Game theory

- The study of multiperson decisions
- Four types of games
    - Static games of complete information
    - Dynamic games of complete information
    - Static games of incomplete information
    - Dynamic games of incomplete information
- Static v. dynamic
    - Simultaneously v. sequentially
- Complete v incomplete information
    - Players' payoffs are public known or private information

Concept of Game Equilibrium

- Nash equilibrium (NE)
  - Static games of complete information
- Subgame perfect Nash equilibrium (SPNE)
  - Dynamic games of complete information
- Bayesian Nash equilibrium (BNE)
  - Static games of incomplete information
- Perfect Bayesian equilibrium (PBE)
  - Dynamic games of incomplete information

Lecture Notes II-1 Static Games of Complete Information

- Normal form game
- The prisoner's dilemma
- Definition and derivation of Nash equilibrium
- Cournot and Bertrand models of duopoly
- Pure and mixed strategies

Static Games of Complete Information

- First the players simultaneously choose actions; then the players receive payoffs that depend on the combination of actions just chosen
- The player's payoff function is common knowledge among all the players

Normal- Form Representation

- The normal-form representation of a games specifies
  - (1) the players in the game
  - (2) the strategies available to each player
  - (3) the payoff received by each player for each combination of strategies that could be chosen by the players

Game definition

- Denotation
  - n player game
  - Strategy space S_i : the set of strategies available to player i
  - $s_i \in S_i$: s_i is a member of the set of strategies S_i
  - Player i's payoff function $u_i(s_1,\ldots,s_n)$: the payoff to player i if players choose strategies $(s_1,\ldots,s_n)$
- Definition
  - The normal-form representation of an n-player game specifies the players' strategy spaces $S_1,\ldots,S_n$ and their playoff functions $u_1,\ldots,u_n$. We denote game by $G=\{S_1,\ldots,S_n ; u_1,\ldots,u_n\}$

Example: The Prisoner's Dilemma

Prisoner 2

Prisoner 1		Mum (silent)	Fink (confess)
	Mum	-1,-1	-9,0
	Fink	0,-9	-6,-6

Strategy sets

S_1=S_2={Mum, Fink }

Payoff functions

u_1(Mum, Mum)= -1, u_1(Mum, Fink)= -9, u_1(Fink, Mum)=0, u_1(Fink, Fink)= -6

u_2(Mum, Mum)= -1, u_2(Mum, Fink)=0, u_2(Fink, Mum)= -9, u_2(Fink, Fink)= -6

Example: The Prisoner's Dilemma

		Prisoner 2	
		Mum (silent)	Fink (confess)
Prisoner 1	Mum	-1,-1 (R,R)	-9,0 (S,T)
	Fink	0,-9 (T,S)	-6,-6 (P,P)

⇨ T (temptation)>R (reward)>P (punishment)>S (suckers)
(Fink, Fink) would be the outcome

Strictly Dominated Strategies

- Definition
  - In the normal-form game $G=\{S_1,\ldots,S_n ; u_1,\ldots,u_n\}$, let s_i' and s_i'' be feasible strategies for player i . Strategies s_i' is strictly dominated by strategy s_i'' if for each feasible combination of the other plays' strategies, i's payoff from playing s_i' is strictly less than i's payoff from paying s_i'' :

$$u_i(s_i,\ldots,s_{i-1}, s_i', s_{i+1},\ldots,s_n) < u_i(s_i,\ldots,s_{i-1}, s_i'', s_{i+1},\ldots,s_n)$$

for each $(s_i,\ldots,s_{i-1},s_{i+1},\ldots,s_n)$ that can be constructed from the other players' strategy spaces $S_1,\ldots,S_{i-1},\ldots,S_n$

Iterated Elimination of Dominated Strategies

	L	M	R
U	1,0	1,2	0,1
D	0,3	0,1	2,0

	L	M	R
U	1,0	1,2	0,1
D	0,3	0,1	2,0

	L	M	R
U	1,0	1,2	0,1
D	0,3	0,1	2,0

	L	M	R
U	1,0	1,2	0,1
D	0,3	0,1	2,0

Outcome =(U,M)

