6
Chapter

STOCHASTIC PROCESS

CHAPTER OUTLINE

After studying this chapter, students will be able to understand the:

- Definition and Classification.
- Markov Process, Markov chain, Matrix approach, Steady-state distribution.
- Counting Process, Binomial process, Poisson process, simulation of stochastic process.
- Queuing system: Main component of queuing system, Little's law.
- Bernulli single sever queing process, system with limited capacity.
- M/M/1 system: Evaluating the system performance.

Definition and Classification

A family of random variables indexed by a parameter such as time is called stochastic process. It is also called chance or random process. Hence stochastic process is a family of random variables $\{X(t)/t \in T\}$ defined on given probability space, indexed by the parameter t where t varies over an index set T. The values assumed by random variable X(t) are called states and the set of all possible values forms the state space of the process. The state space is denoted by I. If the state space of a stochastic process is discrete then it is called a discrete state process often referred as chain. In this case state space is assumed to be {0, 1, 2, 3 ….}. If the state space of stochastic process is continuous then it is called continuous state process. If the index set is discrete then it is called discrete parameter process and if the index set is continuous then it is called continuous parameter process.

		Index set T	
		Discrete	Continuous
State space I	Discrete	Discrete parameter stochastic chain	Continuous parameter stochastic chain
	Continuous	Discrete parameter continuous state process	Continuous parameter continuous state process

Markov Process

Stochastic process X(t) is called Markov process if for any $t_1 < t_2 < t_3 \ldots < t_n < t$ and any sets A_1, A_2, A_3 ….. A_n.

$P\{X(t) \in A\ X(t_1) \in A_1\ X(t_2) \in A_2 \ldots X(t_n) \in A_n\} = P[X(t) \in A, X(t_n) \in A_n\}$

P(future/ past, present) = P(future/present)

A stochastic process such that probability distribution for its future development depend only on the present state and not on how process arrived in that state. If the state space I is discrete then the Markov Process is called Markov Chain. If we further assume that parameter space T is also discrete then we get discrete parameter Markov Chain.

Markov Chain

Let $\{X_n\}$ be a sequence of values describing a mutually exclusive and exhaustive system of events. Let X_n values take only discrete the union I of all possible values of X_n is then a countable set called the state space of the process. Each element $i \in I$ is called a state. The index n is of time. The number of events may be finite or infinite the values of $\{X_n\}$ is said to be a Markov chain or Markov dependent if for all $i_0, i_1, i_2, \ldots i_{n-1}, i_n \in I$ and for all n.

$P(X_n = i_n\ (X_0 = i_0, X_1 = i_1 \ldots\ldots\ldots X_{(n-1)} = i_{(n-1)}) = P(X_n = i_n\ X_{n-1} = i_{n-1})$

The conditional distribution of X_n given the values $X_0, X_1, \ldots\ldots, X_{n-1}$ depends only on X_{n-1} not on the preceding values. If the state space is finite then we have finite Markov Chain.

Transition Probability

The probability of moving from one state to another or remain in the same state in a single period of time is called transition probability. The probability of moving from one state to another depends upon the probability of preceding state. Hence transition probability is a conditional probability.

$P\{X(t+1) = j/X(t) = i\} = P_{ij}(t)$ is transition probability from state i to state j in time t

Transition probability matrix

It is matrix obtained by using transition probabilities of various states.

Let the state space be I = {0, 1, 2, 3, ... n} then

$$P = \begin{pmatrix} P_{00} & P_{01} & P_{02} & P_{0n-2} & P_{0n-1} & P_{0n} \\ P_{10} & P_{11} & P_{12} & P_{1n-2} & P_{1n-1} & P_{1n} \\ P_{20} & P_{21} & P_{22} & P_{2n-2} & P_{2n-1} & P_{2n} \\ P_{n0} & P_{n1} & P_{n2} & P_{nn-2} & P_{nn-1} & P_{nn} \end{pmatrix}$$

$= \{P_{ij}\}$ $i, j \in I$ is called transition probability matrix.

Here P_{ij} is the conditional probability of moving from state i to state j . All conditional one step state probabilities can be represented as elements of a square matrix which is called transition probabilities matrix.

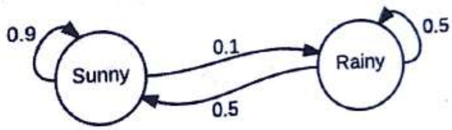
Here, $P_{ij} \geq 0$

$$\sum_i P_{ij} = 1$$

$$\sum_j P_{ij} = 1$$

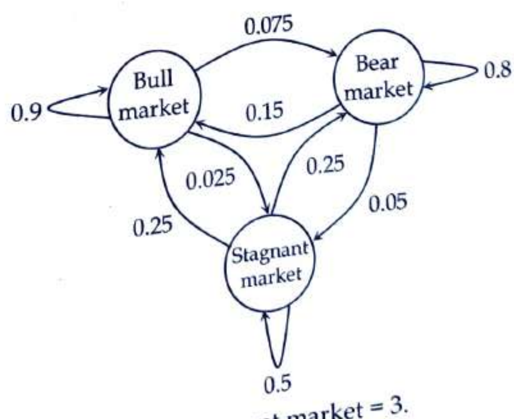
$P_{ij} = P\{X(t+1) = j/X(t) = i\}$ is transition probability

$P^{(n)}_{ij} = P\{X(t+n) = j/X(t) = i\}$ is n step transition probability



Let sunny = 1, Rainy = 2

Transition probability matrix $P = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}$



Let Bull market = 1, Bear mrket = 2 and stagnant market = 3.

Transition probability matrix

$$P = \begin{bmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

n Step Transition Probability

Probability $P_{ij}(t) = P\{X(t+1) = j / X(t) = i\}$
$= P\{X(t+1) = j/X(t) = i, X(t-1) = h, X(t-2) = g \ldots\}$ is called transition probability of moving from state i to state j is n step transition probability

Probability $P_{ij}(t)^n = P\{X(t+n) = j / X(t) = i\}$

$$P_{ij}(n) = \sum_{k=1}^{m} P_{ik}^{(n-1)} P_{kj}^{(1)}$$

$$P_{ij}(2) = \sum_{k=1}^{m} P_{ik} P_{kj}$$

$$P_{ij}(3) = \sum_{k=1}^{m} \sum_{l=1}^{m} P_{jk} P_{kl} P_{lj}$$

Example 1: Find 2 step and 3 step transition probability matrix from the transition probability matrix

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution:

2 step transition probability matrix

$$P^{(2)} = P\,P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3 step transition probability matrix

$$P^{(3)} = P^{(2)}\,P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 2: Find 2 step and 3 step transition probability matrix from the transition probability matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix}$$

Solution:

