Tutorial - 7

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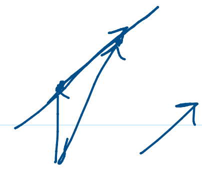


(1) Let $F(x,y,z) = x^2 + 2xy - y^2 + z^2$. Find the gradient of F at $(1,-1,3)$ and the equations of the tangent plane and the normal line to the surface $F(x,y,z) = 7$ at $(1,-1,3)$.

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)$$

$$\nabla F = (2x + 2y,\ 2x - 2y,\ 2z)$$

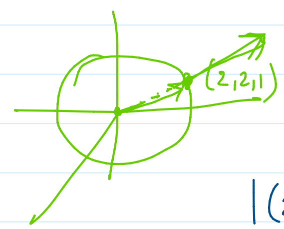
$$\nabla F(1,-1,3) = (0, 4, 6) \equiv 4\hat{j} + 6\hat{k}$$ → perpendicular to tangent plane at pt

$$\text{Normal} = (1,-1,3) + t(0,4,6) = (1, -1+4t, 3+6t)$$

$$\text{Tangent Plane} = 0(x-1) + 4\cdot(y+1) + 6(z-3) = 0$$

$$2y + 3z = 7$$

(2) Find $D_{\underline{u}}F(2,2,1)$, where $F(x,y,z) = 3x - 5y + 2z$, and $\underline{u}$ is the unit vector in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 9$ at $(2,2,1)$.



$$|(2,2,1)| = 3$$

$$D_{\bar{u}} F = \overline{(\nabla F)} \cdot (\bar{u})$$

$$\nabla F = (3, -5, 2) \quad ; \quad \bar{u} = \left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3} \right)$$

$$D_{\bar{u}} F = 2 + \frac{-10}{3} + \frac{2}{3} = 2 - \frac{8}{3} = \frac{-2}{3}$$

(3) Given $\sin(x+y) + \sin(y+z) = 1$, find $\dfrac{\partial^2 z}{\partial x \partial y}$, provided $\cos(y+z) \neq 0$.

$$z = f(x,y)$$

Apply partial der. wrt. x

(1) $$\cos(x+y) + \cos(y+z)\cdot \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{\cos(x+y)}{\cos(y+z)}$$

Apply partial der. wrt. y

$$② = \cos(x+y) + \cos(y+z)\cdot\left(1+\frac{\partial z}{\partial y}\right) = 0 \Rightarrow 1+\frac{\partial z}{\partial y} = \frac{-\cos(x+y)}{\cos(y+z)}$$

Diff. ① w.r.t y

$$-\sin(x+y) - \sin(y+z)\left(1+\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} + \cos(y+z)\frac{\partial^2 z}{\partial x\partial y} = 0$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{1}{\cos(y+z)}\left[\sin(x+y) + \sin(y+z)\left(1+\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x}\right]$$

$$= \frac{1}{\cos(y+z)}\left[\sin(x+y) + \sin(y+z)\frac{\cos^2(x+y)}{\cos^2(y+z)}\right]$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\sin(x+y)}{\cos(y+z)} + \tan(y+z)\frac{\cos^2(x+y)}{\cos^2(y+z)}$$

(4) If $f(0,0) = 0$ and

$$f(x,y) = xy\frac{x^2-y^2}{x^2+y^2} \text{ for } (x,y) \neq (0,0),$$

show that both f_{xy} and f_{yx} exist at $(0,0)$, but they are not equal. Are f_{xy} and f_{yx} continuous at $(0,0)$?

$$f_y(0,0) = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}$$

$$f_{xy}(0,0) = (f_x)_y = (g)_y$$

$$= \lim_{h\to 0}\frac{f_x(0,h)-f_x(0,0)}{h}$$

$$f_x(0,h) = \lim_{k\to 0}\frac{f(k,h)-f(0,h)}{k} = \lim_{k\to 0}\frac{\cancel{k}h}{\cancel{k}}\frac{k^2-h^2}{k^2+h^2} = -h$$

$$f_x(0,h) = -h, \quad f_x(0,0) = 0$$

$$f_{xy}(0,0) = \lim_{h\to 0}\frac{-h-0}{h} = -1$$

$$f_{yx}(0,0) = 1$$

$$f_{xy}(0,0) \neq f_{yx}(0,0)$$

(5) Show that the following functions have local minima at the indicated points.

