Unit-3

Induction & Recursion

Mathematical Induction is a technique of proving a statement, theorem or formula which is thought to be true, for each and every natural number n. By generalizing this in form of a principle which we would use to prove any mathematical statement is '**Principle of Mathematical Induction**'.

For example: $1^3 + 2^3 + 3^3 + \ldots.. + n^3 = (n (n+1) / 2)^2$, the statement is considered here as true for all the values of natural numbers.

Principle of Mathematical Induction Solution and Proof

Consider a statement P(n), where n is a natural number. Then to determine the validity of P(n) for every n, use the following principle:

Step 1: Check whether the given statement is true for n = 1.

Step 2: Assume that given statement P(n) is also true for n = k, where k is any positive integer.

Step 3: Prove that the result is true for P(k+1) for any positive integer k.

If the above-mentioned conditions are satisfied, then it can be concluded that P(n) is true for all n natural numbers.

Proof:

The first step of the principle is a *factual statement* and the second step is a *conditional one*. According to this if the given statement is true for some positive integer k only then it can be concluded that the statement P(n) is valid for n = k + 1.

This is also known as the *inductive step* and the assumption that P(n) is true for n=k is known as the *inductive hypothesis.*

Solved problems

Example 1: Prove that the sum of cubes of n natural numbers is equal to($[n(n+1)]/2)^2$ for all n natural numbers.

Solution:

In the given statement to prove:

$1^3+2^3+3^3+\cdots+n^3 = ([n(n+1)]/2)^2$

Step 1: Now with the help of the principle of induction in Math's, let us check the validity of the given statement P(n) for n=1.

$P(1) = ([1(1+1)]/2)^2 = (2/2)^2 = 1^2 = 1$.

This is true.

Step 2: Now as the given statement is true for n=1, we shall move forward and try proving this for n=k, i.e.

$1^3+2^3+3^3+\cdots+k^3 = ([k (k+1)]/2)^2$.

Step 3: Let us now try to establish that P(k+1) is also true.

$1^3+2^3+3^3+\ldots k^3+ (k+1)^3 = [k (k+1)/2]^2 + (k+1) (k+1)^2$
$= (k+1)^2[k^2/4+ (k+1)]$
$=(k+1)^2(k^2+4k+4/4)$
$=(k+1)^2(k+2/2)^2$
$= [(k+1) (k+2)/2]^2$

Example 2: **Show that $1 + 3 + 5 + \ldots + (2n-1) = n^2$**

Solution:

Step 1: Result is true for n = 1

That is $1 = (1)^2$ (True)

Step 2: Assume that result is true for n = k

$1 + 3 + 5 + \ldots + (2k-1) = k^2$

Step 3: Check for n = k + 1

i.e. $1 + 3 + 5 + \ldots + (2(k+1) -1) = (k+1)^2$

We can write the above equation as,

$1 + 3 + 5 + \ldots + (2k-1) + (2(k+1) - 1) = (k+1)^2$

Using step 2 result, we get

$k^2 + (2(k+1) - 1) = (k+1)^2$

$k^2 + 2k + 2 - 1 = (k+1)^2$

$k^2 + 2k + 1 = (k+1)^2$

$(k+1)^2 = (k+1)^2$

L.H.S. and R.H.S. are same.

So the result is true for n = k+1

By mathematical induction, the statement is true.

We see that the given statement is also true for n=k+1. Hence we can say that by the principle of mathematical induction this statement is valid for all natural numbers n.

Example 3:

Show that $2^{2n}-1$ is divisible by 3 using the principles of mathematical induction.

To prove: $2^{2n}-1$ is divisible by 3

Assume that the given statement be P (k)

Thus, the statement can be written as P (k) = $2^{2n}-1$ is divisible by 3, for every natural number

Step 1: In step 1, assume n= 1, so that the given statement can be written as

P (1) = $2^{2(1)}-1 = 4-1 = 3$. So 3 is divisible by 3.(i.e. 3/3 = 1)

Step 2: Now, assume that P (n) is true for all the natural numbers, say k

Hence, the given statement can be written as

P (k) = $2^{2k}-1$ is divisible by 3.

It means that $2^{2k}-1 = 3a$ (where a belongs to natural number)

Now, we need to prove the statement is true for n= k+1

Hence,

$P(k+1) = 2^{2(k+1)}-1$

$P(k+1)= 2^{2k+2}-1$

$P(k+1)= 2^{2k}.\ 2^2 - 1$

$P(k+1)= (2^{2k}.\ 4)-1$

$P(k+1) =3.2^{2k}+(2^{2k}-1)$

The above expression can be written as

$P(k+1) =3.2^{2k}+ 3a$

Now, take 3 outside, we get

$P(k+1) = 3(2^{2k}+ a)= 3b$, where "b" belongs to natural number

It is proved that p(k+1) holds true, whenever the statement P(k) is true.

