

AI1103-Challenging problem

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QUESTION

Prove by properties of Q-function the following inequality,

$$1 - e^{-2\pi} \geq \left(1 - 2Q\left(\frac{1}{2}\right)\right)^2$$

SOLUTION

Simplifying the above inequality.

$$1 - e^{-2\pi} \geq \left(1 - 2Q\left(\frac{1}{2}\right)\right)^2 \quad (0.0.1)$$

$$1 - e^{-2\pi} \geq 1 + 4Q^2\left(\frac{1}{2}\right) - 4Q\left(\frac{1}{2}\right) \quad (0.0.2)$$

$$-e^{-2\pi} \geq 4Q\left(\frac{1}{2}\right)\left(Q\left(\frac{1}{2}\right) - 1\right) \quad (0.0.3)$$

$$4Q\left(\frac{1}{2}\right)\left(1 - Q\left(\frac{1}{2}\right)\right) \geq e^{-2\pi} \quad (0.0.4)$$

Using,

$$Q(x) + Q(-x) = 1 \quad (0.0.5)$$

$$4Q\left(\frac{1}{2}\right)Q\left(\frac{-1}{2}\right) \geq e^{-2\pi} \quad (0.0.6)$$

Let $erfc(x)$ denote the Complementary Error function.

Lets define $f(x)$ as,

$$f(x) = \alpha e^{-\beta x^2} \quad (0.0.7)$$

where α and β are parameters.

Property of Q-function :

$$2Q(x\sqrt{2}) = erfc(x) \quad (0.0.8)$$

Chernoff lower bound property of Complementary Error function :

$$erfc(x) \geq f(x) \quad (0.0.9)$$

$$2Q(x\sqrt{2}) \geq f(x) \quad (0.0.10)$$

$$\text{for, } \beta > 1, 0 < \alpha \leq \frac{\sqrt{2e} \sqrt{\beta - 1}}{\sqrt{\pi}\beta}$$

Since $f(x)$ is even,

$$4Q(x\sqrt{2}) \cdot Q(-x\sqrt{2}) \geq \alpha^2 e^{-2\beta x^2} \quad (0.0.11)$$

Putting $x = \frac{1}{2\sqrt{2}}$, we get,

$$4Q\left(\frac{1}{2}\right) \cdot Q\left(-\frac{1}{2}\right) \geq \alpha^2 \exp\left(\frac{-\beta}{4}\right) \quad (0.0.12)$$

For $\beta = 8\pi$,

$$0 < \alpha \leq \frac{\sqrt{2e} \sqrt{8\pi - 1}}{\sqrt{\pi} 8\pi} \quad (0.0.13)$$

$$(0.0.14)$$

Clearly, denominator is greater than numerator, therefore, $\alpha < 1$

$$4Q\left(\frac{1}{2}\right) \cdot Q\left(-\frac{1}{2}\right) \geq \alpha^2 \exp(-2\pi) \quad (0.0.15)$$

where $\alpha < 1$

Therefore,

$$4Q\left(\frac{1}{2}\right) \cdot Q\left(-\frac{1}{2}\right) \geq \exp(-2\pi) \quad (0.0.16)$$

Hence proved