

A blue wavy background graphic with a gradient from light blue at the top to a darker blue at the bottom, featuring a wavy line across the top.

# Matrices

Section 2.6



## Section Summary

- Definition of a Matrix
- Matrix Arithmetic
- Transposes and Powers of Arithmetic
- Zero-One matrices



# Matrices

- Matrices are useful discrete structures that can be used in many ways. For example, they are used to:
  - describe certain types of functions known as linear transformations.
  - Express which vertices of a graph are connected by edges (see Chapter 10).
- In later chapters, we will see matrices used to build models of:
  - Transportation systems.
  - Communication networks.
- Algorithms based on matrix models will be presented in later chapters.
- Here we cover the aspect of matrix arithmetic that will be needed later.

# Matrix

**Definition:** A *matrix* is a rectangular array of numbers. A matrix with  $m$  rows and  $n$  columns is called an  $m \times n$  matrix.

- The plural of matrix is *matrices*.
- A matrix with the same number of rows as columns is called *square*.
- Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

$3 \times 2$  matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$$

# Notation

- Let  $m$  and  $n$  be positive integers and let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- The  $i$ th row of  $\mathbf{A}$  is the  $1 \times n$  matrix  $[a_{i1}, a_{i2}, \dots, a_{in}]$ . The  $j$ th column of  $\mathbf{A}$  is the  $m \times 1$  matrix:

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix}$$

- The  $(i,j)$ th *element* or *entry* of  $\mathbf{A}$  is the element  $a_{ij}$ . We can use  $\mathbf{A} = [a_{ij}]$  to denote the matrix with its  $(i,j)$ th element equal to  $a_{ij}$ .



# Matrix Arithmetic: Addition

**Defintion:** Let  $\mathbf{A} = [a_{ij}]$  and  $\mathbf{B} = [b_{ij}]$  be  $m \times n$  matrices. The sum of  $\mathbf{A}$  and  $\mathbf{B}$ , denoted by  $\mathbf{A} + \mathbf{B}$ , is the  $m \times n$  matrix that has  $a_{ij} + b_{ij}$  as its  $(i,j)$ th element. In other words,  $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$ .

**Example:**

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

Note that matrices of different sizes can not be added.

# Matrix Multiplication

**Definition:** Let  $\mathbf{A}$  be an  $n \times k$  matrix and  $\mathbf{B}$  be a  $k \times n$  matrix. The *product* of  $\mathbf{A}$  and  $\mathbf{B}$ , denoted by  $\mathbf{AB}$ , is the  $m \times n$  matrix that has its  $(i,j)$ th element equal to the sum of the products of the corresponding elements from the  $i$ th row of  $\mathbf{A}$  and the  $j$ th column of  $\mathbf{B}$ . In other words, if  $\mathbf{AB} = [c_{ij}]$  then  $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{kj}b_{kj}$ .

**Example:**

$$\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

The product of two matrices is undefined when the number of columns in the first matrix is not the same as the number of rows in the second.

# Illustration of Matrix Multiplication

- The Product of  $\mathbf{A} = [a_{ij}]$  and  $\mathbf{B} = [b_{ij}]$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ik} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mk} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{k1} & b_{k2} & \dots & b_{kj} & \dots & b_{kn} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & c_{ij} & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{ik}b_{kj}$$



## Matrix Multiplication is not Commutative

**Example:** Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Does  $\mathbf{AB} = \mathbf{BA}$ ?

**Solution:**

$$\mathbf{AB} = \begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix} \quad \mathbf{BA} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

$$\mathbf{AB} \neq \mathbf{BA}$$

## Identity Matrix and Powers of Matrices

**Definition:** The *identity matrix* of order  $n$  is the  $m \times n$  matrix  $\mathbf{I}_n = [\delta_{ij}]$ , where  $\delta_{ij} = 1$  if  $i = j$  and  $\delta_{ij} = 0$  if  $i \neq j$ .

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\mathbf{A}\mathbf{I}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$$

when  $\mathbf{A}$  is an  $m \times n$  matrix

Powers of square matrices can be defined. When  $\mathbf{A}$  is an  $n \times n$  matrix, we have:

$$\mathbf{A}^0 = \mathbf{I}_n \quad \mathbf{A}^r = \underbrace{\mathbf{A}\mathbf{A}\mathbf{A}\dots\mathbf{A}}_{r \text{ times}}$$

# Transposes of Matrices

**Definition:** Let  $\mathbf{A} = [a_{ij}]$  be an  $m \times n$  matrix. The *transpose* of  $\mathbf{A}$ , denoted by  $\mathbf{A}^t$ , is the  $n \times m$  matrix obtained by interchanging the rows and columns of  $\mathbf{A}$ .