2 step transition probability matrix

$$P^{(2)} = P\,P = \begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & p \\ q & q+p & 0 \\ 0 & 0 & p \end{bmatrix} = \begin{bmatrix} q & 0 & p \\ 0 & 1 & 0 \\ q & 0 & p \end{bmatrix}$$

3 step transition probability matrix

$$P^{(3)} = P^{(2)}\,P = \begin{bmatrix} q & 0 & p \\ 0 & 1 & 0 \\ q & 0 & p \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & q+p & 0 \\ q & 0 & p \\ 0 & q+p & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix}$$

Example 3: If $P = \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix}$

Find $P_{12}(2)$ $P_{22}(2)$.

$$P_{12}(2) = \sum_{k=1}^{m} P_{1k}P_{k2} = P_{11}P_{12} + P_{12}P_{22}$$

$$= 0.5 \times 0.5 + 0.5 \times 0.6$$

$$= 0.25 + 0.30 = 0.55$$

$$P_{22}(2) = \sum_{k=1}^{m} P_{2k}P_{k2} = P_{21}P_{12} + P_{22}P_{22}$$

$$= 0.4 \times 0.5 + 0.6 \times 0.6$$

$$= 0.2 + 0.36 = 0.56$$

Example 4: If $P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$

Find $P_{11}(3)$, $P_{21}(3)$.

$$P_{11}(3) = \sum_{k=1}^{m} \sum_{l=1}^{m} P_{1k}P_{kl}\,P_{l1}$$

$$= P_{11}P_{11}P_{11} + P_{21}P_{12}P_{21} + P_{12}P_{21}P_{11} + P_{12}P_{22}P_{21}$$

$$= 0.7 \times 0.7 \times 0.7 + 0.7 \times 0.3 \times 0.4 + 0.3 \times 0.4 \times 0.7 + 0.3 \times 0.6 \times 0.4$$

$$= 0.583$$

$$P_{21}(3) = \sum_{k=1}^{m} \sum_{l=1}^{m} P_{2k}P_{kl}\,P_{l1}$$

$$= P_{21}P_{11}P_{11} + P_{21}P_{12}P_{21} + P_{22}P_{21}P_{11} + P_{22}P_{22}P_{21}$$

$$= 0.4 \times 0.7 \times 0.7 + 0.4 \times 0.3 \times 0.4 + 0.6 \times 0.4 \times 0.7 + 0.6 \times 0.6 \times 0.4$$

$$= 0.196 + 0.048 + 0.168 + 0.144$$

$$= 0.556$$

Example 5: In some town each day is either sunny or rainy. A sunny day is followed by another sunny day with probability 0.7, whereas a rainy day is followed by sunny day with probability 0.4. It rains on Monday Make forecast for Tuesday, Wednesday and Thursday.

Solution:

Let state 1 = sunny and state 2 = rainy

Transition probabilities are

$P_{11} = 0.7$, $P_{12} = 0.3$, $P_{21} = 0.4$, $P_{22} = 0.6$

One step transition probability matrix

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

For Tuesday rainy chance $(P_{22}) = 0.6 = 60\%$

For Tuesday sunny chance $(P_{21}) = 0.4 = 40\%$

For Wednesday

$$P^{(2)} = PP = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}\begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.49 + 0.12 & 0.21 + 0.18 \\ 0.28 + 0.24 & 0.12 + 0.36 \end{bmatrix} = \begin{bmatrix} 0.61 & 0.40 \\ 0.52 & 0.48 \end{bmatrix}$$

For Wednesday rainy chance $(P_{22}) = 0.48 = 48\%$

For Wednesday sunny chance $(P_{21}) = 0.52 = 52\%$

For Thursday

$$P^{(3)} = P^{(2)}\,P = \begin{bmatrix} 0.61 & 0.4 \\ 0.52 & 0.48 \end{bmatrix}\begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.427 + 0.16 & 0.183 + 0.24 \\ 0.364 + 0.192 & 0.156 + 0.288 \end{bmatrix} = \begin{bmatrix} 0.632 & 0.423 \\ 0.556 & 0.444 \end{bmatrix}$$

For Thursday rainy chance $(P_{22}) = 0.444 = 44.4\%$

For Thursday sunny chance $(P_{21}) = 0.556 = 55.6\%$

Steady State Distribution

A collection of limiting probabilities $\pi_x = \lim\limits_{h\to 0} P_h(x)$ is called steady state distribution of a markov chain $X(t)$.

When limit exists, it can be used as a forecast of the distribution of X after many transitions.

When steady state distribution exists $\pi P = \pi$ and $\sum\limits_x \pi_x = 1$.

Example 6: Obtain steady state distribution of a Markov Chain having transition probability matrix $\begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix}$

Solution:

Here $P = \begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix}$

Let $\pi = (\pi_1 \quad \pi_2)$

Now, $\pi P = \pi$

or $\quad [\pi_1 \quad \pi_2]\begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix} = [\pi_1 \quad \pi_2]$

or $\quad [0.2\pi_1 + 0.5\pi_2 \quad 0.8\pi_1 + 0.5\pi_2] = [\pi_1 \quad \pi_2]$

Hence $0.2\pi_1 + 0.5\pi_2 = \pi_1$ .....(i)

$0.8\pi_1 + 0.5\pi_2 = \pi_2$ ......(ii)

From (i) $0.5\pi_2 = \pi_1 - 0.2\pi_1$

or $\quad 0.5\pi_2 = 0.8\pi_1$

or $\quad \pi_2 = 8/5\,\pi_1$

$= 1.6\,\pi_1$

Since, $\pi_1 + \pi_2 = 1$

or $\quad \pi_1 + 1.6\pi_1 = 1$

or $\quad 2.6\,\pi_1 = 1$

or $\quad \pi_1 = \dfrac{1}{2.6} = \dfrac{5}{13}$

$\pi_2 = \dfrac{1.6}{2.6} = \dfrac{8}{13}$

Hence in long-run probability of state 1 is 5/13 and state 2 is 8/13.

Counting Process

Counting process includes counts of arrived jobs, completed task, transmitted messages, detected errors, scored goals etc.

A stochastic process X is called counting if X(t) is the number of items counted by the time t. On passes of time additional items can not be counted. Counts are non negative integers. $X(t) \in \{0, 1, 2, 3, \ldots\}$. Hence counting are discrete state.

Hence $\{X(t), t \geq 0\}$ is called a counting process if

(i) $X(0) = 0$

(ii) For $0 < t_1 < t_2 < \ldots < t_n < t$

$$X(0) < X(t_1) < X(t_2) < X(t_n) < \ldots X(t_n) < X(t)$$

(iii) $X(t_i) - X(t_{(i-1)})$ denotes the number of events in (t_{i-1}, t_i)

(iv) $X(t_i) - X(t_{(i-1)})$ are independently distributed.