Weakness of Iterated Elimination

- Assume it is common knowledge that the players are rational
  - All players are rational and all players know that all players know that all players are rational.
- The process often produces a very imprecise prediction about the play of the game
- Example

	L	*C*	*R*
T	0,4	4,0	5,3
M	4,0	0,4	5,3
B	3,5	3,5	6,6

⇨ No strictly dominated strategy was eliminated

Concept of Nash Equilibrium

- Each player's predicted strategy must be that player's best response to the predicated strategies of the other players
- Strategically stable or self–enforcing
  - No single wants to deviate from his or her predicated strategy
- A unique solution to a game theoretic problem, then the solution must be a Nash equilibrium

Definition of Nash Equilibrium

- Definition
  - In the n-player normal-form game G={S_1,….,S_n ; u_1,….,u_n}, the strategies (s_1*,….,s_n*) are a Nash equilibrium if, for each play i, s_i* is player i's best response to the strategies specified for the n-1 other players, (s_1*,…, s_{i-1}* , s_{i+1}*,….,s_n*)

$$u_i(s_1^*,\ldots, s_{i\text{-}1}^*, s_i^* , s_{i+1}^*,\ldots,s_n^*) \geq u_i(s_1^*,\ldots, s_{i\text{-}1}^*, s_i, s_{i+1}^*,\ldots,s_n^*)$$

for every feasible strategy s_i in S_i ; that is s_i* solves

$$\max_{s_i \in S_i} u_i(s_1^*,\ldots, s_{i\text{-}1}^*, s_i, s_{i+1}^*,\ldots,s_n^*)$$

Examples of Nash Equilibrium

	Mum	Fink
Mum	-1,-1	-9,<u>0</u>
Fink	<u>0</u>,-9	<u>-6</u>,<u>-6</u>

	L	C	R
T	0,<u>4</u>	<u>4</u>,0	5,3
M	<u>4</u>,0	0,<u>4</u>	5,3
B	3,5	3,5	<u>6</u>,<u>6</u>

	Opera	Fight
Opera	<u>2</u>,<u>1</u>	0,0
Fight	0,0	<u>1</u>,<u>2</u>

Examples of Nash Equilibrium

Player 2 's strategy
(best response function)

$BR_2(T)=L$
$BR_2(M)=C$
$\mathbf{BR_2(B)=R}$

Player 1's strategy
(best response function)

$BR_1(L)=M$
$BR_1(C)=T$
$\mathbf{BR_1(R)=B}$

	L	C	R
T	$0,\underline{4}$	$\underline{4},0$	$5,3$
M	$\underline{4},0$	$0,\underline{4}$	$5,3$
B	$3,5$	$3,5$	$\underline{6},\underline{6}$

Application 1
Cournot Model of Duopoly

- q_1, q_2 denote the quantities (of a homogeneous product) produced by firm 1 and 2
- Demand function $P(Q)=a-Q$
  - $Q=q_1+q_2$
- Cost function $C_i(q_i)=cq_i$
- Strategy space $S_i=[0,\infty)$
- Payoff function $\pi_i(q_i,q_j)=q_i[P(q_i+q_j)-c]=q_i[a-(q_i+q_j)-c]$

Firm i 's decision

$$\max_{0\le q_i\le\infty}\pi_i(q_i,q_j^*)=\max_{0\le q_i\le\infty}q_i[a-(q_i+q_j^*)-c]$$

