## Unit-2

## Set Theory, Relations and Functions

German mathematician **G. Cantor** introduced the concept of sets. He had defined a set as a collection of definite and distinguishable objects selected by the means of certain rules or description.

**Set** theory forms the basis of several other fields of study like counting theory, relations, graph theory and finite state machines. In this chapter, we will cover the different aspects of **Set Theory**.

### Set - Definition

A set is an unordered collection of different elements. A set can be written explicitly by listing its elements using set bracket. If the order of the elements is changed or any element of a set is repeated, it does not make any changes in the set.

Some Example of Sets

- A set of all positive integers
- A set of all the planets in the solar system
- A set of all the states in India
- A set of all the lowercase letters of the alphabet

### Representation of a Set

Sets can be represented in two ways −

- Roster or Tabular Form
- Set Builder Notation

### Roster or Tabular Form

The set is represented by listing all the elements comprising it. The elements are enclosed within braces and separated by commas.

**Example 1** − Set of vowels in English alphabet, A= {a, e, i, o, u}

**Example 2** − Set of odd numbers less than 10, B= {1, 3, 5, 7, 9}

### Set Builder Notation

The set is defined by specifying a property that elements of the set have in common. The set is described as A={x: p(x)}

**Example 1** − The set {a, e, i, o, u} is written as −

A={x: x is a vowel in English alphabet}

**Example 2** − The set {1, 3, 5, 7, 9} is written as −

B={x: $1 \le x < 10$ and (x%2) $\ne 0$}

If an element x is a member of any set S, it is denoted by $x \in S$ and if an element y is not a member of set S, it is denoted by $y \notin S$.

**Example** − If S= {1, 1.2, 1.7, 2}, $1 \in S$ but $1.5 \notin S$

**Some Important Sets**

**N** − the set of all natural numbers = {1, 2, 3, 4 ...}

**Z** − the set of all integers = {....., −3, −2, −1, 0, 1, 2, 3,}

**$Z^+$** − the set of all positive integers

**Q** − the set of all rational numbers

**R** − the set of all real numbers

**W** − the set of all whole numbers

**Cardinality of a Set**

Cardinality of a set S, denoted by |S| is the number of elements of the set. The number is also referred as the cardinal number. If a set has an infinite number of elements, its cardinality is ∞.

**Example** − $|\{1,4,3,5\}|=4, |\{1,2,3,4,5,\ldots\}|=\infty$

If there are two sets X and Y,

- $|X|=|Y|$ denotes two sets X and Y having same cardinality. It occurs when the number of elements in X is exactly equal to the number of elements in Y. In this case, there exists a bijective function 'f' from X to Y.
- $|X|\le|Y|$ denotes that set X's cardinality is less than or equal to set Y's cardinality. It occurs when number of elements in X is less than or equal to that of Y. Here, there exists an injective function 'f' from X to Y.
- $|X|<|Y|$ denotes that set X's cardinality is less than set Y's cardinality. It occurs when number of elements in X is less than that of Y. Here, the function 'f' from X to Y is injective function but not bijective.
- If $|X|\le|Y|$ and $|X|\ge|Y|$ then $|X|=|Y|$. The sets X and Y are commonly referred as equivalent sets.

**Types of Sets**

Sets can be classified into many types. Some of which are finite, infinite, subset, universal, proper, singleton set, etc.

**Finite Set**

A set which contains a definite number of elements is called a finite set.

**Example** − S= {x|x∈N and 70>x>50}

**Infinite Set**

A set which contains infinite number of elements is called an infinite set.

**Example** − S= {x|x∈N and x>10}

**Subset**

A set X is a subset of set Y (Written as X⊆YX⊆Y) if every element of X is an element of set Y.

**Example 1** − Let, X={1,2,3,4,5,6} and Y={1,2}. Here set Y is a subset of set X as all the elements of set Y is in set X. Hence, we can write Y⊆X.

**Example 2** − Let, X= {1, 2, 3} and Y= {1, 2, 3}. Here set Y is a subset (Not a proper subset) of set X as all the elements of set Y is in set X. Hence, we can write Y⊆X.