Binomial Process

It is discrete time discrete space counting stochastic process. Binomial process X(n) is the number of successes in the first n independent Bernoulli trials, where n = 0, 1, 2, 3.............

Let λ = arrival rate

Δ = frame size

P = probability of success (arrival) during one frame (trial)

$X(t/\Delta)$ = Number of arrivals by the time t

T = inter arrival time

The inter arrival period consists of a Geometric number of frames Y, each frame taking Δ seconds. Hence the inter arrival time can be computed as $T = Y\Delta$. It is rescaled Geometric random variable taking possible values $\Delta, 2\Delta, 3\Delta \ldots\ldots$

$$\lambda = p/\Delta$$

$$n = t/\Delta$$

$$X(n) = \text{Binomial } (n,p)$$

$$Y = \text{Geometric } (p)$$

$$T = Y\Delta$$

$$E(T) = E(Y\Delta) = \Delta\, E(Y) = \Delta/p = 1/\lambda$$

$$V(T) = V(Y\Delta) = \Delta^2\, V(Y) = (1-p)\left(\frac{\Delta}{p}\right)^2 = \frac{1-p}{\lambda^2}$$

Example 7: Suppose that a number of defects coming from an assembly line can be modeled as a Binomial counting process with frames of one-half-minute length and probability p = 0.02 of a defect during each frame.

i) Find the probability of going more than 20 minutes without a defect.

ii) Determine the arrival rate in units of defects per hour.

iii) If the process is stopped for inspection each time a defect is found, on average how long will the process run until it is stopped?

Solution:

Let Xn = Number of defects in n frames

Here, $p = 0.02$

$\Delta = 0.5$ minutes

T = time between two successive defects

Solution:

i) For $t = 20$ minutes $n = \frac{t}{\Delta} = \frac{20}{0.5} = 40$

$P\{X(40) = 0\} = C(40,0)\ 0.02^{0}(1-0.02)^{40} = 0.446$

ii) Since 1 hour has 120 frames, λ = no. of defects per hour = np = 120(0.02) = 2.4 defects per hour.

iii) $E(T) = \frac{\Delta}{p} = \frac{0.5}{0.02} = 250.5$

Example 8: Customers come to a self-service gas station at the rate of 20 per hour. Their arrivals are modeled by a Binomial counting process.

i) How many frames per hour should we choose, and what should be the length of each frame if the probability of an arrival during each frame is to be 0.05?

ii) With this frames, find the expected value and standard deviation of the time between arrivals at the gas station.

Solution:

Arrival rate $\lambda = 20\ hr^{-1}$.

i) $p = 0.05$.

Δ = duration of 1 frame = $p/\lambda = 0.05/(20\ hrs^{-1}) = (1/400)$ hrs = 9 sec.

Also, n = number of frames in 1 hr = 1 hr/Δ = 400 frames.

ii) Let T = inter-arrival time. $E(T) = \Delta/p = 9/0.05 = 180$ sec $= 180/60 = 3$ min.

$SD(T) = (\Delta/p)\sqrt{1-p} = 180\sqrt{1-0.05} = 175.44$ sec $= 175.44/60 = 2.92$ min.

Example 9: Jobs are sent to a mainframe computer at a rate of 4 jobs per minute. Arrivals are modeled by a Binomial counting process.

i. Choose a frame size that makes the probability of a new jobs received during each frame equal to 0.1

ii. Using the chosen frame compute the probability of more than 4 jobs received during one minute

iii. What is probability of more than 20 jobs during 5 minutes

iv. What is average inter arrival time and variance?

v. What is probability that next job does not arrive during next 30 seconds?

Solution:

Here, $\lambda = 4$ per minute, $p = 0.1$

(i) $\Delta = p/\lambda = 0.1/4 = 0.025$ min

For $t = 1$, $n = t/\Delta = 1/.025 = 40$ frames

$n = 40$, $p = 0.1$

(ii) $P(X(n) > 4) = 1 - P(X(n) \le 4) = 1 - \left[\sum_{x=0}^{4} {}^{40}C_x (0.1)^x (0.9)^{40-x}\right]$

$= 1 - [(0.9)^{40} + 40 \times 0.1 \times (0.9)^{39} + 780 \times (0.1)^2 (0.9)^{38} + 9880 \times (0.1)^3 (0.9)^{37}$
$+ 91350 \times (0.1)^4 (0.9)^{36}] = 0.37$

(iii) $P(X(n) > 20) = P(X(n) > 20.5)$

Using continuity correction

$= P\left\{\frac{x(n) - np}{\sqrt{npq}} > \frac{20.5 - 200 \times 0.1}{\sqrt{200 \times 0.1 \times 0.9}}\right\}$

$= P(Z > 0.12) = 0.5 - P(0 < z < 0.12) = 0.5 - 0.0478 = 0.4522$

(iv) $E(T) = 1/\lambda = 1/4 = 0.25$ min $= 15$ sec

$V(T) = \frac{1-p}{\lambda^2} = \frac{0.9}{4^2} = 0.056$

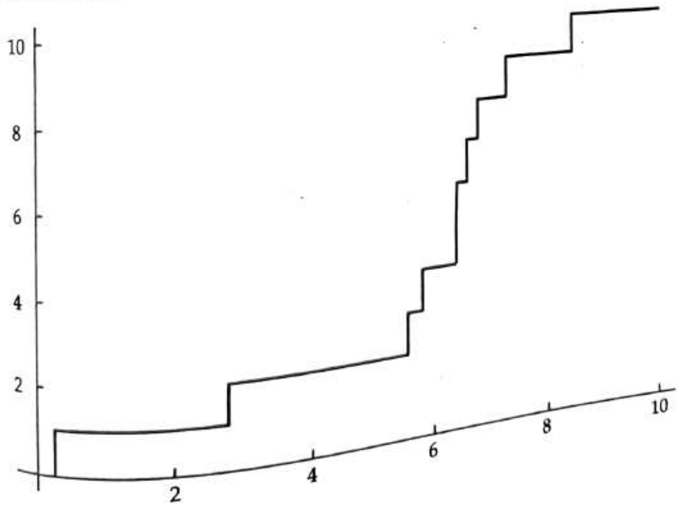
(v) $T = \Delta Y = 0.025\, Y$

$P(T > 30 \text{ sec}) = P(T > 0.5 \text{ min}) = P[Y(0.025) > 0.5]$

$= P(Y > 20) = (1 - P)^n = (1 - 0.1)^{20} = 0.1215$

Poisson Process

It is limiting case of Binomial process as binomial counting process Δ tends to 0. Poisson process is a continuous time counting stochastic process obtained from Binomial counting process when its frame size Δ decreases to 0 while the arrival rate λ remains constant.