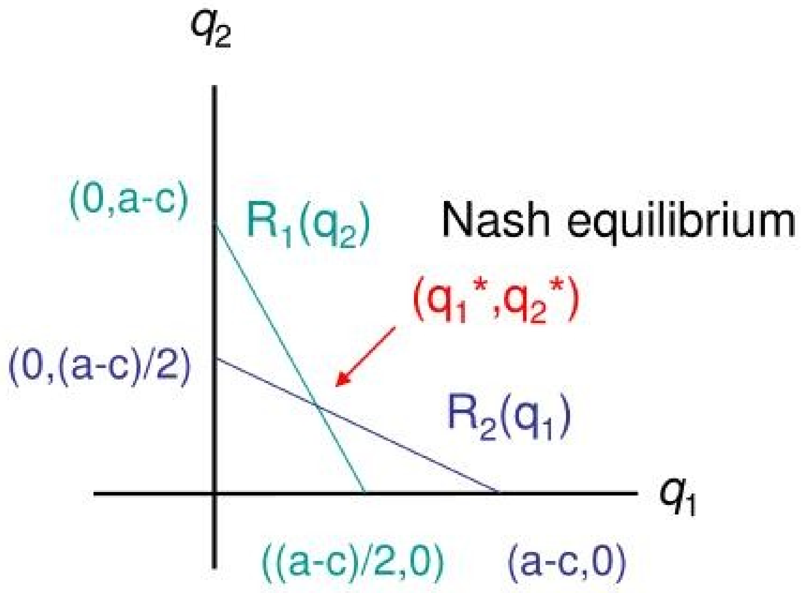
First order condition $q_i=\frac{1}{2}\left(a-q_j^*-c\right)$

$$\Rightarrow\quad q_1^*=\frac{1}{2}\left(a-q_2^*-c\right)\qquad q_2^*=\frac{1}{2}\left(a-q_1^*-c\right)\qquad\Rightarrow\quad q_1^*=q_2^*=\frac{a-c}{3}$$

Cournot Model of Duopoly (cont')

- Best response functions

$$R_2(q_1) = \frac{1}{2}(a - q_1 - c) \qquad R_1(q_2) = \frac{1}{2}(a - q_2 - c)$$



Application 2
Bertrand Model of Duopoly

- Firm 1 and 2 choose prices p_1 and p_2 for differentiated products
- Quantity that customers demand from firm *i* is $q_i(p_{i,}p_j) = a - p_i + bp_j$
- Pay off (profit) functions

$$\pi_i(p_{i,}p_j) = q_i(p_i, p_j)[p_i - c] = [a - p_i + bp_j][p_i - c]$$

- Firm *i*'s decision

$$\max_{0 \le p_i \le \infty} \pi_i(p_{i,}p_j^*) = \max_{0 \le p_i \le \infty} [a - p_i + bp_j^*][p_i - c]$$

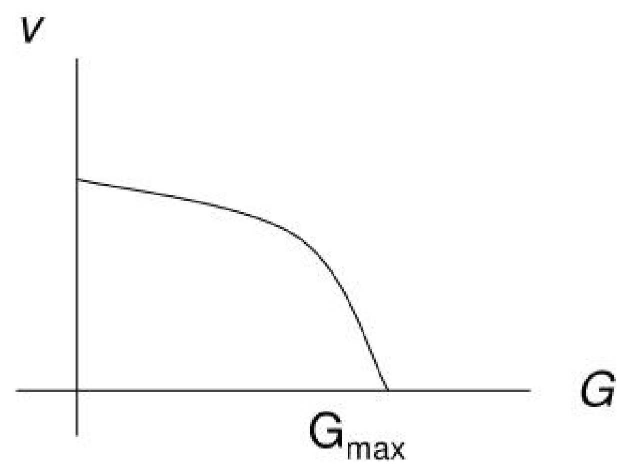
First order condition

$$p_i^* = \frac{1}{2}\left(a + bp_j^* + c\right)$$

$$\Rightarrow \quad \begin{aligned} p_1^* &= \frac{1}{2}\left(a + bp_2^* + c\right) \\ p_2^* &= \frac{1}{2}\left(a + bp_1^* + c\right) \end{aligned} \quad \Rightarrow \quad p_1^* = p_2^* = \frac{a+c}{2-b}$$

Application 3
The problem of Commons

- n farmers in a village
- g_i :The number of goats owns by farmer i
- The total numbers of goats $G= g_1+\ldots+g_n$
- The value of a goat = $v(G)$
- $v'(G)<0$, $v''(G)<0$



The Problem of Commons (cont')

