A Mathematical Introduction to Nash Equilibrium and Its Existence

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Abstract

John Nash *(1928-2015)* was an American mathematician known for his contributions to game theory and the concept of Nash equilibrium. Although Game Theory is heavily utilized in economics, its foundations are quite mathematical. This investigation assignment provides a mathematical introduction to Nash Equilibrium and its existence. The Kakutani Fixed Point theorem is used to prove its existence.

1 Introduction

A *game* in game theory is defined by a set of players, a set of actions or strategies available to each player, and a payoff function that assigns a payoff to each player for each possible combination of actions.

Formally, let $N = 1, 2, \ldots, n$ be the set of players. For each player $i \in N$, let A_i be the set of possible actions or strategies available to player i. Let $A = A_1 \times A_2 \times \cdots \times A_n$ be the set of possible strategy profiles, where $a = (a_1, a_2, \ldots, a_n)$ is an element of A and a_i is the strategy chosen by player i.

For each player i, let $u_i : A \to \mathbb{R}$ be the payoff function, where $u_i(a)$ is the payoff that player i receives when the strategy profile is $a = (a_1, a_2, \ldots, a_n)$. a is the "outcome" of the game and the payoff is the "utility gained" from the outcome. The payoff function represents the preferences of each player over the set of possible outcomes. a_i represents the *pure strategy* of player i, $a = (a_1, a_2, \ldots, a_n)$ is *strategy profile* of n players and $A_i = \{a_i^1, a_i^2, \ldots, a_i^m\}$ represents *strategy space* for player i. $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ is the strategy profile for $n-1$ players. We write $a = (a_i, a_{-i})$ whenever it is convenient.

Defination 1 (Normal-form game) A game is defined as the tuple $G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$, where N is the set of players, A_i is the set of possible strategy for player i, and u_i is the payoff function for player i.

		Player G	
		$SE(q)$	$AI(1-q)$
Player M	$SE(p)$	$(3,3)$	$(1,5)$
	$AI(1-p)$	$(5,1)$	$(0,0)$

Table 1: Example game

Here is a simple example of a game. Suppose Microsoft and Google have choices to allocate their resources to Search Engines or Artificial Intelligence. In Table 1, M and G are players, hence N=2. Let M be player 1 and G be player 2. A_1, or strategy space of M is $\{SE, AI\}$. The top left cell contains $a = (SE, SE)$, meaning both players have allocated their resources to the search engine, which is one of the strategy profiles, where $a_1 = SE$ and $a_2 = SE$ are pure strategies. The set of all strategy profile, A = $\{(SE, SE), (SE, AI), (AI, SE), (AI, AI)\}$. Each of the strategy profiles corresponds to a cell in the table. For player M, the payoff function for the top left cell is $u_1(SE, SE) = 3$. $a_1 = SE$ is an example of pure strategy played by player 1 and $A_1 = \{SE, AI\}$ is the pure strategy profile of player 1.

2 Nash Equilibrium

Nash equilibrium is a concept in game theory that describes a situation in which each player in a game chooses a strategy that is optimal given the strategies of the other players. In other words, at a Nash equilibrium, no player has an incentive to unilaterally change their strategy, given the other players' strategies. There are two kinds of Nash equilibrium in a game.

1. Pure Strategy Nash Equilibrium

2. Mixed Strategy Nash Equilibrium

A pure strategy Nash equilibrium is a set of strategies, one for each player, such that each player's strategy is the best response to the strategies chosen by all the other players. In our example game, there exists two pure strategy Nash Equilibrium, namely (SE, AI) and (AI, SE). Neither companies have the incentive to switch, meaning if Microsoft was working on Search Engine and Google was aware of it then the highest payoff Google will have is by working on Artificial Intelligence and vice versa. And similarly, if Microsoft was working on Artificial Intelligence and Google was aware of it then the highest payoff Google will have is by working on Search Engine and vice versa.

