Tutorial - 7

04 May 2022 14:53

Q1. $A \to$ nilpotent $\ni$ $A^k = 0$ for some k

$A^{k+1} = 0, \ldots$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$$(1-x)(1+x+x^2+\cdots) = 1$$

a) Consider: $I + A + A^2 + \cdots = B$

$$B = I + A + A^2 + \cdots A^{k-1}$$

$$(I - A)\cdot B = I$$

$$B - AB$$

$$I + A + A^2 + \cdots A^{k-1} - A - A^2 - \cdots A^k = I$$

$\therefore$ $I - A$ invertible

b) assume λ is eigenvalue with eigenvector x

$$\underset{}{A}^{\in n \times n} x = \lambda x$$

$$A^2 x = \lambda A x = \lambda^2 x$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$0 = A^k x = \lambda^k x$$

★ Why $k \leq n$? See at end

$\vdots$ multiply $n-k$ more times

$$0 = \lambda^n x$$

$\to 0$ eigenvalue with multiplicity n

A only has 0 as an eigenvalue

$$\chi_A = x^n$$

c) Geometric multiplicity $=$ nullity$(\lambda I - A)$ $=$ nullity$(A - \lambda I)$

$$= \text{nullity}(A)$$

$$= n - \text{rank}(A)$$

$$= 4 - \text{rank}(A)$$

d) A & B commute $\Rightarrow$ $AB = BA$

$A, B \to$ nilpotent

$A^i = 0$, $B^j = 0$ for some i, j

$k = \max(i, j)$

$$A^k = B^k = 0$$

$k = \max(i, j)$ $\quad A^k = B^k = 0$

$$(AB)^k = \underbrace{AB\,AB \cdots\cdots AB}_{AB \text{ rep. } k \text{ times}} = A^k \cdot B^k = 0$$

$\therefore AB$ is nilpotent

Q2. $P^2 = P$

a) $(I - P)^2 = (I-P)(I-P)$

$$= I - 2P + P^2$$
$$= I - 2P - P$$
$$= I - P$$

$\rightarrow$ $I - P$ is idempotent

b) Let $\lambda \rightarrow$ eigenvalue
$x \rightarrow$ eigenvector

$Px = \lambda x \longrightarrow (1)$

$P^2 x = \lambda P x = \lambda^2 x$

$\hookrightarrow Px = \lambda^2 x \longrightarrow (2)$

$(1) = (2)$ $\quad \lambda = \lambda^2$ $\rightarrow 0$, $\rightarrow 1$

c) $P^2 = P \Rightarrow P^2 - P = 0$

$$= P(P - I) = 0$$

P is inv, P^{-1} exists
pre multiply with P^{-1}

$$P - I = 0 \Rightarrow P = I$$

d) P is not inv
basis for nullspace : $\{v_1, v_2, \dots v_k\}$
$\hookrightarrow$ extend to basis $\mathbb{R}^{n \times n}$

$$\{v_1, v_2, v_3, \dots v_n\}$$

$$\underbrace{Pv_{k+1}}_{u_1}, \underbrace{Pv_{k+2}}_{u_2}, \ldots \underbrace{Pv_n}_{u_{n-k}}$$

$$Pu_1 = P^2 v_{k+1} = Pv_{k+1} = u_1$$

u_1 is an eigenvector of P

$u_2, u_3, \ldots u_{n-k}$ are eigenvectors of P

Eigenvectors are lin. indep.

$$\{Pv_{k+1}, \ldots Pv_n\} \to \text{lin. indep.}$$

Q3. $H = I - 2nn^T$ $\qquad m - 2n\underbrace{n^T m}_{\bar{n}\cdot\bar{m}} = m$

$H_0 = I - nn^T$

$Hn = -n \quad \to \quad -1$ is an eigenvalue

n is the eigenvector

Let $m \perp n$

then $Hm = m \quad \to \quad 1$ is an eigenvalue

m as the eigenvector

$$H^2 = (I - 2nn^T)(I - 2nn^T)$$
$$= I - 4nn^T + 4n\underbrace{n^T n}n^T$$
$$= I$$

$$H_0^2 = (I - nn^T)(I - nn^T)$$
$$= I - 2nn^T + n\underbrace{n^T n}n^T = I - nn^T = H_0$$

$H_0 \to$ idempotent

from Q2(d) it is diagonalizable

$\Rightarrow$ $2H_0$ diagonalizable

$2H_0 - I$ diago.

$\hookrightarrow H$ diag.

Q4. $A = \begin{bmatrix} 2 & 10^5 & 10^9 \\ 0 & 1 & \pi \\ 0 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 & 0 \\ e & 2 & 0 \\ 10^8 & 10^{10} & 1 \end{bmatrix}$

$$\chi_A = |\lambda I - A| = (\lambda - 1)(\lambda - 2)(\lambda - 3)$$

$$\chi_B = |\lambda I - B| = (\lambda-1)(\lambda-2)(\lambda-3) = \chi_A$$

$\exists\, P$ s.t.