**Proper Subset**

The term "proper subset" can be defined as "subset of but not equal to". A Set X is a proper subset of set Y (Written as X⊂Y if every element of X is an element of set Y and |X|<|Y|.

**Example** − Let, X={1,2,3,4,5,6} and Y={1,2}. Here set Y⊂X since all elements in Y are contained in X too and X has at least one element is more than set Y.

**Universal Set**

It is a collection of all elements in a particular context or application. All the sets in that context or application are essentially subsets of this universal set. Universal sets are represented as U.

**Example** − We may define U as the set of all animals on earth. In this case, set of all mammals is a subset of U, set of all fishes is a subset of U, and set of all insects is a subset of U and so on.

**Empty Set or Null Set**

An empty set contains no elements. It is denoted by ∅. As the number of elements in an empty set is finite, empty set is a finite set. The cardinality of empty set or null set is zero.

**Example** − $S = \{x \mid x \in N \text{ and } 7 < x < 8\} = \emptyset$

**Singleton Set or Unit Set**

Singleton set or unit set contains only one element. A singleton set is denoted by {s}.

**Example** − $S = \{x \mid x \in N, 7 < x < 9\} = \{8\}$

**Equal Set**

If two sets contain the same elements they are said to be equal.

**Example** − If A={1,2,6} and B={6,1,2}, they are equal as every element of set A is an element of set B and every element of set B is an element of set A.

**Equivalent Set**

If the cardinalities of two sets are same, they are called equivalent sets.

**Example** − If A= {1, 2, 6} and B= {16, 17, 22}, they are equivalent as cardinality of A is equal to the cardinality of B. i.e. $|A|=|B|=3$.

**Overlapping Set**

Two sets that have at least one common element are called overlapping sets.

In case of overlapping sets −

- $n(A \cup B)=n(A)+n(B)-n(A \cap B)$
- $n(A \cup B)=n(A-B)+n(B-A)+n(A \cap B)$
- $n(A)=n(A-B)+n(A \cap B)$
- $n(B)=n(B-A)+n(A \cap B)$

**Example** − Let, A= {1, 2, 6} and B= {6, 12, 42}. There is a common element '6', hence these sets are overlapping sets.

**Disjoint Set**

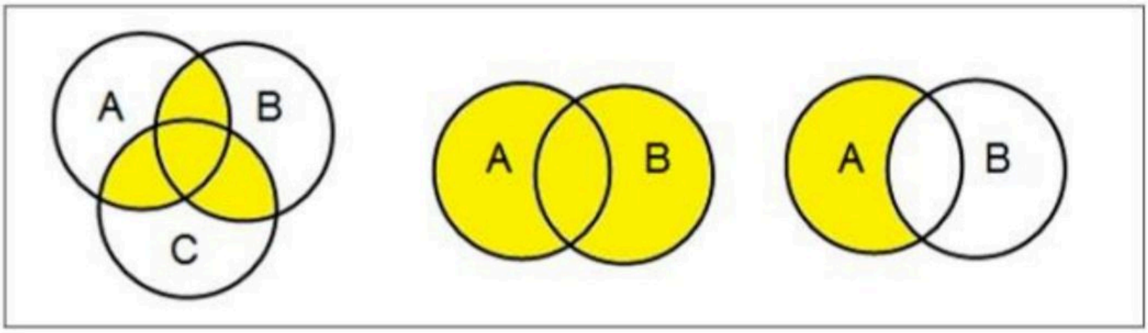
Two sets A and B are called disjoint sets if they do not have even one element in common. Therefore, disjoint sets have the following properties −

- $n(A \cap B)=\emptyset$
- $n(A \cup B)=n(A)+n(B)$
- **Example** − Let, A={1, 2,6} and B={7,9,14}, there is not a single common element, hence these sets are disjoint sets.

**Venn Diagrams**

Venn diagram, invented in 1880 by John Venn, is a schematic diagram that shows all possible logical relations between different mathematical sets.

**Examples**




**Set Operations**

Set Operations include Set Union, Set Intersection, Set Difference, Complement of Set, and Cartesian product.