A mixed strategy Nash equilibrium is a set of probability distributions over the possible actions of each player in a game, where each player's strategy is optimal given the other players' strategies. Our example game has a mixed strategy Nash equilibrium, namely for $p = \frac{1}{3}$ and $q = \frac{1}{3}$. We shall return to the calculations later but what the values are implying is if Microsoft were to allocate their resources to Search Engine $p = \frac{1}{3}$ and Artificial Intelligence $(1-p) = \frac{2}{3}$ of the time then Google would be indifferent of their own choices and vice versa.

Nash's Theorem (1950) Every finite normal form game has at least one Nash Equilibrium in mixed strategies.[5]

The theorem states that every finite game with a finite number of players has at least one Nash equilibrium in mixed strategies. This means that even if a game has no pure strategy Nash equilibrium, it will still have a Nash equilibrium in which players randomize their actions according to some probability distribution over their pure strategies.

Therefore, Nash's Theorem implies that any game that has a pure strategy Nash equilibrium will also have at least one mixed strategy Nash equilibrium. In other words, if a game has a pure strategy Nash equilibrium, then that Nash equilibrium can be found as a special case of the mixed strategy Nash equilibrium.

Defination 2 (Mixed Strategy) Let $G = (N, (A_i)_{i\in N}, (u_i)_{i\in N})$ be a normal form game and for any set X let $\prod(X)$ be set of all probability distributions over X. Then the set of mixed strategies for player i is $\sigma_i = \prod(A_i)$.

In other words, mixed strategy for player i is $\sigma_i = (\sigma_i(a_i^1), \sigma_i(a_i^2), \ldots, \sigma_i(a_i^m))$ where each σ represents the probability of playing each strategy of the game and $\sum_j \sigma(a_i^j) = 1$. Note also that a pure strategy can be expressed as a mixed strategy that places probability 1 on a single pure strategy and probability 0 on each of the other pure strategies. In our example, the mixed strategy for player M is $\sigma_1 = \{p, 1-p\}$ which implies player M chooses option SE and AI with probability p and $(1-p)$ respectively.

Definition 3 (Mixed strategy profile) The set of mixed strategy profiles is simply the Cartesian product of the individual mixed strategy sets $\sigma_1 \times \cdots \times \sigma_n$.

In our example, the mixed strategy profile $\sigma = \{(p, 1-p), (q, 1-q)\}$. $\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n)$ is the mixed strategy profile of other n-1 players. Thus we write $\sigma = (\sigma_i, \sigma_{-i})$ whenever convenient.

Definition 4 (Support) The support of a mixed strategy σ_i for a player i is the set of pure strategies $\{a_i | \sigma_i(a_i) > 0\}$.

$\sigma_i(a_i)$ denotes the probability that a strategy a_i will be played under mixed strategy σ_i. This gives arise to support, i.e the support of a mixed strategy is the set of pure strategies that are played with non-zero probability under the mixed strategy. We may note that a pure strategy is a special case of a mixed strategy, in which the support is a single action.

Definition 5 (The set ΔS_i) The set ΔS_i represents the set of all the possible mixed strategies available to player i i.e the set of all probability distributions over the pure strategies of player i. It is an infinite set because each distribution contains values in the range [0,1]. We write ΔS_{-i} whenever convenient to represent the distribution of players other than player i. Furthermore, we use '$\sum$' to represent $\prod \Delta S_i$, which is the product of all players' possible distribution.

Definition 6 (Expected Payoff in mixed strategy) Given a normal form game $G =$

$(N, (A_i)_{i \in N}, (u_i)_{i \in N})$, the expected playoff of player i when he chooses $a_i \in A_i$ and his oppenents play the mixed strategy $\sigma_{-i} \in \Delta S_{-i}$ is

$$u_i(a_i, \sigma_{-i}) = \sum_{a_{-i} \in A_{-i}} \sigma_{-i}(a_{-i}).u_i(a_i, a_{-i}) \tag{1}$$

Similarly, the expected payoff of player i when he chooses $\sigma_i \in \Delta S_i$ and his opponents play the mixed strategy $\sigma_{-i} \in \Delta S_{-i}$ is

$$u_1(\sigma_i, \sigma_{-i}) = \sum_{a_i \in A_i} \sigma_i(a_i).u_i(a_i, \sigma_{-i}) = \sum_{a_i \in A_i} \left(\sum_{a_{-i} \in A_{-i}} \sigma_i(a_i).\sigma_{-i}(a_{-i}).u_i(a_i, a_{-i}) \right) \tag{2}$$

We had previously talked about payoff in pure strategy in order to define a normal form game. Now have extended the definition for a mixed strategy. In other words, $u_i(\sigma_i, \sigma_{-i})$ is the expected payoff function of the mixed strategy profile played by n players in the game. The outcome of the game is now random. Now we will look at games from an individual agent's point of view, rather than from the vantage point of an outside observer.

