Tutorial - 3

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1. Show that the cubic $x^3 - 6x + 3$ has all roots real.

$$f(x) = x^3 - 6x + 3$$

$$f'(x) = 3x^2 - 6$$

$$f'(x) = 0 \Rightarrow x_1 = \sqrt{2}, \quad x_2 = -\sqrt{2}$$

$$f(\sqrt{2}) = 2\sqrt{2} - 6\sqrt{2} + 3 = -4\sqrt{2} + 3 < 0$$

$$f(-\sqrt{2}) = -2\sqrt{2} + 6\sqrt{2} + 3 = 4\sqrt{2} + 3 > 0$$

$$x \to -\infty, \ f(x) \to -\infty, \quad x \to \infty, \ f(x) \to \infty$$

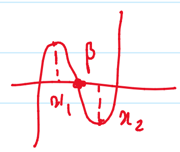
Atleast 1 root $\in (-\infty, -\sqrt{2})$
" " " $\in (-\sqrt{2}, \sqrt{2})$
" " " $\in (\sqrt{2}, \infty)$

} Atleast 3 roots + Cubic has atmost 3 real roots

$\Rightarrow$ $f(x)$ has 3 real roots

4. Consider the cubic $f(x) = x^3 + px + q$, where p and q are real numbers. If $f(x)$ has three distinct real roots, show that $4p^3 + 27q^2 < 0$ by proving the following:
   (i) $p < 0$.
   (ii) f has maximum/minimum at $\pm\sqrt{-p/3}$.
   (iii) The maximum/minimum values are of opposite signs.

$$\alpha < \beta < \gamma$$



$$f(\alpha) = f(\beta) = f(\gamma) = 0$$

By Rolle's Th. $\exists\ x_1$ s.t $x_1 \in (\alpha, \beta)$, $f'(x_1) = 0$
" $\exists\ x_2$ s.t. $x_2 \in (\beta, \gamma)$, $f'(x_2) = 0$

$$f'(x) = 3x^2 + p$$

Since $f'(x)$ has 2 roots $\quad -4 \times 3 \times p > 0$
$\Rightarrow p < 0$

$$f'(x) = 0 \Rightarrow x_1 = -\sqrt{\frac{-p}{3}}, \quad x_2 = +\sqrt{\frac{-p}{3}}$$

$f''(x_1) < 0 \Rightarrow x_1$ is max.
$f''(x_2) > 0 \Rightarrow x_2$ is min

$$f(x_1) = q + \sqrt{\frac{-4p^3}{27}} \quad ; \quad f(x_2) = q - \sqrt{\frac{-4p^3}{27}}$$

$$f(x_1)\cdot f(x_2) < 0 \Rightarrow q^2 - \left(\frac{-4p^3}{27}\right) < 0$$

$$= \frac{27q^2+4p^3}{27} < 0 \Rightarrow 27q^2+4p^3 < 0$$

5. Use the MVT to prove $|\sin a - \sin b| \leq |a-b|$ for all $a, b \in \mathbb{R}$.

For $a = b$

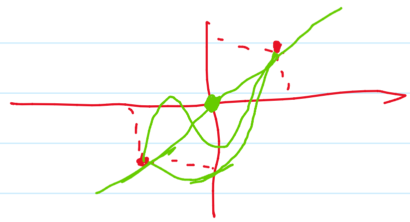
This is trivial, $\because 0 \leq 0$

For $a \neq b$, Take $f(x) = \sin x$ in $[a, b]$

$\therefore \exists c \in [a,b]$ s.t.

$$\left|\frac{f(a)-f(b)}{a-b}\right| = f'(c) = \cos c \leq 1 \Rightarrow |f(a)-f(b)| \leq |a-b|$$

7. Let $a > 0$ and f be continuous on $[-a, a]$. Suppose that $f'(x)$ exists and $f'(x) \leq 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$.