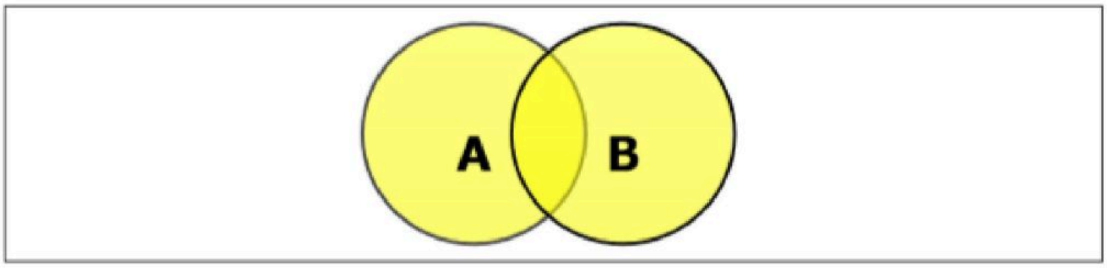
**Set Union**

The union of sets A and B (denoted by $A \cup B$) is the set of elements which are in A, in B or in both A and B. Hence, $A \cup B = \{x | x \in A \text{ OR } x \in B\}$.

**Example−**

If $A=\{10,11,12,13\}$ and $B = \{13,14,15\}$ then $A \cup B=\{10,11,12,13,14,15\}$.

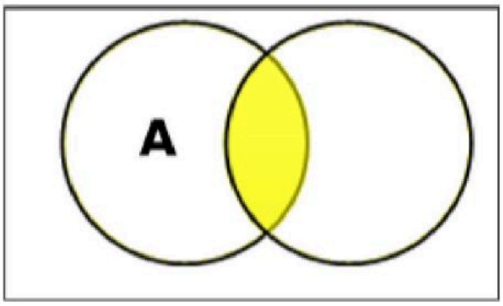
(The common element occurs only once)




**Set Intersection**

The intersection of sets A and B (denoted by $A \cap B$) is the set of elements which are in both A and B. Hence, $A \cap B= \{x | x \in A \text{ AND } x \in B\}$.

**Example** − If $A=\{11, 12,13\}$ and $B=\{13,14,15\}$, then $A \cap B=\{13\}$.



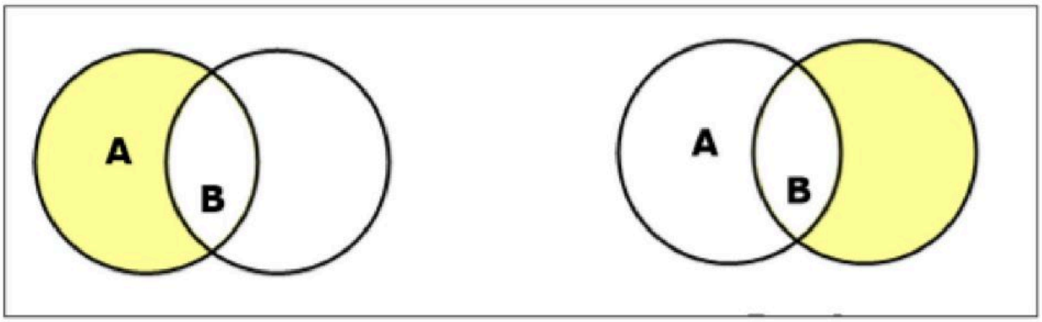

**Set Difference/ Relative Complement**

The set difference of sets A and B (denoted by A–B) is the set of elements which are only in A but not in B. Hence, A−B= {x|x∈A AND x∉B}.

**Example**

If A={10,11,12,13} and B={13,14,15} then (A−B)={10,11,12} and (B−A)={14,15}.

Here, we can see (A−B) ≠ (B−A)



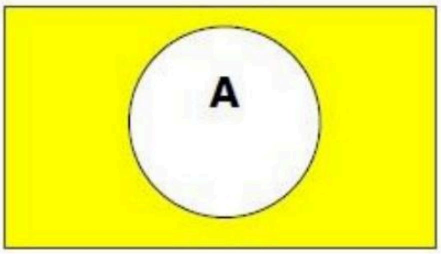

**Complement of a Set**

The complement of a set A (denoted by A′) is the set of elements which are not in set A. Hence, A′= {x|x∉A}.

