Tutorial - 4

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6. (a) Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0$ for all $x \in [a,b]$. Show that $\int_a^b f(x)dx \geq 0$. Further, if f is continuous and $\int_a^b f(x)dx = 0$, show that $f(x) = 0$ for all $x \in [a,b]$.

   (b) Give an example of a Riemann integrable function on $[a,b]$ such that $f(x) \geq 0$ for all $x \in [a,b]$ and $\int_a^b f(x)dx = 0$, but $f(x) \neq 0$ for some $x \in [a,b]$.

a) $f(x) \geqslant 0$

$$P_0 = \{a = x_0, x_1, \dots, x_n = b\}$$

$$m_i(f) = \inf_{x \in [x_{i-1}, x_i]} f(x) \geqslant 0 \quad [\because f(x) \geqslant 0 \quad x \in [a,b]]$$

$$L(f, P_0) \geqslant 0 \qquad \because \quad L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

$$L(f) = \sup_{\text{for all } P} L(f,P) = \int_a^b f(x)\,dx$$

$$I = L(f) \geqslant L(f, P_0) \geqslant 0 \quad \Rightarrow \int_a^b f(x)\,dx \geqslant 0$$

$$F(x) = \int_a^x f(t)\,dt \quad \Longrightarrow \quad F'(x) = f(x) \geqslant 0 \ [\text{FTC}]$$

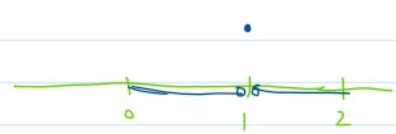
$F = \int$

$F(x)$ is inc.

$$\text{For } x \in [a,b], \quad \underset{0}{\underset{\|}{F(a)}} \leq F(x) \leq \underset{0 \text{ (given)}}{\underset{\|}{F(b)}} \quad \Rightarrow \underline{F(x) = 0}$$

$$f(x) = 0$$

b) $f(x) = \begin{cases} 0 & ; x \neq 1 \\ 1 & ; x = 1 \end{cases} \quad f : [0,2] \to \mathbb{R}$



7. Evaluate $\lim_{n\to\infty} S_n$ by showing that S_n is an approximate Riemann sum for a suitable function over a suitable interval:

   (i) $S_n = \dfrac{1}{n^{5/2}} \sum_{i=1}^{n} i^{3/2}$

   (iii) $S_n = \sum_{i=1}^{n} \dfrac{1}{\sqrt{in + n^2}}$

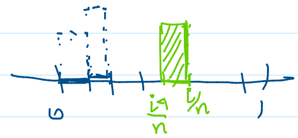
   (iv) $S_n = \dfrac{1}{n} \sum_{i=1}^{n} \cos \dfrac{i\pi}{n}$

   (v) $S_n = \dfrac{1}{n} \left\{ \sum_{i=1}^{n} \left(\dfrac{i}{n}\right) + \sum_{i=n+1}^{2n} \left(\dfrac{i}{n}\right)^{3/2} + \sum_{i=2n+1}^{3n} \left(\dfrac{i}{n}\right)^2 \right\}$

(v) $S_n = \frac{1}{n}\left\{\sum_{i=1}^{n}\left(\frac{i}{n}\right) + \sum_{i=n+1}^{2n}\left(\frac{i}{n}\right)^{3/2} + \sum_{i=2n+1}^{3n}\left(\frac{i}{n}\right)^2\right\}$

i) Consider $P = \{0, 1/n, 2/n, \dots n/n\}$

tags t_i $\frac{i}{n}$ $\quad i = 1, 2, \dots n$



$f : [0,1] \to \mathbb{R}$: $f(x) = \frac{2}{5} x^{5/2}$; $f'(x) = x^{3/2}$

$$S_n = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^{3/2}\left(\frac{i}{n} - \frac{i-1}{n}\right) \qquad \|P\| \to 0 , \quad n \to \infty$$

$$\lim_{n\to\infty} S_n = \int_0^1 x^{3/2}\,dx = \int_0^1 f'(x)\,dx$$

By FTC, $f(1) - f(0) = \frac{2}{5}$ //

iii) $S_n = \sum_{i=1}^{n} \frac{1}{n} \frac{1}{\sqrt{\left(\frac{i}{n}\right)+1}} \Rightarrow S_n \to \int_0^1 \frac{1}{\sqrt{x+1}}\,dx$

$= 2(\sqrt{2} - 1)$

iv) $S_n \to \int_0^1 \cos \pi x = \left.\frac{\sin \pi x}{\pi}\right|_0^1 = \underline{\underline{0}}$

v) $S_n \to \int_0^1 x\,dx + \int_1^2 x^{3/2}\,dx + \int_2^3 x^2\,dx = \frac{1}{2} + \frac{2}{5}(4\sqrt{2} - 1) + \frac{19}{3}$

(b) $\frac{dF}{dx}$, if for $x \in \mathbb{R}$ (i) $F(x) = \int_1^{2x} \cos(t^2)dt$ (ii) $F(x) = \int_0^{x^2} \cos(t)dt$.