(i) $f(x, y) = x^4 + y^4 + 4x - 32y - 7, \quad (x_0, y_0) = (-1, 2)$

H = Hessian Matrix $= \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

For minima $\begin{cases} \det(H)(x_0, y_0) > 0 \ \checkmark \\ f_{xx}(x_0, y_0) > 0 \ \checkmark \end{cases}$

$f_x = 4x^3 + 4 \quad ; \quad f_y = 4y^3 - 32$

$f_{xx} = 12x^2, \quad f_{yy} = 12y^2, \quad f_{xy} = f_{yx} = 0$

$H = \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{bmatrix} \Rightarrow \det(H) = 144\, x^2 y^2$

$\left.\begin{aligned} \det(H)(-1,2) &= 144 \times 1 \times 4 = 576 > 0 \\ f_{xx}(-1,2) &= 12 > 0 \end{aligned}\right\} (x_0, y_0) \to$ local minima

(6) Analyze the following functions for local maxima, local minima and saddle points :

(ii) $f(x, y) = x^3 - 3xy^2$

$\left. f_x = 3x^2 - 3y^2 \quad ; \quad f_y = -6xy \right\}$ Critical points need $f_x = f_y = 0$ $(0,0)$

$f_{xx} = 6x, \quad f_{yy} = -6x, \quad f_{xy} = f_{yx} = -6y$

$H = \begin{bmatrix} 6x & -6y \\ -6y & -6x \end{bmatrix} \Rightarrow \det(H) = -36x^2 - 36y^2$

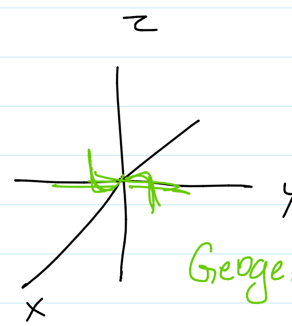
$\det(H)(0,0) = 0 \longrightarrow$ inconclusive

$f(\delta, 0) = \delta^3$

$\delta > 0, \quad f(\delta, 0) > 0$

$\delta < 0, \quad f(\delta, 0) < 0$

Saddle pt



Geogebra

(7) Find the absolute maximum and the absolute minimum of

$f(x, y) = (x^2 - 4x)\cos y$ for $1 \le x \le 3$, $-\pi/4 \le y \le \pi/4$.

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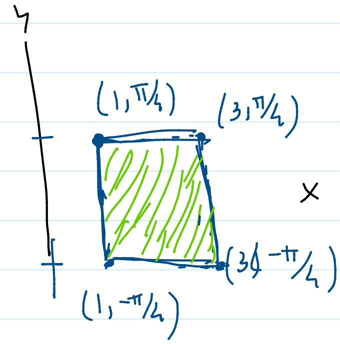

i) Critical pt.
ii) Boundary

$$f_x = (2x-4)\cos y$$

$$f_y = -(x^2-4x)\sin y$$

at $x=2$, $f_x = 0$, at $y=0$, $f_y=0$

Critical pt $\rightarrow (2,0)$ $\quad f(2,0) = 4-8 = \underline{-4}$

$$f(x, \pm\pi/4) = \frac{x^2-4x}{\sqrt{2}} \rightarrow \text{critical pt. } x=2$$

$$f(2, \pm\pi/4) = \underline{-\frac{4}{\sqrt{2}}} \; ; \; f(1, \pm\pi/4) = \underline{\frac{-3}{\sqrt{2}}} \; ; \; f(3, \pm\pi/4) = \underline{\frac{-3}{\sqrt{2}}}$$

$$f(1,y) = -3\cos y \rightarrow \text{critical pt} = y=0$$

$$f(1,0) = \underline{-3}$$

$$f(3,y) = -3\cos y \rightarrow \text{critical pt } y=0$$

$$f(3,0) = \underline{-3}$$

Maximum $= -\frac{3}{\sqrt{2}}$ at $(1, \pi/4)$ $(1, -\pi/4)$ $(3, \pi/4)$ $(3, -\pi/4)$

Minimum $= -4$ at $(2,0)$