Cheapest Flights Within K Stops

Aayush Adhikari

June 23, 2024

Problem Statement

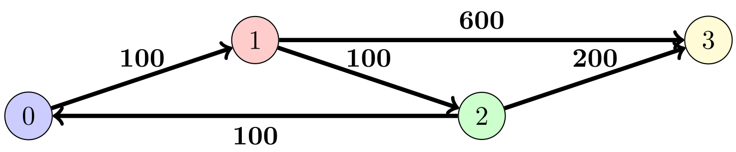
There are n cities connected by some number of flights. You are given an array flights where flights[i] = [from_i, to_i, price_i] indicates that there is a flight from city from_i to city to_i with cost price_i.

You are also given three integers src, dst, and k. Return the cheapest price from src to dst with at most k stops. If there is no such route, return -1.

Examples

Example 1

- **Input**: $n = 4$, flights = [[0,1,100],[1,2,100],[2,0,100],[1,3,600],[2,3,200]], src = 0, dst = 3, k = 1
- **Output**: 700

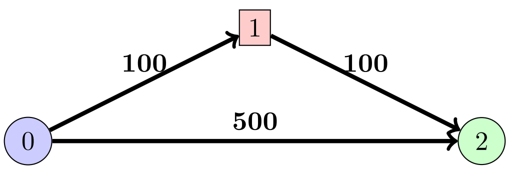


Iteration Table

Step	Current City	Path Taken	Current Cost	Current Stops
1	0	Start	0	-1
2	1	$0 \to 1$	100	0
3	2	$0 \to 2$	100	0
4	3	$0 \to 1 \to 3$	700	1

Example 2

- **Input**: $n = 3$, flights = [[0,1,100],[1,2,100],[0,2,500]], src = 0, dst = 2, k = 1
- **Output**: 200

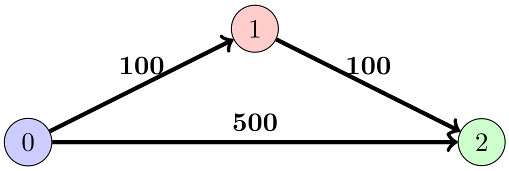


Iteration Table

Step	Current City	Path Taken	Current Cost	Current Stops
1	0	Start	0	-1
2	1	$0 \to 1$	100	0
3	2	$0 \to 2$	500	0

Example 3

- **Input**: $n = 3$, flights = [[0,1,100],[1,2,100],[0,2,500]], src = 0, dst = 2, k = 0
- **Output**: 500



Iteration Table

Step	Current City	Path Taken	Current Cost	Current Stops
1	0	Start	0	-1
2	1	$0 \to 1$	100	0
3	2	$0 \to 2$	500	0

Algorithm and Approach

To solve this problem, we can use a modified version of Dijkstra's algorithm, which is typically used to find the shortest path in terms of distance. Here, we need to consider the number of stops as an additional constraint.

Approach

To find the cheapest price from a source city to a destination city with at most K stops, we use a graph representation of flights and employ a priority queue (min-heap) for efficient path exploration. Here's the structured approach:

1. **Graph Representation**: Represent the flights using an adjacency list where each city points to its neighboring cities along with the cost of the flight.
2. **Priority Queue**: Use a priority queue to explore paths in order of increasing cost. This ensures that once we reach the destination city, it's done with the minimum possible cost.
3. **Algorithm Steps**:
   - Initialize the priority queue with the source city, a cost of 0 (starting cost), and -1 stops (initially).
   - Dequeue from the priority queue and explore each city along with its current accumulated cost and stops taken.
   - If the dequeued city is the destination city, return the accumulated cost as the cheapest price found.
   - If the number of stops taken is less than K, explore all neighboring cities:
     - Calculate the cost to reach each neighboring city by adding the cost of the flight to the current accumulated cost.
     - Enqueue these neighbors with updated costs and increment the number of stops by 1.
   - Continue this process until either the priority queue is empty (indicating no more paths to explore) or the destination city is reached.
4. **Termination**: If no path is found within K stops, return -1 indicating that no valid route exists.

