AI1103-Challenging problem

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Question

Prove by properties of Q-function the following inequality,

$$1 - \exp(-2\pi) \geq \left(1 - 2Q\left(\frac{1}{2}\right)\right)^2$$

Solution

Some Properties of Q function:

Lemma 0.1. $Q(x) + Q(-x) = 1$

Lemma 0.2. *Chernoff Lower Bound Property*

$$Q(x) \geq f(x)$$

where $f(x) = \alpha \exp(-\beta x^2)$ *and*
$\beta > 1, 0 < \alpha \leq \dfrac{\sqrt{2e}\sqrt{\beta - 1}}{\sqrt{\pi}\beta}$

Simplifying the inequality given in question,

$$1 - \exp(-2\pi) \geq \left(1 - 2Q\left(\frac{1}{2}\right)\right)^2 \quad (0.0.1)$$

$$1 - \exp(-2\pi) \geq 1 + 4Q^2\left(\frac{1}{2}\right)^2 - 4Q\left(\frac{1}{2}\right) \quad (0.0.2)$$

$$-\exp(-2\pi) \geq 4Q\left(\frac{1}{2}\right)\left(Q\left(\frac{1}{2}\right) - 1\right) \quad (0.0.3)$$

$$\exp(-2\pi) \leq 4Q\left(\frac{1}{2}\right)\left(1 - Q\left(\frac{1}{2}\right)\right) \quad (0.0.4)$$

$$4Q\left(\frac{1}{2}\right)\left(1 - Q\left(\frac{1}{2}\right)\right) \geq \exp(-2\pi) \quad (0.0.5)$$

Using Lemma 0.1 of Q-functions,

$$Q(x) + Q(-x) = 1 \quad (0.0.6)$$

$$4Q\left(\frac{1}{2}\right)Q\left(-\frac{1}{2}\right) \geq \exp(-2\pi) \quad (0.0.7)$$

Lemma 0.3. $4Q\left(\frac{1}{2}\right)Q\left(-\frac{1}{2}\right) \geq \exp(-2\pi)$

Proof. Using Lemma 0.2 of Q-functions and the fact that $f(x)$ is even.

$$4Q(x\sqrt{2})Q(-x\sqrt{2}) \geq \alpha^2 \exp(-2\beta x^2) \quad (0.0.8)$$

Putting $x = \dfrac{1}{2\sqrt{2}}$, we get,

$$4Q\left(\frac{1}{2}\right)Q\left(-\frac{1}{2}\right) \geq \alpha^2 \exp\left(\frac{-\beta}{4}\right) \quad (0.0.9)$$

For $\beta = 8\pi$,

$$0 < \alpha \leq \frac{\sqrt{2e}\sqrt{8\pi - 1}}{\sqrt{\pi}8\pi} \quad (0.0.10)$$

$$(0.0.11)$$

Clearly, denominator is greater than numerator, therefore, $\alpha < 1$

$$4Q\left(\frac{1}{2}\right)Q\left(-\frac{1}{2}\right) \geq \alpha^2 \exp(-2\pi) \quad (0.0.12)$$

where $\alpha < 1$

Therefore,

$$4Q\left(\frac{1}{2}\right)Q\left(-\frac{1}{2}\right) \geq \exp(-2\pi) \quad (0.0.13)$$

$\square$

Hence proved