Let $X(t)$ = No. of arrivals occurring until time t

T = inter arrival time

T_k = time of k^{th} arrival.

$X(t) = \text{Poisson}(\lambda t)$

$T = \text{Exponential}(\lambda)$

$$T_k = \text{Gamma}(k, \lambda)$$

$$E\,X(t) = np = \frac{t}{\Delta}p = \lambda t$$

$$V\,X(t) = \lambda t \qquad F_T(t) = 1 - e^{-\lambda_t}$$

Prob. of k^{th} arrival before time t

$$P(T_k \le t) = P[X(t) \ge k]$$
$$P(T_k > t) = P[X(t) < k]$$

Example 10: The number of hits to a certain web site follows Poisson process with 5 hits per minute.0

i) What is time required to get 5000 hits?

ii) What is probability that hitting occurs within 12 hours?

Solution:

Number of hits $k = 5000$, $\lambda = 5\ \text{min}^{-1}$

$$\text{Expected time} = \frac{k}{\lambda} = \frac{5000}{5} = 1000 \text{ minutes.}$$

$$\text{Standard deviation } (\sigma) = \frac{\sqrt{k}}{\lambda} = 14.14$$

$$P(T_k < 12\text{ hrs.}) = P(T_k < 720) = P\left(\frac{T_k - \mu}{\sigma} < \frac{720 - \mu}{\sigma}\right)$$

$$= P\left(z < \frac{720 - 1000}{14.14}\right) = P(Z < -19.44) = 0$$

Example 11: Customers arrive at a shop at the rate of 2 per minute. Find (i) expected number of customers in a 5 minute period (ii) the variance of the number of customers in the same period (iii) the probability that there will be at least one customer.

Solution:

Here, $\lambda = 2$

$t = 5$

(i) $E(X) = \lambda t = 5 \times 2 = 10$

(ii) $V(X) = \lambda t = 5 \times 2 = 10$

(iii) $P\{X(5) \ge 1\} = 1 - P\{X(5) < 1\}$

$= 1 - P\{X(5) = 0\}$

$= 1 - e^{-10} = 0.999$

Example 12: Shipments of paper arrive at a printing shop according to a Poisson process at a rate of 0.5 shipments per day.

i) Find the probability that the printing shop receives more than two shipments in a day.

ii) If there are more than 4 days between shipments, all the paper will be used up and the presses will be idle. What is the probability that this will happen?

Solution:

Arrival rate $\lambda = 0.5$ per day.

$X(t)$ = number of arrivals (shipments) in t days, it is Poisson $(0.5t)$

T = inter-arrival time measured in days, it is Exponential (0.5).

i) $P[X(1) > 2] = 1 - P[X(1) \le 2] = 1 - [\,P\{X(1) = 0\} + P\{X(1)=1\} + P\{X(1) = 2\}\}$

$$= 1 - \left\{\frac{e^{-0.5}\, 0.5^0}{0!} + \frac{e^{(-0.5)}\,(0.5)^1}{1!} + \frac{e^{(-0.5)}\,(0.5)^2}{2!}\right\}$$

$$= 1 - e^{-0.5}\{1 + 0.5 + 0.125\}$$

$$= 1 - 0.6065 \times 1.625 = 0.014$$

(ii) $P[T > 4] = \int_4^\infty 0.5e^{-0.5t}\,dt = 0.5\left[\frac{e^{-0.5t}}{-0.5}\right]_4^\infty = e^{-0.5\times 4} = 0.135$

Simulation of Stochastic Process

It requires lengthy computation process to determine different characteristics of stochastic process.

It is used in prediction of future behaviour of stochastic process. Monte Carlo method is used for simulation process. Simulation is carried out for discrete time process, market chain continuous time process, possion process etc.

Queuing system

It is facility consisting of one or several servers designed to perform certain tasks or process certain jobs and a queue of jobs waiting to be processed.

Jobs arrive at the queuing system, wait for an available server, get processed by the server and leave.

Examples are

- A medical office serving patients
- An internet service provider whose customers connect to the internet, browse and disconnect
- A printer processing job sent to it from different computers
- A TV channel viewed by many people at various times
- A personal or shared computer executing tasks sent by its users

Features of Queue

Calling population:
It is infinite population with independent arrivals and not influenced by queuing system.

Arrival process:
Arrival rate follows Poisson distribution with parameter λ.

Queuing configuration:
Queue is single waiting line with unlimited space

Queue discipline:
Queue discipline is based upon first come first serve (FCFS).

Service process:
Service rate follows exponential distribution with parameter μ.

Main Components of Queuing System

Arrival

Job arrives to the queuing system at random times. A counting process A(t) tells the number of arrivals that occurred by time t . In stationary queuing system arrivals occur at arrival rate

λ = Average number of arrivals per unit time = $\frac{EA(t)}{t}$ for any $t > 0$

Queuing and Routing to Servers

Arrived jobs are processed according to the order of their arrivals, on a first come first serve basis. When new job arrives it may find the system in different states. If one server is available at a time it will certainly take a new job. If several servers are available the job may be randomized to one of them or server may be chosen according to some rule.

Service

Once a server becomes available, it immediately starts processing the next assigned job. In practice service time are random because they depend upon amount of work required by each task. The average service time is μ. It varies from one server to other. The service rate is defined as the average number of jobs processed by a continuously working server during one unit of time. S(t) tells the number of customers served by time t

μ = Average number of customers served per unit time = $\frac{ES(t)}{t}$, $t > 0$

Departure

When the service is completed, the job leaves the system

Little's law

It is Law given by John Little.

It gives the relationship between the expected number of jobs, the expected response time and the arrival rate. It is valid for any stationary queuing system .It is applied to M/M/1 queuing system and its components, the queue and server. Assuming the system is functional, all the jobs go through the entire system and thus each component is subject to the same arrival rate λ. According to Little law

$$\lambda E(R) = E(X)$$
$$\lambda E(S) = E(X_s)$$
$$\lambda E(W) = E(X_w)$$

Bernoulli Single Server Queuing Process

It is discrete time queuing process with one server, unlimited capacity, arrivals occur according to Bernoulli counting process. The probability of a new arrival during each frame is P_A, during each frame each busy server completes its job Bernoulli counting process with probability P_s independent of other server and independent of the process arrivals. There is geometric number of frames (P_A) between successive arrivals, each service takes a Geometric number of frames (P_S), service of any job takes at least one frame. Arrival X(t) increases by 1 and departure decreases by 1

$$P_A = \lambda\Delta$$
$$P_S = \mu\Delta$$

It is homogeneous Markov chain as P_A and P_S never change. The number of jobs in system increases by 1 with each arrival or decreases by 1 in each departure. It guarantees at most one arrival and one departure during each frame.