More specifically, A′= (U−A) where U is a universal set which contains all objects.

**Example** –

If A={x: x belongs to set of odd integers} then A′= {y|y does not belong to set of odd integers}



**Cartesian product / Cross Product**

The Cartesian product of n number of sets A1, A2…$A_n$ denoted as A1×A2⋯×$A_n$ can be defined as all possible ordered pairs (x1, x2,…$x_n$) where x1∈A1,x2∈A2,…$x_n$∈$A_n$.

**Example** − If we take two sets A= {a, b} and B= {1, 2},

The Cartesian product of A and B is written as − A×B= {(a,1),(a,2),(b,1),(b,2)}

The Cartesian product of B and A is written as − B×A={(1,a),(1,b),(2,a),(2,b)}.

**Power Set**

Power set of a set S is the set of all subsets of S including the empty set. The cardinality of a power set of a set S of cardinality n is $2^n$. Power set is denoted as P(S).

**Example −**

For a set S= {a, b,c,d}

Let us calculate the subsets −

- Subsets with 0 elements − {∅} (the empty set)
- Subsets with 1 element − {a},{b},{c},{d}
- Subsets with 2 elements − {a,b},{a,c},{a,d},{b,c},{b,d},{c,d}
- Subsets with 3 elements − {a,b,c},{a,b,d},{a,c,d},{b,c,d}
- Subsets with 4 elements − {a,b,c,d}

Hence,P(S)=

{{∅},{a},{b},{c},{d},{a,b},{a,c},{a,d},{b,c},{b,d},{c,d},{a,b,c},{a,b,d},{a,c,d},{b,c,d},{a,b,c,d}}

$|P(S)|=2^4=16$.

**Note** − The power set of an empty set is also an empty set.

$|P(\{\emptyset\})|=2^0=1$

**Relations**

**Relations** may exist between objects of the same set or between objects of two or more sets.

**Definition and Properties**

A binary relation R from set x to y (written as xRy or R(x,y) is a subset of the Cartesian product x×y. If the ordered pair of G is reversed, the relation also changes.

Generally an n-ary relation R between sets A1,…., and $A_n$ is a subset of the n-ary product A1×⋯×$A_n$. The minimum cardinality of a relation R is Zero and maximum is $n^2$ in this case.

A binary relation R on a single set A is a subset of A×A.

For two distinct sets, A and B, having cardinalities *m* and *n* respectively, the maximum cardinality of a relation R from A to B is *mn*.

Domain and Range

If there are two sets A and B, and relation R have order pair (x, y), then −

- The **domain** of R, Dom(R), is the set {x|(x,y)∈R for some y in B}
- The **range** of R, Ran(R), is the set {y|(x,y)∈R for some x in A}

Examples

Let, A= {1, 2, 9} and B= {1, 3, 7}

- Case 1 − If relation R is 'equal to' then R={(1,1),(3,3)}

Dom(R) = {1, 3}, Ran(R) = {1, 3}, Ran(R)={1,3}

- Case 2 − If relation R is 'less than' then R={(1,3),(1,7),(2,3),(2,7)}

Dom(R) = {1,2},Ran(R)={3,7}{1,2},Ran(R)={3,7}

- Case 3 − If relation R is 'greater than' then R={(2,1),(9,1),(9,3),(9,7)}

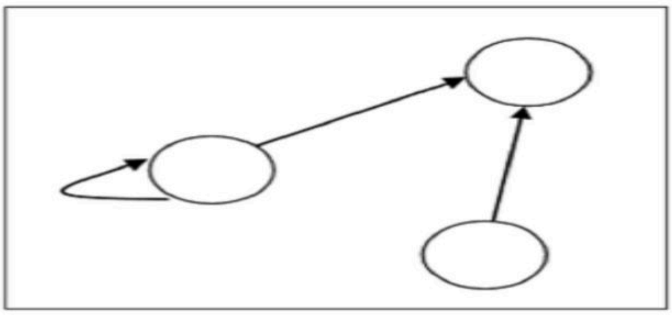
Dom(R) = {2,9},Ran(R)={1,3,7}{2,9},Ran(R)={1,3,7}

**Representation of Relations using Graph**

A relation can be represented using a directed graph.

