AI1103-Challenging problem

Name : Aayush Patel, Roll No.: CS20BTECH11001

Question

Prove by properties of Q-function the following inequality, we get,

$$1 - e^{-2\pi} \geq \left(1 - 2Q\left(\frac{1}{2}\right)\right)^2$$

Solution

Simplifying the above inequality.

$$1 - e^{-2\pi} \geq \left(1 - 2Q\left(\frac{1}{2}\right)\right)^2 \tag{0.0.1}$$

$$1 - e^{-2\pi} \geq 1 + 4Q^2\left(\frac{1}{2}\right)^2 - 4Q\left(\frac{1}{2}\right) \tag{0.0.2}$$

$$-e^{-2\pi} \geq 4Q\left(\frac{1}{2}\right)\left(Q\left(\frac{1}{2}\right) - 1\right) \tag{0.0.3}$$

$$4Q\left(\frac{1}{2}\right)\left(1 - Q\left(\frac{1}{2}\right)\right) \geq e^{-2\pi} \tag{0.0.4}$$

Using,

$$Q(x) + Q(-x) = 1 \tag{0.0.5}$$

$$4Q\left(\frac{1}{2}\right)Q\left(\frac{-1}{2}\right) \geq e^{-2\pi} \tag{0.0.6}$$

Let $erfc(x)$ denote the Error function.
Lets define $f(x)$ as,

$$f(x) = \alpha e^{-\beta x^2} \tag{0.0.7}$$

where α and β are parameters and $x \geq 0$

Chernoff lower bound property of Error function :

$$erfc(x) \geq f(x) \tag{0.0.8}$$

for, $\beta > 1, 0 < \alpha \leq \dfrac{\sqrt{2e}\sqrt{\beta - 1}}{\sqrt{\pi}\beta}$

Property of Q-function :

$$2Q(x\sqrt{2}) = erfc(x) \tag{0.0.9}$$

Since $f(x)$ is even,

$$erfc(x) \cdot erfc(-x) \geq f^2(x) \tag{0.0.10}$$

$$4Q(x\sqrt{2}) \cdot Q(-x\sqrt{2}) \geq \alpha^2 e^{-2\beta x^2} \tag{0.0.11}$$

Putting $x = \dfrac{1}{2\sqrt{2}}$, we get,

$$4Q\left(\frac{1}{2}\right) \cdot Q\left(-\frac{1}{2}\right) \geq \alpha^2 exp\left(\frac{-\beta}{4}\right) \tag{0.0.12}$$

For $\beta = 8\pi$,

$$0 < \alpha \leq \frac{\sqrt{2e}\sqrt{8\pi - 1}}{\sqrt{\pi}8\pi} \tag{0.0.13}$$

$$\tag{0.0.14}$$

Clearly, denominator is greater than numerator, therefore, $\alpha < 1$

$$4Q\left(\frac{1}{2}\right) \cdot Q\left(-\frac{1}{2}\right) \geq \alpha^2 exp(-2\pi) \tag{0.0.15}$$

where $\alpha < 1$

Therefore,

$$4Q\left(\frac{1}{2}\right) \cdot Q\left(-\frac{1}{2}\right) \geq exp(-2\pi) \tag{0.0.16}$$

Hence, Proved