In this case;

P_{00} = P(no arrivals) = $1 - P_A$

P_{01} = P(new arrivals) = P_A

For all $i \geq 1$

$P_{i,i-1}$ = P(no arrivals ∩ one departure) = $(1 - P_A)P_S$

$P_{i,i}$ = P{(no arrivals ∩ no departure) ∪ (no arrivals ∩ no departure)} = $(1 - P_A)(1 - P_S) + P_AP_S$

$P_{i,i+1}$ = P(one arrivals ∩ no departure) = $P_A(1 - P_s)$

Now transition probability matrix is

$$P = \begin{bmatrix} 1 - P_A & P_A & 0 \\ (1 - P_A)P_S & (1 - P_A)(1 - P_S) + P_AP_S & P_A(1 - P_S) \\ 0 & (1 - P_A)P_S & (1 - P_A)(1 - P_S) + P_AP_S \\ 0 & 0 & (1 - P_A)P_S \\ \vdots & \vdots & \vdots \end{bmatrix}$$

Example 13: Laptop computers arrive at a repair shop at the rate of four per day. Assume an 8-hour working day. The expected time to complete service on a laptop is 1.25 hours. Model this process as a single-server Bernoulli queuing process with 15-minute frames. a) Find the service rate. b) Find the arrival and service probabilities.

Solution:

Here, λ = 4 per day = 4 per 8 hour = ½ = 0.5 per hour

Δ = 15 minutes = 15/60 = ¼ hr

Service time of 1 laptop = 1.25 hrs

1/1.25 laptop = 1 hr

Hence μ = 0.8 per hour

$$P_A = \lambda\,\Delta = 0.5 \times \frac{1}{4} = 0.125$$

$$P_S = \mu\,\Delta = 0.8 \times \frac{1}{4} = 0.2$$

Example 14: A barbershop has one barber and two chairs for waiting. The expected time for a barber to cut customer's hair is 15 minutes. Customers arrive at the rate of two per hour provided the barbershop is not full. However, if the barbershop is full (three customers), potential customers go elsewhere. Assume that the barbershop can be modeled as single-server Bernoulli queuing process with limited capacity. Use frame size of 3 minutes. a) Derive the one-step transition probability matrix for this process. b) Find steady-state probabilities and interpret them.

Solution:

Service time for 1 customer = 15 minutes

Service time for 4 customers = 1 hr

Hence μ = 4 per hour

Arrival of customers = 2 per hour

Hence $\lambda = 2$ per hour

Frame size $\Delta = 3$ minutes $= 3/60 = 1/20$ hr $= 0.05$ hr

Capacity $C = 3$

$P_A = \lambda\,\Delta = 2 \times 0.05 = 0.1$

$P_S = \mu\,\Delta = 4 \times 0.05 = 0.2$

$P_{00} = 1 - P_A = 1 - 0.1 = 0.9$

$P_{01} = P_A = 0.1$

For all $i \geq 1$

$P_{i,i-1} = (1 - P_A)\,P_S = 0.9 \times 0.2 = 0.18$

$P_{i,i} = (1 - P_A)\,(1 - P_S) + P_A P_S = 0.9 \times 0.8 + 0.1 \times 0.2 = 0.72 + 0.02 = 0.74$

$P_{i,\,i+1} = P_A\,(1 - P_S) = 0.1 \times 0.8 = 0.08$

Now transition probability matrix is

$$P = \begin{bmatrix} 0.9 & 0.1 & 0 & 0 \\ 0.18 & 0.74 & 0.08 & 0 \\ 0 & 0.18 & 0.74 & 0.08 \\ 0 & 0 & 0.18 & 0.82 \end{bmatrix}$$

For steady state distribution

Now, $\pi p = \pi$

or $$[\pi_0 \quad \pi_1 \quad \pi_2 \quad \pi_3]\begin{bmatrix} 0.9 & 0.1 & 0 & 0 \\ 0.18 & 0.74 & 0.08 & 0 \\ 0 & 0.18 & 0.74 & 0.08 \\ 0 & 0 & 0.18 & 0.82 \end{bmatrix}$$

$$= [\pi_0 \quad \pi_1 \quad \pi_2 \quad \pi_3]$$

or $$[0.9\pi_0 + 0.18\pi_1 \quad 0.1\pi_0 + 0.74\pi_1 + 0.08\pi_2 \quad 0.08\pi_1 + 0.74\pi_2 + 0.18\pi_3 \quad 0.08\pi_2 + 0.82\pi_3]$$

$$= [\pi_0 \quad \pi_1 \quad \pi_2 \quad \pi_3]$$

Hence $0.9\pi_0 + 0.18\pi_1 = \pi_0$ ....(i)

$0.1\pi_0 + 0.74\pi_1 + 0.08\pi_2 = \pi_1$ ....(ii)

$0.08\pi_1 + 0.74\pi_2 + 0.18\pi_3 = \pi_2$ ......(iii)

$0.08\pi_2 + 0.82\pi_3 = \pi_3$ ......(iv)

Also $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$ ......(v)

From (i) $0.18\pi_1 = 0.1\pi_0$

From (ii) $0.18\pi_1 + 0.74\pi_1 + 0.08\pi_2 = \pi_1$

or $0.08\pi_2 = 0.02\pi_1$

or $0.16\pi_2 = 0.08\pi_1$

From (iii) $0.1\pi_2 + 0.74\pi_2 + 0.18\pi_3 = \pi_2$

or $0.18\pi_3 = 0.1\pi_2$

Now from (v) $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$

or $1.8\,\pi_1 + \pi_1 + \pi_2 + \pi_3 = 1$

or $2.8\pi_1 + \pi_2 + \pi_3 = 1$

or $2.8 \times 2\pi_2 + \pi_2 + \pi_3 = 1$

$$6.6\pi_2 + \pi_3 = 1$$

or $$6.6 \times 1.8\pi_3 + \pi_3 = 1$$

or $$12.88\pi_3 = 1$$

or $$\pi_3 = \frac{1}{12.88} = 0.077$$

or $$\pi_2 = 1.8\pi_3 = \frac{1.8}{12.88} = 0.139$$

$$\pi_1 = 2\pi_2 = 2 \times \frac{1.8}{12.88} = \frac{3.6}{12.88} = 0.279$$

$$\pi_0 = 1.8\pi_1 = 1.8 \times \frac{3.6}{12.88} = 0.503$$

According to steady state probabilities 50.3% of time there are no customers in barber shop. 27.9% of time there is no waiting line but barber is working 13.9% of time barber is working and one more customer is waiting and 7.7% of time barber shop is completely full and no vacant seats for waiting.

