Tutorial - 5

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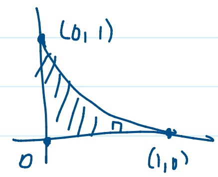
(1) Find the area of the region bounded by the given curves in each of the following cases.

(i) $\sqrt{x}+\sqrt{y}=1$, $x=0$ and $y=0$



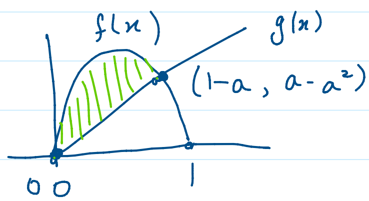
$\sqrt{x}=1 \Rightarrow x=1$

$x=0$ to $x=1$

$y=(1-\sqrt{x})^2 = 1+x-2\sqrt{x}$

$$\int_0^1 1+x-2\sqrt{x}\,dx = x+\frac{x^2}{2}-\frac{4}{3}x^{3/2}\Big|_0^1 = 1+\frac{1}{2}-\frac{4}{3} = \frac{1}{6}$$

(2) Let $f(x)=x-x^2$ and $g(x)=ax$. Determine a so that the region above the graph of g and below the graph of f has area 4.5



$x-x^2=ax \Rightarrow (1-a)x = x^2$

$$\int_0^{1-a} x-x^2-ax\,dx = \int_0^{1-a}(1-a)x-x^2\,dx$$

$$\left[(1-a)\frac{x^2}{2}-\frac{x^3}{3}\right]_0^{1-a} = \frac{(1-a)^3}{2}-\frac{(1-a)^3}{3} = \frac{(1-a)^3}{6} = 4.5$$

$(1-a)^3 = 27 \rightarrow 1-a=3 \Rightarrow \underline{a=-2}$

(3) Find the area of the region inside the circle $r_1=6a\cos\theta$ and outside the cardioid $r_2=2a(1+\cos\theta)$.

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta \quad ; r=f(\theta)$$

$r_1=r_2$; $6a\cos\theta = 2a(1+\cos\theta)$

$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{-\pi}{3}, \frac{\pi}{3}$

$$\text{reqd area} = \frac{1}{2}\int_{-\pi/3}^{\pi/3} r_1^2 - r_2^2\,d\theta$$

$4(2\cos^2\theta-1)$

$$2a^2\int_{-\pi/3}^{\pi/3} 8\cos^2\theta - 2\cos\theta - 1\,d\theta = 2a^2\int_{-\pi/3}^{\pi/3} 4\cos 2\theta - 2\cos\theta + 3\,d\theta$$

$$= 2a^2\Big[2\sin 2\theta - 2\sin\theta + 3\theta\Big]_{-\pi/3}^{\pi/3}$$

$$= 2a^2\left[\,2\sin 2\theta - 2\sin\theta + 3\theta\,\right]_{-\pi/3}^{\pi/3}$$

$$= 2a^2\left[2\cdot\cancel{\frac{\sqrt{3}}{2}} - 2\cdot\cancel{\frac{\sqrt{3}}{2}} + \pi - \left(-2\cdot\cancel{\frac{\sqrt{3}}{2}} + 2\cdot\cancel{\frac{\sqrt{3}}{2}} - \pi\right)\right]$$

$$= \underline{4a^2\pi}$$

(5) For the following curve, find the arc length as well as the the area of the surface generated by revolving it about the line $y = -1$.

$$y = \frac{x^3}{3} + \frac{1}{4x}, \quad 1 \le x \le 3$$

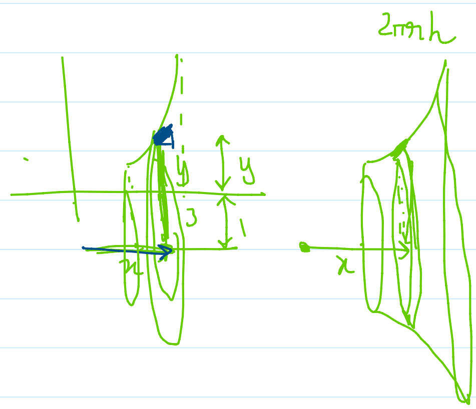
$$\frac{dy}{dx} = x^2 - \frac{1}{4x^2} \quad ; \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\frac{ds}{dx} = \sqrt{1 + x^4 + \frac{1}{16x^4} - \frac{1}{2}} = \sqrt{x^4 + \frac{1}{16x^4} + \frac{1}{2}} = \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}$$

