Tutorial - 6

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Q1.

Equation	Surface	Eigenvalues of A
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	ellipsoid	all three positive
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z}{c}=0$	elliptic paraboloid	two positive, one zero
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$	elliptic cone	two positive, one negative
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	1-sheeted hyperboloid	two positive, one negative
$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	2-sheeted hyperboloid	one positive, two negative
$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z}{c}=0$	hyperbolic paraboloid	one positive, one negative, one zero.

$ax^2+by^2+cz^2+2dxy+2eyz+2fzx$

$$[x\ y\ z]\begin{bmatrix} a & d & \\ d & b & \\ & & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

i) $2xy + 2yz + 2zx$

$$[x\ y\ z]\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Diag: find eigenvalue

$T = [V_1\ V_2\ V_3]$ with $\lambda_1, \lambda_2, \lambda_3$, $V_i \in \mathbb{R}^{n\times 1}$

$T^{-1}AT = D$ — diag entries eigenvalues

for $A \rightarrow$ eigenvalues : $2, -1, -1$

eigenvectors : $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$$T = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow [x\ y\ z]T^{-1}AT\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = D \quad [X\ Y\ Z] \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Transformed Quadric $\quad 2X^2 - Y^2 - Z^2 = 1$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{aligned} X &= x - y - z = 0 \\ Y &= x + z = 0 \\ Z &= x + y = 0 \end{aligned}$$

2-sheeted hyperboloid

ii) $x^2 - 2y^2 + 4z^2 + 6yz = 1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 3 \\ 0 & 3 & 4 \end{bmatrix} \rightarrow \text{eigenvalues: } \underset{+ve}{1},\ \underset{+ve}{1+3\sqrt{2}},\ \underset{-ve}{1-3\sqrt{2}}$$

1-sheeted hyperboloid

$$\begin{bmatrix} \cdots & \cdots & \cdots \\ 0 & 3 & 4 \end{bmatrix} \Rightarrow \text{eigenvalues} \quad 1,\ 1+3\sqrt{2},\ 1-3\sqrt{2}$$

1 sheeted hyperboloid

iii) $-x^2 - y^2 + 2z^2 + 8xy - 4xz + 4yz = 1$

$$\begin{bmatrix} -1 & 4 & -2 \\ 4 & -1 & 2 \\ -2 & 2 & 2 \end{bmatrix} \rightarrow -6, 3, 3$$

1-sheeted hyperboloid

Q2. $$\iiint_V e^{-(2x^2+5y^2+2z^2-4xy-2xz+4yz)}\,dx\,dy\,dz$$

$$A = \begin{bmatrix} 2 & -2 & -1 \\ -2 & 5 & 2 \\ -1 & 2 & 2 \end{bmatrix} \longrightarrow \text{eigenvalues} : 1, 1, 7$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad T^{-1}AT = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Quadric: $X^2 + Y^2 + 7Z^2$; $X = x + 2y - z$, $Y = y + 2z$, $Z = x + z$

$dX = dx$
$dY = dy$
$dZ = dz$

$$\iiint e^{-(X^2+Y^2+7Z^2)}\,dX\,dY\,dZ$$

$$\star \quad \int_{-\infty}^{\infty} e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{\sqrt{a}}$$

$$\Rightarrow \quad \frac{\sqrt{\pi}}{\sqrt{1}} \times \frac{\sqrt{\pi}}{\sqrt{1}} \times \frac{\sqrt{\pi}}{\sqrt{7}} = \frac{\pi^{3/2}}{\sqrt{7}}$$

Q3. $ax^2 + by^2 + cz^2 + 2hxy + 2gxz + 2fyz = 0$

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \rightarrow \det = 0$$

$A \rightarrow \lambda_1, \lambda_2, \lambda_3$

$$T^{-1} A T = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$|T^{-1}|\,|A|\,|T| = \lambda_1\lambda_2\lambda_3$$

$\Rightarrow |A| = \lambda_1\lambda_2\lambda_3 \rightarrow$ det of a matrix

= product of eigenvalues

$|A| = 0 \Rightarrow$ atleast 1 eigenvalue $= 0$

$$T^{-1} A T = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \lambda X^2 + \mu Y^2$$

WLOG

$X, Y \rightarrow$ linear comb^n of x, y, z

$$(\sqrt{\lambda}\,X + i\sqrt{\mu}\,Y)(\sqrt{\lambda}\,X - i\sqrt{\mu}\,Y)$$

write X in terms of x, y, z , Y interms x, y, z

$\rightarrow$ factorized

Q4. Let $\lambda \rightarrow$ eigen value

$x \rightarrow$ eigen vector

$$Ax = \lambda x \rightarrow (1)$$

Transpose on both sides : $(Ax)^T = \lambda x^T$

$$x^T A^T = \lambda x^T \rightarrow (2)$$

(2) × (1) $\quad x^T \underbrace{A^T \cdot A}\, x = \lambda^2 x^T x$

$$\rightarrow \quad x^T \overset{I}{\ } x = \lambda^2 x^T x \longrightarrow \lambda = \pm 1$$

Cayley Hamilton Th. $\quad f(x) \rightarrow$ roots are eigenvalues

$$f(A) = 0 \qquad f(x) = (x-p_1)(x-p_2)(x-p_3)$$

$$(A - p_1 I)(A - p_2 I)(A - p_3 I) = 0$$

$$A = p_1 I, \quad A = p_2 I, \quad A = p_3 I$$

When $|A| = 1$, $\quad p_1 = 1$ or $p_2 = 1$ or $p_3 = 1$

Let $p_1 = 1$

$$f(x) = (x-1)(x-p_2)(x-p_3) \longrightarrow 1 \text{ is an eigenvalue}$$

Q5. $A \rightarrow$ v eigenvector with eigenvalue $+1/-1$

$\underset{\mathbb{R}^{n\times 1}}{\rho + i\sigma}$ " " " $\underset{\mathbb{R}}{\alpha + i\beta}$

$$A(\rho + i\sigma) = (\alpha + i\beta)(\rho + i\sigma)$$

$$A\rho + iA\sigma = \alpha\rho - \beta\sigma + i(\beta\rho + \alpha\sigma)$$

$A \rightarrow$ real matrix

Equate Real col^n vectors & Imag. col^m vectors

$$A\rho = \alpha\rho - \beta\sigma$$

$$A\sigma = \beta\rho + \alpha\sigma$$

Eigen vectors of A, $\quad v, \quad \rho + i\sigma; \quad \rho - i\sigma$

values $\quad +1/-1, \quad \alpha + i\beta; \quad \alpha - i\beta$

☆ Eigenvectors of distinct eigenvalues are orthogonal

$$v \perp \rho + i\sigma, \quad v \perp \rho - i\sigma, \quad \rho + i\sigma \perp \rho - i\sigma$$

$$\frac{(\rho + i\sigma) + (\rho - i\sigma)}{2} = \rho := u_2$$

$$\frac{(\rho + i\sigma) - (\rho - i\sigma)}{2i} = \sigma := u_3$$ → normalize

$$V := u_1$$

$\{u_1, u_2, u_3\}$ → orthonormal

$\ \ \ \ \parallel$

O → transformation matrix

$$O^{-1} = O^T$$

$O^{-1} A O$ transform