Example 15: Any printer represents a single server system; the job is sent to a printer at the rate of 10 per hour and takes an average of 50 seconds to print a job. Printer is printing a job and there is another job stored in queue. Assuming single server queuing process with 10 seconds frame. Find out transition probability matrix.

Solution:

Here $\lambda = 10$ per hour $= 1/6$ per minute

$\mu = 1$ per 50 sec $= 6/5$ per minute

$\Delta = 10$ sec $= 1/6$ minute

$P_A = \lambda\Delta = 1/36$

$P_S = \mu\Delta = 1/5$

$P_{00} = 1 - P_A = 1 - 1/36 = 35/36 = 0.972$

$P_{01} = P_A = 1/36 = 0.028$

$P_{i,\,i-1} = (1 - P_A)P_S = 35/36 \times 1/5 = 7/36 = 0.195$

$P_{i,i} = (1 - P_A)(1 - P_S) + P_AP_S = 35/36 \times 4/5 + 1/36 \times 1/5 = 7/9 + 1/180 = 141/180 = 0.783$

$P_{i,\,i+1} = P_A(1 - P_S) = 1/36 \times 4/5 = 1/45 = 0.022$

Now transition probability matrix is

$$P = \begin{bmatrix} 0.972 & 0.028 & 0 & 0 & 0 \\ 0.195 & 0.783 & 0.022 & 0 & 0 \\ 0 & 0.195 & 0.783 & 0.022 & 0 \\ 0 & 0 & 0.195 & 0.783 & 0.022 \end{bmatrix}$$

Parameters of queuing system

λ = arrival rate

μ = service rate

$\rho = \dfrac{\lambda}{\mu}$ = utilization or arrival to service ratio

If $\lambda > \mu$ then queue is infinite, service provider is busy, service system is failure.

If $\lambda = \mu$ then customers come and served, there will be no queue, server will be busy

If $\lambda < \mu$ then there will be no queue, there is idle time for server

Random variables in queuing system

$X_S(t)$ = Number jobs receiving service at time t

$X_W(t)$ = Number of jobs waiting in a queue at time t

$X(t)$ = Total number of jobs in the system

S_k = Service time of k^{th} job

W_k = Waiting time of k^{th} job

R_k = Total time a job spends in the system from its arrival until departure

N = Number of customers in the system (waiting and in service)

L_s = Mean (average) number of customers in the system

L_q = Mean (average) number of customers in the queue

L_b = Mean length of non empty queue

W_S = Mean waiting time in the system

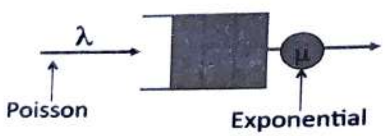
W_q = Mean waiting time in queue

P_w = Probability that an arriving customer has to wait

System with limited capacity

M/M/1 System

It is continuous time Markov process with one server, unlimited capacity, exponential inter arrival times with arrival rate λ, exponential service time with service rate μ, service time and interarrival time are independent. It is limiting case of a Bernoulli queuing process. When frame Δ is small then Δ^2 is negligible.



It is both death process in which only one customer is served at time. Transition are due to arrival or departure of customer. Only nearest neighbour transitions are allowed.

$a_{i,i+1} = \lambda$, $a_{i,i-1} = \mu$

$a_{ij} = 0$ for $|i - j| > 1$.

Transition probabilities for Bernoulli single server queuing process can be obtained as

$$P_{00} = 1 - P_A = 1 - \lambda\Delta$$

$$P_{10} = P_A = \lambda\Delta$$

For $I \geq 1$

$p_{i,i-1} = (1 - P_A) P_S = (1 - \lambda \Delta) \mu \Delta = \mu\Delta - \lambda\mu\Delta^2 = \mu\Delta$

$p_{i,i+1} = P_A (1 - P_S) = \lambda\Delta (1 - M\Delta) = \lambda\Delta - \lambda\mu\Delta^2 = \lambda\Delta$

$p_{i,i} = (1 - P_A)(1 - P_S) + P_A P_S = (1 - \lambda\Delta)(1 - \mu\Delta) + \lambda\Delta.\mu\Delta$

$= 1 - \lambda\Delta - \lambda\Delta + \lambda\mu\Delta^2 + \lambda\mu\Delta^2$

$= 1 - \lambda\Delta - \mu\Delta$

Transition probability matrix is

$$P = \begin{bmatrix} 1 - \lambda\Delta & \lambda\Delta & 0 & 0 \\ \mu\Delta & 1 - \lambda\Delta - \mu\Delta & \lambda\Delta & 0 \\ 0 & \mu\Delta & 1 - \lambda\Delta - \mu\Delta & \lambda\Delta \\ 0 & 0 & \mu\Delta & 1 - \lambda\Delta - \mu\Delta \end{bmatrix}$$

Steady state distribution of M/M/1 system

$\pi_0 = 1 - \rho$

$\pi_1 = \rho(1 - \rho)$

$\pi_2 = \rho^2 (1 - \rho)$

and so on.

Evaluating the System Performance

Many important characteristics of the system are as follows;

Utilization rate $= \dfrac{\text{Arrival rate}}{\text{Service rate}} = \dfrac{\lambda}{\mu} = \rho$

Idle rate = 1 – utilization rate $= 1 - \dfrac{\lambda}{\mu} = 1 - \rho$

Probability of no customer in queue

$P_0 = 1 - \dfrac{\lambda}{\mu} = 1 - \rho$

Probability of one customer in queue

$P_1 = \rho P_0 = \rho(1 - \rho)$

Probability of two customers in queue

$P_2 = \rho P_1 = \rho^2(1 - \rho)$

Probability of n customers in queue

$P_n = \rho^n(1 - \rho), \rho < 1, n = 0, 1, 2, 3, 4, 5.........$

Probability of server being busy $= 1 - P_0 = \rho$

Expected (average) number of customers in the system

$L_S = \dfrac{\text{Utilization rate}}{\text{Idle rate}} = \dfrac{\rho}{1 - \rho}$

Expected queue length (Expected number of customers waiting in the queue)

$L_q = L_s$ – Utilization factor

$= L_s - \rho = \dfrac{\rho}{1 - \rho} - \rho = \dfrac{\rho - \rho(1 - \rho)}{1 - \rho} = \dfrac{\rho^2}{1 - \rho}$

Expected (average) waiting time of a customer in the queue

$$W_q = \frac{\text{Average number of customer in queue}}{\text{Arrival rate}} = \frac{L_q}{\lambda}$$

Expected (average) waiting time of a customer in the system

$$W_S = \frac{\text{Average number of customer in system}}{\text{Arrival rate}} = \frac{L_s}{\lambda}$$

Probability of k or more customers in the system

$$P(n \geq k) = \rho^k$$

Variance of queue length

$$V(n) = \frac{\rho}{(1-\rho)^2}$$

Expected number of customers served per busy period

$$L_b = \frac{L_S}{1 - P_0} = \frac{1}{1-\rho}$$

Expected length of non empty queue

$$L'_q = \frac{\text{Expected number of customers in the queue}}{\text{P(more than one customer in queue)}} = \frac{L_q}{P(n > 1)}$$

Example 16: In computer network of a software company arrival of printing message follows Poisson law on an average at every 10 minutes, printer prints messages takes on an average 6 minutes to print following exponential law. Find (i) average arrival rate and average service rate for 1 hour (ii) average arrival rate and average service rate for 15 minutes.

