Generalization of Fibonacci Sequences

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December 5, 2022

Abstract

The Fibonacci Sequence is a well known sequence with many unique occurrences, including the number of petals flowers have, the shape of hurricanes with the Fibonacci Spiral, and in the proportions of artwork. This report focuses on generalizing the famous Fibonacci Sequence by using two arbitrary starting values and examines the general consequences, including the ratios of consecutive terms, a recursive relationship between consecutive ratios, and the limit the ratios approach.

1 Generalized Fibonacci Sequence

Definition 1.1. The Fibonacci sequence is defined recursively as $F_1 = 1, F_2 = 2$ and for $n \geq 3$

$$F_n = F_{n-1} + F_{n-2}$$

Thus, $\{F_n\} = \{1, 2, 3, 5, 8, ..\}$

Let us now consider a different sequence $\{a_n\}$ where the starting values are arbitrary.

Definition 1.2. Let a_1 and a_2 be arbitrary real values with $a_1 \leq a_2$. We define $\{a_n\}$ by the same recursive formula as the Fibonacci sequence, namely for $n \geq 3$,

$$a_n = a_{n-1} + a_{n-2}$$

For example, we have $a_3 = a_2 + a_1, a_4 = a_3 + a_2 = 2a_2 + a_1$, etc. Note that we cannot compute actual values of a_n since a_1 and a_2 are assumed to be arbitrary.

Next, we define a sequence $\{r_n\}$ by considering the ratios of consecutive terms of $\{a_n\}$

Definition 1.3. Let $\{a_n\}$ be defined previously above. Define the sequence of ratios $\{r_n\}$ by

$$r_n = \frac{a_{n+1}}{a_n}$$

For example we have $r_1 = \frac{a_2}{a_1}, r_2 = \frac{a_3}{a_2}$, etc

Definition 1.4 A sequence is $\{a_n\}$ is called a Cauchy sequence if for all $\epsilon > 0$, there exists an integer $N > 0$ such that any $m, n \geq N$ implies $|a_m - a_n|$ ¡ϵ.

1.1 Some proofs

1. Find and prove a formula for a_n in terms of the initial values a_1, a_2 and the Fibonacci numbers F_n.

   Proof We know F_n=$\{1, 2, 3, 5, 8, 13, 21, 34, ...\}$, where $a_1 = 1$ and $a_2 = 2$

   A table listing first few terms of F_n with their formula in terms of a_1 and a_2

F_n	x	a_n
F_1	1	a_1
F_2	2	a_2
F_3	3	$a_1 + a_2$
F_4	5	$a_1 + 2a_2$
F_5	8	$2a_1 + 3a_2$
F_6	13	$3a_1 + 5a_2$
F_7	21	$5a_1 + 8a_2$

We notice that $a_n = F_{n-3}a_1 + F_{n-2}a_2$ for $n \geq 4$.

We prove that using induction.

(a) Base property ($n = 4$): Since $a_1 = 1$ and $a_2 = 2$, $a_4 = F_1 a_1 + F_2 a_2 = 1 * 1 + 2 * 2 = 5$

(b) Inductive property: Assume that $a_n = F_{n-3}a_1 + F_{n-2}a_2$ holds for some $n = k$ (inductive assumption). It suffices to show that $a_{k+1} = F_{k-2}a_1 + F_{k-1}a_2$. It follow from a_k that

$$a_k = F_{k-3}a_1 + F_{k-2}a_2$$

$$\implies a_k + F_{k-4}a_1 + F_{k-3}a_2 = F_{k-3}a_1 + F_{k-2}a_2 + F_{k-4}a_1 + F_{k-3}a_2$$

