Tutorial - 4

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Q1. $A \in \mathbb{R}^{2\times n} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \\ v_1 & v_2 & \cdots & v_n \end{bmatrix}$

$$AA^T = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \\ v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ \vdots & \vdots \\ u_n & v_n \end{bmatrix} = \begin{bmatrix} \sum u_i^2 & \sum u_i v_i \\ \sum u_i v_i & \sum v_i^2 \end{bmatrix}$$

$$\therefore \det(AA^T) = (\sum u_i^2)(\sum v_i^2) - (\sum u_i v_i)^2 \quad \checkmark$$

Look at $A^T A$:

$$\begin{bmatrix} \boxed{\begin{matrix} u_1^2+v_1^2 & u_1u_2+v_1v_2 \\ u_2u_1+v_2v_1 & u_2^2+v_2^2 \end{matrix}} & \cdots & u_1u_n+v_1v_n \\ & \cdots & u_2u_n+v_2v_n \\ \vdots \quad\quad \vdots & & \vdots \\ & & u_n^2+v_n^2 \end{bmatrix}$$

$u_1^2+v_1^2 \quad u_1u_3+v_1v_3$

$u_1u_3+v_1v_3 \quad u_3u_3+v_3v_3$

Principle minors : example : $\begin{bmatrix} u_1^2+v_1^2 & u_1u_2+v_1v_2 \\ u_2u_1+v_2v_1 & u_2^2+v_2^2 \end{bmatrix} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$

both have same det

$$\therefore \text{Sum of principle minors} = \sum_{1\le i<j\le n} \begin{vmatrix} u_i & u_j \\ v_i & v_j \end{vmatrix}^2$$

$\begin{bmatrix} u_i & u_j \\ v_i & v_j \end{bmatrix}^T \begin{bmatrix} u_i & u_j \\ v_i & v_j \end{bmatrix}$

★ Fact : $(u_1^2+v_1^2)(u_2^2+v_2^2) - (u_1u_2+v_1v_2)^2 = (u_1v_2 - u_2v_1)^2$

$\cancel{u_1^2u_2^2} + u_1^2v_2^2 + v_1^2u_2^2 + \cancel{v_1^2v_2^2} - \cancel{u_1^2u_2^2} - \cancel{v_1^2v_2^2} - 2u_1u_2v_1v_2$

Q3. $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \langle y,y\rangle + \langle x,y\rangle + \langle y,x\rangle$ (1)

$\|x-y\|^2 = \langle x-y, x-y\rangle = \langle x,x\rangle + \langle y,y\rangle - \langle x,y\rangle - \langle y,x\rangle$ (2)

$\|x+iy\|^2 = \langle x+iy, x+iy\rangle = \langle x,x\rangle + \langle iy,iy\rangle + \langle x,iy\rangle + \langle iy,x\rangle$

$= \langle x,x\rangle + \langle y,y\rangle - i\langle x,y\rangle + i\langle y,x\rangle$ (3)

$\|x-iy\|^2 = \langle x-iy, x-iy\rangle = \langle x,x\rangle + \langle y,y\rangle + i\langle x,y\rangle - i\langle y,x\rangle$ (4)

$\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$

$\langle x, y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$

$\langle \alpha x, y\rangle = \alpha\langle x,y\rangle$

$\langle x, \alpha y\rangle = \bar{\alpha}\langle x,y\rangle$

$\langle x,y\rangle = \overline{\langle y,x\rangle}$

(1) − (2) : $2\langle x,y\rangle + 2\langle y,x\rangle$

So (1) − (2) + i(3) − i(4)

$= 4\langle x,y\rangle$

$i((3)-(4))$ $\quad i(-2i\langle x,y\rangle + 2i\langle y,x\rangle)$

Q5. A is orthonormal in rows

$\therefore AA^* = I$ (where $A^* = \bar{A}^T$)

$$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \begin{bmatrix} \bar{r}_1 & \bar{r}_2 & \cdots & \bar{r}_n \end{bmatrix} = \begin{bmatrix} r_1\bar{r}_1 & r_1\bar{r}_2 & \cdots & r_1\bar{r}_n \\ \vdots & \vdots & & \vdots \\ r_n\bar{r}_1 & \cdots & \cdots & r_n\bar{r}_n \end{bmatrix} = I$$

$r_i \in \mathbb{R}^{1\times n}$

$\Rightarrow A^* = A^{-1}$ ✓

$\therefore A^* A = I$

$\begin{bmatrix} c_1^* \\ c_2^* \\ \vdots \\ c_n^* \end{bmatrix}$ $\quad [c_1, c_2 \cdots c_n]$ $\quad c_i \in \mathbb{R}^{n\times 1}$

