Tutorial - 1

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(iii) $\lim_{n\to\infty} \frac{n^{2/3}\sin(n!)}{n+1} = 0$

$$\left| \frac{n^{2/3}\sin(n!)}{n+1} - 0 \right| < \varepsilon \quad \forall\ n \geq n_0 \in \mathbb{N}$$

$a_n \to L$ (check)
For any ε, $\exists\, n_0 \in \mathbb{N}$ s.t
$|a_n - L| < \varepsilon \ \forall n \geq n_0$

$$\left| \frac{n^{2/3}\sin(n!)}{n+1} \right| \leq \left| \frac{n^{2/3}}{n+1} \right| \leq \left| \frac{n^{2/3}}{n} \right| = \left| \frac{1}{n^{1/3}} \right| < \varepsilon$$

$$n > \frac{1}{\varepsilon^3}$$

$$n_0 = \left\lfloor \frac{1}{\varepsilon^3} \right\rfloor$$

(iv) $\lim_{n\to\infty} \left(\frac{n}{n+1} - \frac{n+1}{n} \right) = 0$

$$\left| \frac{n^2 - (n+1)^2}{n(n+1)} \right| = \left| \frac{2n+1}{n(n+1)} \right| \leq \left| \frac{2n+2}{n(n+1)} \right| = \left| \frac{2}{n} \right| \leq \varepsilon$$

$$n \geq \frac{2}{\varepsilon}$$

$$\Rightarrow n_0 = \left\lfloor \frac{2}{\varepsilon} \right\rfloor$$

(i) $\lim_{n\to\infty} \left(\underbrace{\frac{n}{n^2+1}}_{a_1} + \underbrace{\frac{n}{n^2+2}}_{a_2} + \cdots + \underbrace{\frac{n}{n^2+n}}_{a_n} \right) = S_n$

$$\underset{\text{const}}{a_n} \leq a_i \leq \underset{\text{const}}{a_1} \implies \sum^{n} a_n \leq \sum_{i=1}^{n} a_i \leq \sum^{n} a_1$$

$$n a_n \leq S_n \leq n a_1$$

$$\lim_{n\to\infty} \quad \underset{\downarrow\ 1}{\frac{n^2}{n^2+n}} \leq S_n \leq \underset{\downarrow\ 1}{\frac{n^2}{n^2+1}} \quad \Rightarrow S_n = 1$$

$$\left| \frac{n^2}{n^2+n} - 1 \right|$$

$$\left| \frac{n}{n^2+n} \right| = \left| \frac{1}{n+1} \right| \leq \left| \frac{1}{n} \right| < \varepsilon$$

(iv) $\lim_{n\to\infty} (n)^{1/n}$

$$n^{1/n} \geq 1 \Rightarrow n \geq 1$$

$$a_n = n^{1/n} - 1 \geq 0$$

$$n = (1+a_n)^n = 1 + {}^nC_1 a_n + \underbrace{{}^nC_2 a_n^2 \cdots}_{\text{exclude}}$$

$$n \geq \underbrace{1 + {}^nC_1 a_n}_{\text{exclude}} + {}^nC_2 a_n^2$$

$$\cancel{n} \geq {}^nC_2 a_n^2 = \frac{\cancel{n}(n-1)}{2} a_n^2$$

$$0 \leq a_n \leq \sqrt{\frac{2}{n-1}} \qquad \text{Apply Sandwich} \Rightarrow \quad a_n \to 0$$

$$n^{1/n} \to 1$$

(v) $\lim\limits_{n\to\infty} \left(\frac{\cos \pi\sqrt{n}}{n^2}\right)$

$$-\frac{1}{n^2} \leq \frac{\cos \pi\sqrt{n}}{n^2} \leq \frac{1}{n^2} \qquad \text{Sandwich}$$

$$n \to \infty, \quad \frac{\cos \pi\sqrt{n}}{n^2} \to 0$$

(vi) $\lim\limits_{n\to\infty} (\sqrt{n}(\sqrt{n+1} - \sqrt{n}))$ multiply $(\sqrt{n+1} + \sqrt{n})$ both Num, Den.

