Unit-5
Counting & Discrete Probability

Counting

Basic of Counting

Basic Counting Principles

"How many different 8-letter passwords are there?"

"How many possible ways are there to pick 11 soccer players out of a 20-player team?"

Most importantly, counting is the basis for computing probabilities of discrete events. ("What is the probability of winning the lottery?")

The sum rule:

If a task can be done in n1 ways and a second task in n2 ways, and if these two tasks cannot be done at the same time, then there are n1 + n2 ways to do either task?

Example:

The department will award a free computer to either a CS student or a CS professor. How many different choices are there, if there are 530 students and 15 professors?

There are = 545 choices.

Generalized sum rule:

If we have tasks T1, T2, …, Tm that can be done in n1, n2, …, nm ways, respectively, and no two of these tasks can be done at the same time, then there are n1 + n2 + … + nm ways to do one of these tasks.

The product rule:

Suppose that a procedure can be broken down into two successive tasks. If there are n1 ways to do the first task and n2 ways to do the second task after the first task has been done, then there are n1n2 ways to do the procedure.

Example: How many different license plates are there that containing exactly three English letters?

Solution: There are 26 possibilities to pick the first letter, then 26 possibilities for the second one and 26 for the last one. So there are $26 \cdot 26 \cdot 26 = 17576$ different license plates.

Generalized product rule:

If we have a procedure consisting of sequential tasks T1, T2, …., Tm that can be done in n1, n2, …, nm ways, respectively, then there are n1 · n2 · … · n*m ways to carry out the procedure.

The sum and product rules can also be phrased in terms of set theory.

Sum rule:

Let A1, A2,….., Am be disjoint sets. Then the number of ways to choose any element from one of these sets is $|A1 \cup A2 \cup \dots \cup Am| = |A1| + |A2| + \dots + |Am|$.

Product rule: Let A1, A2, …, Am be finite sets. Then the number of ways to choose one element from each set in the order A1, A2, …, Am is $|A1 \times A2 \times \dots \times Am| = |A1| \cdot |A2| \cdot \dots \cdot |Am|$.

Inclusion- Exclusion

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How many bit strings of length 8 either start with a 1 or end with 0?

Task 1: Construct a string of length 8 that starts with a 1.There is one way to pick the first bit (1), two ways to pick the second bit (0 or 1), two ways to pick the third bit (0 or 1),.two ways to pick the eighth bit (0 or 1).

Product rule: Task 1 can be done in $1 \cdot 2^7 = 128$ ways.

Task 2: Construct a string of length 8 that ends with 0.There are two ways to pick the first bit (0 or 1),two ways to pick the second bit (0 or 1),. .two ways to pick the sixth bit (0 or 1), one way to pick the seventh bit (0), and one way to pick the eighth bit (0).Product rule: Task 2 can be done in $2^6 = 64$ ways.

Tree Diagrams

How many bit strings of length four do not have two consecutive 1s? Task 1 Task 2 Task 3 Task 4 (1st bit) (2nd bit) (3rd bit) (4th bit) 11 1 1 1 1 1 1 1 .There are 8 strings.

The Pigeonhole Principle

Assume 13 pigeons fly into 12 pigeonholes to roost.

A least one of 12 pigeonholes must have at least two pigeons in it.

Statement:

The pigeonhole principle:

If (k + 1) or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Example 1:

If there are 11 players in a soccer team that wins 12-0, there must be at least one player in the team who scored at least twice.

Example 2:

If you have 6 classes from Monday to Friday, there must be at least one day on which you have at least two classes.

Example 3

Show that among any group of 367 people, there must be at least two with the same birthday.

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Proof:

To use pigeonhole principle, first find boxes and objects.

Suppose that for each day of a year, we have a box that contains a birthday that occurs on that day.

The number of boxes is 366 and the number of objects is 367.

By the pigeonhole principle, at least one of these boxes contains two or more birthdays.

So, there must be at least two people with the same birthday.

The Pigeonhole Principle

The generalized pigeonhole principle:

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ of the objects.

Proof by contradiction:

Suppose that none of the boxes contains more than $\lceil N/k \rceil$ objects. Then, the total number of objects is at most $\lceil N/k \rceil$-1 objects.

$$\lceil N/k \rceil < (N/k)+1$$

$$k(\lceil N/k \rceil - 1) < k[(\lceil N/k \rceil + 1) - 1] = N$$

This is a contradiction because there are a total of N objects.

Example 4

In our 60-student class, at least 12 students will get the same letter grade (A, B, C, D, or F).

Solution: Try itself

Example 5

If you have 5 pigeons sitting in 2 pigeonholes, then one of the pigeonholes must have at least 5/2 = 2.5 pigeons—but since (hopefully) the boxes can't have half-pigeons, then one of them must in fact contain 3 pigeons.

Example 6

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D and F.

Solution: To use pigeonhole principle, first find boxes and objects.

Suppose that for each grade, we have a box that contains students who got that grade. The number of boxes is 5, by the generalized pigeonhole principle, to have at least 6 (= N/5) students at the same box, the total number of the students must be at least N = 5. 5 + 1 = 26.

Permutations

A permutation of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an r-permutation.

Formula

The formula for permutation of n objects for r selection of objects is given by: **P (n,r) = n!/(n-r)!**

For example, the number of ways 3rd and 4th position can be awarded to 10 members is given by:

P(10, 2) = 10!/(10-2)! = 10!/8! = (10.9.8!)/8! = 10 x 9 = 90

Example 1

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P (8, 3) = 8·7·6 = 336= (8·7·6·5·4·3·2·1)/(5·4·3·2·1)

General formula: P (n, r) = n!/(n – r)!Knowing this, we can return to our initial question:

Example 2

How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?