$c_i c_i^* = 1$

but $c_i c_j^* = 0 \quad i \ne j$

$\therefore$ Columns are orthonormal as well

Q6. $v = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\ i\\ -1\end{bmatrix}$, $w = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\ -2i\\ -1\end{bmatrix}$

$\langle v, w\rangle = w^* v$
$= \frac{1}{\sqrt{6}}[1 \ \ 2i \ -1] \cdot \frac{1}{\sqrt{3}}\begin{bmatrix}1\\ i\\ -1\end{bmatrix} = \frac{1}{\sqrt{18}}(1-2+1) = 0$

Inner Product $:= \langle x, y\rangle = y^* x = \sum x_i \bar{y_i}$

Note that v, w are already orthonormal

Let $u = [1 \ 0 \ 0]^T$

$u - \langle v, u\rangle v - \langle w, u\rangle w$

$\therefore \tilde{u} = u - \langle v, u\rangle v - \langle w, u\rangle w$

$= [1 0 0]^T - \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}[1 \ i \ -1]^T - \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}}[1 \ -2i \ -1]^T$

$= [\,1/2 \ \ 0 \ \ 1/2\,]^T$

$u^* v = 0$?

$\frac{1}{\sqrt{3}} \cdot [1/2 \ 0 \ 1/2]\begin{bmatrix}1\\ i\\ -1\end{bmatrix} = 1/2 + 0 - 1/2 = 0$

By normalizing, we get $u = \begin{bmatrix}1/\sqrt{2}\\ 0\\ 1/\sqrt{2}\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$

Without GSOP?

Assume $u = [a \ b \ c]^T$

3 eqn, 3 var :

$a - ib - c = 0$
$a + 2ib - c = 0$
$\Big\}$ $b = 0$, $a = c$

$|a|^2 + |b|^2 + |c|^2 = 1 \rightarrow |a| = |c| = \frac{1}{\sqrt{2}}$

$a = c = \frac{1}{\sqrt{2}} e^{i\theta}$

Tutorial Sheet:

Q4. $\overset{v_1}{[1 \ 1 \ 0 \ 0]}$, $\overset{v_2}{[1 \ 0 \ 1 \ 0]}$, $\overset{v_3}{[1 \ 0 \ 0 \ 1]}$, $\overset{v_4}{[0 \ 1 \ 1 \ 0]}$, $\overset{v_5}{[0 \ 1 \ 0 \ 1]}$, $v_6\,[0 \ 0 \ 1 \ 1]$

$\mathbb{R}^{1\times 4} \rightarrow$ 4 basis vectors
max 4 vectors that are mutually orthonormal

$u_1 = [1 \ 1 \ 0 \ 0]$

$u_2 = [1 \ 0 \ 1 \ 0] - \frac{\langle u_1, v_2\rangle}{\langle u_1, u_1\rangle} u_1$

$= [1 \ 0 \ 1 \ 0] - \frac{1}{2} \cdot [1 \ 1 \ 0 \ 0]$

$= [\,1/2 \ -1/2 \ \ 1 \ \ 0\,] = \frac{1}{2}[1 \ -1 \ 2 \ 0]$

$u_3 = \frac{1}{3}[1, -1, -1, 3]$

$u_4 = \frac{1}{2}[-1, 1, 1, 1]$

$u_5 = u_6 = [0 \ 0 \ 0 \ 0]$

Will try for easier soln

Will try for easier sol^n

Q2 :
$$\begin{vmatrix} \cos\alpha & 1 & & & 0 \\ 1 & 2\cos\alpha & 1 & & \\ & & 2\cos\alpha & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & 2\cos\alpha \end{vmatrix}$$

inductive

$2\cos\alpha$

$$\cos\alpha \begin{vmatrix} 2\cos\alpha & 1 & & \\ 1 & 2\cos\alpha & 1 & \\ & \ddots & \ddots & 1 \\ & & 1 & 2\cos\alpha \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & & & \\ 0 & 2\cos\alpha & 1 & & \\ & 1 & 2\cos\alpha & 1 & \\ & & 1 & 2\cos\alpha & 1 \\ & & & 1 & 2\cos\alpha \end{vmatrix}$$

D_{n-1} $\qquad$ $C_2 \to C_2 - C_1$

$$\cos\alpha \cdot D_{n-1} - D_{n-2}$$

$$D_k = a(e^{i\alpha})^k + b(e^{-i\alpha})^k$$

$$D_1 = 2\cos\alpha = a(e^{i\alpha}) + b e^{-i\alpha}$$

$$a = b = 1$$

$$D_k = e^{i\alpha k} + e^{-i\alpha k} = \underline{2\cos(k\alpha)}$$

$$\det = \cos\alpha \cdot 2\cos((n-1)\alpha) - \cos((n-2)\alpha)$$

$$= \cos(n\alpha) + \cos((n-2)\alpha) - \cos((n-2)\alpha)$$

$$= \cos(n\alpha)$$