Add $(F_{k-4}a_1 + F_{k-3}a_2)$ to Both Sides

$$\implies a_k + a_{k-1} = a_1(F_{k-3} + F_{k-4}) + a_2(F_{k-2} + F_{k-3})$$

Because $(a_{k-1} = F_{k-4}a_1 + F_{k-3}a_2)$

$$\implies a_k + a_{k-1} = F_{k-2}a_1 + F_{k-1}a_2$$

Because $(F_{k-2} = F_{k-3} + F_{k-4}$ and $F_{k-1} = F_{k-2} + F_{k-3})$

$$\implies a_{k+1} = F_{k-2}a_1 + F_{k-1}a_2 \qquad \text{Because } (a_{k+1} = a_k + a_{k-1}) \tag{1}$$

Hence, from induction we conclude that $a_n = F_{n-3}a_1 + F_{n-2}a_2$ holds for all $n \geq 4$.

2. Find and prove a recursive formula for r_n.

Proof We know,

$$\begin{aligned} r_n &= \frac{a_{n+1}}{a_n} \\ &= \frac{a_n + a_{n-1}}{a_n} \\ &= \frac{a_n}{a_n} + \frac{a_{n-1}}{a_n} \\ &= 1 + \frac{1}{r_{n-1}} \qquad \text{Because } r_{n-1} = \frac{a_n}{a_{n-1}} \end{aligned} \tag{2}$$

Hence, $r_n = 1 + \frac{1}{r_{n-1}}$ with $r_1 = \frac{a_2}{a_1}$.

3. Prove that r_n is bounded, namely $\frac{3}{2} \leq r_n \leq 2$ for all $n \geq 3$.

Proof : We use induction to prove that r_n is bounded.

(a) Base property ($n = 3$):

$$\begin{aligned}
r_3 &= \frac{a_4}{a_3} \\
&= \frac{a_3 + a_2}{a_3} \\
&= \frac{2a_2 + a_1}{a_2 + a_1} && \text{(Because } a_3 = a_2 + a_1\text{)} \\
&\leq \frac{2a_2 + 2a_1}{a_2 + a_1} && \text{(Because } a_1 \geq 0\text{)} \\
&\leq 2
\end{aligned}$$

$$\begin{aligned}
r_3 &= \frac{a_4}{a_3} \\
&= \frac{a_3 + a_2}{a_3} \\
&= \frac{2a_2 + a_1}{a_2 + a_1} && \text{(Because } a_3 = a_2 + a_1\text{)} \\
&\geq \frac{2a_1 + a_1}{a_1 + a_1} && \text{(Because } a_2 \geq 0\text{)} \\
&\geq \frac{3a_1}{2a_1} \\
&\geq \frac{3}{2}
\end{aligned}$$

Hence, $\frac{3}{2} \leq r_3 \leq 2$.

(b) Inductive property: Assume that $\frac{3}{2} \leq r_k \leq 2$ holds for some $n = k$ (inductive assumption). It suffices to prove that $\frac{3}{2} \leq r_{k+1} \leq 2$ for $n = k + 1$. This follows from the chain of implications

$$\begin{aligned}
&\Longrightarrow \frac{3}{2} \leq r_k \leq 2 && \text{(inductive assumption)} \\
&\Longrightarrow \frac{2}{3} \geq \frac{1}{r_k} \geq \frac{1}{2} && \text{(reciprocal both sides)} \\
&\Longrightarrow \frac{1}{2} \leq \frac{1}{r_k} \leq \frac{2}{3} \\
&\Longrightarrow \frac{1}{2} + 1 \leq 1 + \frac{1}{r_k} \leq \frac{2}{3} + 1 && \text{(add 1 to all sides)} \\
&\Longrightarrow \frac{3}{2} \leq r_{k+1} \leq \frac{5}{3} \\
&\Longrightarrow \frac{3}{2} \leq r_{k+1} \leq 2 && \text{(by transitivity since } 2 \leq 5/3\text{)}
\end{aligned}$$

Hence, proved.