		Player G		
		$SE(q)$	$AI(1-q)$	M's Expected Payoff
Player M	$SE(p)$	$(3,3)$	$(1,5)$	$3q + (1-q)1 = 2q + 1$
	$AI(1-p)$	$(5,1)$	$(0,0)$	$5q$
	G's Expected Payoff	$3p + 1(1-p) = 2p + 1$	$5p$	

Table 2: Example game with Expected Payoff

The table above shows the expected payoffs of both players in our example. For example, Microsoft's expected payoff when they choose strategy SE is

$$\begin{aligned} u_i(a_1, \sigma_{-1}) &= 3q + 1(1-q) \\ &= 1 + 2q \end{aligned}$$

Similarly, Microsoft's Expected payoff in mixed strategy is:

$$\begin{aligned} u_i(\sigma_1, \sigma_{-1}) &= (2q+1)p + 5q(1-p) \\ &= 2pq + p + 5q - 5pq \\ &= p + 5q - 3pq \end{aligned}$$

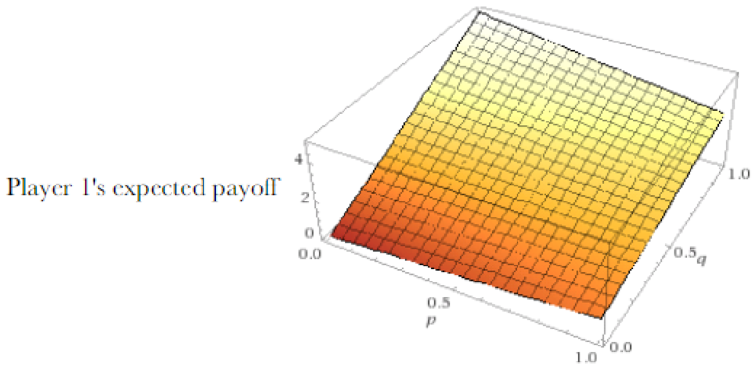

Figure 1: Graph of Microsoft's Expected Payoff in mixed strategy

Definition 7 (Best Response) For player i, a strategy σ_i is a best response to the strategy profile σ_{-i} if $u_i(\sigma_i, \sigma_{-i}) \geq u_i(a_i^{'}, \sigma_{-i})$ for all $a_i^{'} \in A_i$.

The best response is a multi-valued function that maps a mixed strategy σ_i to one or more mixed strategies, meaning, σ_i may not be the only best response to σ_{-i}. We will call $BR_i(\sigma_{-i})$ the set of best responses for player i and note that $\sigma_i \in BR_i(\sigma_{-i})$.

In our example, for Microsoft, mixed strategy $\sigma_1 = (p = 0, 1-p = 1)$ is the best response when $\sigma_{-1} = (q = 1, 1-q = 0)$ which means choosing AI is Microsoft's best response when Google chooses the strategy SE. In fact, any p is Microsoft's BR to $q = \frac{1}{3}$, because when Google chooses SE with probability $\frac{1}{3}$, Microsoft becomes indifferent between its two options. Therefore $BR_1(\sigma_{-1})$ becomes an infinite set when $\sigma_{-1} = (q = \frac{1}{3}, 1-q = \frac{2}{3})$.

Definition 8 (Nash Equilibrium in mixed strategy) A mixed strategy profile $\sigma^* = \{\sigma_1^*, \ldots, \sigma_n^*\}$ is a Nash Equilibrium if for each player σ_i^* is a best response to σ_{-i}^*. That is for all $i \in N$,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*) \forall \sigma_i \in \Delta S_i \tag{3}$$

In other words, a mixed strategy profile is a Nash Equilibrium if each player's strategy is a best response to the strategy profile played by the other players in the game. I.e, σ^* is a Nash Equilibrium if $\sigma_i^* \in BR_i(\sigma_{-i})$ for all players i.