Consider $(-a, 0)$, $(0, a)$

Apply MVT on $(-a, 0)$. $\therefore \exists c_1 \in (-a, 0)$ s.t.

$$\frac{f(0)-f(-a)}{0-(-a)} = f'(c_1) \Rightarrow f(0) + a = f'(c_1)\cdot a$$

Because $f'(x) \leq 1$, $f'(c_1) \leq 1 \Rightarrow f(0) + a \leq a \Rightarrow f(0) \leq 0$ (1)

Apply MVT on $(0, a)$. $\exists c_2 \in (0, a)$ s.t.

$$\frac{f(a)-f(0)}{a-0} = f'(c_2) \Rightarrow a - f(0) = f'(c_2)\cdot a \leq a$$

$f(0) \geq 0$ (2)

From (1) & (2), $f(0) = 0$

8. In each case, find a function f which satisfies all the given conditions, or else show that no such function exists.

(i) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 1$

(ii) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 2$

(iii) $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 100$ for all $x > 0$

(iv) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 1$ for all $x < 0$

i) Not Possible , Let $f'(x) = g(x)$

$g(0) = 1$, $g(1) = 1$

$\therefore$ By Rolle's Theorem, $\exists\, x_1 \in (0,1)$ s.t $g'(x_1) = 0$

$= f''(x_1) = 0$

This is a contradiction

ii) $f(x) = \dfrac{x^2}{2} + x$

iii) Not Possible

$f''(x) \geq 0 \Rightarrow f'(x)$ is inc.

For all $c > 0$, $f'(c) \geq 1$

Apply MVT between $(0, x)$

$\dfrac{f(x) - f(0)}{x} = f'(c) \Rightarrow f(x) - f(0) \geq x$

$\Rightarrow f(x) \geq f(0) + x$ for all $x > 0$

Let $f(0) = \alpha$

10

$f(x) \geq \alpha + x$ for all $x > 0$

for any $x_1 \geq 100 - \alpha$, $f(x_1) \geq 100 \longrightarrow$ contradiction

iv) $f(x) = \begin{cases} 1 + x + x^2 & ; x \geq 0 \\ e^x & ; x < 0 \end{cases}$

9. Let $f(x) = 1 + 12|x| - 3x^2$. Find the absolute maximum and the absolute minimum of f on $[-2, 5]$. Verify it from the sketch of the curve $y = f(x)$ on $[-2, 5]$.

For $x < 0$, $f(x) = 1 - 12x - 3x^2 \rightarrow f'(x) = 0 \rightarrow x = -2$

$x \geq 0$, $f(x) = 1 + 12x - 3x^2 \rightarrow f'(x) = 0 \rightarrow x = +2$

Points to check : $\underbrace{-2, 5}_{\text{bounds}}$, $\underset{\text{non diff}}{0}$, $\underset{\text{max/min}}{+2}$,

$f(-2) = 1 + 24 - 12 = 13$

$f(5) = 1 + 60 - 75 = -14$

$f(0) = 1$

$\underbrace{\text{bounds}}$ non diff   max/min

Abs. max : $x = \pm 2$, Abs. min : $x = 5$

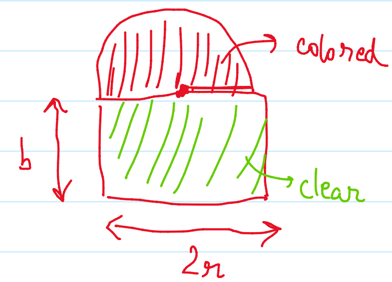
$f(5) = 1 + 60 - 75 = -14$

$f(0) = 1$

$f(2) = 1 + 24 - 12 = 13$

10. A window is to be made in the form of a rectangle surmounted by a semicircular portion with diameter equal to the base of the rectangle. The rectangular portion is to be of clear glass and the semicircular portion is to be of colored glass admitting only half as much light per square foot as the clear glass. If the total perimeter of the window frame is to be p feet, find the dimensions of the window which will admit the maximum light.

$l = kA$



Let clear glass allow k W/sq.foot
Colored glass allow $\frac{k}{2}$ W/sq foot

$$l(r,b) = 2rb \times k + \frac{k\pi r^2}{4}$$

$$p = 2r + 2b + \pi r \Rightarrow b = \frac{1}{2}\left(p - (2+\pi)r\right)$$

$$l = rk\left(p - (2+\pi)r\right) + \frac{k\pi r^2}{4} = prk - 2r^2k - \pi r^2 k + \frac{k\pi r^2}{4}$$

$$l = k\left(pr - 2r^2 - \frac{3\pi r^2}{4}\right)$$

$$l'(r) = p\cancel{k} - 4r\cancel{k} - \frac{3\pi r\cancel{k}}{2} = 0$$

$$r_0 = \frac{2p}{8+3\pi}$$

Check: $l''(r_0) < 0 \rightarrow$ max light

$$b = \frac{1}{2}\left(\frac{4+\pi}{8+3\pi}\right)p$$