Solution:

i) 1 message arrives in 10 minutes

6 messages arrive in 60 minutes

Hence $\lambda = 6$ per hour (average arrival rate)

1 message serves in 6 minutes

10 messages serve in 60 minutes

Hence $\mu = 10$ per hour (average service rate)

ii) 1 message arrives in 10 minutes

1.5 messages arrive in 15 minutes

Hence $\lambda = 1.5$ per 15 minutes (average arrival rate)

1 message serves in 6 minutes

2.5 messages serve in 15 minutes

Hence $\mu = 2.5$ per 15 minutes (average service rate)

Example 17: A computer repairman finds that the time spent on his jobs has an exponential distribution with mean 20 minutes. If he repairs sets in an order in which they come in and if the arrival of computers follow Poisson distribution with an average rate of 8 per 6 hour. (i) Find repairs idle time each day (ii) Find average number of jobs brought in?

Solution:

Here, service rate of computer is 1 computer per 20 minute
3 computers per 1 hour

Hence $\mu = 3$ per hour

Arrival rate of computer is 8 per 6 hour
4/3 per1 hour

Hence $\lambda = 4/3$ per hour

Utility rate $(\rho) = \frac{\lambda}{\mu} = \frac{\frac{4}{3}}{3} = \frac{4}{9}$

Ideal rate $= 1 - \rho = 1 - \frac{4}{9} = \frac{5}{9}$

Ideal time in 6 hours $= 6 \times$ ideal rate $= 6 \times \frac{5}{9} = \frac{10}{3}$ hrs

Average length of system $(L_s) = \frac{\rho}{1-\rho} = \frac{4/9}{5/9} = \frac{4}{5}$

Example 18: Telephone calls arrive at telephone booth following Poisson distribution at an average time of 5 minutes between one and next. The length of phone call is assumed to be exponentially distributed with an average of 4 minutes. (i) What is probability that a person arriving at booth will have to wait? (ii) What is average length of queue that forms from time to time?

Solution:

$\lambda = 1$ call per 5 minute $= 1/5$ per minute
$\mu = 1$ call per 4 minute $= 1/4$ per minute

(i) Probability that server is busy $(\rho) = \frac{\lambda}{\mu} = \frac{1/5}{1/4} = \frac{4}{5} = 0.8$

(ii) Average length of queue $(L_q) = \frac{\rho^2}{1-\rho} = \frac{0.64}{1-0.8} = 3.2$

Example 19: In a health clinic, the average rate of arrival of patients is 12 patients per hour. On an average, a doctor can serve patients at the rate of one patient every four minutes. Assume, the arrival of patients follows a Poisson distribution and service to patients follows an exponential distribution. (i) Find the average number of patients in the waiting line and in the clinic (ii) Find the average waiting time in the waiting line or in the queue and (iii) average waiting time in the clinic.

Solution:

Arrival rate of patient, $\lambda = 12$ patients per hour
Service rate of patient, $\mu = 1$ in 4 minutes $= 15$ patients per hour

Now, $\rho = \frac{\lambda}{\mu} = \frac{12}{15} = 0.8$

Average number of patients in system $L_s = \frac{\rho}{1-\rho} = \frac{0.8}{1-0.8} = 4$ patients

Average number of patients in queue $L_q = \frac{\rho^2}{1-\rho} = \frac{0.64}{1-0.8} = 3.2$ patients

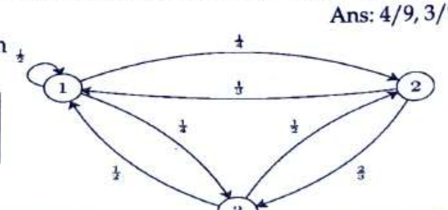
Average waiting time in queue $W_q = \frac{L_q}{\lambda} = \frac{3.2}{12} = 0.26$ hrs

Average waiting time in the system $W_s = \frac{L_s}{\lambda} = \frac{4}{12} = 0.33$ hrs

EXERCISE

1. Define stochastic process and classify.
2. Differentiate between Markov Process and Markov chain.
3. What is steady state distribution?
4. Discuss Binomial process and Poisson process.
5. Describe Bernoulli single server queuing system.
6. What is queuing? Describe M/M/1 queuing system.
7. Define the terms; transition probability, Markov process, Markov chain, n step transition probability.
8. A computer system can operate in two different modes. Every hour it remains in the same mode or switches to a different mode according to the transition probability matrix $\begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$.
   (i) Compute 2 step transition probability matrix
   (ii) If the system is in mode I at 5:30 in a particular time then what is probability that it will be on same mode on the same day at 8:30. **Ans:** $\begin{bmatrix} 0.37 & 0.63 \\ 0.36 & 0.64 \end{bmatrix}$, 0.363
9. An offspring of a tall man is short with probability 0.4 and tall with probability 0.6. An offspring of short man is tall with probability 0.3 and short with probability 0.7.
   (i) Write transition probability matrix of the Markov chain
   (ii) What is probability that grand child of tall man is short? **Ans:** $\begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$, 0.52
10. A Markov chain has transition probability matrix $\begin{bmatrix} 0.4 & \dots & 0 \\ 0 & 0.7 & \dots \\ 0.2 & \dots & 0.6 \end{bmatrix}$
    (i) Fill in the blanks
    (ii) Compute steady state probabilities **Ans:** 0.6, 0.3, 0.2; 0.16, 0.48, 0.36
11. A Markov chain has 3 possible states A, B and C. Every hour it makes a transition to a different state. From state A transition to State B and C are equally likely. From state B transitions to state A only and from state C transition to state A and B are equally likely. Find steady states distribution of states. Ans: 4/9, 3/9, 2/9
12. Obtain transition probability matrix from the state transition diagram.

