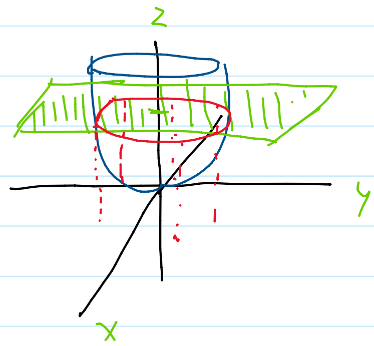
Tutorial - 6

08 January 2022 15:20

(2) Describe the level curves and the contour lines for the following functions corresponding to the values $c = -3, -2, -1, 0, 1, 2, 3, 4$:
(i) $f(x, y) = x - y$ (ii) $f(x, y) = x^2 + y^2$



i) $x - y = c$ → Level curve

$x - y = c$ at $z = c$ → Contour line

ii) For $c < 0$
Level curve, Contour line dont exist
For $c = 0$
Both $= (0, 0, 0)$
For $c > 0$
Level curve ⇒ $x^2 + y^2 = c$
Contour line ⇒ $x^2 + y^2 = c$ at $z = c$

(3) Using definition, examine the following functions for continuity at $(0, 0)$. The expressions below give the value at $(x, y) \neq (0, 0)$. At $(0, 0)$, the value should be taken as zero:

$$(i)\ \frac{x^3 y}{x^6 + y^2} \quad (ii)\ xy\frac{x^2 - y^2}{x^2 + y^2}$$

i) Not continuous
Sequential continuity
$(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n^3}\right)$

$(x_n, y_n) \to (0, 0)$

$$f(x_n, y_n) = \frac{1/n^6}{2/n^6} = \frac{1}{2} \Rightarrow f(x_n, y_n) \to \frac{1}{2} \neq 0 = f(0,0)$$

ii) For f to be cont.

$$\|(x,y) - (0,0)\| < \delta \Rightarrow |f - 0| < \varepsilon$$

$$\sqrt{x^2 + y^2} < \delta$$
$$2xy < x^2 + y^2 < \delta^2 \quad ; \quad \frac{x+y}{2} \geq \sqrt{xy}$$

$$\Rightarrow xy < \frac{\delta^2}{2} = \varepsilon$$

$$\Rightarrow \quad xy < \frac{\delta^2}{2} = \varepsilon$$

$$\left| xy \overbrace{\frac{x^2-y^2}{x^2+y^2}}^{<1} \right| < |xy| < \varepsilon = \frac{\delta^2}{2}$$

(6) Examine the following functions for the existence of partial derivatives at $(0,0)$. The expressions below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero.

(i) $xy\dfrac{x^2-y^2}{x^2+y^2}$

y const

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0-0}{h} = 0$$

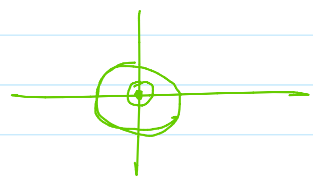
$\therefore$ $f_x(0,0)$ exists and value is 0

Similarly $f_y(0,0) = 0$

(7) Let $f(0,0) = 0$ and

$$f(x,y) = (x^2+y^2)\sin\frac{1}{x^2+y^2} \quad \text{for } (x,y) \neq (0,0).$$

Show that f is continuous at $(0,0)$, and the partial derivatives of f exist but are not bounded in any disc (howsoever small) around $(0,0)$.



For cont. $\|(x,y) - (0,0)\| < \delta$, $|f(x,y) - 0| < \varepsilon$

$$\sqrt{x^2+y^2} < \delta \Rightarrow x^2+y^2 < \delta^2 = \varepsilon$$

$$|f(x,y) - 0| = \left|(x^2+y^2)\sin\frac{1}{x^2+y^2}\right| < |x^2+y^2| < \varepsilon \rightarrow \text{Cont.}$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h^2\sin(1/h^2) - 0}{h}$$

$$= \lim_{h \to 0} h\sin\left(\frac{1}{h^2}\right) = 0$$

$f_x(0,0) = 0$

Similar $\quad f_y(0,0) = 0$

$$f_x(x,y) = \underbrace{2x \sin \frac{1}{x^2+y^2}} - \underbrace{\frac{2x}{x^2+y^2} \cos\left(\frac{1}{x^2+y^2}\right)}$$

bounded ∵ sin bounded
disc around (0,0) $2x$ is bounded

unbounded
cos is bounded
$\frac{2x}{x^2+y^2}$ is unbounded at (0,0)

(8) Let $f(0,0) = 0$ and

$$f(x,y) = \begin{cases} x\sin(1/x) + y\sin(1/y), & \text{if } x \neq 0,\ y \neq 0 \\ x\sin 1/x, & \text{if } x \neq 0,\ y = 0 \ ✓ \\ y \sin 1/y, & \text{if } y \neq 0,\ x = 0. \ ✓ \end{cases}$$

Show that none of the partial derivatives of f exist at $(0,0)$ although f is continuous at $(0,0)$.

