CS5623

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HW1

DATE

1. $6x + 10y = 2 \Rightarrow 3x + 5y = 1$

$\boxed{x = 2,\ y = -1 \text{ satisfies}}$

other soln: x_0, y_0

$3(2) + 5(-1) = 1$

$3(x_0) + 5(y_0) = 1$

$3(x_0 - 2) + 5(y_0 + 1) = 0$

$5 \mid (x_0 - 2)$ & $3 \mid (y_0 + 1)$

$x_0 = 2 + 5d_1$ $\quad y_0 = 3d_2 - 1$

$\forall d_1 \in \mathbb{Z}$ $\quad \forall d_2 \in \mathbb{Z}$

$3(x_0) + 5(y_0) = 1$

$6 + 15d_1 + 15d_2 - 5 = 1$

$15(d_1 + d_2) = 0 \Rightarrow \boxed{d_1 = -d_2}$

$\therefore \boxed{(x_0, y_0) = \begin{bmatrix} -5d_2 + 2 \\ 3d_2 - 1 \end{bmatrix} \quad \forall d_2 \in \mathbb{Z}}$

2. $6x + 10y + 15z = 1$

$\boxed{(1, 1, -1) \text{ satisfies}}$

Let other be (x_0, y_0, z_0)

$6(x_0 - 1) + 10(y_0 - 1) + 15(z_0 + 1) = 0$

$2 \cdot 3 \cdot (x_0 - 1) + 2 \cdot 5 (y_0 - 1) + 3 \cdot 5 (z_0 + 1) = 0$

$5 \mid x_0 - 1$ & $3 \mid y_0 - 1$ & $2 \mid z_0 + 1$

$x_0 = 5d_1 + 1$ $\quad y_0 = 3d_2 + 1$ $\quad z_0 = 2d_3 - 1$ $\quad \forall d_1, d_2, d_3 \in \mathbb{Z}$

$6x_0 + 10y_0 + 15z_0 = 1 \Rightarrow \boxed{d_1 + d_2 + d_3 = 0}$

$$\therefore \begin{bmatrix} 5d_1+1 \\ 3d_2+1 \\ -2(d_1+d_2)-1 \end{bmatrix} \quad \forall\, d_1, d_2 \in \mathbb{Z} \text{ satisfies}$$

3. WLOG let $m \geq n$

Induction on $(m+n)$

Base Case : True for $(m,n) = (1,1)$ & $(2,1)$

Induction hypothesis: $\gcd(a^m-1, a^n-1) = a^{\gcd(m,n)}-1$ $\forall\, m+n \leq k$, $m,n \in \mathbb{N}$

Now consider m,n s.t. $m+n = k+1$

$$LHS = \gcd(a^m-1, a^n-1)$$

$$= \gcd(a^m-a^n, a^n-1) \quad [\gcd(a,b) = \gcd(a-b,b)]$$

$$= \gcd(a^n(a^{m-n}-1), a^n-1)$$

(a^n, a^n-1 can't share any prime factors for $a>1$, say $p_1, p_2, p_3 \ldots p_k \mid a^n$ then $p_1, p_2 \ldots p_k \nmid a^n-1$)

$$\therefore = \gcd(a^{m-n}-1, a^n-1)$$

$$m-n+n = m \leq k$$

By Induction Hypothesis

$$= a^{\gcd(m-n,n)}-1 = a^{\gcd(m,n)}-1 = RHS$$

Hence Proved

4. $2x + 3y + 5z = 0$

consider $\boxed{z = -1}$, $2x + 3y = 5 \Rightarrow 2(x-1) + 3(y-1) = 0$

$3 \mid x-1$ & $2 \mid y-1$

$\boxed{x = 3k+1 \Rightarrow y = -2k + 1 \quad \forall k \in \mathbb{Z}}$

$\boxed{z = -2}$, $2x + 3y = 10 \Rightarrow 2(x-2) + 3(y-2) = 0$

$\boxed{x = 3k+2 \Rightarrow y = -2k + 2 \quad \forall k \in \mathbb{Z}}$

$\therefore \forall z = -d$ when $d \in \mathbb{Z}$

$x = 3k + d$, $y = -2k + d$ is only solution

$\therefore \begin{bmatrix} 3k+d \\ -2k+d \\ -d \end{bmatrix} \quad \forall k, d \in \mathbb{Z}$ is the soln set.