The number of vertices in the graph is equal to the number of elements in the set from which the relation has been defined. For each ordered pair (x, y) in the relation R, there will be a directed edge from the vertex 'x' to vertex 'y'. If there is an ordered pair (x, x), there will be self- loop on vertex 'x'.

Suppose, there is a relation R= {(1,1),(1,2),(3,2)} on set S={1,2,3}, it can be represented by the following graph −



**Types of Relations**

- The **Empty Relation** between sets X and Y, or on E, is the empty set ∅

- The **Full Relation** between sets X and Y is the set X×Y
- The **Identity Relation** on set X is the set {(x,x)|x∈X}
- The Inverse Relation R' of a relation R is defined as − R′={(b,a)|(a,b)∈R}

**Example** − If R= {(1, 2), (2, 3)} then R′ will be {(2, 1),(3,2)}

- A relation R on set A is called **Reflexive** if ∀ a∈A is related to a (aRa holds)

**Example** − The relation R= {(a, a), (b, b)} on set X= {a ,b} is reflexive.

- A relation R on set A is called **Irreflexive** if no a∈A is related to a (aRa does not hold).

**Example** − The relation R= {(a, b), (b, a)} on set X= {a,b} is irreflexive.

- A relation R on set A is called **Symmetric** if xRy implies yRx, ∀x∈A and ∀y∈A.

**Example** − The relation R={(1,2),(2,1),(3,2),(2,3)} on set A={1,2,3} is symmetric.

- A relation R on set A is called **Anti-Symmetric** if xRy and yRx implies x=y, ∀x∈A and ∀y∈A.

**Example**− The relation R={(x,y)→N| x≤y} is anti-symmetric since x≤y and y≤x implies x=y.

- A relation R on set A is called **Transitive** if xRy and yRz implies xRz, ∀x, y,z∈A.

**Example** − The relation R={(1,2),(2,3),(1,3)} on set A={1,2,3} is transitive.

- A relation is an **Equivalence Relation** if it is reflexive, symmetric and transitive.

**Example** − The relation R={(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)} on set A={1,2,3} is an equivalence relation since it is reflexive, symmetric and transitive.

## Functions

Functions are an important part of discrete mathematics. This article is all about functions, their types, and other details of functions. A function assigns exactly one element of a set to each element of the other set. Functions are the rules that assign one input to one output. The function can be represented as f: A ⇢ B. A is called the *domain of the function* and B is called the *codomain function.*

**Functions:**

- A function assigns exactly one element of one set to each element of other sets.
- A function is a rule that assigns each input exactly one output.
- A function f from A to B is an assignment of exactly one element of B to each element of A (where A and B are non-empty sets).

- A function f from set A to set B is represented as **f: A ⇢ B** where A is called the domain of f and B is called as codomain of f.
- If b is a unique element of B to element a of A assigned by function F then, it is written as f(a) = b.
- Function f maps A to B means f is a function from A to B i.e. f: A ⇢ B

**Domain of a function:**

- If f is a function from set A to set B then, A is called the domain of function f.
- The set of all inputs for a function is called its domain.

**Codomain of a function:**

- If f is a function from set A to set B then, B is called the codomain of function f.
- The set of all allowable outputs for a function is called its codomain.

**Pre-image and Image of a function:**

A function f: A ⇢ B such that for each a ∈ A, there exists a unique b ∈ B such that (a, b) ∈ R then, a is called the pre-image of f and b is called the image of f.

**Types of function:**

**One-One function (or Injective Function):**

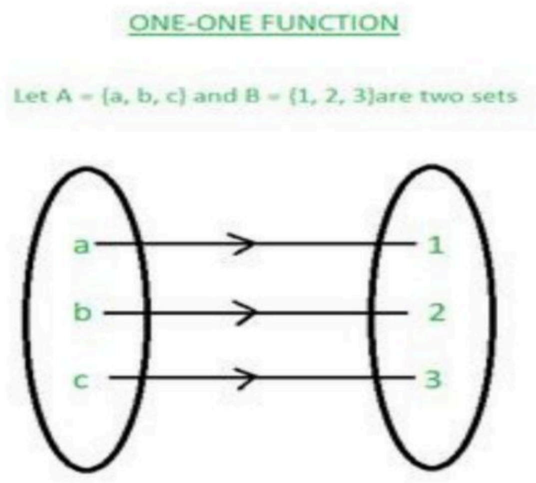
A function in which one element of the domain is connected to one element of the codomain.

