Tutorial - 5

25 April 2022 18:52

Q1. Characteristic eq^n $\rightarrow$ χ

$$\chi_A = |A - \underset{\text{var}}{x} I| \rightarrow \text{fun}^n \text{ of } x$$

$$B = P^{-1} A P$$

$$\chi_B = |B - xI| = |P^{-1}AP - xI|$$

Write $I = P^{-1}P \rightarrow \chi_B = |P^{-1}AP - xP^{-1}P|$

$$= |P^{-1}(A - xI)P| = |P^{-1}||A - xI||P|$$

reciprocal

$$\chi_B = |A - xI| = \chi_A$$

☆ Matrices A & B are similar iff $\exists$ invertible matrix T s.t.

$$T^{-1}AT = B$$

↳ similarity transform

Prop:

$\det(B) = \det(A)$

$\chi_B = \chi_A$

eigenvalues are same

☆ eigenvectors are not exactly same

V is eigenvector of A for eigenvalue λ

$T^{-1}V$ is " of B " " "

Q2. We will prove $I - AB$ inv $\Rightarrow$ $I - BA$ inv (rev is similar)

Let C to be the inverse $I - AB$

$$C(I - AB) = I \Rightarrow C - CAB = I$$

Post multiply with A

$$CA - CABA = A$$

$$CA - CABA = A$$

Premultiply with B

$$BCA - BCABA = BA$$

$$\rightarrow \quad BCA(I - BA) = I - (I - BA)$$

$$\rightarrow \quad (I - BA)(I + BCA) = I$$

$\therefore$ $I - BA$ is inv & $I + BCA$ is inverse ✓

Assume that λ is eigenvalue of AB & eigenvector is $x \in \mathbb{R}^{n \times 1}$

$$ABx = \lambda x$$

Let $\underline{Bx = y} \in \mathbb{R}^{n \times 1}$

Look at $BAy = BABx = B(ABx)$

$$= B \cdot (\lambda x) = \lambda \cdot Bx$$

$$BAy = \lambda \cdot y$$

$\therefore$ AB & BA have same eigenvalues

Q3. $A \in \mathbb{R}^{n \times n}$

nullity$(A) = k \Rightarrow$ From Rank-Nullity Th. rank $= n-k$

☆ rank $= n-k \Rightarrow$ some $(n-k) \times (n-k)$ subdet of A is non zero *

and all $(n-k+1) \times (n-k+1)$ " " " is zero

Char. Poly $= |A - xI|$

$$\chi_A = \begin{vmatrix} a_{11}-x & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ a_{n1} & - & - & - & a_{nn}-x \end{vmatrix}$$

$\rightarrow \chi_A(0) = 0$

$\because$ matrix is not full rank

$\det(A) = 0$

$\hookrightarrow a_0 = 0$

Differentiate

$$\chi_A' = \begin{vmatrix} -1 & 0 & 0 & 0 & 0 & \cdots \\ a_{21} & a_{22}-x & \cdots & & & \\ \vdots & \vdots & & \vdots & & \\ \vdots & \vdots & & \vdots & & \\ \vdots & \vdots & & & & a_{nn}-x \end{vmatrix} + \begin{vmatrix} a_{11}-x & a_{12} & \cdots & a_{1n} \\ 0 & -1 & \cdots & \\ \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \\ & & & a_{nn}-x \end{vmatrix} + \cdots$$

when you diff once more $\rightarrow$ $\hookrightarrow$ give rise n more $\rightarrow$ $\hookrightarrow$ give n more

Look at the first term in χ_A' = $(n-1)\times(n-1)$ sub det

and $x=0$ from $*$ this $= 0$

and all other terms go to 0

$\chi_A'(0) = 0$

recall that char poly : $a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$

$\chi_A'(0) \rightarrow a_1 = 0 \rightarrow x$ has no coeff

Similarly, $\chi_A'' \rightarrow$ all terms are basically $(n-2)\times(n-2)$ sub det

$\chi_A''(0) = 0 \rightarrow a_2 = 0$

Continue until $(k-1)^{th}$ derivative the value $= 0$

and for k^{th} derivative, non zero value

$a_0, a_1, \cdots a_{k-1} = 0 \;;\; a_k \neq 0$

$$\chi_A = a_n x^n + a_{n-1}x^{n-1} + \cdots a_k x^k$$
$$= x^k(\cdots)$$

atleast x^k divides χ_A

Qn. $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = \lambda^3 - 3\lambda^2 - 9\lambda - 5$$