    Ans: $\begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 1/2 & 0 \end{bmatrix}$



13. Tasks are sent to computers at an average rate of 2 tasks per minute. Arrival of tasks module by binomial counting process with $\Delta = 2$ second frames
    (i) Find probability of more than 3 tasks sent during 8 seconds?
    (ii) Find probability of more than 10 tasks sent during 50 seconds.
14. Printing jobs are sent to printer at the average rate of 3 jobs per minute. Using binomial counting process (i) what frame length gives probability 0.2? (ii) Using frame length compute expectation and standard deviation for the number of printing jobs sent to printer during an minute period? **Ans:** $\frac{1}{15}, \frac{1}{3}$, 0.29

15. For a Binomial counting process with 2-second frames and the arrival rate of 10 arrivals per hour, calculate the probability of at least three new arrivals during an interval of 15 minutes. **Ans:** 0.2643

16. Customers of certain internet service provider connect to the internet at the average rate of 10 new connections per minute. Connection are modeled by binomial counting process,
(i) What frame length gives the probability 0.1 of an arrival during given frame?
(ii) What is expectation and standard deviation for the number of seconds between two consecutive connections? **Ans:** $\frac{1}{6000}, \frac{1}{600}, 0.0015$

17. Messages arrive at message center according to binomial counting process with average inter arrival time of 10 seconds. Choosing frame size of 2 seconds compute probability that during 100 minutes operation no more than 500 messages arrive. **Ans:** 0

18. Messages arrive at mail server at the average rate of 5 messages every 10 minutes. The arrival of message is modeled by binomial counting process.
(i) What frame length makes the probability of a new message arrival during a given frame 0.15.
(ii) If 20 messages arrive during an hour. Find arrival rate is increased or decreased as compared to frame of (i). **Ans:** 0.3 min, decreased

19. Telephone calls arrive at customer service center according to Poisson process with rate of 2 calls every 5 minutes. Find the probability of getting more than 10 calls during next 15 minutes.

20. The number of baseball games rained out in Mudville is a Poisson process with the arrival rate of 5 per 30 days. a) Find the probability that there are more than 5 rained-out games in 15 days. b) Find the probability that there are no rained-out games in seven days. **Ans:** 0.042, 0.311

21. An internet service provider offers discount on connecting internet. If customers connect internet according to Poisson process with rate of 2 customers per minute. (i) What is probability that no offer is made during first 3 minutes (ii) Find expectation and variance of the time of the first offer. **Ans:** 0.0024, 6, 6

22. Customers arrive at ATM at the rate of 9 customers per hour and spend 3 minutes on average. The system is modeled by Bernoulli single server queuing system with 15 seconds frame. Find the transition probability matrix for the number of customers at the ATM at the end of each frame.

$$\textbf{Ans:} \begin{bmatrix} 0.9625 & 0.0375 & 0 & 0 & 0 & - \\ 0.08 & 0.88511 & 0.034 & 0 & 0 & - \\ 0 & 0.08 & 0.88511 & 0.034 & 0 & - \\ 0 & 0 & 0.08 & 0.8854 & 0.034 & - \\ 0 & 0 & 0 & 0.08 & 0.8854 & 0.0343 \end{bmatrix}$$

23. A printer is a single-server queuing system because it can process only one job at a time while other jobs are stored in a queue. Suppose the jobs are sent to a printer at the rate of 20 per hour, and that it takes an average of 40 seconds to print each job. Currently a printer is printing a job, and there is another job stored in a queue. Assuming Bernoulli single-server queuing process with 20-second frames. Find transition probability matrix.

$$\textbf{Ans:} \begin{bmatrix} 8/9 & 1/9 & 0 & 0 & 0 & - \\ 4/9 & 9/18 & 1/18 & 0 & 0 & - \\ 0 & 4/9 & 9/18 & 1/18 & 0 & - \\ 0 & 0 & 4/9 & 9/18 & 1/18 & - \end{bmatrix}$$

24. Jobs arrive at the server at the rate of 8 jobs per hour. The service takes 3 minutes, on the average. This system is modeled by the single-server Bernoulli queuing process with 5-second frames and capacity limited by 3 jobs. Write the transition probability matrix for the number of jobs in the system at the end of each frame.

Ans: $\begin{bmatrix} 0.988 & 0.012 & 0 & 0 \\ 0.027 & 0.962 & 0.011 & 0 \\ 0 & 0.027 & 0.962 & 0.011 \\ 0 & 0 & 0.027 & 0.973 \end{bmatrix}$

25. A customer service with limited capacity C = 2 can have one person getting service and one more person" on hold". When the capacity is reached and someone tries to call, (s)he gets a busy signal or voice mail. Suppose the representative gets an average of 10 calls per hour, and the average phone conversation lasts 4 minutes. Modeling this by a Bernoulli single-server queuing process with limited capacity and 1-minute frames, a) Compute the steady-state distribution and interpret it. **Ans:** 0.438, 0.3512, 0.2106

26. In queuing system with single server average inter arrival time is 6 minutes and average service time is 4 minutes. Find (i) expected response time (ii) fraction of time when the server is busy. **Ans:** 12 min, 2/3

27. For an M/M/1 queuing system with the average inter arrival time of 5 minutes and the average service time of 3 minutes, compute a) the expected time; b) the customer spent in system probability that there are fewer than 2 jobs in the system; c) the fraction of customers who have to wait before their service starts. **Ans:** 7.5 min, 0.64, 0.9

28. Cars arrive at fast food drive through window according to a Poisson process with the average rate of 1 car every 10 minutes. The time each customer spends ordering and getting food is Exponential with the average time of 3 minutes. When a customer is served, the other arrived customers stay in a line waiting for their turn. Compute (a) the expected number of cars in the line at any time. (b) The proportion of time when nobody is served at the drive through window. (c) The expected time it takes to follow the drive through lane, from arrival till departure. **Ans:** 9/70, 0.7, 4.28 min

29. Jobs sent to a printer are held in a buffer until they can be printed. Jobs are printed sequentially on a first-come, first-serve basis. Jobs arrive at the printer at the rate of four per minute. The average time to print a job is 10 seconds. Assuming an M/M/1 system, (a) Find the expected value and standard deviation of the number of jobs in this system at any time. (b) When a job is submitted, what is the probability that it will begin printing immediately? **Ans:** 2.44, 0.33

30. In a bank cheques are cashed at a single teller counter. Customers arrive at the counter in a Poisson manner at an average rate of 30 customers per hour. The teller takes on average a minute and half to cash cheque. The service time has been shown to be exponentially distributed. (i) Find the percentage of time teller is busy. (ii) Find average time a customer is expected to spend in system. **Ans:** 75%, 3, 6

31. In a service department managed by a server, on average one customer arrive per 10 minutes. It has been found that each customer requires 6 minutes to be served. Find out (i) Average queue length (ii) Average time spent in the system (iii) Probability that there will be two customers in the queue. **Ans:** 0.9, 15 min, 0.144

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