Tutorial - 2

15 December 2021 11:35

2. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\underbrace{\lim_{x\to\alpha} f(x)}_{} = L$ exists for $\alpha \in \mathbb{R}$. Show that

$$\lim_{h\to 0}[f(\alpha+h) - f(\alpha-h)] = 0.$$

Analyze the converse. $\longrightarrow$ False

$\{h_n\} \longrightarrow 0$

$\alpha + h_n \longrightarrow \alpha$

$f(\alpha + h_n) \longrightarrow L$ RHL = L

Similarly for $\alpha - h_n \longrightarrow \alpha$ LHL = L

$\lim_{h\to 0} f(\alpha+h) = L$, $\lim_{h\to 0} f(\alpha-h) = L$

Hence, proved

Take func^n : $f = \begin{cases} 1 & , x = 0 \\ 1/|x| & , x \neq 0 \end{cases}$ $\left.\begin{matrix} f(0+h) = 1/|h| \\ f(0-h) = 1/|h| \end{matrix}\right\}$ this is a contradiction

3. Discuss the continuity of the following functions :

(i) $f(x) = \sin\frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$

(ii) $f(x) = x\sin\frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$

i) For $x \neq 0$, continuous (comp. of 2 cont. func^n)
For $x = 0$,

Look at $\{a_n\} = \frac{2}{(4n+1)\pi} \longrightarrow 0$ as $n \to \infty$

$\sin\left(\frac{1}{a_n}\right) = \sin\left((4n+1)\frac{\pi}{2}\right) = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1$

$a_n \to 0$, $f(a_n) \to 1 \neq f(0) \Rightarrow$ Not cont. at 0

ii) For $x \neq 0$, continuous
For $x = 0$, continuous :
For any $\epsilon > 0$, $\exists\, \delta$) $\delta(\epsilon)$

$\left| x\sin\frac{1}{x} - 0 \right| \leq |x| < \epsilon$ $\Rightarrow$ satisfied for $0 < |x - 0| < \delta = \epsilon$

4. Let $f:\mathbb{R}\to\mathbb{R}$ satisfy $f(x+y)=f(x)+f(y)$ for all $x,y\in\mathbb{R}$. If f is continuous at 0, show that f is continuous at every $c\in\mathbb{R}$.

$$f(x+y) = f(x) + f(y) \Rightarrow f(0) = 2f(0) = 0$$

$$\lim_{x\to 0} f(x) = 0 \quad ✓$$

Show that $\lim_{x\to c} f(x) = f(c)$

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0} f(c) + f(h) = f(c) + \lim_{h\to 0} f(h)$$

$$\lim_{h\to 0} f(c+h) = f(c) + 0 = f(c) \qquad \therefore \underline{\text{Continuous}}$$

5. Let $f(x)=x^2\sin(1/x)$ for $x\neq 0$ and $f(0)=0$. Show that f is differentiable on $\mathbb{R}$. Is f' a continuous function?

For $x \neq 0$, func^n is cont, diff

$$f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

For $x = 0$,

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2\sin(1/x) - 0}{x} = 0$$

f is diff.

$\{a_n\} = 1/2n\pi$ Let $f'(x) = g(x)$

$a_n \to 0$

$g(a_n) \to -1 \neq g(0) \quad \therefore g(x)$ not cont.

$f'(x)$ not cont

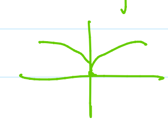
7. If $f:(a,b)\to\mathbb{R}$ is differentiable at $c\in(a,b)$, then show that

$$\lim_{h\to 0^+}\frac{f(c+h)-f(c-h)}{2h}$$

exists and equals $f'(c)$. Is the converse true ? [Hint: Consider $f(x)=|x|$.] ✓

Another example to disprove converse

$\sqrt{x}$, for $x \geq 0$, $\sqrt{-x}$ for $x \leq 0$



f is diff : $\lim_{h\to 0^+} \frac{f(c+h) - f(c-h) + f(c) - f(c)}{2h}$

$$\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{2h} - \lim_{h\to 0^+} \frac{f(c-h)-f(c)}{2h} \quad h \to -h$$

$$= \frac{f'(c)}{2} + \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{2h} = f'(c)$$