4. Prove that for each $n \geq 4$, $|r_{n+1} - r_n| \leq \left(\frac{2}{3}\right)^2 |r_{n-1} - r_n|$

$$
\begin{aligned}
|r_{n+1} - r_n| &= \left| \left(1 + \frac{1}{r_n}\right) - \left(1 + \frac{1}{r_{n-1}}\right) \right| && \text{(From 2)} \\
&= \left| \frac{1}{r_n} - \frac{1}{r_{n-1}} \right| \\
&= \left| \frac{r_{n-1} - r_n}{r_n r_{n-1}} \right| \\
&\leq \left| \frac{r_{n-1} - r_n}{\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)} \right| && \text{(Because } r_n \geq \frac{3}{2}, r_{n-1} \geq \frac{3}{2}\text{)} \\
&\leq \left(\frac{2}{3}\right)^2 |r_{n-1} - r_n| && (3)
\end{aligned}
$$

Hence, proved.

5. Prove that for each $n \geq 3$, $|r_{n+1} - r_n| \leq \left(\frac{2}{3}\right)^{2(n-3)} |r_4 - r_3|$.
   We prove

   (a) Base property ($n = 3$):

   $$
   \begin{aligned}
   |r_4 - r_3| &\leq \left(\frac{2}{3}\right)^{2(3-3)} |r_4 - r_3| \\
   &\leq |r_4 - r_3| && \text{(True)}
   \end{aligned}
   $$

   (b) Inductive property: Assume that $|r_{k+1} - r_k| \leq \left(\frac{2}{3}\right)^{2(k-3)} |r_4 - r_3|$ holds for some $n = k$ (inductive assumption). It suffices to prove that $|r_{k+2} - r_{k+1}| \leq \left(\frac{2}{3}\right)^{2(k-2)} |r_4 - r_3|$. This follows from the chain of implications

   $$
   \begin{aligned}
   &\Longrightarrow |r_{k+1} - r_k| \leq \left(\frac{2}{3}\right)^{2(k-3)} |r_4 - r_3| && \text{(inductive assumption)} \\
   &\Longrightarrow |r_{k+1} - r_k| \leq \left(\frac{2}{3}\right)^{2k-6} |r_4 - r_3| \\
   &\Longrightarrow |r_{k+1} - r_k| \left(\frac{2}{3}\right)^2 \leq \left(\frac{2}{3}\right)^{2k-6} \left(\frac{2}{3}\right)^2 |r_4 - r_3| && \text{(Multiply both sides by } \left(\frac{2}{3}\right)^2 \text{)} \\
   &\Longrightarrow |r_{k+1} - r_k| \left(\frac{2}{3}\right)^2 \leq \left(\frac{2}{3}\right)^{2k-4} |r_4 - r_3| \\
   &\Longrightarrow |r_{k+2} - r_{k+1}| \leq \left(\frac{2}{3}\right)^{2(k-2)} |r_4 - r_3| && \text{(by transitivity from 3)}
   \end{aligned}
   $$

   Hence, proved.

6. Prove that $\{r_n\}$ is a Cauchy sequence using $\epsilon - N$ argument.

   Let $\epsilon > 0$ be arbitrarily given. Choose any $N > \frac{\frac{\ln(\epsilon/2|r_4-r_3|)}{\ln(2/3)}+6}{2}$.Assume without loss of generality that $m \geq n$. Then for any $m, n \geq N$, we have ,