Firm *i*'s decision

$$\pi_i(g_i, g_{-i}{}^*) = g_i v(g_i + g_{-i}{}^*) - c g_i$$

Fist order condition

$$\partial \pi_i(g_i, g_{-i}{}^*)/\partial g_i = v(g_i + g_{-i}{}^*) + g_i v'(g_i + g_{-i}{}^*) - c = 0, \forall i \in \{1, \ldots, n\}$$

Summarize all equations

$$\Rightarrow \quad v(G^*) + \frac{1}{n} G^* v'(G^*) - c = 0$$

Social optimum

$$\max_{0 \le G \le \infty} G v(G) - Gc$$

$$\Rightarrow \quad v(G^{**}) + G^{**} v'(G^{**}) - c = 0$$

$\Rightarrow$ Implication : $G^* > G^{**}$ (over grows)

Mixed Strategies

- Matching pennies

	Head	*Tails*
Head	-1,1	1,-1
Tails	1,-1	-1,1

- In any game in which each player would like to outguess the other(s), there is no *pure strategy* Nash equilibrium
  - E.g. poker, baseball, battle
  - The solution of such a game necessarily involves *uncertainty* about what the players will do
  - Solution : *mixed strategy*

Definition of Mixed Strategies

- Definition
  - In the normal-form game $G=\{S_1,\ldots,S_n ; u_1,\ldots,u_n\}$, suppose $S_i=\{s_{i1},\ldots,s_{iK}\}$. Then the mixed strategy for player i is a probability distribution $p_i=(p_{i1},\ldots,p_{ik})$, where $0 \le p_{ik} \le 1$ for $k=1,\ldots,K$ and $p_{i1}+,\ldots,+p_{iK}=1$
- Example
  - In penny matching game, a mixed strategy for player i is the probability distribution $(q,1-q)$, where q is the probability of playing Heads, $1-q$ is the probability of playing Trail, and $0 \le q \le 1$

Mixed strategy in Nash Equilibrium

- Strategy set $S_1=\{s_{11},\ldots,s_{1j}\}$, $S_2=\{s_{21},\ldots,s_{2k}\}$
- Player 1 believes that player 2 will play the strategies $(s_{21},\ldots,s_{2k})$ with probabilities $(p_{21},\ldots,p_{2k})$, then player 1's expected payoff from playing the pure strategy s_{1j} is

$$\sum_{k=1}^{k} p_{2k} u_1(s_{1j}, s_{2k})$$

- Player 1's expected payoff from paying the mixed strategy $p_1=(p_{11},\ldots,p_{1j})$ is

$$v_1(p_1, p_2) = \sum_{j=1}^{J}\sum_{k=1}^{k} p_{1j}\cdot p_{2k} u_1(s_{1j}, s_{2k}) \quad v_2(p_1, p_2) = \sum_{j=1}^{J}\sum_{k=1}^{k} p_{1j}\cdot p_{2k} u_2(s_{1j}, s_{2k})$$

- Definition
  - In the two player normal-form game $G=\{S_1,S_2;u_1,u_2\}$, the mixed strategies (p_1^*,p_2^*) are a Nash equilibrium if each player's mixed strategy is a best response to the other player's mixed strategy. That is

$$v_1(p_1^*, p_2^*) \geq v_1(p_1, p_2^*) \qquad v_2(p_1^*, p_2^*) \geq v_2(p_1^*, p_2)$$

Mixed Strategy

			Player 2	
			q	$1-q$
			Head	*Tails*
Player 1	r	*Head*	-1,1	1,-1
	$1-r$	*Tails*	1,-1	-1,1