$$= x^2 + \frac{1}{4x^2}$$

$$L = \int_1^3 \frac{ds}{dx}\,dx = \int_1^3 x^2 + \frac{1}{4x^2}\,dx = \left[\frac{x^3}{3} - \frac{1}{4x}\right]_1^3$$

$$L = \frac{53}{6}$$

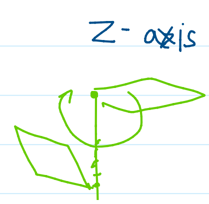


$$dA = 2\pi(y+1)\,\frac{ds}{dx}\,dx$$

$$A = \int_1^3 2\pi\left(\frac{x^3}{3} + \frac{1}{4x} + 1\right)\cdot\left(x^2 + \frac{1}{4x^2}\right)dx$$

$$= \frac{1823}{18}\cdot\pi$$

(8) A fixed line L in 3-space and a square of side r in a plane perpendicular to L are given. One vertex of the square is on L. As this vertex moves a distance h along L, the square turns through a full revolution with L as the axis. Find the volume of the solid generated by this motion.



$$A(z) = r^2$$

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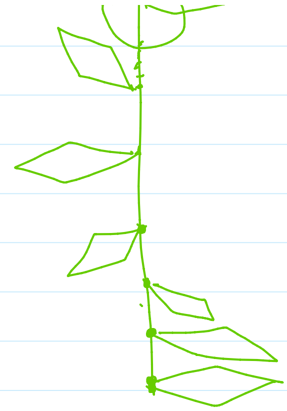
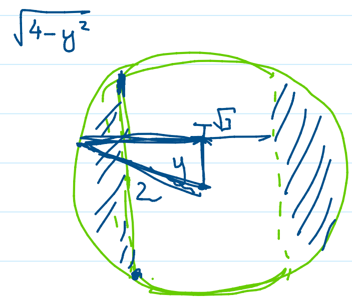
$$dV = A(z)\cdot dh = r^2\, dh$$

$$V = \int_0^h r^2\, dh = \underline{\underline{r^2 h}}$$



(10) A round hole of radius $\sqrt{3}$ cms is bored through the center of a solid ball of radius 2 cms. Find the volume cut out.



$$A(y) = \pi\left(\sqrt{4-y^2}\right)^2 - \pi\left(\sqrt{3}\right)^2$$

$$= \pi(1-y^2)$$

$$dV = \pi(1-y^2)\, dy$$

$$\text{Volume left} = \int_{-1}^{1} \pi(1-y^2)\, dy$$

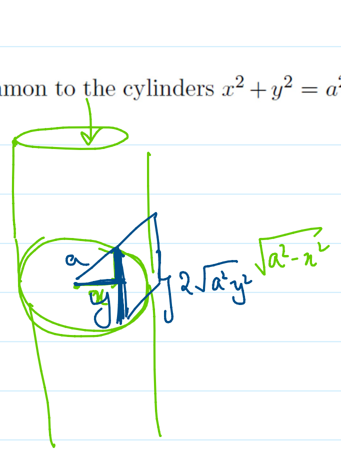
$$= \pi\left(y - \frac{y^3}{3}\right)\Bigg|_{-1}^{1} = \pi\left(2 - \frac{2}{3}\right)$$

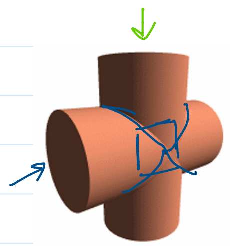
$$= \frac{4\pi}{3}$$

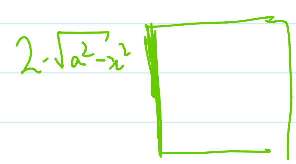
$$\text{Volume of sphere} = \frac{4}{3}\pi(2)^3 = \frac{32}{3}\pi$$

$$\text{Vol. cut out} = \frac{28}{3}\pi$$

(7) Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$.







$$\text{At any } y,\quad A(y) = \left(2\sqrt{a^2-y^2}\right)^2 = 4(a^2-y^2)$$

$$dV = A(y)\,dy$$

$$V = \int_{-a}^{a} 4(a^2-y^2)\,dy = 4\left[a^2y - \frac{y^3}{3}\right]_{-a}^{a}$$

$$= 4\left[2a^3 - \frac{2a^3}{3}\right] = \frac{16a^3}{3}$$