$$
\begin{aligned}
|r_m - r_n| &\leq |r_m - r_{m-1}| + |r_{m-1} - r_{m-2}| + \dots \\
&\quad \dots + |r_{n+2} - r_{n+1}| + |r_{n+1} - r_n| \qquad \text{(Triangle Inequality)} \\
&\leq \left(\frac{2}{3}\right)^{2m-8} |r_4 - r_3| + \left(\frac{2}{3}\right)^{2m-10} |r_4 - r_3| + \dots \\
&\quad \dots + \left(\frac{2}{3}\right)^{2n-4} |r_4 - r_3| + \left(\frac{2}{3}\right)^{2n-6} |r_4 - r_3| \\
&\leq \left(\frac{2}{3}\right)^{2n-6} |r_4 - r_3| \left\{ \left(\frac{2}{3}\right)^{2m-2n-2} + \left(\frac{2}{3}\right)^{2m-2n-4} + \dots + \left(\frac{2}{3}\right)^{2} + \left(\frac{2}{3}\right)^{1} \right\} \\
&\leq 2\left(\frac{2}{3}\right)^{2n-6} |r_4 - r_3| \qquad \text{(Because of } \textbf{Result 1}\text{)} \\
&\leq 2\left(\frac{2}{3}\right)^{2N-6} |r_4 - r_3| \qquad \text{(Since } n \geq N\text{)} \\
&< 2\left(\frac{2}{3}\right)^{2\left(\frac{\frac{\ln(\epsilon/2|r_4-r_3|)}{\ln(2/3)}+6}{2}\right)-6} |r_4 - r_3|\,| \qquad \left(\text{Since } N > \frac{\frac{\ln(\epsilon/2|r_4-r_3|)}{\ln(2/3)}+6}{2}\right) \\
&< 2\left(\frac{2}{3}\right)^{\left(\frac{\ln(\epsilon/2|r_4-r_3|)}{\ln(2/3)}\right)} |r_4 - r_3| \\
&< 2\left(\frac{2}{3}\right)^{\log_{2/3}(\epsilon/2|r_4-r_3|)} |r_4 - r_3| \\
&< 2\left(\frac{\epsilon}{2\,|r_4 - r_3|}\right) |r_4 - r_3| \\
&< \epsilon
\end{aligned}
$$

   Hence, $\{r_n\}$ is a Cauchy sequence.

7. Find the limit r to which $\{r_n\}$ converges. Does the value of $\{r_n\}$ depend upon the values of a_1 and a_2?

   Since we have already proven that $\{r_n\}$ is a Cauchy sequence, we can determine its limit r. We denote $r = \lim_{n\to\infty} r_n$. We apply limit laws to its recursive definition to obtain the solution for r:

$$
\begin{aligned}
&\Longrightarrow r_n = 1 + \frac{1}{r_{n-1}} \qquad \text{(from 2)} \\
&\Longrightarrow \lim_{n\to\infty} r_n = \lim_{n\to\infty} 1 + \frac{1}{r_{n-1}} \\
&\Longrightarrow \lim_{n\to\infty} r_n = 1 + \frac{1}{\lim_{n\to\infty} r_{n-1}} \\
&\Longrightarrow r = 1 + \frac{1}{r} \\
&\Longrightarrow r^2 - r - 1 = 0 \\
&\quad \therefore r = \frac{1 \pm \sqrt{5}}{2}
\end{aligned}
$$

Since we have shown that $r_n \geq 3/2$, then only valid solution for r among the two derived r is $\frac{1+\sqrt{5}}{2} \approx 1.61803$.

We will show that value of r does not depend upon a_1 and a_2. Let a_n be some arbitrary term in a arbitrary Fibonacci sequence.

We know as $n \to \infty$,

$$\begin{aligned}
&\Longrightarrow r_{n-1} = r_{n-2} \\
&\Longrightarrow \frac{a_n}{a_{n-1}} = \frac{a_{n-1}}{a_{n-2}} = \phi && \text{(where } \phi \text{ is a value to be found)} \\
&\Longrightarrow \frac{a_{n-1} + a_{n-2}}{a_{n-1}} = \frac{a_{n-1}}{a_{n-2}} = \phi && \text{(since } a_n = a_{n-1} + a_{n-2}\text{)} \\
&\Longrightarrow 1 + \frac{a_{n-2}}{a_{n-1}} = \frac{a_{n-1}}{a_{n-2}} = \phi \\
&\Longrightarrow 1 + \frac{1}{\phi} = \frac{a_{n-1}}{a_{n-2}} = \phi \\
&\Longrightarrow 1 + \frac{1}{\phi} = \phi \\
&\therefore \phi = \frac{1 \pm \sqrt{5}}{2} && \text{(solving for } \phi\text{)}
\end{aligned}$$

Since we have shown that $r_n \geq 3/2$, then only valid solution for ϕ among the two derived ϕ is $\frac{1+\sqrt{5}}{2} \approx 1.61803$.

