AI1103-Challenging problem

Name : Aayush Patel, Roll No.: CS20BTECH11001

Question

Prove by properties of Q-function the following inequality,

$$1-\exp(-2\pi) \geq \left(1-2Q\left(\frac{1}{2}\right)\right)^2$$

Solution

Some Properties of Q function:

Property 1:

$$Q(x)+Q(-x)=1$$

Property 2(Chernoff Lower Bound Property):

$$2Q(x\sqrt{2}) \geq f(x)$$

Where $f(x)=\alpha\exp(-\beta x^2)$ and
$\beta > 1, 0 < \alpha \leq \frac{\sqrt{2e}\sqrt{\beta-1}}{\sqrt{\pi}\beta}$

Simplifying the inequality given in question,

$$1-\exp(-2\pi) \geq \left(1-2Q\left(\frac{1}{2}\right)\right)^2 \quad (0.0.1)$$

$$1-\exp(-2\pi) \geq 1+4Q^2\left(\frac{1}{2}\right)^2-4Q\left(\frac{1}{2}\right) \quad (0.0.2)$$

$$-\exp(-2\pi) \geq 4Q\left(\frac{1}{2}\right)\left(Q\left(\frac{1}{2}\right)-1\right) \quad (0.0.3)$$

$$4Q\left(\frac{1}{2}\right)\left(1-Q\left(\frac{1}{2}\right)\right) \geq \exp(-2\pi) \quad (0.0.4)$$

Using Property 1 of Q-functions,

$$Q(x)+Q(-x)=1 \quad (0.0.5)$$

$$4Q\left(\frac{1}{2}\right)Q\left(\frac{-1}{2}\right) \geq \exp(-2\pi) \quad (0.0.6)$$

Lemma 1: $4Q\left(\frac{1}{2}\right)Q\left(\frac{-1}{2}\right) \geq \exp(-2\pi)$

Proof:
Using Property 2 of Q-functions and the fact that $f(x)$ is even.

$$4Q(x\sqrt{2})\cdot Q(-x\sqrt{2}) \geq \alpha^2\exp(-2\beta x^2) \quad (0.0.7)$$

Putting $x=\frac{1}{2\sqrt{2}}$, we get,

$$4Q\left(\frac{1}{2}\right)\cdot Q\left(-\frac{1}{2}\right) \geq \alpha^2 exp\left(\frac{-\beta}{4}\right) \quad (0.0.8)$$

For $\beta = 8\pi$,

$$0 < \alpha \leq \frac{\sqrt{2e}\sqrt{8\pi-1}}{\sqrt{\pi}8\pi} \quad (0.0.9)$$

$$(0.0.10)$$

Clearly, denominator is greater than numerator, therefore, $\alpha < 1$

$$4Q\left(\frac{1}{2}\right)\cdot Q\left(-\frac{1}{2}\right) \geq \alpha^2 exp(-2\pi) \quad (0.0.11)$$

where $\alpha < 1$

Therefore,

$$4Q\left(\frac{1}{2}\right)\cdot Q\left(-\frac{1}{2}\right) \geq exp(-2\pi) \quad (0.0.12)$$

Hence proved