $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ for any $a, b \in R$

c) $L(x) = \begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix}$

$x_1 = 0$ & x_2 & x_3 = any real no

kernel set of vectors $= \begin{bmatrix} 0 \\ b \\ c \end{bmatrix}$

for any $b, c \in R$

$\begin{bmatrix} a \\ a \\ a \end{bmatrix}$ for any $a \in R$

49) a) $L(x) = \begin{bmatrix} x_1 + x_2 \\ 0 \end{bmatrix}$

$L(x) = Ax$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

b) $L(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

c) $L(x) = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$

$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

50) $L(x) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 3x_1 + 2x_2 \end{bmatrix}$

$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$L(e_1) = L\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

$L(e_2) = L\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix}$

51) $L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y - 3z \\ 4x - 5y - 6z \\ 7x + 8y + 9z \end{bmatrix}$

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$L(e_1) = L\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0-0 \\ 4-0-0 \\ 7+0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$

$L(e_2) = L\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 8 \end{bmatrix}$

$L(e_3) = L\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 & -3 \\ 4 & -5 & -6 \\ 7 & 8 & 9 \end{bmatrix}$

52) $L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + 3y - z \\ 4y + 5z \end{bmatrix}$

Standard basis: $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$,

$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

1

$L(e_1) = L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\end{bmatrix}$

$L(e_2) = L\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix}$

$L(e_3) = L\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}-1\\5\end{bmatrix}$

$$A = \begin{bmatrix}2 & 3 & -1\\0 & 4 & 5\end{bmatrix}$$

53.a) $A = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\-y\end{pmatrix}$$

reflection through x-axis
maps point (x,y) to $(x,-y)$

b) $A = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-x\\y\end{bmatrix}$$

reflection through y-axis
maps point $(x,y$ to $(-x,y)$

c) $y = x$ $\quad A = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}y\\x\end{bmatrix}$$

x & y coordinates are swapped
reflection through line $y = x$
maps point (x,y) to (y,x)

d) $y = -x$

$$A = \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$$

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-y\\-x\end{bmatrix}$$

x & y coordinates are swapped
& both change their sign
point (x,y) to $(-y,-x)$

e) $A = \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-x\\-y\end{pmatrix}$$

x & y coordinates change their sign.
(x,y) to $(-x,-y)$

f) $A = \begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix}$

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix}x\cos\theta - y\sin\theta\\x\sin\theta + y\cos\theta\end{bmatrix}$$

Rotation by angle θ maps
point (x,y) to $(x\cos\theta - y\sin\theta,$
$x\sin\theta + y\cos\theta)$

g) $A = \begin{pmatrix}k & 0\\0 & 1\end{pmatrix}$

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}k & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}kx\\y\end{pmatrix}$$

x-coordinate is scaled,
y-coordinate remains unchanged
x-axis is stretch /
compressed
Horizontal scaling

h) $A = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$

stretches/compresses an object vertically relative to x-axis

i) $A = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

changes an object's size relative to origin without distorting its shape.

j) $A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

Horizontal lines remain horizontal & vertical lines are slanted.

k) $A = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$

Vertical lines remain vertical, horizontal line are slanted

54) $R_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$R_{\pi/2} = \begin{pmatrix} \cos \pi/2 & -\sin \pi/2 \\ \sin\pi/2 & \cos \pi/2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$D_3 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

$M = D_3 \cdot R_{\pi/2} \cdot R_x$

$= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$M = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$

$V = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ $\quad V' = MV \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$V' = \begin{pmatrix} 3 \\ 12 \end{pmatrix}$

56. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $V \in \mathbb{R}^2$

$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$p = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ $\quad R_\theta x = p$

$\det(R_\theta) = \cos^2\theta - (-\sin^2\theta) = 1$

$R_{\pi/2} = \begin{pmatrix} \cos\pi/2 & -\sin\pi/2 \\ \sin\pi/2 & \cos\pi/2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$q = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$q' = R_{\pi/2}\, q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$q' = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

a) Yes, preimage of $(1,-3)$ exists for any θ because rotation matrix is invertible

b) Yes, image of $(3,-1)$ $\theta = \frac{\pi}{2}$ exists & point $(1,3)$