In our example $\sigma = ((p = \frac{1}{3}, 1 - p = \frac{2}{3}), (q = \frac{1}{3}, 1 - q = \frac{2}{3})$ is a Nash Equilibrium. Because when Microsoft chooses SE and AI with $(p = \frac{1}{3}, 1 - p = \frac{2}{3})$ respectively and Google chooses SE and AI with $(q = \frac{1}{3}, 1 - q = \frac{2}{3})$, both Microsoft and Google become indifferent of what strategy the other player chooses, meaning they will have same expected payoff playing either strategy.

3 Existance of Nash Equilibrium

We know that Best Response(BR) is a multi-valued function $BR : \sum \rightrightarrows \sum$ such that for all $\sigma \in \sum$, we have $BR(\sigma) = [BR_i(\sigma_{-i})]$. The existence of Nash equilibrium is then equivalent to the existence of a mixed strategy σ such that $\sigma \in BR(\sigma)$: i.e, the **existence of a fixed point of the mapping B**.

3.1 More definitions and theorem

- A set in a Euclidean space is compact if and only if it is bounded and closed.
- A set S is convex if for any $x, y \in S$ and any $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y \in S$.
- (Weierstrass Theorem) Let A be a nonempty compact subset of a finite dimensional Euclidean space and let $f : A \to R$ be a continuous function. Then there exists an optimal solution to the optimization problem $min_{x \in A} f(x)$.

3.2 Kakutani's Fixed Point Theorem

Let A be a non-empty subset of a finite dimensional Euclidean space. Let $f : A \rightrightarrows A$ be a correspondence, with $x \in A \mapsto f(x) \subseteq A$, satisfying the following conditions:

1. A is a compact and convex set.
2. $f(x)$ is a non-empty for all $x \in A$

3. $f(x)$ is a convex-valued correspondence: for all $x \in A, f(x)$ is a convex set

4. $f(x)$ has a closed graph: that is, if $\{x^n, y^n\} \longrightarrow \{x, y\}$ with $y^n \in f(x^n)$, then $y \in f(x)$.

Then, f has a fixed point, that is, there exists some $x \in A$, such that $x \in f(x)$ [2].

The idea is to apply Kakutani's theorem to the best response correspondence $B : \sum \rightrightarrows \sum$.

We will show that $B(\sigma)$ satisfies the conditions of Kakutani's theorem.

(1) $\sum$ is compact, convex and non-empty. By definition

$$\sum = \prod_{\forall i} \Delta S_i \tag{4}$$

where each $\Delta S_i = \{x | \sum_j x_j = 1\}$ is a *simplex* of dimension $|A_i| - 1$, thus each ΔS_i is closed and bounded, and thus compact[4]. Their product set is also compact.

Let's think of it in terms of our example. $\sum$ is a closed set because $\sum$ contains its boundaries, or its limiting points, which is the definition of a closed set. See the black boundary of $\sum$ in figure 2. And $\sum$ is a bounded set because any probability p has a lower bound of 0 and an upper bound of 1.

Lastly, $\sum$ is convex because if we take any two points $\sigma_1, \sigma_2 \in \sum$ and connect σ_1 and σ_2 by a linear line, then each point on the linear line is also in $\sum$. Thus fulfills the definition of convexity. This can also be verified in figure 2.