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(h,0)}{h}$$

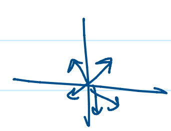
$$\lim_{h\to 0} \frac{h \sin(1/h)}{h} = \lim_{h \to 0} \sin\left(\frac{1}{h}\right) \rightarrow \text{Does not exist}$$

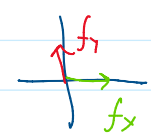
$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h \to 0} \frac{f(0,h)}{h}$$

$$= \lim_{h\to 0} \frac{h \sin 1/h}{h} = \lim_{h \to 0} \sin \frac{1}{h} \rightarrow \text{Does not exist}$$

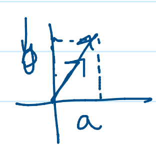
(9) Examine the following functions for the existence of directional derivatives and differentiability at $(0,0)$. The expressions below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero:

(i) $xy\dfrac{x^2-y^2}{x^2+y^2}$





$\bar{v} = (a,b) \qquad (ta, tb) \equiv t\bar{v}$



$$D_{\bar{v}}(0,0) = \lim_{t\to 0} \frac{f(ta,tb) - f(0,0)}{t}$$

$$= \lim_{t \to 0} t^{\cancel{2}} ab \left(\frac{\cancel{t^2}a^2 - \cancel{t^2}b^2}{\cancel{t^2}a^2 + \cancel{t^2}b^2}\right) \cdot \frac{1}{\cancel{t}} = \lim_{t\to 0} tab\left(\frac{a^2-b^2}{a^2+b^2}\right)$$

$$= \lim_{t\to 0} t^2 ab\left(\frac{\cancel{t^2}a^2-\cancel{t^2}b^2}{\cancel{t^2}a^2+\cancel{t^2}b^2}\right)\cdot\frac{1}{\cancel{t}} = \lim_{t\to 0} tab\left(\frac{a^2-b^2}{a^2+b^2}\right)$$

$$D_{\bar v}(0,0) = 0$$

$$\star\begin{cases} f_x = D_{\bar v_1} \rightarrow \bar V_1 = (1,0) \\ f_y = D_{\bar v_2} \rightarrow \bar V_2 = (0,1) \end{cases}$$

(10) Let $f(x,y) = 0$ if $y = 0$ and

$$f(x,y) = \frac{y}{|y|}\sqrt{x^2+y^2} \text{ if } y \neq 0.$$

Show that f is continuous at $(0,0)$, $D_{\underline{u}}f(0,0)$ exists for every vector $\underline{u}$, yet f is not differentiable at $(0,0)$.

Continuity :

$$\|(x,y)-(0,0)\| < \delta \rightarrow |f(x,y)-0| < \varepsilon$$

$$\sqrt{x^2+y^2} < \delta = \varepsilon$$

$$\left|\frac{y}{|y|}\sqrt{x^2+y^2}\right| = \left|\sqrt{x^2+y^2}\right| < \delta = \varepsilon$$

Continuous ✓

Directional derivative

$$\bar u = (a,b)$$

$$D_{\bar u}(0,0) = \lim_{t\to 0} \frac{f(ta,tb)-f(0,0)}{t} = \lim_{t\to 0}\frac{tb}{|tb|}\sqrt{t^2a^2+t^2b^2}\cdot\frac{1}{t}$$

$$= \lim_{t\to 0}\sqrt{a^2+b^2}\,\frac{\cancel{t}b}{|\cancel{t}b|} \rightarrow \text{exists} = \frac{b}{|b|}\sqrt{a^2+b^2}$$

$$f_x(0,0) = 0$$
$$f_y(0,0) = 1$$

To show diff. show that

To show diff. show that

$$\lim_{h,k\to 0,0} \left| \frac{f(h,k) - f(0,0) - h f_x - k f_y}{\sqrt{h^2+k^2}} \right| = 0$$

$$\lim_{h,k\to 0,0} \left| \frac{k/|k| \sqrt{h^2+k^2} - k}{\sqrt{h^2+k^2}} \right|$$

$$\lim_{h,k\to 0,0} \left| \frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}} \right| \rightarrow \text{does not exist}$$

Why?

Look at path $k = mh$, $m > 0$

$$\lim_{h,k\to 0,0} \left| \frac{h}{|h|} - \frac{m}{\sqrt{1+m^2}} \right| \rightarrow \text{diff for diff values of } m$$

$f(x,y)$ is not differentiable