$$\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{1+\frac{1}{n}} + 1}$$

$\frac{1}{n} \to$ tends to 0

$$= \frac{1}{1+1} = \frac{1}{2}$$

(i) $\{\frac{n^2}{n+1}\}_{n\geq 1}$, Wait

(i) $\left\{\frac{n}{n^2+1}\right\}_{n\geq 1}$

$$a_n = \frac{n}{n^2+1} = \frac{1}{n + 1/n}$$

$$b_n = n + \frac{1}{n}$$

$$b_{n+1} - b_n = n+1 + \frac{1}{n+1} - n - \frac{1}{n}$$

$$b_{n+1} - b_n = n+1 + \frac{1}{n+1} - n - \frac{1}{n}$$

$$= 1 - \frac{1}{n(n+1)}$$

For all $n \geq 1$, $\frac{1}{n(n+1)} \leq \frac{1}{2} \Rightarrow b_{n+1} - b_n > 0$, b_n inc.

a_n dec.

(iii) $\left\{\frac{1-n}{n^2}\right\}_{n\geq 2}$ $= a_n = \frac{1}{n^2} - \frac{1}{n}$

$$a_{n+1} - a_n = \frac{1}{(n+1)^2} - \frac{1}{(n+1)} - \frac{1}{n^2} + \frac{1}{n} = \frac{n^2 - n - 1}{(n+1)^2 n^2}$$

$$= \frac{n(n-1) - 1}{(n+1)^2 n^2}$$

$n \geq 2 \Rightarrow n(n-1) \geq 2$

For all $n \geq 2$, $a_{n+1} - a_n > 0 \Rightarrow a_n$ is inc.

(ii) $a_1 = \sqrt{2}, a_{n+1} = \sqrt{2 + a_n} \; \forall \; n \geq 1$

Wait

7. If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2} \quad \text{for all } n \geq n_0.$$

$|a_n - L| < \varepsilon \quad \forall n \geq n_0$

Rev Δ Ineq. $\quad ||a_n| - |L|| \leq |a_n - L|$ ✓

$||a_n| - |L|| < \varepsilon \quad \Rightarrow |L| - \varepsilon \leq |a_n| \leq |L| + \varepsilon$

$$||a_n| - |L|| < \varepsilon \quad \Rightarrow |L| - \varepsilon \leqslant |a_n| \leqslant |L| + \varepsilon$$

Take $\varepsilon = \frac{|L|}{2} \quad \Rightarrow \quad \frac{|L|}{2} \leqslant |a_n| \leqslant 3\frac{|L|}{2}$

8. If $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$, show that $\lim_{n\to\infty} a_n^{1/2} = 0$.
Optional: State and prove a corresponding result if $a_n \to L > 0$.

For any ε, $|a_n - 0| < \varepsilon \quad \forall\ n \geqslant n_0$

$|a_n| < \varepsilon$

$\varepsilon > 0 \rightarrow \quad \varepsilon = \delta^2$

$|a_n| < \delta^2 \quad \forall\ n \geqslant n_0$

For any δ, $|\sqrt{a_n}| < \delta \quad \forall\ n \geqslant n_0$

$\therefore \sqrt{a_n} \rightarrow 0$

10. Show that a sequence $\{a_n\}_{n\geq 1}$ is convergent if and only if both the subsequences $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ are convergent to the same limit.

$\{a_n\}$ conv. $\Longleftrightarrow$ $\{a_{2n}\}$ conv, $\{a_{2n+1}\}$ conv

$\{a_n\}$ conv $\Rightarrow$ $\left.\begin{array}{l}\{a_{2n}\} \text{ conv} \\ \{a_{2n+1}\} \text{ conv}\end{array}\right\}$ Trivial

Rev. Implication:

For any $\varepsilon > 0$, $\exists\ n_1, n_2$

$|a_{2n} - L| < \varepsilon \quad \forall\ n > n_1$

$|a_{2n+1} - L| < \varepsilon \quad \forall\ n > n_2$

$n_0 = \max(n_1, n_2)$

$\therefore$ For any $\varepsilon > 0$,

$|a_n - L| < \varepsilon \quad \forall\ n \geqslant 2n_0 + 1$