# Stochastic Dominance

## First Order Stochastic Dominance (FSD)

Let $X$ and $Y$ be two random variables with cumulative distribution functions $F_X(x)$ and $F_Y(x)$, respectively.
We say that $X$ first-order stochastically dominates $Y$ if:

$$F_X(x) \leq F_Y(x) \quad \text{for all } x \in \mathbb{R}, \text{ with strict inequality for some } x$$

Equivalently, for all non-decreasing utility functions $u$, we have:

$$\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$$

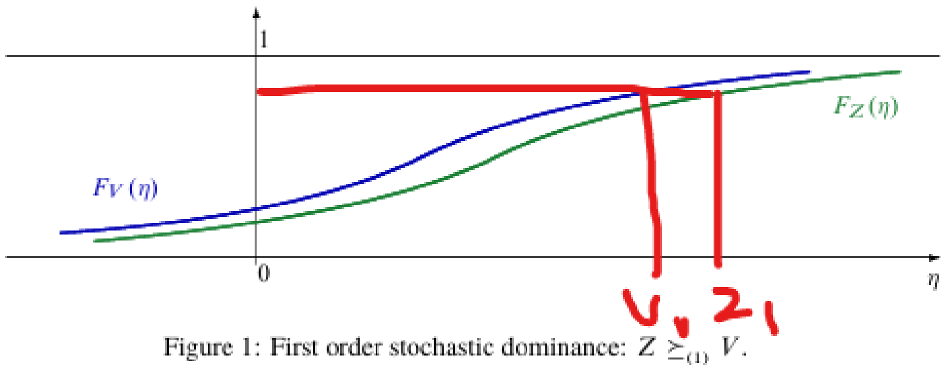


Figure 1: FSD

## Second Order Stochastic Dominance (SSD)

We say that $X$ second-order stochastically dominates $Y$ if:

$$\int_{-\infty}^{x} F_X(t)\, dt \leq \int_{-\infty}^{x} F_Y(t)\, dt \quad \text{for all } x \in \mathbb{R}, \text{ with strict inequality for some } x$$

Equivalently, for all increasing and concave utility functions $u$, we have:

$$\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$$

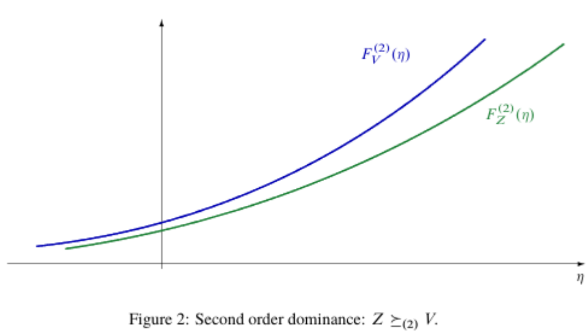


Figure 2: SSD

$$F_Z^{(2)}(\eta) = \int_{-\infty}^{\eta} F_Z(\xi)\, d\xi$$

$$\begin{aligned}
F_Z^{(2)}(\eta) &= \int_{-\infty}^{\eta} \left( \int_{-\infty}^{\xi} P_Z(dt) \right) d\xi = \int_{-\infty}^{\eta} \int_{t}^{\eta} \mathbf{1}_{t \leq \xi}\, d\xi\, P_Z(dt) \\
&= \int_{-\infty}^{\eta} (\eta - t)\, P_Z(dt) = \int_{-\infty}^{\infty} \max(\eta - t, 0)\, P_Z(dt) \\
&= \mathbb{E}\left[\max(\eta - Z, 0)\right]
\end{aligned} \tag{1}$$

We can express the function $F_Z^{(2)}(\cdot)$ as the *expected shortfall*: for each target value $\eta$ we have

$$F_Z^{(2)}(\eta) = \mathbb{E}\left[(\eta - Z)_+\right] \tag{2}$$

# Optimization Problem

$$\begin{aligned}
\max \quad & \mathbb{E}\left[R^\top x\right] \\
\text{s.t.} \quad & R^\top x \succeq_{(2)} Y, \\
& x_1 + x_2 + \cdots + x_n = S, \\
& x \in X_0.
\end{aligned}$$