Player 1's expected playoff $=q(-1)+(1-q)(1)=1-2q$ when he play *Head*

$=q(1)+(1-q)(-1)=2q-1$ when he play *Tail*

Compare $1-2q$ and $2q-1$

If $q<1/2$, then player 1 plays *Head*

If $q>1/2$, then play 1 plays *Tail*

If $q=1/2$, player 1 is indifferent in *Head* and *Tail*

Mixed strategy in Nash Equilibrium: example

			Player 2	
			q Head	$1-q$ Tails
Player 1	r	Head	-1,1	1,-1
	$1-r$	Tails	1,-1	-1,1

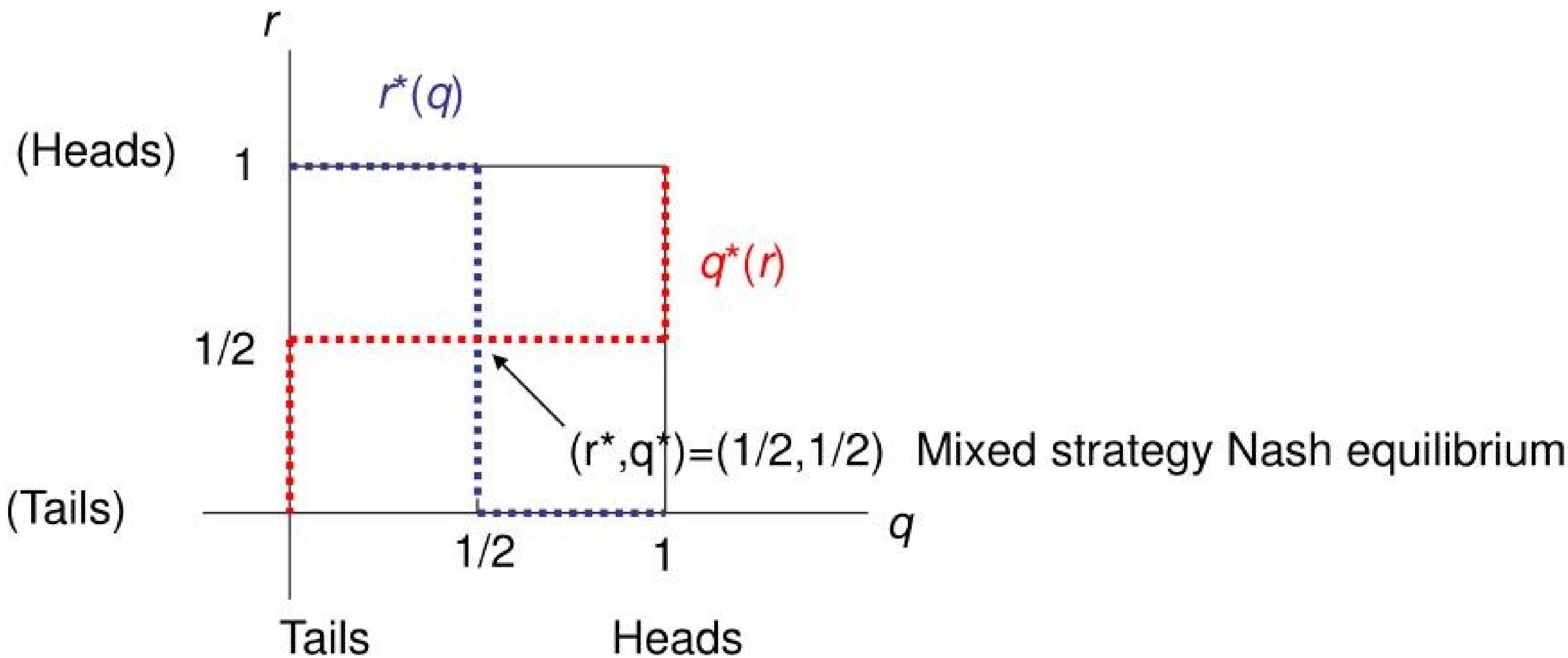
Player 1's expected playoff $=rq^*(-1)+r(1-q^*)(1)+(1-r)q^*(1)+(1-r)(1-q^*)(-1)$
$=(2q^*-1)+r(2-4q^*)$

Player 2's expected playoff $=qr^*(1)+q(1-r^*)(-1)+(1-q)r^*(-1)+(1-q)(1-r^*)(1)$
$=(2r^*-1)+q(2-4r^*)$

$r^*=1$ if $q^*<1/2$
$r^*=0$ if $q^*>1/2$
$r^*=$ any number in $(0,1)$ if $q^*=1/2$

$q^*=1$ if $r^*<1/2$
$q^*=0$ if $r^*>1/2$
$q^*=$ any number in $(0,1)$ if $r^*=1/2$