A function f: A ⇢ B is said to be a one-one (injective) function if different elements of A have different images in B.

**f: A ⇢ B is one-one**

**$\Rightarrow a \neq b \Rightarrow f(a) \neq f(b)$ for all a, b ∈ A**

**$\Rightarrow f(a) = f(b) \Rightarrow a = b$ for all a, b ∈ A**




*ONE-ONE FUNCTION*

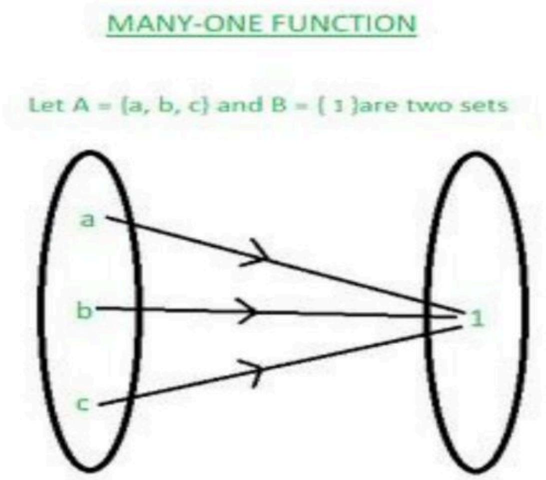
**Many-One function:**

A function f: A ⇢ B is said to be a many-one function if two or more elements of set A have the same image in B.

A function f: A ⇢ B is a many-one function if it is not a one-one function.

**f: A ⇢ B is many-one**

**⇒ a ≠ b but f(a) = f(b)   for all a, b ∈ A**



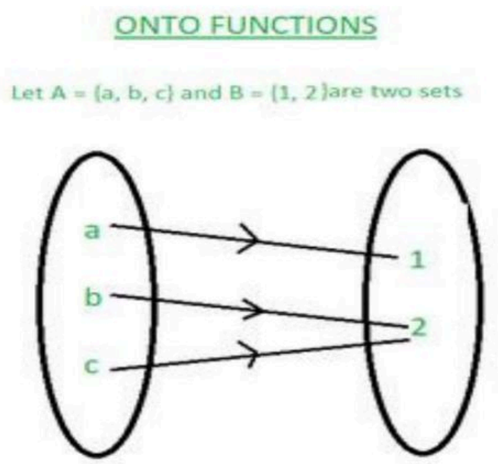

*MANY-ONE FUNCTION*

**Onto function (or Surjective Function):**

A function f: A -> B is said to be onto (surjective) function if every element of B is an image of some element of A i.e. f(A) = B or range of f is the codomain of f.

A function in which every element of the codomain has one pre-image.

**f: A ⇢ B is onto if for each b∈ B, there exists a∈ A such that f(a) = b.**

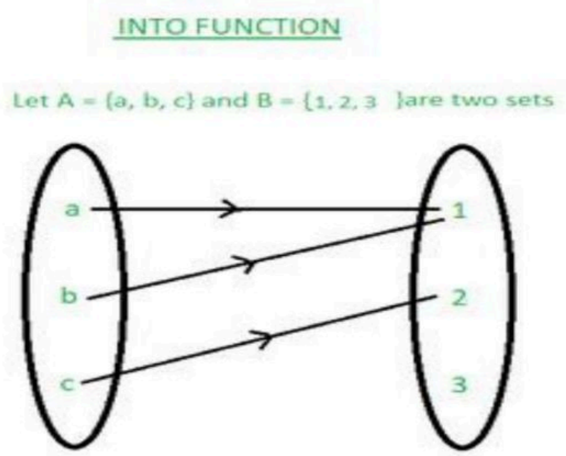



***ONTO FUNCTION***

**Into Function:**

A function f: A ⇢ B is said to be an into a function if there exists an element in B with no pre-image in A.