Solution:

Try itself

Example 3

In how many ways can 6 persons occupy 3 vacant seats? Solution: Given n = 6, r = 3 Total number of ways = nPr = $6P3$ ways = 6 × 5 × 4 = 120 ways

Example 4

How many permutations of the letters ABCDEFGH contain the string ABC?

Solution:

Combination

A combination is a mathematical technique that determines the number of possible arrangements in a collection of items where the order of the selection does not matter. In combinations, you can select the items in any order.

Formula for Combination

Mathematically, the formula for determining the number of possible arrangements by selecting only a few objects from a set with no repetition is expressed in the following way:

$$C(n, k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Example 1

How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?

C(6, 3) = 6!/(3!·3!) = 720/(6·6) = 720/36 = 20

There are 20 different ways, that is, 20 different groups to be picked.

Example 2

In how many ways can 5 persons be selected from amongst 10 persons?

Solution: The selection can be done in $10C5$ ways. = 10×9×8×7×6 1×2×3×4×5 = 252 ways

Example 3

How many ways are there to select 5 players from a 10- member tennis team to make a trip to a match at another school?

How many ways are there to select five players from 10 member tennis team to make a trip to match to another school? Solution: 5 members can be selected from 10 members in $10C5$ ways. Now $10C5$ = 10×9×8×7×6×5×4×3×2×1 5×4×3×2×1 = 252 ways

Example 4

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A soccer club has 8 female and 7 male members. For today's match, the coach wants to have 6 female and 5 male players on the grass. How many possible configurations are there?

$C(8, 6) \cdot C(7, 5) = 8!/(6!\cdot 2!) \cdot 7!/(5!\cdot 2!) = 28\cdot 21 = 588$

Binomial Coefficients

Expressions of the form C(n, k) are also called binomial coefficients. How come? A binomial expression is the sum of two terms, such as (a + b).

Now consider $(a + b)^2 = (a + b)(a + b)$. When expanding such expressions, we have to form all possible products of a term in the first factor and a term in the second factor: $(a + b)^2$

Then we can sum identical terms: $(a + b)^2 = a^2 + 2ab + b^2$

For $(a + b)^3 = (a + b)(a + b)(a + b)$ we have(a + b)3 = $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

The Binomial Theorem

Binomial theorem, statement that for any positive integer *n*, the *n*th power of the sum of two numbers *a* and *b* may be expressed as the sum of *n* + 1 terms of the form

$$\binom{n}{r} a^{n-r} b^r;$$

in the sequence of terms, the index *r* takes on the successive values 0, 1, 2,..., *n*. The coefficients, called the binomial coefficients, are defined by the formula

$$\binom{n}{r} = n!/(n-r)!r!,$$

Binomial Theorem

Binomial theorem primarily helps to find the expanded value of the algebraic expression of the form $(x + y)^n$. Finding the value of $(x + y)^2$, $(x + y)^3$, $(a + b + c)^2$ is easy and can be obtained by algebraically multiplying the number of times based on the exponent value. But finding the expanded form of $(x + y)^{17}$ or other such expressions with higher exponential values involves too much calculation. It can be made easier with the help of the binomial theorem.

The exponent value of this binomial theorem expansion can be a negative number or a fraction. Here we limit our explanations to only non-negative values. Let us learn more about the terms, formula and the properties of coefficients in this binomial expansion article.

What is Binomial Theorem?

The first mention of the binomial theorem was in the 4th century BC by a famous Greek mathematician by name of Euclids. The binomial theorem states the principle for expanding the algebraic expression $(x + y)^n$ and expresses it as a sum of the terms involving individual exponents of variables x and y. Each term in a binomial expansion is associated with a numeric value which is called coefficient.

Statement: According to the binomial theorem, it is possible to expand any non-negative power of binomial (x + y) into a sum of the form,

$(x+y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 + \dots + {}^nC_{n-1}\, x^1 y^{n-1} + {}^nC_n\, x^0 y^n$

where, $n \geq 0$ is an integer and each ${}^nC_k$ is a positive integer known as a binomial coefficient.

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Note: When an exponent is zero, the corresponding power expression is 1. This multiplicative factor is often omitted from the term, therefore often the right hand side is directly written as ${}^nC_0\, x^n + ...$ This formula is also referred to as the binomial formula or the binomial identity. Using summation notation, the binomial theorem can be given as,

$(x+y)^n = \sum^n_{k=0} {}^nC_k\, x^{n-k}y^k = \sum^n_{k=0} {}^nC_k\, x^k y^{n-k}$

Example: Let us expand $(x+3)^5$ using the binomial theorem. Here $y = 3$ and $n = 5$. Substituting and expanding, we get:

$(x+3)^5 = {}^5C_0\, x^5 3^0 + {}^5C_1\, x^{5-1} 3^1 + {}^5C_2\, x^{5-2}\, 3^2 + {}^5C_3\, x^{5-3}\, 3^3 + {}^5C_4\, x^{5-4}\, 3^4 + {}^5C_5\, x^{5-5}\, 3^5$

$= x^5 + 5\, x^4.\ 3 + 10\, x^3\ .\ 9 + 10\, x^2\ .\ 27 + 5x\ .81 + 3^5$

$= x^5 + 15\, x^4 + 90x^3 + 270\, x^2 + 405\, x + 243$

Example 1: What is the binomial expansion of $(x^2 + 1)^5$ using the binomial theorem?

Solution:

The following formula derived from the Binomial Theorem is helpful to find the expansion.

