Tutorial - 2

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11. List all possibilities for the reduced row echelon matrices of order 4×4 having exactly one pivot. Count the number of free parameters (degrees of freedom) in each case. For example one of the possibilty is $\begin{bmatrix} 0 & \boxed{1} & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ wherein there are 2 degrees of freedom. Repeat for 0, 2, 3 and 4 pivots.

$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

0 pivots

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1 pivot

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3 dof

2 dof

1 dof

0 dof

2 pivots

~~$^4P_2$~~

$^4C_2 = 6$

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4 dof

3 dof

2 dof

3 pivots

$^4C_3 = 4$

$$\begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2 dof

1 dof

4 pivots

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Find whether the following sets of vectors are linearly dependent or independent:

(i) $[1, -1, 1], [1, 1, -1], [-1, 1, 1], [0, 1, 0]$.

(ii) $[1, 9, 9, 8], [2, 0, 0, 3], [2, 0, 0, 8]$.

$\begin{bmatrix} 0 & 1 & 1 & -2 \end{bmatrix}$

$au_1 + bu_2 + cu_3 + du_4 = 0$

2. Find whether the following sets of vectors are linearly dependent or independent:

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(ii) $[1, 9, 9, 8], [2, 0, 0, 3], [2, 0, 0, 8]$.

$$a u_1 + b u_2 + c u_3 + d u_4 = 0$$

i) $[1\ 1\ -1] + [-1\ 1\ 1] = [0\ 2\ 0] = 2[0\ 1\ 0]$

Linear Dependent

ii) $a[1\ 9\ 9\ 8] + b[2\ 0\ 0\ 3] + c[2\ 0\ 0\ 8] = [0\ 0\ 0\ 0]$, $[a, b, c] \neq [0\ 0\ 0]$

Assume linearly dep

$$a + 2b + 2c = 0 \rightarrow b + c = 0$$
$$\left.\begin{array}{l} 9a + 0b + 0c = 0 \\ 9a + 0b + 0c = 0 \end{array}\right\} a = 0$$
$$8a + 3b + 8c = 0 \rightarrow 3b + 8c = 0$$

$b = 0$
$c = 0$

Assumption is false $\Rightarrow$ linearly indep.

3. Find the ranks of the following matrices:

(i) $\begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}$, (ii) $\begin{bmatrix} m & n \\ n & m \\ p & p \end{bmatrix}$ $(m^2 \neq n^2)$,

i) $\begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix} \xrightarrow[R_1 \to R_1 - R_3]{} \begin{bmatrix} 2 & -1 \\ -2 & 1 \\ 6 & -3 \end{bmatrix} \xrightarrow[R_2 \to R_2 + R_1]{} \begin{bmatrix} 2 & -1 \\ 0 & 0 \\ 6 & -3 \end{bmatrix}$

$\xrightarrow[R_3 \to R_3 - 3R_1]{} \begin{bmatrix} \boxed{2} & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ ← 1 non zero row

rank = 1

ii) $\begin{bmatrix} m & n \\ n & m \\ p & p \end{bmatrix}$ $(m^2 \neq n^2)$

rank = k $\begin{cases} \text{some } k \times k \rightarrow \det \neq 0 \checkmark \\ \text{all } (k+1) \times (k+1) \rightarrow \det = 0 \checkmark \end{cases}$

$R_1 \to R_1 + R_2$ $\begin{bmatrix} m+n & m+n \\ n & m \\ p & p \end{bmatrix} \xrightarrow{R_1 \to \frac{R_1}{m+n}} \begin{bmatrix} 1 & 1 \\ n & m \\ p & p \end{bmatrix}$

$\begin{bmatrix} \boxed{1} & 1 \\ 0 & \boxed{m-n} \\ 0 & 0 \end{bmatrix} \xleftarrow{R_2 \to R_2 - nR_1} \begin{bmatrix} 1 & 1 \\ n & m \\ 0 & 0 \end{bmatrix} \xleftarrow{R_3 \to R_3 - pR_1}$

$m = n$ rank 1

$m \neq n$ rank 2

8. For $a < b$, consider the system of equations:

$$\begin{aligned} x &+ y &+ z &= 1 \\ ax &+ by &+ 2z &= 3 \\ a^2x &+ b^2y &+ 4z &= 9. \end{aligned}$$

Find the pairs (a, b) for which the system has infinitely many solutions.