Converse is not true, Look $|x|$ at $c = 0$

$f(0+h) = f(0-h)$ $\Rightarrow$ Limit exists

But f is not diff at 0

9. Using the Theorem on derivative of inverse function. Compute the derivative of
(i) $\cos^{-1} x,\ -1 < x < 1$. (ii) $\operatorname{cosec}^{-1} x,\ |x| > 1$.

$f(x) = \cos^{-1} x$

$g(x) = \cos x \Rightarrow g'(x) = -\sin x$

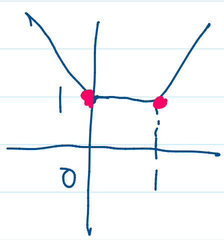
$g(f(x)) = x$

$g'(f(x))\, f'(x) = 1 \Rightarrow f'(x) = 1/g'(f(x))$

$$f'(x) = \frac{1}{-\sin(\cos^{-1} x)} = \frac{-1}{\sqrt{1 - \cos^2(\cos^{-1} x)}} = \frac{-1}{\sqrt{1 - x^2}}$$

11. Construct an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous every where and is differentiable everywhere except at 2 points.

$|x - 1| + |x|$



12. Let $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$
Show that f is discontinuous at every $c \in \mathbb{R}$.

$\{a_n\} \in \mathbb{Q}$, $\{b_n\} \in \mathbb{R} \setminus \{\mathbb{Q}\}$

$a_n \to c$ $\quad$ $b_n \to c$

$f(a_n) \to 1 \neq f(b_n) \to 0$

f is discontinuous.

We will use

$\lim_{x \to 0} f(x) = 0 \Leftrightarrow \lim_{x \to 0} |f(x)| = 0$

15. **(Optional)** Let $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$. Show that the following are equivalent :

(i) f is differentiable at c.

(ii) There exist $\delta > 0$ and a function $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ such that $\lim_{h \to 0} \epsilon_1(h) = 0$ and

$$f(c + h) = f(c) + \alpha h + h\epsilon_1(h) \text{ for all } h \in (-\delta, \delta).$$

(iii) There exists $\alpha \in \mathbb{R}$ such that

$$\lim_{h \to 0} \left(\frac{|f(c + h) - f(c) - \alpha h|}{|h|} \right) = 0.$$

$(i) \equiv (ii) \equiv (iii)$

$i \Rightarrow ii$

$ii \Rightarrow iii$

$iii \Rightarrow i$

✓

$i \Rightarrow ii$

Let $\delta > 0$, $(c-\delta, c+\delta) \subseteq (a, b)$

$\alpha = f'(c)$

Define $\varepsilon_1(h) : (-\delta, \delta) \longrightarrow \mathbb{R}$

$= \frac{f(c+h) - f(c)}{h} - \alpha \quad ; \quad h \neq 0$

$= 0 \quad ; \quad h = 0$

We see the exp. (ii) is satisfied

$\lim_{h \to 0} \varepsilon_1(h) = \left(\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \right) - \alpha$

$= f'(c) - \alpha = 0$ Hence Proved

$(ii) \Rightarrow (iii)$

$\frac{|f(c+h) - f(c) - \alpha h|}{|h|} = \frac{|h \varepsilon_1(h)|}{|h|} = |\varepsilon_1(h)|$

$\lim_{h \to 0} \varepsilon_1(h) = 0 \Rightarrow \lim_{h \to 0} |\varepsilon_1(h)| = 0$

(iii) $\Rightarrow$ (i)

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} - \alpha = 0$$

$$\underbrace{\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}}_{\text{Derivative}} = \alpha$$

Derivative exists & is $= \alpha$

$\therefore$ f is diff at c