$(x + y)^n = (x+y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1}y^1 + {}^nC_2\, x^{n-2}\, y^2 + ... + {}^nC_k\, x^{n-k}y^k + ... + {}^nC_n\, x^0 y^n$

$(x^2 + 1)^5 = {}^5C_0\, (x^2)^5 1^0 + {}^5C_1\, (x^2)^{5-1} 1^1 + {}^5C_2\, (x^2)^{5-2}\, 1^2 + {}^5C_3\, (x^2)^{5-3}\, 1^3 + {}^5C_4\, (x^2)^{5-4}\, 1^4 + {}^5C_5\, (x^2)^{5-5}\, 1^5$

$= x^{10} + 5\, x^8 + 10\, x^6 + 10\, x^4 + 5\, x^2 + 1$

Answer: $(x^2 + 1)^5 = x^{10} + 5\, x^8 + 10\, x^6 + 10\, x^4 + 5\, x^2 + 1$

Example 2: Find the 7^{th} term in the expansion of $(x + 2)^{10}$

Solution:

The general term in the expansion of $(x+a)^n$ using the binomial theorem formula is

$T_{r+1} = {}^nC_r\, x^{n-r}y^r$.

Here $r = 6$, $n = 10$, $a = 2$

Thus by substituting, we get

$T_7 = T_{6+1} = {}^{10}C_6\, x^{10-6}\, 2^6$

$T_7 = 210\, x^4\ .\ 64$

$= 13440\, x^4$

Answer: 7^{th} term in $(x + 2)^{10}$ is $13440\, x^4$

Example 3: Find the coefficient of x^2 in $(x + (1/x))^8$

Solution:

Using the binomial theorem formula in the expansion of $(x + 1/x)^8$, we have x^2 as the fourth term.

${}^8C_0\, x^8(1/x)^0 + {}^8C_1\, x^7(1/x)^1 + {}^8C_2\, x^6(1/x)^2 + {}^8C_3\, x^5(1/x)^3 + {}^8C_4\, x^4(1/x)^4 + {}^8C_5\, x^3(1/x)^5 + {}^8C_6\, x^2(1/x)^6 + {}^8C_7\, x^1(1/x)^7 + {}^8C_8\, x^0(1/x)^8 +$

The coefficient of the fourth term is ${}^8C_3 = 56$

Answer: The coefficient of x^2 in the expansion of $(x + (1/x))^8$ is 56

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Binomial Expansion

The binomial theorem is also known as the binomial expansion which gives the formula for the expansion of the exponential power of a binomial expression. Binomial expansion of $(x + y)^n$ by using the binomial theorem is as follows,

$(x+y)^n = {}^nC_0\, x^ny^0 + {}^nC_1\, x^{n-1}y^1 + {}^nC_2\, x^{n-2}\, y^2 + \ldots + {}^nC_{n-1}\, x^1y^{n-1} + {}^nC_n\, x^0y^n$

Binomial Expansion Formula

$$(x+y)^n = {}^nC_0x^ny^0 + {}^nC_1x^{n-1}y^1 + {}^nC_2x^{n-2}y^2 + \ldots\ldots + {}^nC_{n-1}xy^{n-1} + {}^nC_nx^0y^n$$

Binomial Theorem Formula

The binomial theorem formula is used in the expansion of any power of a binomial in the form of a series. The binomial theorem formula is $(a+b)^n= \sum^n_{r=0} {}^nC_r\, a^{n-r}b^r$, where n is a positive integer and a, b are real numbers, and $0 < r \leq n$. This formula helps to expand the binomial expressions such as $(x + a)^{10}$, $(2x + 5)^3$, $(x - (1/x))^4$, and so on. The binomial theorem formula helps in the expansion of a binomial raised to a certain power. Let us understand the binomial theorem formula and its application in the following sections.

The binomial theorem states: If x and y are real numbers, then for all $n \in N$,

$(x+y)^n = {}^nC_0\, x^ny^0 + {}^nC_1\, x^{n-1}y^1 + {}^nC_2\, x^{n-2}\, y^2 + \ldots + {}^nC_k\, x^{n-k}y^k + \ldots + {}^nC_n\, x^0y^n$

$\Rightarrow (x + y)^n = \sum^n_{k=0} {}^nC_k\, x^{n-k}y^k$

where, ${}^nC_r = n! / [r!\,(n - r)!]$

Binomial Theorem Expansion Proof

Let $x, a, n \in N$. Let us prove the binomial theorem formula through the principle of mathematical induction. It is enough to prove for $n = 1$, $n = 2$, for $n = k \geq 2$, and for $n = k+ 1$.

It is obvious that $(x +y)^1 = x +y$ and

$(x +y)^2 = (x + y)\,(x +y)$

$= x^2 + xy + xy + y^2$ (using distributive property)

$= x^2 + 2xy + y^2$

Thus the result is true for $n = 1$ and $n = 2$. Let k be a positive integer. Let us prove the result is true for $k \geq 2$.

Assuming $(x + y)^n = \sum^n_{r=0} {}^nC_r\, x^{n-r}y^r$,

$(x + y)^k = \sum^k_{r=0} {}^kC_r\, x^{k-r}y^r$

$\Rightarrow (x+y)^k = {}^kC_0\, x^ky^0 + {}^kC_1\, x^{k-1}y^1 + {}^kC_2\, x^{k-2}\, y^2 + \ldots + {}^kC_r\, x^{k-r}y^r + \ldots + {}^kC_k\, x^0y^k$

$\Rightarrow (x+y)^k = x^k + {}^kC_1\, x^{k-1}y^1 + {}^kC_2\, x^{k-2}\, y^2 + \ldots + {}^kC_r\, x^{k-r}y^r + \ldots + y^k$

Thus the result is true for $n = k \geq 2$.

Now consider the expansion for $n = k + 1$.