$\begin{bmatrix} 1 & 1 & 1 \\ a & b & 2 \\ a^2 & b^2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$

$A x = b$

necessary cond^n for inf sol^n $|A| = 0$

$\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$

$|A| = (b-a)(c-a)(c-b)$

$(b-a)(2-a)(2-b) = 0$

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{bmatrix}$

$a = 2$ or $b = 2$

Case 1 $a = 2$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & b & 2 & 3 \\ 4 & b^2 & 4 & 9 \end{array}\right]$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & b-2 & 0 & 1 \\ 0 & b^2-4 & 0 & 5 \end{array}\right]$

$R_3 \to R_3 - (b+2)R_1$

$$\left[\begin{array}{ccc:c} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{b-2} & 0 & 1 \\ 0 & 0 & 0 & \boxed{3-b} \end{array}\right]$$

$b = 3$

$(a,b) = \underline{(2,3)}$

<u>Case 2</u> $b = 2$, $a = 3$ ✗

9. Show that the row space of a matrix does not change by row operations. Show that the dimension of the column space is unchanged by row operations.

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

$r = ar_1 + br_2 + cr_3$, $a,b,c \in \mathbb{R}$

$R_2 \to R_2 - R_1 + 6R_3$

$r_2 \to r_2 + 6r_3 - r_1 \equiv r_2 = \alpha r_1 + \beta r_2 + \gamma r_3$

like computer syntax

$[\alpha\ \beta\ \gamma] \neq [0\ 0\ 0]$

$r = ar_1 + b(\alpha r_1 + \beta r_2 + \gamma r_3) + cr_3$

$$r = \underset{\substack{|||\\ ar_1}}{(a + \alpha b)r_1} + \underset{\substack{|||\\ br_2}}{b\beta r_2} + \underset{\substack{|||\\ cr_3}}{(c + \gamma b)r_3}$$

Say that there were k indep columns

$C_{i_1}, C_{i_2}, \ldots, C_{i_k}$

$i_1 \quad i_2 \quad i_3$

$A = [C_{i_1}\ C_{i_2} \ldots C_{i_k}] \to |A| \neq 0$

$n \times n \quad n \times 1 = n \times 1$

$[\ ][\ ]$

$A \longmapsto EA$

$[C_{i_1}\ C_{i_2} \cdots C_{i_k}] \longrightarrow [EC_{i_1}\ EC_{i_2} \ldots EC_{i_k}]$

$A \to |A| \neq 0$

$E \to |E| \neq 0$

$EA \to |EA| \neq 0 \Rightarrow$ columns of EA are indep

$EC_{i_1} \quad EC_{i_2} \quad EC_{i_k}$ indep

$EA \to |EA| \neq 0 \Rightarrow$ columns of EA are indep

$\underbrace{EC_{i_1}, EC_{i_2}, \ldots, EC_{i_k}}_{k}$ indep

$R_2 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \Rightarrow \begin{bmatrix} r_1 \\ r_2 + r_1 \\ r_3 \end{bmatrix}$$

Handwritten Problem:

1) Try following operations:

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - 2R_1$
$R_2 \to R_2/2$
$R_3 \to R_3 + R_2$
$R_4 \to R_4 - 2R_2$

you finally get:

$\mathbb{R}^{n \times m}$, $n = 4$, $m = 7$

$$\left[\begin{array}{ccccccc:c} \boxed{1} & 0 & 2 & -1 & 1 & -2 & -1 & 1 \\ 0 & \boxed{1} & 1 & 1 & 1 & 3 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

rank $= 2 \Rightarrow$ rank + nullity $= 7 \Rightarrow$ nullity $= 5$

2) Yes, solvable $\because$ augmented column $= 0$ when row $= 0$

3) Take the k ($= 2$) pivotal columns. Form $n \times k$ matrix
Find k indep. row vectors through Elementary column op.
(or do EROs on transpose of this matrix)

$\Rightarrow$ $k \times k$ matrix of non zero det.
Here, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (first 2 col
2 row)

5) Basis of column space $\to$ pivotal columns (since they are linearly indep)

$$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \& \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

7) Free var $\to$ non pivotal columns $\to$ x_3, x_4, x_5, x_6, x_7

6). We write $Ax = b$ as

$$\begin{bmatrix} \boxed{1} & 0 & 2 & -1 & 1 & -2 & -1 \\ 0 & \boxed{1} & 1 & 1 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + 2x_3 - x_4 + x_5 - 2x_6 - x_7 = 1$$
$$x_2 + x_3 + x_4 + x_5 + 3x_6 + 3x_7 = 4$$

Let $x_3, x_4, \dots x_7 = a, b, c, d, e$ (parameters)

$$\therefore \quad x_1 = 1 - 2a + b - c + 2d + e$$
$$x_2 = 4 - a - b - c - 3d - 3e$$

Sol^n :

$$\begin{bmatrix} 1 - 2a + b - c + 2d + e \\ 4 - a - b - c - 3d - 3e \\ a \\ b \\ c \\ d \\ e \end{bmatrix}$$

→ 7×1 col^n vector

$$\Rightarrow \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + a \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ -3 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + e \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

→ Complete set of sol^n

4) Basis of null space (A)

$\because$ Combinations of these vectors satisfy

$$Ax = 0$$

$\therefore$ Sol^n of $Ax = b \Rightarrow$ particular sol^n + linear comb^n of nullspace vectors