If  $\mathbf{A}^t = [b_{ij}]$ , then  $b_{ij} = a_{ji}$  for  $i = 1, 2, \dots, n$   
and  $j = 1, 2, \dots, m$ .

The transpose of the matrix  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$  is the matrix  $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ .

# Transposes of Matrices

**Definition:** A square matrix  $A$  is called symmetric if  $A = A^t$ . Thus  $A = [a_{ij}]$  is symmetric if  $a_{ij} = a_{ji}$  for  $i$  and  $j$  with  $1 \leq i \leq n$  and  $1 \leq j \leq n$ .

The matrix  $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$  is square.

Square matrices do not change when their rows and columns are interchanged.



# Zero-One Matrices

**Definition:** A matrix all of whose entries are either 0 or 1 is called a *zero-one matrix*. (These will be used in Chapters 9 and 10.)

Algorithms operating on discrete structures represented by zero-one matrices are based on Boolean arithmetic defined by the following Boolean operations:

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases} \quad b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

# Zero-One Matrices

**Definition:** Let  $\mathbf{A} = [a_{ij}]$  and  $\mathbf{B} = [b_{ij}]$  be an  $m \times n$  zero-one matrices.

- The *join* of  $\mathbf{A}$  and  $\mathbf{B}$  is the zero-one matrix with  $(i,j)$ th entry  $a_{ij} \vee b_{ij}$ . The *join* of  $\mathbf{A}$  and  $\mathbf{B}$  is denoted by  $\mathbf{A} \vee \mathbf{B}$ .
- The *meet* of  $\mathbf{A}$  and  $\mathbf{B}$  is the zero-one matrix with  $(i,j)$ th entry  $a_{ij} \wedge b_{ij}$ . The *meet* of  $\mathbf{A}$  and  $\mathbf{B}$  is denoted by  $\mathbf{A} \wedge \mathbf{B}$ .

## Joins and Meets of Zero-One Matrices

**Example:** Find the join and meet of the zero-one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

**Solution:** The join of **A** and **B** is

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The meet of **A** and **B** is

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

## Boolean Product of Zero-One Matrices

**Definition:** Let  $\mathbf{A} = [a_{ij}]$  be an  $m \times k$  zero-one matrix and  $\mathbf{B} = [b_{ij}]$  be a  $k \times n$  zero-one matrix. The *Boolean product* of  $\mathbf{A}$  and  $\mathbf{B}$ , denoted by  $\mathbf{A} \odot \mathbf{B}$ , is the  $m \times n$  zero-one matrix with  $(i,j)$ th entry

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \dots \vee (a_{ik} \wedge b_{kj}).$$

**Example:** Find the Boolean product of  $\mathbf{A}$  and  $\mathbf{B}$ , where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

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## Boolean Product of Zero-One Matrices

**Solution:** The Boolean product  $\mathbf{A} \odot \mathbf{B}$  is given by

$$\begin{aligned}\mathbf{A} \odot \mathbf{B} &= \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix} \\ &= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.\end{aligned}$$

## Boolean Powers of Zero-One Matrices

**Definition:** Let  $\mathbf{A}$  be a square zero-one matrix and let  $r$  be a positive integer. The  $r$ th Boolean power of  $\mathbf{A}$  is the Boolean product of  $r$  factors of  $\mathbf{A}$ , denoted by  $\mathbf{A}^{[r]}$ . Hence,

$$\mathbf{A}^{[r]} = \underbrace{\mathbf{A} \odot \mathbf{A} \odot \dots \odot \mathbf{A}}_{r \text{ times}}.$$

We define  $\mathbf{A}^{[r]}$  to be  $\mathbf{I}_n$ .

(The Boolean product is well defined because the Boolean product of matrices is associative.)

## Boolean Powers of Zero-One Matrices

**Example:** Let  $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ .

Find  $\mathbf{A}^n$  for all positive integers  $n$ .

**Solution:**

$$\mathbf{A}^{[2]} = \mathbf{A} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \mathbf{A}^{[3]} = \mathbf{A}^{[2]} \odot \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A}^{[4]} = \mathbf{A}^{[3]} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A}^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \mathbf{A}^{[n]} = \mathbf{A}^5 \quad \text{for all positive integers } n \text{ with } n \geq 5.$$