AI1103-Challenging problem

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Question

Prove by properties of Q-function the following inequality,

$$1 - e^{-2\pi} \geq \left(1 - 2Q\left(\frac{1}{2}\right)\right)^2$$

Solution

Simplifying the above inequality.

$$1 - e^{-2\pi} \geq \left(1 - 2Q\left(\frac{1}{2}\right)\right)^2 \tag{0.0.1}$$

$$1 - e^{-2\pi} \geq 1 + 4Q^2\left(\frac{1}{2}\right)^2 - 4Q\left(\frac{1}{2}\right) \tag{0.0.2}$$

$$-e^{-2\pi} \geq 4Q\left(\frac{1}{2}\right)\left(Q\left(\frac{1}{2}\right) - 1\right) \tag{0.0.3}$$

$$4Q\left(\frac{1}{2}\right)\left(1 - Q\left(\frac{1}{2}\right)\right) \geq e^{-2\pi} \tag{0.0.4}$$

Using,

$$Q(x) + Q(-x) = 1 \tag{0.0.5}$$

$$4Q\left(\frac{1}{2}\right)Q\left(\frac{-1}{2}\right) \geq e^{-2\pi} \tag{0.0.6}$$

Lets define $f(x)$ as,

$$f(x) = \alpha e^{-\beta x^2} \tag{0.0.7}$$

where α and β are parameters.

Chernoff lower bound property of Q-function :

$$2Q(x\sqrt{2}) \geq f(x) \tag{0.0.8}$$

for, $\beta > 1, 0 < \alpha \leq \dfrac{\sqrt{2e}\sqrt{\beta - 1}}{\sqrt{\pi}\beta}$

Since $f(x)$ is even,

$$4Q(x\sqrt{2}) \cdot Q(-x\sqrt{2}) \geq \alpha^2 e^{-2\beta x^2} \tag{0.0.9}$$

Putting $x = \dfrac{1}{2\sqrt{2}}$, we get,

$$4Q\left(\frac{1}{2}\right) \cdot Q\left(-\frac{1}{2}\right) \geq \alpha^2 exp\left(\frac{-\beta}{4}\right) \tag{0.0.10}$$

For $\beta = 8\pi$,

$$0 < \alpha \leq \frac{\sqrt{2e}\sqrt{8\pi - 1}}{\sqrt{\pi}8\pi} \tag{0.0.11}$$

$$\tag{0.0.12}$$

Clearly, denominator is greater than numerator, therefore, $\alpha < 1$

$$4Q\left(\frac{1}{2}\right) \cdot Q\left(-\frac{1}{2}\right) \geq \alpha^2 exp(-2\pi) \tag{0.0.13}$$

where $\alpha < 1$

Therefore,

$$4Q\left(\frac{1}{2}\right) \cdot Q\left(-\frac{1}{2}\right) \geq exp(-2\pi) \tag{0.0.14}$$

Hence proved