Mixed Strategy in Nash Equilibrium: example (cont')



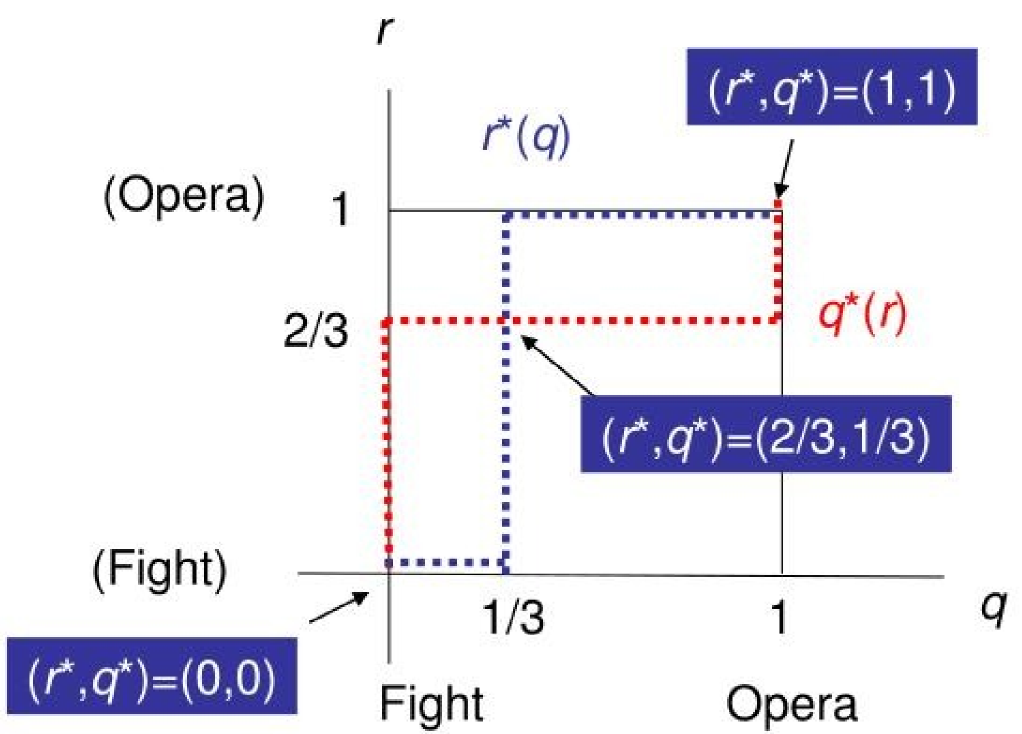
Example 2

		q	$1-q$
		Opera	*Fight*
r	*Opera*	2,1	0,0
$1-r$	*Fight*	0,0	1,2

Player 1's expected playoff $= rq^*(2)+r(1-q^*)(0)+ (1-r)q^*(0)+(1-r)(1-q^*)(1)$
$= r(3q^*-1)-(1+q^*)$

Player 2's expected playoff $= qr^*(1)+q(1-r^*)(0)+ (1-q)r^*(0)+(1-q)(1-r^*)(2)$
$= q(3r^*-2)+1-r^*$

Example 2



Payer 1's mixed strategies

$(r^*, 1-r^*) = (2/3, 1/3)$

Payer 2's mixed strategies

$(q^*, 1-q^*) = (1/3, 2/3)$

pure strategies $(r^*, 1-r^*) = (1,0), (0,1)$

pure strategies $(q^*, 1-q^*) = (1,0), (0,1)$

Theorem: Existence of Nash Equilibrium

- (Nash 1950): In the *n*-player normal-form game $G=\{S_1,\ldots.,S_n ; u_1,\ldots.,u_n\}$, *if* n is finite and S_i is finite for every I then there exists at least one Nash equilibrium, possibly involving mixed strategies

Homework #1

- Problem set
  - 1.3, 1.5, 1.6, 1.7, 1.8, 1.13(from Gibbons)
- Due date
  - two weeks from current class meeting
- Bonus credit
  - Propose new applications in the context of IT/IS or potential extensions from Application 1-4