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\exists\, Q$ s.t. $Q^{-1}BQ =$

$$P^{-1}AP = Q^{-1}BQ$$

$$= Q P^{-1} A \underbrace{P Q^{-1}}_{T} = \underbrace{T^{-1} A T = B}_{A \text{ is similar to } B}$$

Q5. $A^T = -A \longrightarrow$ A to be real

Let λ be eigenvalue

$$Ax = \lambda x$$

Take conjugate : $A^* x^* = \lambda^* x^*$

$$= A x^* = \lambda^* x^*$$

Take transpose :

$$(A x^*)^T x = \lambda^* (x^*)^T x$$

$$= (x^*)^T A^T x = \lambda^* \underbrace{(x^*)^T x}_{}$$

$\|x\|^2 = \langle x, x\rangle$
$(x^*)^T x$

$$= -(x^*)^T (Ax) = \lambda^* \|x\|^2$$

$$= -\lambda (x^*)^T x = \lambda^* \|x\|^2$$

$$= -\lambda \cancel{\|x\|^2} = \lambda^* \cancel{\|x\|^2}$$

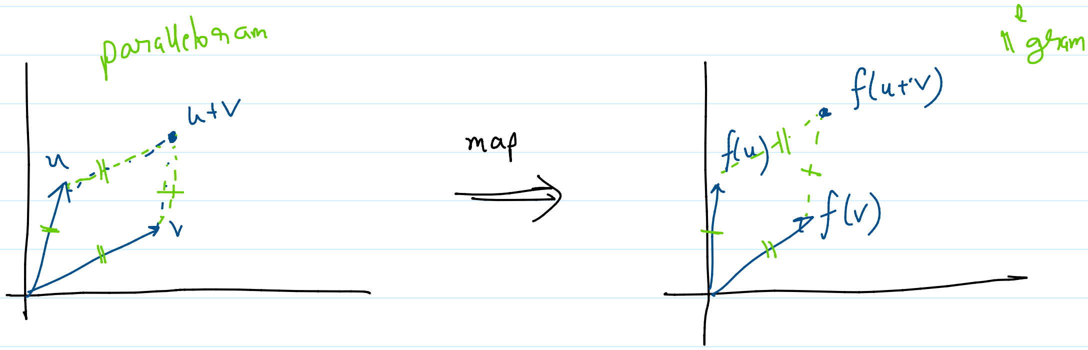
$$= \lambda^* = -\lambda \longrightarrow$$ λ is purely imag.
$\mathrm{Re}(\lambda) = 0$

About diagonalization : Over $\mathbb{R}$, skew symm. may not diag.
Over $\mathbb{C}$, " " is diag.

Q6. f : map. $\mathbb{R}^3 \to \mathbb{R}^3$

Q6. f : map. $\mathbb{R}^3 \to \mathbb{R}^3$

$$\|f(x) - f(y)\| = \|x - y\| \quad \to \text{dist. preserving map (isometry)}$$



$$\underbrace{f(u+v) = f(u) + f(v)}_{\text{linear map}} \quad \text{(Parallelogram prop.)}$$

You can find $A \in \mathbb{R}^{3\times 3}$

$f : Ax$

old basis : e_1, e_2, e_3

new basis : e_1, e_2, e_3

$e_1 = ae_1 + be_2 + ce_3$

Nature of A

$$\|f(x)\| = \|x\|$$

$$\|Ax\|^2 = \|x\|^2$$

$$(Ax)^T Ax = x^T x$$

$$x^T A^T A x = x^T x = x^T I x$$

$$x^T (A^T A - I) x = 0$$

$$\to A^T A - I = 0$$

$$\Rightarrow A^T A = I$$

$A \to$ orthogonal matrix

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

☆ On Q1.(b) Why $k \leq n$?

Consider $\chi_A = a_0 + a_1 x + \cdots a_n x^n$

From Cayley Hamilton, matrix A also satisfies this

$$I a_0 + a_1 A + \cdots a_n A^n = 0 \longrightarrow (1)$$

Suppose $k > n$,

Let $k = n + i$; $i \in \mathbb{N}$; note that $i < k$ (obvious)

Multiply A^i on both sides,

$$a_0 A^i + a_1 A^{i+1} + \cdots \underbrace{a_n A^{n+i}}_{\text{equal to } 0} = 0$$

$$\Rightarrow a_0 A^i + a_1 A^{i+1} + \cdots a_{n-1} A^{n-1+i} = 0$$

$$\Rightarrow A^i (a_0 I + a_1 A + \cdots a_{n-1} A^{n-1}) = 0$$

Since $i < k$, $A^i \neq 0$

$$\therefore a_0 I + a_1 A + \cdots a_{n-1} A^{n-1} = 0$$

BUT these are the first n terms in Cayley Hamilton Th.

Substituting in (1),

$$0 + a_n A^n = 0 \Rightarrow a_n A^n = 0$$

$$\Rightarrow A^n = 0$$

However, this contradicts our assumption that nilpotency index $k > n$, since here it is n

$\therefore$ Our assumption is FALSE $\Rightarrow k \leq n$