Hence, proved that the value of r does not depend upon a_1 and a_2.

8. Prove that for $n \geq 2$, $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1}$.
We prove this using induction.

(a) Base Case: For $n = 2$,

$$\begin{aligned}
&F_2^2 - F_3F_1 = (-1)^{2-1} \\
&1^2 - 2 = -1 \\
&-1 = -1 && \text{(True)}
\end{aligned}$$

(b) Inductive Case: Assume that $F_k^2 - F_{k+1}F_{k-1} = (-1)^{k-1}$ holds for some $n = k$ (inductive assumption). It suffices to prove that $F_{k+1}^2 - F_{k+2}F_k = (-1)^k$ for $n = k+1$. This follows from the chain of implications

$$\begin{aligned}
F_k^2 - F_{k+1}F_{k-1} &= F_{k+1}^2 - \left(F_{k+1} + F_k\right) F_k \\
&= F_{k+1}^2 - F_{k+1}F_k - F_k^2 \\
&= F_{k+1}\left(F_k + F_{k-1}\right) - F_{k+1}F_k - F_k^2 \\
&= F_{k+1}F_k + F_{k+1}F_{k-1} - F_{k+1}F_k - F_k^2 \\
&= F_{k+1}F_{k-1} - F_k^2 \\
&= (-1)(F_k^2 - F_{k+1}F_{k-1}) \\
&= (-1)(-1^{k-1}) \\
&= (-1)^k
\end{aligned}$$

Hence, proved.

1.2 Conclusion

:

We rigorously analyzed the general Fibonacci sequence on its term by term ratios in the report. However, we did not show an iterative formula for the original Fibonacci sequence until now. We will employ generating functions to make the formula.

Definition 1.5: Given a sequence $a_1, a_2, a_3, \ldots, a_k$, the function

$$A(x) = \sum_{k\geq 0}^{a} a_k x^k \tag{4}$$

is called the ordinary generating function (OGF) of the sequence.

Let us generalize the original Fibonacci sequence using OGF. Let $F_n$be defined as $F_{n+2} = F_{n+1} + F_n$ with $F_1 = F_2 = 1$.

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} F_n x^n = F_1 x + F_2 x^2 + F_3 x^3 + F_4 x^4 + \ldots \\
xf(x) &= \qquad F_1 x^2 + F_2 x^3 + F_3 x^4 + \ldots && \text{(Multiplying with } x\text{)} \\
x^2 f(x) &= \qquad\qquad F_1 x^3 + F_2 x^4 + \ldots && \text{(Multiplying with } x^2\text{)}
\end{aligned}$$

Subtracting second and third equations from the first equation above, we get:

$$\begin{aligned}
&\Longrightarrow (1 - x - x^2) f(x) = x \\
&\Longrightarrow f(x) = \frac{x}{1 - x - x^2} \\
&\Longrightarrow f(x) = \frac{x}{-(x-\phi)(x+\phi')} = \frac{\alpha}{x+\phi} + \frac{\beta}{x+\phi'} && \text{(Where } \phi = \frac{1+\sqrt{5}}{2}, \phi' = \frac{1}{\phi}\text{)}
\end{aligned}$$

$$-x = \alpha(x + \phi') + \beta(x + \phi) \tag{5}$$

To solve 5, let $x = -\phi$,

$$\begin{aligned}
&\Longrightarrow \phi = \alpha(-\phi + \phi') = \alpha(-\sqrt{5}) \\
&\Longrightarrow \alpha = \frac{-\phi}{\sqrt{5}}
\end{aligned}$$