Thus, $2^{2n}-1$ is divisible by 3 is proved using the principles of mathematical induction

Example 11

Prove that De Morgan's law by induction

Inductive proof of Generalized De Morgan's law for sets:

$$\overline{A_1 \cap A_2 \cap \ldots \cap A_n} = \overline{A}_1 \cup \overline{A}_2 \cup \ldots \cup \overline{A}_n$$

Base case: n = 2 $\quad \overline{A_1 \cap A_2} = \overline{A}_1 \cup \overline{A}_2$ $\quad$ De Morgans's law

Inductive step:

Assume: $\overline{A_1 \cap A_2 \cap \ldots \cap A_k} = \overline{A}_1 \cup \overline{A}_2 \cup \ldots \cup \overline{A}_k$

Prove: $\overline{A_1 \cap A_2 \cap \ldots \cap A_k \cap A_{k+1}} = \overline{A}_1 \cup \overline{A}_2 \cup \ldots \cup \overline{A}_k \cup \overline{A}_{k+1}$

Define: $B = A_1 \cap A_2 \ldots \cap A_k$

$$\begin{aligned}
\overline{A_1 \cap A_2 \cap \ldots \cap A_k \cap A_{k+1}} &= \overline{B \cap A_{k+1}} && \text{Definition of B} \\
&= \overline{B} \cup \overline{A}_{k+1} && \text{De Morgans's law} \\
&= \overline{A_1 \cap A_2 \cap \ldots \cap A_k} \cup \overline{A}_{k+1} && \text{Definition of B} \\
&= \overline{A}_1 \cup \overline{A}_2 \cup \ldots \cup \overline{A}_k \cup \overline{A}_{k+1} && \text{Inductive hypothesis}
\end{aligned}$$

Equal

Strong induction & Well-Ordering

Another form of mathematical induction called strong induction.

Strong induction

To prove that P (n) is true for all positive integers n where P(n) is a propositional function, we complete two steps.

Basis step: We verify that the proposition P (1) is true.

Inductive step:

We show that the conditional statement $[P(1) \wedge P(2) \wedge \ldots\ldots \wedge P(k)] \rightarrow P(k+1)$ is true for all +ve integers k.

Strong induction is sometimes called the second principle of Mathematical induction or complete induction.

Example 2

Show that if n is an integer greater than 1, then n can be written as the product of primes.

Let n and d denote integers. We say that d is a divisor of n if n=cd for some integer c. An integer n is called a prime if n>1 and if the only positive divisors of n are 11 and n. Prove, by induction, that every integer n>1 is either a prime or a product of primes.

My try: First, that there's nothing to prove because a number is always a prime or not, so do not what to think. Step: P (n): n is either a prime or a product of primes. If n=2 then 2 is prime. P (n): True.

I want to see P (n) →P (n+1). If n is a prime then 22 is a divisor of p+1, then is a product of primes. If n is a product of primes. I can't say anything about n+1.

Proofs using the Well –Ordering Property

The validity of both the principle of mathematical induction & strong induction follows from a fundamental axiom of the set of integers, the well-ordering property. The well ordering property states that every non –empty set can be used directly in proofs.

The Well-Ordering Property

Every non-empty set of non-negative integers has a least element.

Example 5

Use the well- ordering property to prove the division algorithm.

Division algorithm states that if a is an integer, then there are unique integers q & r with $0 <= r < d$ & $a = dq + r$.

Solution:

Let S be the set of non-negative integers of the form a – dq, where q is an integer. This set is non-empty because –dq can be made as large as desired (taking q to be a negative integer with large absolute value). By the well-ordering property, S has a least element r = a – dq0. The integer r is nonnegative. It is also the case that r < d. If it were not, then there would be a smaller nonnegative element in S, namely, a – d(q0 + 1). To see this, suppose that r ≥ d. Because a=dq0+r, it follows that a–d(q0+1)=(a–dq0)–d=r–d≥0. Consequently, there are integers q and r with 0 ≤ r < d.

Recursive Definitions & Structural induction

It may be easy to define this object in terms of itself. This process is called recursion. When we define a set recursively, we specify some initial elements in a basis step & provide a rule for constructing new elements from those we already have in the recursive step.

Recursively defined functions

Basis step: Specify the value of the function at zero.

Recursive step: Give a rule for finding its value at an integer from its values at smaller integers.

Such a definition is called a recursive or inductive definition.

Example 1

Suppose that f is defined recursively by

f(0)=3

f(n+1)=2(n)+3

Find f(1),f(2),f(3) & f(4).

Solution:

f(1)=2f(0)+3=2.3+3=9

f(2)=2f(1)+3=2.9+3=21

f(3)=2f(2)+3=2.21+3=45

f(4)=2f(3)+3=2.45+3=93

Recursively defined functions are well-defined.

Fibonacci numbers f_0, f_1, f_2,……f_n are defined by the equations $f_0=0$,$f_1=1$ &

$f_n=f_{n-1}+f_{n-2}$, $n=2,3,4\ldots\ldots$

$f_2=?$ $f_3=?$

Recursively defined sets & Structures

Recursively defined sets have two parts.

1. Basis step &
2. Recursively step

In basis step, an initial collection of elements is specified.

2. Recursive step

In the recursive step, rules for forming new elements in the set from those already known to be in the set are provided.

Example 5

Consider the subset S of the set of integers recursively defined by

Basis step:

$3 \in s$

Recursive step:

If $x \in s$ & $y \in s$, then $x+y \in s$.

The new elements found to be in s are 3 by the basis step 3+3=6 at the first application of the recursive step. 3+6=6+3=9& 6+6=12 at the second application of the recursive step & so on.

Recursive Algorithms

Definition 1

A process in which a function calls itself as a subroutine. This technique allows the function to leverage solutions to smaller instances of the same problem, thereby solving the problem through repetition until a base condition is met.

Example 1

To calculate factorial of n number using recursion

```
int factorial(n);
   if (n == 1)
      return 1;
   else
```

return (n * factorial(n-1));

Example 2

Give a recursive algorithm for finding the sum of first n positive integers.

Solution:

Define sum (n) by

sum(1) = 1

sum(n + 1) = sum(n) + n, where n > 1

Algorithm

Procedure sum (n: positive integer)

if (n = 1)

sum(n) = 1

else

n = n + sum(n – 1)