A function f: A⇢ B is into function when it is not onto.



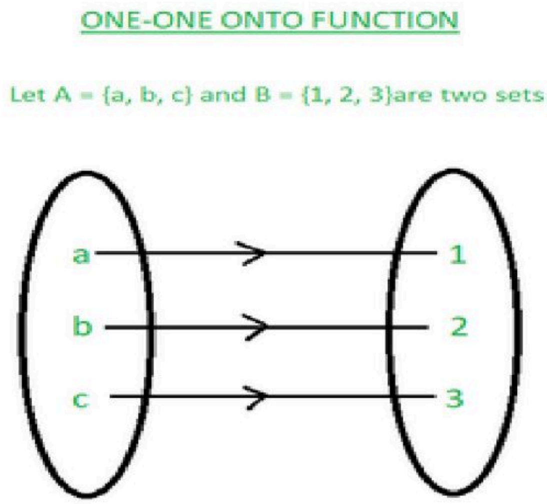
*INTO FUNCTION*

**One-One Correspondent function (or Bijective Function or One-One Onto Function):**

A function which is both one-one and onto (both injective and surjective) is called one-one correspondent (bijective) function.

**f : A ⇢ B is one-one correspondent (bijective) if:**

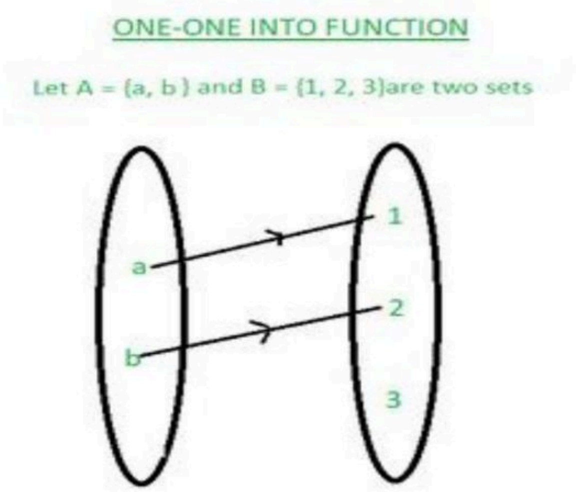
- **one-one i.e. $f(a) = f(b) \Rightarrow a = b$ for all $a, b \in A$**
- **Onto i.e. for each $b \in B$, there exists $a \in A$ such that $f(a) = b$.**




*ONE-ONE CORRESPONDENT FUNCTION*

**One-One Into function:**

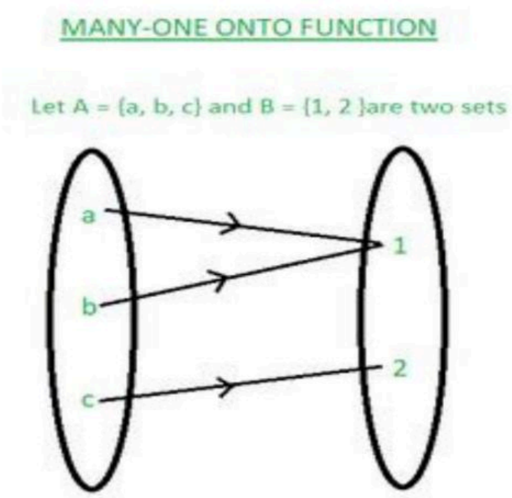
A function that is both one-one and into is called one-one into function.