$\downarrow \quad \downarrow \quad \downarrow$

$-1 \quad -1 \quad 5$

eigen values : $-1, -1, 5$

eigenvector satisfies : $\underbrace{(A - \lambda I)}_{P} x = 0$

$Px = 0$

For $\lambda = -1$

augmented matrix : $\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right]$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} \boxed{2} & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

basis of eigenvector space = $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

any other eigenvector of -1 is a LC of these 2

For $\lambda = 5$

eigenvector : $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For B , eigenvalues : 1, 2, 3

eigenvectors : $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Quiz Discussion

Q1. $\bar{a} \times \bar{p} = \bar{b}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ p_1 & p_2 & p_3 \end{vmatrix} = \hat{i}(\underbrace{a_2p_3 - a_3p_2}_{b_1}) + \hat{j}(\underbrace{a_3p_1 - a_1p_3}_{b_2}) + \hat{k}(\underbrace{a_1p_2 - a_2p_1}_{b_3})$$

$$\underbrace{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Look at $\bar{a} \cdot \bar{b} = 1 \times 1 \times \cos(\pi/3) = \frac{1}{2} = a_1b_1 + a_2b_2 + a_3b_3$

$\text{rank}(A) = 2$

$\text{rank}(A|B) = 3$

Harshil

$\bar{a} \times \bar{p} = \bar{b}$ if sol^n exists

$0 = \bar{a} \cdot (\bar{a} \times \bar{p}) = \bar{a} \cdot \bar{b} = |a|\,|b| \cos \pi/3 \neq 0$

no sol^n : $3 \geq \text{rank}(A|B) > \text{rank}(A)$

$2 \quad ; \quad 1$

$3 \quad ; \quad 2$?

Q6. $A \in \mathbb{R}^{n \times n}$

$$\mathbb{R}^{n^2} = \mathbb{R}^{n^2 \times 1}$$

$$\begin{bmatrix} C_1 & C_2 & C_3 \cdots C_n \end{bmatrix} \longrightarrow \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{bmatrix} \rightarrow \mathbb{R}^{n^2 \times 1}$$

a)
$$\begin{bmatrix} a & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & a & x_{23} & \cdots & x_{2n} \\ \vdots & & a & & \\ & & & \ddots & \end{bmatrix} \rightarrow \begin{bmatrix} a \\ x_{21} \\ x_{31} \\ \vdots \\ x_{n1} \\ x_{12} \\ a \\ x_{32} \\ \vdots \end{bmatrix}$$

b)
$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$I \qquad -I \qquad O$

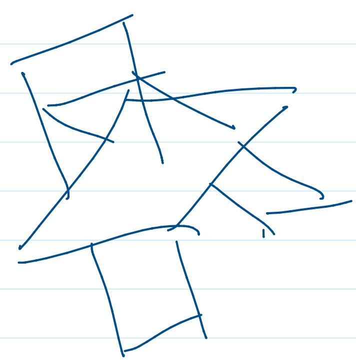
c)
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow \text{orth} \qquad a^2+b^2+c^2 = \lambda^2$$

$$\begin{bmatrix} p & q & r \\ s & t & u \\ v & w & x \end{bmatrix} \rightarrow \text{orth} \qquad p^2+q^2+r^2 = \mu^2$$

$$\begin{bmatrix} a+p & b+q & c+r \\ & & \\ & & \end{bmatrix}$$

$AA^T = I, \quad BB^T = I$

$(A+B)(A^T+B^T) \quad \therefore \quad AA^T + AB^T + BA^T + BB^T = I$



Answers (not official)

1. $2, 3 \rightarrow 5$

2. a, b
3. b, c
4. 21
5. (c) ✗ Answer is (d). We find 2, but linear comb^n of these 2 also satisfy
6. (a), (d)

$$|a|^2 + |b|^2 + |c|^2 = 1 \qquad a, b, c \in \mathbb{C}$$

$$2ai + b + c = 0$$

$$a + 2ic = 0$$

$$[x\ y\ z] \rightarrow [|x|\ y\ z]$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad x \geq 0$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad x < 0$$