Week-2 Assignment: Markov Chains - Solutions

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1 Problem 1: Two-State Loop (10 Marks)

1.1 (a) Transition Matrix Q

From the diagram, the transition probabilities are:

$$\text{State 1: } P(1 \to 2) = 0.5, \quad P(1 \to 3) = 0.5 \tag{1}$$
$$\text{State 2: } P(2 \to 1) = 0.75, \quad P(2 \to 4) = 0.25 \tag{2}$$
$$\text{State 3: } P(3 \to 1) = 0.25, \quad P(3 \to 4) = 0.75 \tag{3}$$
$$\text{State 4: } P(4 \to 2) = 0.75, \quad P(4 \to 3) = 0.25 \tag{4}$$

The transition matrix Q is:

$$Q = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 \\ 0.75 & 0 & 0 & 0.25 \\ 0.25 & 0 & 0 & 0.75 \\ 0 & 0.75 & 0.25 & 0 \end{pmatrix}$$

1.2 (b) Recurrent and Transient States

All states $\{1, 2, 3, 4\}$ are **recurrent**. The chain forms two communicating loops that are connected through state 1, making all states accessible from each other and hence, Recurrent.

1.3 (c) Two Different Stationary Distributions

Due to the chain's structure with two loops, multiple stationary distributions exist:

First stationary distribution (concentrated on states 1,2):

Let $\pi_1 + \pi_2 = 1$, $\pi_3 = \pi_4 = 0$

From $\pi_Q = \pi$:

$$\pi_1 = 0.75\pi_2 \tag{5}$$
$$\pi_2 = 0.5\pi_1 \tag{6}$$

Solving: $\pi_1 = 0.6$, $\pi_2 = 0.4$, $\pi_3 = 0$, $\pi_4 = 0$
Second stationary distribution (concentrated on states 3,4):
Let $\pi_3 + \pi_4 = 1$, $\pi_1 = \pi_2 = 0$
From $\pi Q = \pi$:

$$\pi_3 = 0.25\pi_4 \tag{7}$$
$$\pi_4 = 0.75\pi_3 \tag{8}$$

Solving: $\pi_1 = 0$, $\pi_2 = 0$, $\pi_3 = 0.25$, $\pi_4 = 0.75$

2 Problem 2: Winning Streak (10 Marks)

2.1 (a) Long-run proportion of wins

Let π_W and π_L be the stationary probabilities of winning and losing.
From the stationary equations:

$$\pi_W = 0.8\pi_W + 0.3\pi_L \tag{9}$$
$$\pi_W + \pi_L = 1 \tag{10}$$

Substituting: $\pi_W = 0.8\pi_W + 0.3(1 - \pi_W)$
Solving: $0.5\pi_W = 0.3$, so $\boxed{\pi_W = 0.6}$

2.2 (b) Long-run proportion of games with dinner

$$P(\text{dinner}) = P(\text{dinner}|\text{win}) \times P(\text{win}) + P(\text{dinner}|\text{loss}) \times P(\text{loss}) \tag{11}$$
$$= 0.7 \times 0.6 + 0.2 \times 0.4 \tag{12}$$
$$= 0.42 + 0.08 = \boxed{0.5} \tag{13}$$

2.3 (c) Expected number of games for dinner

Since $P(\text{dinner}) = 0.5$, this follows a geometric distribution.
Expected games $= \frac{1}{0.5} = \boxed{2 \text{ games}}$

3 Problem 3: Cat and Mouse Game (10 Marks)

3.1 (a) Stationary distributions

Cat chain: Moves to other room with probability 0.8

$$\pi_1 = 0.2\pi_1 + 0.8\pi_2 \tag{14}$$
$$\pi_1 + \pi_2 = 1 \tag{15}$$

Solution: $\boxed{\pi_1 = \pi_2 = 0.5}$

Mouse chain: $1 \to 2$ with 0.3, $2 \to 1$ with 0.6

$$\mu_1 = 0.7\mu_1 + 0.6\mu_2 \tag{16}$$
$$\mu_1 + \mu_2 = 1 \tag{17}$$

Solution: $\boxed{\mu_1 = \frac{2}{3}, \quad \mu_2 = \frac{1}{3}}$

3.2 (b) Is Z_n a Markov chain?