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$(x + y)^{k+1} = (x + y)(x + y)^k$

$= (x + y)(x^k + {}^kC_1 x^{k-1}y^1 + {}^kC_2 x^{k-2} y^2 + ... + {}^kC_r x^{k-r}y^r + ... + y^k)$

$= x^{k+1} + (1 + {}^kC_1)x^ky + ({}^kC_1 + {}^kC_2) x^{k-1}y^2 + ... + ({}^kC_{r-1} + {}^kC_r) x^{k-r+1}y^r + ... + ({}^kC_{k-1} + 1) xy^k + y^{k+1}$

$= x^{k+1} + {}^{k+1}C_1x^ky + {}^{k+1}C_2 x^{k-1}y^2 + ... + {}^{k+1}C_r x^{k-r+1}y^r + ... + {}^{k+1}C_k xy^k + y^{k+1}$ [Because ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$]

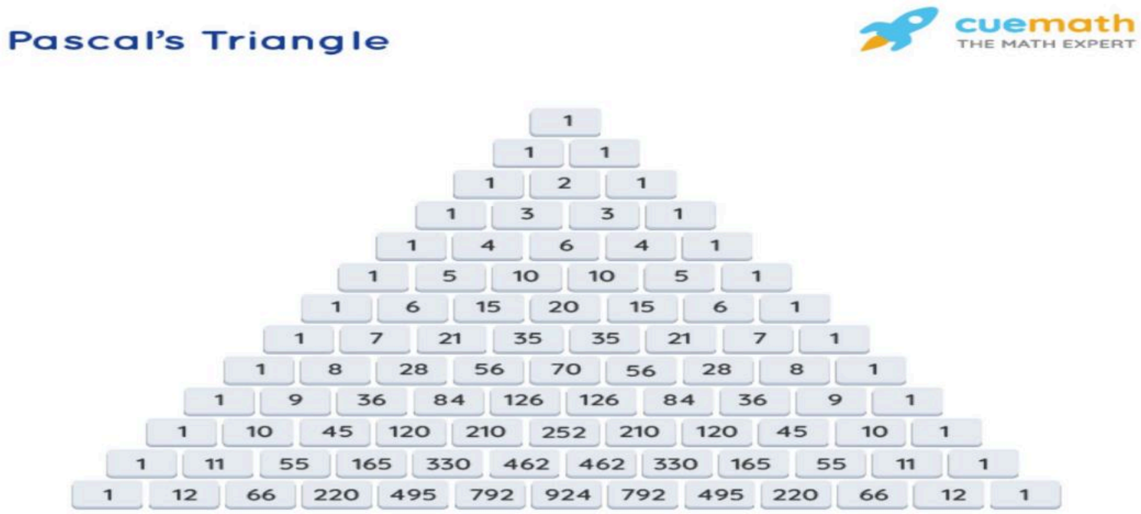
Thus the result is true for n = k+1. By mathematical induction, this result is true for all positive integers 'n'. Hence proved.

Pascal's Triangle Binomial Expansion

The binomial coefficients are the numbers associated with the variables x, and y in the expansion of $(x + y)^n$. The binomial coefficients are represented as ${}^nC_0$, ${}^nC_1$, ${}^nC_2$...... The binomial coefficients are obtained through the Pascal triangle or by using the combinations formula.

Binomial Theorem Coefficients

The values of the binomial coefficients exhibit a specific trend which can be observed in the form of Pascal's triangle. Pascal's triangle is an arrangement of binomial coefficients in triangular form. It is named after the French mathematician Blaise Pascal. The numbers in Pascal's triangle have all the border elements as 1 and the remaining numbers within the triangle are placed in such a way that each number is the sum of two numbers just above the number.



Discrete Probability

Finite Probability Experiment:

A procedure that yields one of outcomes. Sample space of the experiment S: the set of possible outcomes. Event E: a subset of the sample space.

Definition 1. The probability of an event E, which is a subset of a finite sample space S of equally likely outcomes, is p(E)=|E|/|S|.

Example 1

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An urn contains 4 blue balls and 5 red balls. What is the probability that a ball chosen from the urn is blue? There are 9 possible outcomes, and 4 of them produce a blue ball. Hence, the probability that a blue ball is chosen is $P(E) = |E|/|S| = 4/9$.

Solution Experiment: Chose a ball from the urn. Sample space $S = \{b,b,b,b,r,r,r,r,r\}$. $|S| = 9$. Event $E = \{b,b,b,b\}$. $|E| = 4$

Example 2

What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7? There are 36 possible outcomes. Among them there 36 successful outcomes: (1,6),(2,5),(3,4),(4,3),(5,2),(6,1）.

Hence, the probability is $P(E) = |E|/|S| = 6/36=1/6$.

Solution Experiment: Roll two dice.

Sample space $S = \{(x,y) \mid 1 \leq x,y \leq 6\}$. $|S| = 6\times6=36$. Event $E = \{(x,y) \mid x + y = 7\}$. $|E| = 6$.

Example 4

There are lotteries that award enormous prizes to people who correctly choose a set of six numbers out of the first n positive integers, where n between 30 and 50. What is the probability that a person picks the correct six numbers out of 40? The total number of ways to choose 6 numbers out of 40 is $P(E) = |E|/|S| = C(40,6)=40!/(34!6!)=3838380$. Hence, the probability is 1/3838380.

Solution Suppose that the 6 numbers from 1 to 40 for the prize are $\{u,v,w,x,y,z\}$.

Experiment: Select 6 numbers from 1 to 40. Sample space $S = \{\{a,b,c,d,e,f\} \mid 0 \leq a,b,c,d,e,f \leq 40\}$. $|S| = C(40,6)$ Event 1 (large prize) $E = \{ \{u,v,w,x,y,z\} \}$. $|E| = 1$.

Example 5

Find the probability that a hand of five cards in poker contains three cards of one kind (same kind of character: 2, 3,....., K, A). The probability is $P(E) = |E|/|S|$

Solution Experiment: Pick 5 cards from 52 cards. Sample space $S = \{\{a,b,c,d,e\} \mid a,b,c,d,e$ are picked from 52 cards$\}$. $|S| = C(52,5)$ Event $E = \{ \{a,b,c,d,e\} \mid$ three of them are same kind$\}$.