application of FTC

$$F(x) = \int_{u(x)}^{v(x)} f(t)\,dt = \int_a^{v(x)} \cdots - \int_a^{u(x)}$$

$$F'(x) = \underline{v'(x) f(v(x)) - u'(x) f(u(x))}$$

i) Take $u = \frac{t}{2}$

$t = 2x \to u \to x$; $t = 1$, $u = 1/2$

$du = \frac{dt}{2}$

$$F(x) = \int_{1/2}^{x} 2\cos(4u^2)\,du \Rightarrow F'(x) = 2\cos(4x^2)$$

ii) $u = \sqrt{t}$, $F(x) = \int_0^x 2u\cos(u^2)\,du$

$F'(x) = 2x\cos(x^2)$

9. Let p be a real number and let f be a continuous function on $\mathbb{R}$ that satisfies the equation $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that the integral $\int_a^{a+p} f(t)dt$ has the same value for every real number a. (Hint : Consider $F(a) = \int_a^{a+p} f(t)dt,\ a \in \mathbb{R}$.)

$$F(x) = \int_x^{x+p} f(t)\, dt$$

$$F'(x) = 1 \cdot f(x+p) - 1 \cdot f(x) = f(x+p) - f(x) = 0$$

$F(x)$ is constant

$\int_x^{x+p} f(t)\,dt$ has same value $\forall x$

10. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R},\ \lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x-t) dt.$$

Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = 0 = g'(0)$.

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \left[\sin\lambda x \cos\lambda t - \cos\lambda x \sin\lambda t \right]$$

$$= \underbrace{\frac{1}{\lambda} \sin\lambda x}\ \underbrace{\int_0^x f(t) \cos\lambda t\, dt} - \frac{1}{\lambda} \cos\lambda x \int_0^x f(t) \sin\lambda t\, dt$$

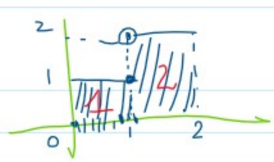
$$= \frac{1}{\lambda} \cos\lambda x \cdot \lambda \int_0^x \cdots dt + \frac{1}{\lambda} \sin\lambda x\, \cancel{f(x) \cos\lambda x} + \frac{1}{\lambda} \sin\lambda x \cdot \lambda \int_0^x \cdots dt - \frac{1}{\lambda} \cancel{\cos\lambda x}\, f(x) \sin\lambda x$$

$$g'(x) = \cos\lambda x \int_0^x f(t) \cos\lambda t\, dt + \sin\lambda x \int_0^x f(t) \sin(\lambda t)$$

$$g''(x) = f(x) - \lambda^2 \left(\frac{1}{\lambda} \int_0^x f(t) \underbrace{(\sin\lambda x \cos\lambda t - \cos\lambda x \sin\lambda t)}_{\sin\lambda(x-t)} \right)$$

$$g''(x) = f(x) - \lambda^2 g(x) \Rightarrow g''(x) + \lambda^2 g(x) = f(x)$$

5. Let $f(x) = 1$ if $x \in [0,1]$ and $f(x) = 2$ if $x \in (1,2]$. Show from the first principles that f is Riemann integrable on $[0,2]$ and find $\int_0^2 f(x)dx$.



$P = \{0 = x_0, x_1, x_2, \cdots x_n = 2\}$

$\exists i_0$ s.t. $x_{i_0} \in [0,1]$, $x_{i_0+1} \in (1,2]$ ✓

$\exists i_0$ s.t. $x_{i_0} \in [0,1]$, $x_{i_0+1} \in (1,2]$ ✓

For $i = 1, 2, 3, \dots i_0$

$$m_i(f) = \inf_{x \in \underbrace{[x_{i-1}, x_i]}} f(x) = 1 \quad ; \quad M_i(f) = \sup_{x \in [x_{i-1}, x_i]} f(x) = 1$$

For $i = i_0+1$

$$m_i(f) = 1 \quad ; \quad M_i(f) = 2$$

For $i = i_0+2, i_0+3, \dots n$

$$m_i(f) = 2 \quad , \quad M_i(f) = 2$$

$$L(P,f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) = \sum_{i=1}^{i_0+1} 1 \cdot (x_i - x_{i-1}) + \sum_{i=i_0+2}^{n} 2 \cdot (x_i - x_{i-1})$$

$$= x_{i_0+1} - x_0 + 2(x_n - x_{i_0+1})$$

$$= \underline{4 - x_{i_0+1}}$$

$$U(P,f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) = \sum_{i=1}^{i_0} 1 \cdot (x_i - x_{i-1}) + \sum_{i=i_0+1}^{n} 2 \cdot (x_i - x_{i-1})$$

$$= x_{i_0} - x_0 + 2(x_n - x_{i_0})$$

$$= \underline{4 - x_{i_0}}$$

$U(P,f) - L(P,f) \to 0$ as $\|P\| \to 0$

$x_{i_0+1} - x_{i_0} \to 0$ $\qquad x_{i_0+1} - x_{i_0} \leq \|P\| \to 0$

$\therefore$ Func^n is Riemann Integrable

$U(P,f) = L(P,f)$

$\|P\| \to 0$, $x_{i_0+1} \to \underset{1}{x_{i_0}}$

$$\int_0^2 f(x)\,dx = 4 - 1 = 3$$