Again to solve 5, let $x = -\phi'$,

$$\begin{aligned}
&\Longrightarrow \phi' = \beta(-\phi' + \phi) = \beta(\sqrt{5}) \\
&\Longrightarrow \beta = \frac{\phi'}{\sqrt{5}}
\end{aligned}$$

Replacing α and β in 5,

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{5}} \left(\frac{-\phi}{x+\phi} + \frac{\phi'}{x+\phi'} \right) \\
&= \frac{1}{\sqrt{5}} \left(\frac{-1}{\frac{x}{\phi}+1} + \frac{1}{\frac{x}{\phi}+1} \right) \\
&= \frac{1}{\sqrt{5}} \left(\frac{-1}{-\phi' x+1} + \frac{1}{-\phi x+1} \right) \\
&= \frac{1}{\sqrt{5}} \left(-\sum_{n=0}^{\infty} (\phi' x)^n + \sum_{n=0}^{\infty} (\phi x)^n \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} (\phi^n - \phi'^n) x^n
\end{aligned}$$

Comparing the equation with 4,

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - \phi'^n) \tag{6}$$

1.3 Appendix

Result 1:

$$\left(\frac{2}{3}\right)^{2m-2n-2} + \left(\frac{2}{3}\right)^{2m-2n-4} + \cdots + \left(\frac{2}{3}\right)^{2} + \left(\frac{2}{3}\right)^{1} = \sum_{i=1}^{2m-2n-2} \left(\frac{2}{3}\right)^{i} \leq \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{i}$$
$$= \frac{1}{1-2/3} - \sum_{i=0}^{1} \left(\frac{2}{3}\right)^{i}$$
$$= 2$$

Result 2:

$$\begin{aligned}
&\implies 2\left(\frac{2}{3}\right)^{2N-6} |r_4 - r_3| < \epsilon \\
&\implies \left(\frac{2}{3}\right)^{2N-6} |r_4 - r_3| < \epsilon/2 \\
&\implies \left(\frac{2}{3}\right)^{2N-6} < \epsilon/2\,|r_4 - r_3| \\
&\implies \ln(\left(\frac{2}{3}\right)^{2N-6}) < \ln(\epsilon/2\,|r_4 - r_3|) \\
&\implies 2N - 6(\ln\left(\frac{2}{3}\right)) < \ln(\epsilon/2\,|r_4 - r_3|) \\
&\implies 2N - 6 > \frac{\ln(\epsilon/2\,|r_4 - r_3|)}{(\ln\left(\frac{2}{3}\right))} \\
&\implies 2N > \frac{\ln(\epsilon/2\,|r_4 - r_3|)}{(\ln\left(\frac{2}{3}\right))} + 6 \\
&\implies N > \frac{\frac{\ln(\epsilon/2|r_4 - r_3|)}{(\ln\left(\frac{2}{3}\right))} + 6}{2}
\end{aligned}$$

References

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[4] Robert Sedgewick and Philippe Flajolet. *An Introduction to the Analysis of Algorithms*. Pearson Education India, 2013.

Investigation Assignment
TEAM PROGRESS REPORT
(Required for teams of 2 students)

Instructions: After completing your Investigation Assignment, each student should separately complete this form to describe their contributions to the assignment. Be sure that major tasks are equally divided and that each student is involved in developing the mathematical solution, performing calculations, and in writing the report. Thus, each student should contribute to all of these components. Team members should agree on the percentage of their overall contribution.

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1. Your First and Last Name: Aayush Kapri

2. Your Teammate(s) Name(s): Tyler Casas

3. Description of tasks that you performed on this project (use bulleted list):

*Defination
*Question 1, 2, 3(Induction case), 5(Induction case), 6, 7
*Latex writing
*Refining conclusion

4. Percentage of your contribution towards the project: 50 %
(Report 50% if evenly divided among a team of two, 33.3% for a team of three, etc.)

5. Signature (Print your name): Aayush Kapri Date: 12/04/2022