$|E| = C(13,1)C(4,3)C(49,2)$.

Probabilities of Complements & Unions of events

Theorem 1

complement of an Event

The **complement** of an event E is the event “E doesn’t happen”

The notation E^{-} is used for the complement of event E.

We can compute the probability of the complement using $P(E^{-})=1-P(E)$

Notice also that $P(E)=1-P(E^{-})$

Example 1

If you pull a random card from a deck of playing cards, what is the probability it is not a heart?

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Solution

There are 13 hearts in the deck, so

P(heart)=13/52=1/4.

The probability of *not* drawing a heart is the complement:

P(not heart)=1−P(heart)=1−1/4=3/4(not heart)

P(A or B)

The probability of either A or B occurring (or both) is P(A or B)=P(A)+P(B)−P(A and B)

Example 3

Suppose we draw one card from a standard deck. What is the probability that we get a Queen or a King?

Solution

There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes. Thus the probability of drawing a Queen or a King is:

P(King or Queen)=8/52(King or Queen)=8/52.

Note that in this case, there are no cards that are both a Queen and a King, so P(King and Queen)=0(King and Queen)=0. Using our probability rule, we could have said:

P(King or Queen)=P(King)+P(Queen)−P(King and Queen)=4/52+4/52−0=8/52(King or Queen)=(King)+(Queen)−(King and Queen)=4/52+4/52−0=8/52

In the last example, the events were **mutually exclusive**, so P(A or B)=P(A)+P(B)

Mutually Exclusive

Two events A and B are **mutually exclusive** if they have no outcomes in common.

Thus, P(A or B)=P(A)+P(B)

Example 4

Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King?

Solution

- Half the cards are red, so P(Red)=26/52
- There are four kings, so P(King)=4/52
- There are two red kings, so P(Red and King)=2/52
- We can then calculate
  P(Red or King)=P(Red)+P(King)−P(Red and King)=26/52+4/52−2/52=28/52=7/13

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Probability Theory

Probability theory is a branch of mathematics that investigates the probabilities associated with a random phenomenon. A random phenomenon can have several outcomes. Probability theory describes the chance of occurrence of a particular outcome by using certain formal concepts.

Probability theory makes use of some fundamentals such as sample space, probability distributions, random variables etc. to find the likelihood of occurrence of an event.

What is Probability Theory?

Probability can be defined as the number of favorable outcomes divided by the total number of possible outcomes of an event.

P(E)=|E|/|S|

Probability Formula

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$$P(A) = \frac{\text{Number of favorable outcomes to A}}{\text{Total number of possible outcomes}}$$

Probability Theory Example

Suppose the probability of obtaining a number 4 on rolling a fair dice needs to be established. The number of favorable outcomes is 1. The possible outcomes of the dice are {1, 2, 3, 4, 5, 6}. This implies that there are a total of 6 outcomes. Thus, the probability of obtaining 4 on a dice roll, using probability theory, can be computed as 1 / 6 = 0.167.

Example 2: What is the probability of drawing a queen from a deck of cards?

Solution: A deck of cards has 4 suits. Each suit consists of 13 cards.

Thus, the total number of possible outcomes = (4)(13) = 52

There can be 4 queens, one belonging to each suit. Hence, the number of favorable outcomes = 4.

The card probability = 4 / 52 = 1 / 13

Answer: The probability of getting a queen from a deck of cards is 1 / 13

Conditional Probability

If A and B are two events associated with the same sample space of a random experiment, **the conditional probability of event A given that B has occurred is given by P(A/B) = P(A ∩ B)/ P (B)**, provided P(B) ≠ 0.

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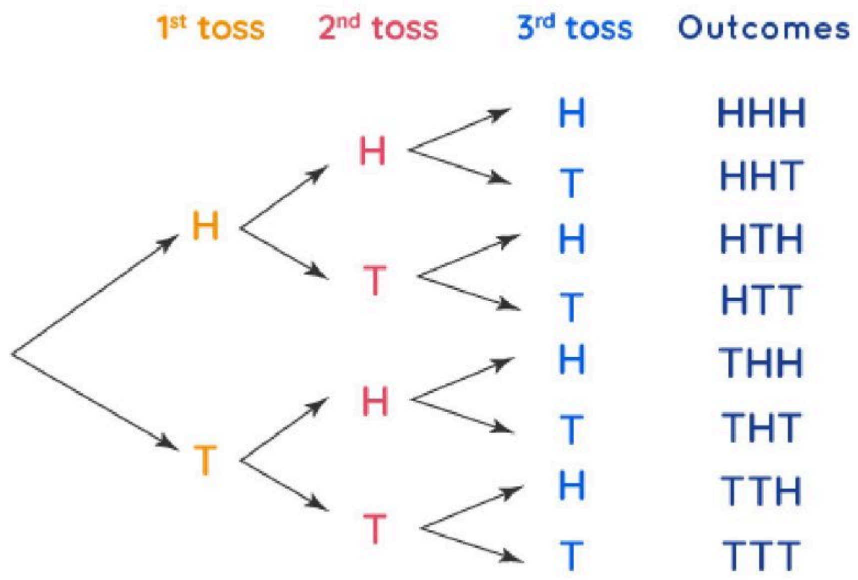
Let us understand conditional probability with an example. Let us find the conditional probability of getting at least two tails given that it is a head on the first toss when 3 coins are tossed. The sample space, S (the list of all outcomes) when 3 coins are tossed is given as follows:



S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Let us assume the two events A and B as follows:

- A = the event of getting at least two tails
- B = the event of getting a head on the first toss

Then, A = {HTT, THT, TTH, TTT} and B = {HHH, HHT, HTH, HTT}.

Then P(A) = 4/8 = 1/2 and P(B) = 4/8 = 1/2.