Algorithm

Input: Number of cities n, flights represented as flights, source city src, destination city dst, maximum stops k
Output: Cheapest price from src to dst with at most k stops, or -1 if no such route exists

```
Function FindCheapestPrice(flights, src, dst, k):
    Initialize a graph represented as an adjacency list;
    Initialize a priority queue to store the current path cost and city;
    Enqueue (src, 0, −1) into the priority queue ;   // Start with source city, cost 0, and -1 stops
    while priority queue is not empty do
        Dequeue (city, cost, stops) from the priority queue;
        if city is dst then
            return cost;
        end
        if stops < k then
            for each neighbor of city in the graph do
                Calculate newCost as cost + price of flight;
                Enqueue (neighbor, newCost, stops + 1) into the priority queue;
            end
        end
    end
    return −1 ;                                        // No valid path found
```

Algorithm 1: Find Cheapest Price with at most K Stops

Solution Code

```python
class Solution:
    def findCheapestPrice(self, n: int, flights: List[List[int]], src: int, dst: int, k: int) -> int:
        graph = defaultdict(list)
        for u, v, price in flights:
            graph[u].append((v, price))

        pq = [(0, src, 0)]
        min_cost = defaultdict(lambda: float('inf'))
        min_cost[(src, 0)] = 0

        while pq:
            cost, current_city, stops = heappop(pq)

            if current_city == dst:
                return cost

            if stops <= k:
                for neighbor, price in graph[current_city]:
                    new_cost = cost + price
                    if new_cost < min_cost[(neighbor, stops + 1)]:
                        min_cost[(neighbor, stops + 1)] = new_cost
                        heappush(pq, (new_cost, neighbor, stops + 1))

        return -1
```

Bellman-Ford Algorithm for Cheapest Flights with K Stops

Problem Statement

There are n cities connected by some number of flights. You are given an array flights where flights[i] = [from_i, to_i, price_i] indicates that there is a flight from city from_i to city to_i with cost price_i.
You are also given three integers src, dst, and k. Return the cheapest price from src to dst with at most k stops. If there is no such route, return -1.

Approach

We use the Bellman-Ford algorithm to find the shortest paths from the source city src to all other cities with at most k stops. The algorithm iteratively relaxes all edges $n-1$ times, where n is the number of cities, to guarantee finding the shortest path in a graph with negative weights.

Algorithm

1. Initialize an array dist with size n to store the minimum cost to reach each city from src. Initialize dist[src] to 0 and all other entries to ∞.
2. Relax all edges $n-1$ times:
   - For each flight $[u, v, w]$ in flights, if dist[u] + w < dist[v], update dist[v] to dist[u] + w.
3. After $n-1$ iterations, dist[dst] contains the minimum cost to reach dst from src with at most k stops, or -1 if no such path exists.

Source Code

```python
class Solution:
    def findCheapestPrice(self, n: int, flights: List[List[int]], src: int, dst: int, k: int) -> int:
        # Step 1: Initialize distances array with infinity and set source distance to 0
        inf = float('inf')
        dist = [inf] * n
        dist[src] = 0

        # Step 2: Relax edges for k + 1 times
        for _ in range(k + 1):
            # Create a copy of dist array for the current iteration
            current_dist = dist[:]

            # Relax all edges (u, v, w)
            for u, v, w in flights:
                if dist[u] != inf and dist[u] + w < current_dist[v]:
                    current_dist[v] = dist[u] + w

            # Update dist array for the next iteration
            dist = current_dist

        # Step 3: Return the shortest distance to dst, or -1 if not reachable
        return dist[dst] if dist[dst] != inf else -1
```

Iteration Table for Example 1

Iteration	State of dist[0]	State of dist[1]	State of dist[2]	State of dist[3]
0	0	∞	∞	∞
1	0	100	∞	600
2	0	100	200	700

Iteration Table for Example 2

Iteration	**State of** dist[**0**]	**State of** dist[**1**]	**State of** dist[**2**]
0	0	∞	∞
1	0	100	∞
2	0	100	600

Iteration Table for Example 3

Iteration	**State of** dist[**0**]	**State of** dist[**1**]	**State of** dist[**2**]
0	0	∞	∞
1	0	100	∞
2	0	100	∞
3	0	100	600