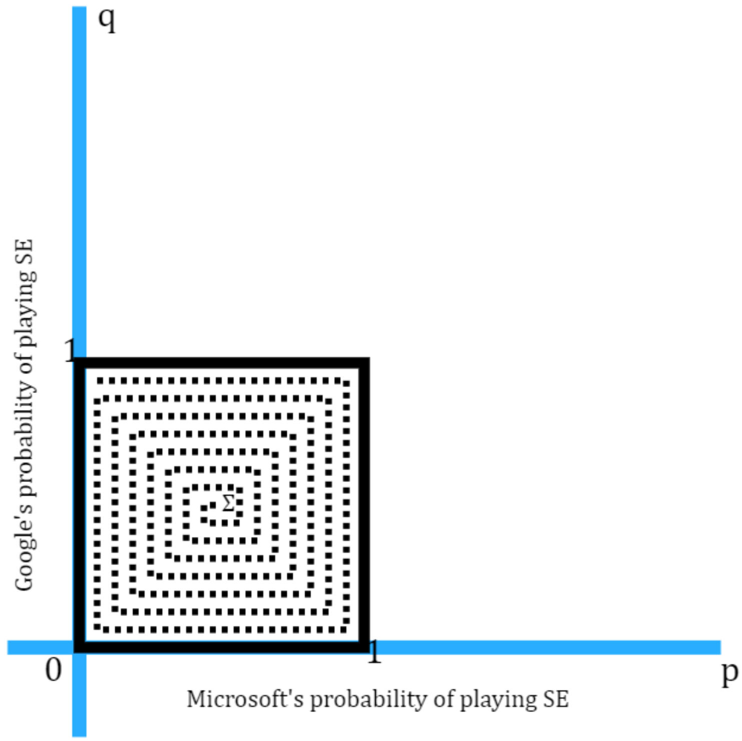

Figure 2: 2D mixed strategy space of the SE-AI game

(2) $BR(\sigma)$ is non empty for any mixed strategy σ.

By definition,

$$BR_i(\sigma_{-i}) = \arg\max_{x \in \Delta S_i} u_i(x, \sigma_{-i}) \tag{5}$$

where σ_i is non-empty and compact, and u_i is linear in x. Hence, u_i is continuous, and by Weierstrass's theorem $BR(\sigma)$ is non-empty.

In our example, this is true because every player has a best response to the other players' strategies-whatever those strategies are.

(3) $BR(\sigma)$ is a convex-valued correspondence.

Equivalently, $BR(\sigma) \subset \sum$ is convex if and only if $BR_i(\sigma_{-i})$ is convex for all i. Let $\sigma_i^{'}, \sigma_i^{''} \in BR_i(\sigma_{-i})$.

Then, for all $\lambda \in [0,1] \in BR_i(\sigma_{-i})$, we have

$$u_i(\sigma_i^{'}, \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}) \qquad \forall \tau_i \in \Delta S_i$$
$$u_i(\sigma_i^{''}, \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}) \qquad \forall \tau_i \in \Delta S_i$$

The relations above imply that $\forall \lambda \in [0,1]$, we have

$$\lambda u_i(\sigma_i^{'}, \sigma_{-i}) + (1-\lambda) u_i(\sigma_i^{''}, \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}) \quad \forall \tau_i \in \Delta S_i$$

And by linearity of u_i,

$$u_i(\lambda \sigma_i^{'} + (1-\lambda)\sigma_i^{''}, \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}) \quad \forall \tau_i \in \Delta S_i$$

Therefore, $\lambda \sigma_i^{'} + (1-\lambda)\sigma_i^{''} \in BR_i(\sigma_{-i})$, showing that $BR(\sigma)$ is convex valued.