We have to find the probability of getting at least two tails given that it is a head on the first toss. It means, out of all elements of B, we have to choose only the ones with two tails. We can see that among the elements of B, there is only one element (which is HTT) with two tails. Thus, the required probability is P(A | B) = 1/4 (only 1 outcome of B is favorable to A out of 4 outcomes of B).

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Conditional Probability Formula

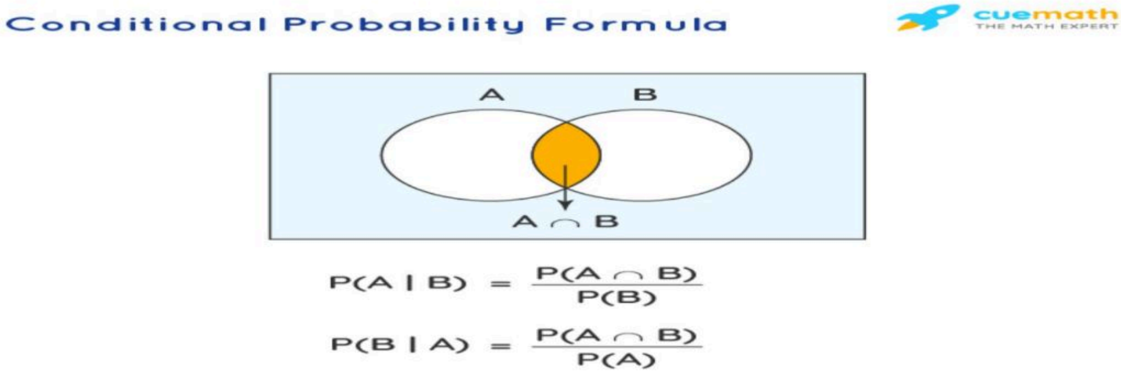
In the above example, we have got $P(A \mid B) = 1/4$, here 1 represents the element HTT which is present both in "A and B" and 4 represents the total number of elements in B. Using this, we can derive the formula of conditional probability as follows.

$P(A \mid B) = P(A \cap B) / P(B)$ (Note that $P(B) \neq 0$ here)

Similarly, we can define $P(B \mid A)$ as follows:

$P(B \mid A) = P(A \cap B) / P(A)$ (Note that $P(A) \neq 0$ here)

These formulas are also known as the "Kolmogorov definition" of conditional probability.



Dependent and Independent Events

The definition of independent and dependent events is connected to conditional probability. Let us see the definitions of independent and dependent events along with their formulas.

Dependent Events

Dependent events, as the name suggests, are any two events in which the happening of one event depends on the happening of the other event.

- If A depends on B, then the probability of A is $P(A \mid B)$.
- If B depends on A, then the probability of B is $P(B \mid A)$.

By the conditional probability formulas,

$P(A \mid B) = P(A \cap B) / P(B) \Rightarrow P(A \cap B) = P(A \mid B) \cdot P(B)$

$P(B \mid A) = P(A \cap B) / P(A) \Rightarrow P(A \cap B) = P(B \mid A) \cdot P(A)$

Thus, two event A and B are said to be dependent events if one of the conditions is satisfied.

- $P(A \cap B) = P(A \mid B) \cdot P(B)$ (or)
- $P(A \cap B) = P(B \mid A) \cdot P(A)$

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Independent Events

Independent events, as the name suggests, are any two events in which the happening of one event does not depend on the happening of the other event. i.e., if A and B are independent then P(A | B) = P(A) and P(B | A) = P(B). Thus, to get the formula of independent events, we just need to replace P(A | B) with P(A) (or P(B | A) with P(B)) in one of the above (dependent events) formulas. Hence, two events are said to be independent if

$P(A \cap B) = P(A) \cdot P(B)$

This is also called as multiplication rule of probability.

What is a Random Variable?

A random variable is a variable that can take on many values. This is because there can be several outcomes of a random occurrence. Thus, a random variable should not be confused with an algebraic variable. An algebraic variable represents the value of an unknown quantity in an algebraic equation that can be calculated. On the other hand, a random variable can have a set of values that could be the resulting outcome of a random experiment.

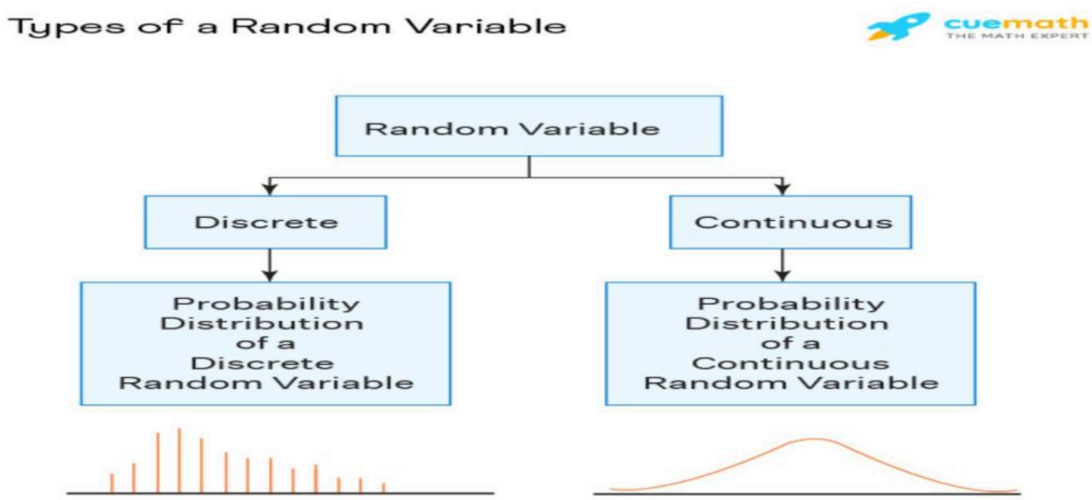
Random Variable Definition

A random variable can be defined as a type of variable whose value depends upon the numerical outcomes of a certain random phenomenon. It is also known as a stochastic variable. Random variables are always real numbers as they are required to be measurable.