Yes, Z_n is a Markov chain. The joint process (cat position, mouse position) has the Markov property because the future state depends only on the current positions of both cat and mouse, not on their history.

4 Problem 4: The Wandering King (20 Marks)

The stationary distribution is proportional to the degree (number of possible moves) from each square type:

Square types and stationary probabilities:

- **Corner squares** (4 squares): 3 possible moves each
- **Edge squares** (24 squares): 5 possible moves each
- **Interior squares** (36 squares): 8 possible moves each

Total degree $= 4 \times 3 + 24 \times 5 + 36 \times 8 = 12 + 120 + 288 = 420$

Stationary probabilities:

$$\text{Corner squares: } \frac{3}{420} = \boxed{\frac{1}{140}} \text{ each} \tag{18}$$
$$\text{Edge squares: } \frac{5}{420} = \boxed{\frac{1}{84}} \text{ each} \tag{19}$$
$$\text{Interior squares: } \frac{8}{420} = \boxed{\frac{2}{105}} \text{ each} \tag{20}$$

5 Problem 5: Stock Price Model (25 Marks)

5.1 (a) Is the stock price recurrent?

No, the stock price is not recurrent. The process has positive drift:

$$E[\text{change}] = 0.1 \times 0.01 - 0.05 \times 0.01 = 0.0005 > 0$$

and no boundaries, so it can drift to infinity.

5.2 (b) Does stationary distribution exist?

No, since the chain is not recurrent, no stationary distribution exists.

5.3 (c) Probability of earning Rs. 5 before 1:00 pm

Need to reach Rs. 130 (from Rs. 120) before 1:00 pm (3 hours = 2160 five-second intervals). This requires the stock to increase by 1000 ticks in at most 2160 steps.

With $p_{\text{up}} = 0.1$, $p_{\text{stay}} = 0.85$, $p_{\text{down}} = 0.05$, this is a biased random walk.

Expected drift per step $= 0.1 - 0.05 = 0.05$ ticks upward.

Simulation recommended for exact probability, but given the positive drift and sufficient time, the probability is positive but less than 1.

6 Problem 6: Substitution Shuffle (25 Marks) - BONUS

6.1 (a) Transition probabilities and stationary distribution

Transition probability: From permutation g to h in one step:

$$P(g \to h) = \begin{cases} \frac{1}{\binom{26}{2}} = \frac{1}{325} & \text{if } h \text{ differs from } g \text{ by exactly one transposition} \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

Stationary distribution: Since the chain is symmetric (all permutations are equivalent), the stationary distribution is **uniform** over all 26! permutations.

6.2 (b) Reversibility proof

For the Metropolis-Hastings chain with scores $s(g)$:

Let $q(g, h)$ be the transition probability from g to h.

For $g \neq h$ where h is reachable from g by one transposition:

$$\text{Proposal probability: } \frac{1}{325} \tag{22}$$

$$\text{Acceptance probability: } \min\left(1, \frac{s(h)}{s(g)}\right) \tag{23}$$

So: $q(g, h) = \frac{1}{325} \times \min\left(1, \frac{s(h)}{s(g)}\right)$

Detailed balance verification:

$$s(g)q(g, h) = s(g) \times \frac{1}{325} \times \min\left(1, \frac{s(h)}{s(g)}\right) \tag{24}$$

$$= \frac{1}{325} \times \min(s(g), s(h)) \tag{25}$$

$$s(h)q(h, g) = s(h) \times \frac{1}{325} \times \min\left(1, \frac{s(g)}{s(h)}\right) \tag{26}$$

$$= \frac{1}{325} \times \min(s(h), s(g)) \tag{27}$$

Therefore: $\boxed{s(g)q(g,h) = s(h)q(h,g)}$, proving the chain is reversible with stationary distribution proportional to $s(g)$.