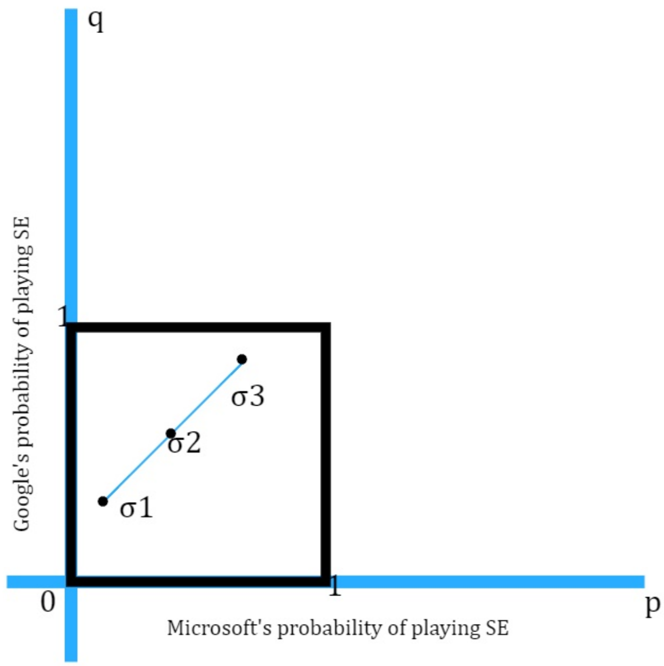

Figure 3: 2D mixed strategy space of the SE-AI game

In order to see this in our example, let σ_1 and σ_2 be any two mixed strategies in $BR(\sigma)$. In other words, σ_1 and σ_2 are two points in $\sum$ with additional property that in σ_1 and σ_2 everyone plays their best response to σ. We will show that if we connect σ_1 and σ_2 with a line, then every point on that line is also the best response to σ, which implies that $BR(x)$ is convex. Let $\sigma_3 = \lambda\sigma_1 + (1 - \lambda)\sigma_2$ be a point on that line, for some λ between 0 and 1, both inclusive. Thus, by definition of expected payoff in mixed strategy, every player gets the same payoff in σ_3 as in either σ_1 and σ_2. Example, Microsoft's expected payoff in $\sigma_3 = \lambda \times$ its expected payoff in σ_1 + $(1 - \lambda) \times$ its expected payoff in $\sigma_2 = (\lambda + 1 - \lambda) \times$ its expected payoff in either σ_1 or σ_2.

(4) $BR(\sigma)$ has a closed graph.

Proof: Suppose on the contrary that $BR(\sigma)$ does not have a closed graph. Then, there exists a sequence $(\sigma^n, \tilde{\sigma}^n) \longrightarrow (\sigma, \tilde{\sigma})$ with $\tilde{\sigma}^n \in BR(\sigma^n)$, but $\tilde{\sigma} \notin BR(\sigma)$, i.e., there exists some i such that $\tilde{\sigma}_i \notin BR_i(\sigma_{-i})$.

This implies that there exists some $\sigma_i^{'} \in \Delta S_i$ and some $\epsilon > 0$ such that

$$u_i(\sigma_i^{'}, \sigma_{-i}) > u_i(\tilde{\sigma}_i, \sigma_{-i}) + 3\epsilon \tag{6}$$

By continuity of u_i and the fact that $\sigma_{-i}^n \longrightarrow \sigma_{-i}$, we have for sufficiently large n,

$$u_i(\sigma_i^{'}, \sigma_{-i}^n) \geq u_i(\sigma_i^{'}, \sigma_{-i}) - \epsilon \tag{7}$$

Combining 6 and 7, we obtain

$$u_i(\sigma_i^{'}, \sigma_{-i}^n) > u_i(\tilde{\sigma}_i, \sigma_{-i}) + 2\epsilon \geq u_i(\tilde{\sigma}_i^n, \tilde{\sigma}_{-i}^n) + \epsilon \tag{8}$$

where the second relation follows from the continuity of u_i. This contradicts the assumption that $\tilde{\sigma}_i^n \in BR_i(\sigma_{-i}^n)$. Thus, proved.

4 Conclusion

We have thus proved the existence of the Nash Equilibrium in mixed strategy using the fixed point theorem. This provides a powerful tool for analyzing the strategic interactions of players in a game and understanding the equilibrium outcomes that may arise. It is worth noting that the Nash's existence theorem also holds for pure strategy existence in finite games, where players have a finite number of actions to choose from. However, in the case of infinite games, the standard proof of the theorem does not apply due to the absence of compactness. Instead, a similar theorem can be extended to address the existence of Nash equilibrium in infinite games, as shown by Sun in his 2015 paper on pure strategy existence[7]. Throughout our investigation, we have covered several important topics in mathematical analysis, including the concept of continuity, boundedness, and closedness of functions. These concepts are crucial for applying the fixed point theorem and proving the existence of Nash equilibrium. Overall, our investigation has demonstrated the power of mathematical analysis and the fixed point theorem in understanding strategic interactions and identifying the equilibrium outcomes that may arise. As such, our findings have important implications for a wide range of fields, from economics and political science to computer science and biology

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