Random Variable Example

Suppose 2 dice are rolled and the random variable, X, is used to represent the sum of the numbers. Then, the smallest value of X will be equal to 2 (1 + 1), while the highest value would be 12 (6 + 6). Thus, X could take on any value between 2 to 12 (inclusive). Now if probabilities are attached to each outcome then the probability distribution of X can be determined.

Types of Random Variables



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Random Variables can be divided into two broad categories depending upon the type of data available. These are given as follows:

- Discrete random variable
- Continuous random variable

A probability mass function is used to describe a discrete random variable and a probability density function describes a continuous random variable. The upcoming sections will cover these topics in detail.

Discrete Random Variable

A discrete random variable is a variable that can take on a finite number of distinct values. For example, the number of children in a family can be represented using a discrete random variable. A probability distribution is used to determine what values a random variable can take and how often does it take on these values. Some of the discrete random variables that are associated with certain special probability distributions will be detailed in the upcoming section.

Mean of a Random Variable

The average value of a random variable is called the mean of a random variable. The mean is also known as the expected value. It is generally denoted by E[X]. where X is the random variable.

The mean or expected value of a random variable can also be defined as the weighted average of all the values of the variable. The formulas for the mean of a random variable are given below:

- Mean of a Discrete Random Variable: $E[X] = \sum xP(X=x)$. Here $P(X = x)$ is the probability mass function.
- Mean of a Continuous Random Variable: $E[X] = \int xf(x)dx$. $f(x)$ is the probability density function

Variance of a Random Variable

The variance of a random variable can be defined as the expected value of the square of the difference of the random variable from the mean. The variance of a random variable is given by Var[X] or σ^2. If μ is the mean then the formula for the variance is given as follows:

- Variance of a Discrete Random Variable: $Var[X] = \sum(x-\mu)^2P(X=x)$
- Variance of a Continuous Random Variable: $Var[X] = \int(x-\mu)^2f(x)dx$

Expected Values

Definition 1

The expected value is also called as the expectation or mean of the random variable X on the sample space S is equal to

$E(S) = \sum_{s\in S} P(s)X(s)$

The derivation of X at s $\in$S is X(s)-E(X), the difference between the value of X & the mean of X.

Example 1: What is the expected value of a dice roll?

Solution: The random variable X can take on values from 1 to 6. The probability of occurrence of each

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value is 1 / 6.

Using the formula, $E[X] = \sum xP(X=x)$

$E[X] = 1\cdot(1 / 6) + 2\cdot(1 / 6) + 3\cdot(1 / 6) + 4\cdot(1 / 6) + 5\cdot(1 / 6) + 6\cdot(1 / 6) = 21 / 6$

Answer: $E[X] = 21 / 6=7/2$

Variance

The variance of a random variable X is defined by

$$\mathrm{var}(X)=E[(X-\mu)^2],\text{where } \mu=E(X)$$

For a discrete random variable X, the variance of X is obtained as follows:

$$\mathrm{var}(X)=\sum(x-\mu)^2 p_X(x),$$

Example

Consider the rolling of a fair six-sided die, with X the number on the uppermost face. We know that the pf of X is

$$p_X(x)=1/6, x=1,2,3,4,5,6,$$

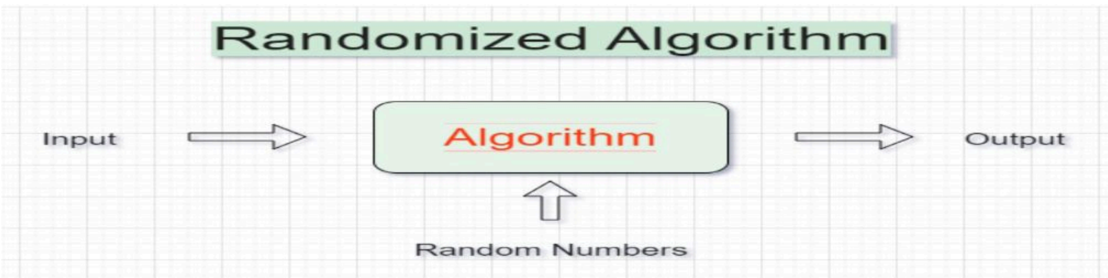
and that $\mu_X=3.5$. The variance of X is given by

$$\mathrm{var}(X)=E[(X-\mu_X)^2]=\sum(x-\mu_X)^2 p_X(x)=\sum(x-\mu_X)^2 1/6=1/6((1-3.5)^2+(2-3.5)^2+(3-3.5)^2+(4-3.5)^2+(5-3.5)^2+(6-3.5)^2)= 1/6\times17.5=35/12\approx2.9167$$

Hence, the standard deviation of X is $\sigma_X=\sqrt{35/12} \approx1.7078$

Randomized Algorithms

A randomized algorithm is a technique that uses a source of randomness as part of its logic. It is typically used to reduce either the running time or time complexity or the memory used or space complexity, in a standard algorithm.



The algorithm works by generating a random number(r) within a specified range of numbers & making decision based on (r) value.

Advantages

Simplicity

Performance

Scopes

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Algebric identities

Data Structures

Mathematical Programming

Graph Algorithms

Counting & enumerations

Probability calculation in Hashing

The probability of success is $(k - 1)/k$. In asking for the expected value of X2, we are asking for expected number of steps until the first success.

We can use our knowledge of probability and expected values to analyze a number of interesting aspects of hashing including:

1. Expected number of items per location
2. Expected time for a search
3. Expected number of collisions
4. Expected number of empty locations
5. Expected time until all locations have at least one item
6. Expected maximum number of items per location.