*ONE-ONE INTO FUNCTION*

**Many-one onto function:**

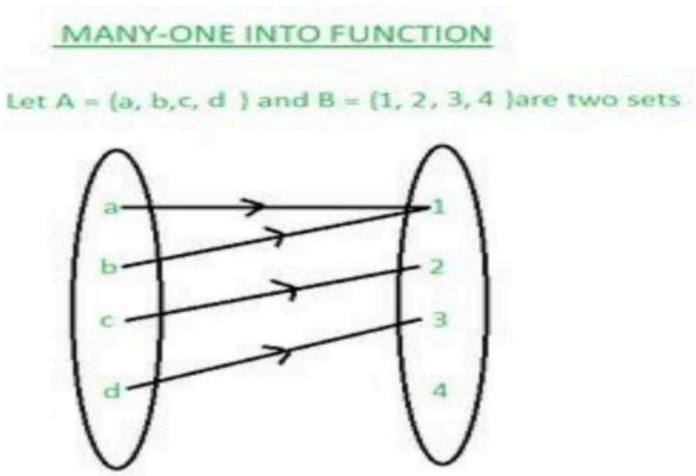
A function that is both many-one and onto is called many-one onto function.




*MANY-ONE ONTO FUNCTION*

**Many-one into a function:**

A function that is both many-one and into is called many-one into function.




*MANY-ONE INTO FUNCTION*

**Inverse of a function:**

Let $f: A \rightarrow B$ be a bijection then, a function $g: B \rightarrow A$ which associates each element $b \in B$ to a different element $a \in A$ such that $f(a) = b$ is called the inverse of f.

**$f(a) = b = f^{-1}g(b) = a$**

**Composition of functions:-**

Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two functions then, a function $gof: A \rightarrow C$ is defined by

**$(gof)(x) = g(f(x))$, for all $x \in A$**

is called the composition of f and g.

**Note:**

For the composition of functions f and g be two functions:

- $fog \neq gof$
- If f and g both are one-one function then fog is also one-one.
- If f and g both are onto function then fog is also onto.
- If f and fog both are one-one function then g is also one-one.
- If f and fog both are onto function then it is not necessary that g is also onto.
- $(fog)^{-1} = g^{-1} o f^{-1}$
- $f^{-1} o f = f^{-1}(f(a)) = f^{-1}(b) = a$
- $fof^{-1} = f(f^{-1}(b)) = f(a) = b$

**Sample Questions:**

**Q 1: Show that the function f : R ⇢ R, given by f(x) = 2x, is one-one and onto.**

**Sol: For one-one:**

Let a, b ∈ R such that f(a) = f(b) then,

f(a) = f(b)

⇒ 2a = 2b

⇒ a = b

**Therefore, f: R ⇢ R is one-one.**

**For onto:**

Let p be any real number in R (co-domain).

f(x) = p

⇒ 2x = p

⇒ x = p/2

p/2 ∈ R for p ∈ R such that f(p/2) = 2(p/2) = p

For each p∈ R (codomain) there exists x = p/2 ∈ R (domain) such that f(x) = y

For each element in codomain has its pre-image in domain.

**So, f: R ⇢ R is onto.**

Since f: R ⇢ R is both one-one and onto.

**f : R ⇢ R is one-one correspondent (bijective function).**

**Q 2: Let f : R ⇢ R ; f(x) = cos x and g : R ⇢ R ; g(x) = $x^3$ . Find fog and gof.**

**Sol:** Since the range of f is a subset of the domain of g and the range of g is a subset of the domain of f. So, fog and gof both exist.

**gof (x)** = g(f(x)) = g(cos x) = $(\cos x)^3 = \cos^3 x$

**fog (x)** = f(g(x)) = f($x^3$) = cos $x^3$

**Q 3: If f : Q ⇢ Q is given by f(x) = $x^2$ , then find $f^{-1}(16)$.**

**Sol:**

Let $f^{-1}(16) = x$

f(x) = 16

$x^2 = 16$

x = ± 4

$f^{-1}(16) = \{-4, 4\}$

**Q 4 :- If f : R ⇢ R; f(x) = 2x + 7 is a bijective function then, find the inverse of f.**

**Sol:** Let x ∈ R (domain), y ∈ R (codomain) such that f(a) = b

f(x) = y

⇒ 2x + 7 = y

⇒ x = (y -7)/2

⇒ $f^{-1}(y)$ = (y -7)/2

**Thus, $f^{-1}$ : R ⇢ R is defined as $f^{-1}(x)$ = (x -7)/2 for all x∈ R.**

**Q 5: If f : A ⇢ B and |A| = 5 and |B| = 3 then find total number of functions.**

**Sol:** Total number of functions = $3^5$ = 243