AI1103-Assignment 6

Name : Aayush Patel, Roll No.: CS20BTECH11001

Latex codes :

https://github.com/Aayush−2492/Assignments/tree/main/Assignment6

UGC/MATH 2018(June math set-a), Q.104

Let X and Y be two random variables satisfying $X \geq 0, Y \geq 0, E(X) = 3, Var(X) = 9, E(Y) = 2$ and $Var(Y) = 4$. Which of the following statements are correct?

A) $0 \leq Cov(X, Y) \leq 4$
B) $E(XY) \leq 3$
C) $Var(X + Y) \leq 25$
D) $E(X + Y)^2 \geq 25$

Solution

$$E(X^2) = Var(X) + (E(X))^2 = 18 \tag{0.0.1}$$

Similarly,

$$E(Y^2) = Var(Y) + (E(Y))^2 = 8 \tag{0.0.2}$$

We can use the Covariance inequality for this question,

$$(Cov(X, Y))^2 \leq Var(X)Var(Y) \tag{0.0.3}$$

The proof of this inequality is as shown,

$$Var\left(\frac{X}{\sigma_X} \pm \frac{Y}{\sigma_Y}\right) = Var\left(\frac{X}{\sigma_X}\right) + Var\left(\frac{\pm Y}{\sigma_Y}\right) + 2Cov\left(\frac{X}{\sigma_X}, \frac{\pm Y}{\sigma_Y}\right) \tag{0.0.4}$$

$$= \frac{1}{{\sigma_X}^2}Var(X) + \frac{1}{{\sigma_Y}^2}Var(Y) + 2Cov\left(\frac{X}{\sigma_X}, \frac{\pm Y}{\sigma_Y}\right) \tag{0.0.5}$$

$$= 2 \pm 2\frac{Cov(X, Y)}{\sigma_X \sigma_Y} \tag{0.0.6}$$

Since Variance is always positive,

$$Var\left(\frac{X}{\sigma_X} \pm \frac{Y}{\sigma_Y}\right) \geq 0 \tag{0.0.7}$$

$$2 \pm 2\frac{Cov(X, Y)}{\sigma_X \sigma_Y} \geq 0 \tag{0.0.8}$$

$$1 \pm 1\frac{Cov(X, Y)}{\sigma_X \sigma_Y} \geq 0 \tag{0.0.9}$$

$$|(Cov(X, Y))| \leq (\sigma_X)(\sigma_Y) \tag{0.0.10}$$

$$(Cov(X, Y))^2 \leq Var(X)Var(Y) \tag{0.0.11}$$

1) Substituting values of variance we get,

$$-6 \leq Cov(X, Y) \leq 6 \tag{0.0.12}$$

Therefore, option A is incorrect.

2) From equation (0.0.12),

$$Cov(X, Y) = E(XY) - E(X)E(Y) \tag{0.0.13}$$

$$-6 \leq E(XY) - E(X)E(Y) \leq 6 \tag{0.0.14}$$

$$0 \leq E(XY) \leq 12 \tag{0.0.15}$$

Also, if X and Y are independent,

$$E(XY) = E(X)E(Y) = 6 \tag{0.0.16}$$

Therefore, Option B is incorrect.

3) Now,

$$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y) \tag{0.0.17}$$

$$= 13 + 2Cov(X, Y) \tag{0.0.18}$$

From equation (0.0.12),

$$1 \leq Var(X + Y) \leq 25 \tag{0.0.19}$$

Therefore, Option C is correct.

4) Now,

$$E(X+Y)^2 = E(X^2) + E(Y^2) + 2E(XY) \tag{0.0.20}$$

$$E(X+Y)^2 = 26 + 2E(XY) \tag{0.0.21}$$

From equation (0.0.15),

$$26 \leq E(X+Y)^2 \leq 50 \tag{0.0.22}$$

Therefore, Option D is correct.