E(collisions) = n − E(occupied locations) = n − k + E(empty locations)

Advanced Counting

Recurrence Relations

Suppose that number of bacteria in a colony doubles every hour. If a colony begins with five bacteria. How many will be present in n hours?

To solve this problem

Let a_n be the number of bacteria at the end of n hours. Because the number of bacteria doubles every hour, $a_n=2a_{n-1}$, n is a +ve integer.

$a_0 =5$

$a_n =5.2^n$

Solving linear Recurrence relations

Theorem 1

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$a_n =c_1a_{n-1}+c_2a_{n-2}+\ldots+c_k a_{n-k}$ where $c_1,c_2,\ldots,c_k$ are real numbers & $c_k \neq 0$.

$a_0=c_0$, $a_1=c_1\ldots a_{k-1}=c_{k-1}$.

Example 3

What is the recurrence relation $a_n=a_{n-1}+2a_{n-2}$ with $a_0=2$ & $a_1=7$?

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Let us find the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$, with $a_0=2$ and $a_1=7$. Let us solve the characteristic equation which is equivalent to $k^2-k-2=0$, and hence by has the solutions $k1=-1$ and $k2=2$. It follows that the solution of the equation is $a_n = C_1\cdot(-1)^n + C_2\cdot 2^n$. We have that

$2=a_0=C_1+C_2$ and $7=-C_1+2C_2$. Therefore, $C_1=-1$ and $C_2=3$. We conclude that

$a_n = (-1)^n+1+3\cdot 2^n$.

Example 4

Find an explicit formula for the Fibonacci numbers.

Recurrence relation $f_n=f_{n-1}+f_{n-2}$ & also satisfies the initial conditions $f_0=0$ & $f_1=1$.The roots of the equation $r^2-r-2=0$ are $r_1= (1+\sqrt{5/2})$ & $r_2= (1-\sqrt{5/2})$.Therefore, from Theorem 1,It follows that the Fibonacci numbers are given by

$f_n= \alpha_1(1+\sqrt{5/2})^n+ \alpha_2(1-\sqrt{5/2})^n$.

$f_0= \alpha_1+ \alpha_2=0$

$f1=\alpha_1 (1+\sqrt{5}/2) +\alpha_2 (1-\sqrt{5}/2)=1$

$\alpha_1= 1/\sqrt{5}$ & $\alpha_2= -1/\sqrt{5}$

The Fibonacci numbers are given by

$f_n =1/\sqrt{5}(1+\sqrt{5}/2)^n -1/\sqrt{5}(1-\sqrt{5}/2)^n$

Theorem 2

Let c_1 & c_2 be real numbers with $c_2 \neq 0$.Suppose that $r^2-c_1r-c_2=0$ has only one root r. A sequence $\{a_n\}$ is a solution of the recurrence relation

$a_n =c_1a_{n-1}+c_2a_{n-2}$ if & only if $a_n= \alpha_1 r_0^n+ \alpha_2 n r_0^n$, n=0,1,2...... where $\alpha 1$ & $\alpha 2$ are constants.

Example 5

What is the solution of the recurrence relation $a_n=6a_{n-1}-9a_{n-2}$ with initial conditions $a_0=1$ & $a_1=6$?

Solution: $r^2 - 6r + 9 = 0$ has only 3 as a root. So the format of the solution is $a_n = \alpha_1 3^n + \alpha_2 n3^n$. Need to determine α_1 and α_2 from initial conditions: $a_0 = 1 = \alpha_1 = 6 = \alpha_1 \cdot 3^n + \alpha_2 3$ Solving these equations we get $\alpha_1 = 1$ and $\alpha_2 = 1$. Therefore, $a_n = 3^n + n\, 3^n$.

Theorem 3

Let c1, c2, . . . , ck be real numbers. Suppose that the characteristic equation $r^k - c1r^{k-1} - \cdots - ck = 0$ has k distinct roots r1, r2, . . . , rk. Then, a sequence {an} is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$ for n = 0, 1, 2, . . . , where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Example 6

Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$, with the initial conditions $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$.

Solution:

$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$

$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$

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$\Rightarrow a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n$

where r_1 , r_2 and r_3 are roots of

$r^3 - 6r^2 + 11r - 6 = 0$

$\Rightarrow (r - 1)(r^2 - 5r + 6) = 0$

$\Rightarrow (r - 1)(r - 2)(r - 3) = 0$

Hence

$a_n = \alpha_1 + \alpha_2 2^n + \alpha_3 3^n$

$a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 2$ Eq1

$a_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 = 5$ Eq2

$a_2 = \alpha_1 + 4\alpha_2 + 9\alpha_3 = 15$ Eq3

Eq2 - Eq1

$\Rightarrow \alpha_2 + 2\alpha_3 = 3$

Eq3 - Eq2

$\Rightarrow 2\alpha_2 + 6\alpha_3 = 10 \Rightarrow \alpha_2 + 3\alpha_3 = 5$

$\Rightarrow \alpha_3 = 2$

$\Rightarrow \alpha_2 = -1$

Hence, $\alpha_1 = 1$

$\Rightarrow a_n = 1 - 2^n + 2.\ 3^n$

$a_n = 2.\ 3^n - 2^n + 1$

$a_0 = 2 - 1 + 1 = 2$ Verified

$a_1 = 6 - 2 + 1 = 5$ Verified

$a_2 = 18 - 4 + 1 = 15$ Verified

$a_3 = 54 - 8 + 1 = 47$

$a_3 = 6a_2 - 11a_1 + 6a_0$

$\Rightarrow a_3 = 6(15) - 11(5) + 6(2)$

$\Rightarrow a_3 = 90 - 55 + 12$

$\Rightarrow a_3 = 47$ Verified

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$a_n = 2.\ 3^n - 2^n + 1$ **is the required solution .**

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