1. Total number of ways to put the letter $= N!$

Total number of Derangements $= N!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \ldots\right)$

Total number of ways in which at least one $= N! - \text{Dear}(N)$

Probability $= 1 - \dfrac{\text{Dear}(N)}{N!}$

$$= 1 - \sum_{k=0}^{N} \frac{(-1)^k}{k!}$$

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} \ldots \rightarrow$ put $x = -1$

$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} + \ldots \rightarrow$ approx probability $= 1 - \frac{1}{e}$

1: ~~Door~~ gift 1 has 1000 dollars → not good to switch

~~2. Initial Choice: Door #1 Door #2~~

2: Initial Choice: 1 2 3

Initial Choice:	1	2	3
Reality:	1000	0	0
Host opens:	#2 or #3	#3	#2
Remaining:	#3 or #2	#1	#1
Good to swith	No	Yes	Yes

Switching is good $\frac{2}{3}$ times.

Expected winning = $1000 \times \frac{2}{3} = 666\$$

③ a) $P(A \cap B \mid C) = \frac{P(A \cap B \cap C)}{P(C)}$

$P(A \mid B \cap C) P(B \mid C) = \frac{P(A \cap B \cap C)}{P(B \cap C)} \cdot \frac{P(B \cap C)}{P(C)} = \frac{P(A \cap B \cap C)}{P(C)}$

True

b) If independant $P(A \cap B) = P(A) \cdot P(B)$

$P(A \cap B \mid C) = \frac{P(A \cap B \cap C)}{P(C)} \neq P(A/C) \cdot P(B \mid C)$

is not always

False

~~than that of P(A|B)~~ contributing to P(A|B^c)

(c) $P(A|B) = P(A|D \cap B)P(D|B) + P(A|D^c \cap B)P(D^c|B)$

$P(A|B^c) = P(A|D \cap B^c)P(D|B^c) + P(A|D^c \cap B^c) \cdot P(D^c|B^c)$

We can't know for sure unless $P(D|B)$, $P(D^c|B)$, $P(D|B^c)$, & $P(D^c|B^c)$ are given

so false

(4) (a) Let $X = \{1, 2, 3, \dots \infty\}$

& $P(X = x) = \frac{c}{x^3}$ ~~function~~ is the given pmf

$$E(X) = \sum_{x=1}^{\infty} x \cdot \frac{c}{x^3} = c\sum_{x=1}^{\infty}\frac{1}{x^2} < \infty$$

$$E(X^2) = \sum_{x=1}^{\infty} x^2 \cdot \frac{c}{x^3} = c\sum_{x=1}^{\infty}\frac{1}{x} = \infty$$

(x^3 because for x^4 even x^2 will be finite)

(b) Again take $P(X = x) = \frac{c}{2x^3}$ $X \geq 1$

$$E(X) = \int_1^\infty \frac{c}{x^2}\,dx \to \text{finite}$$

$$E(X^2) = \int_1^\infty \frac{c}{x}\,dx \to \text{infinite}$$

~~(b) Let PMF be f(x)~~

$$\cancel{\int_0^\infty x f(x) = 1 \quad \& \quad \int_0^\infty e^{-x} f(x) \leq \frac{1}{3}}$$

(c) By Jensenn's inequality:

$$\phi(E[X]) \leq E[\phi(x)]$$

for e^{-x}

$$e^{-E[X]} \leq E[e^{-X}]$$

↳ $$\frac{1}{e} \leq E(e^{-x})$$

$$\frac{1}{e} > \frac{1}{3} \to \text{so not possible.}$$

(5) $P(M \le m)$

all draws are atmost m.

↳ $P(\text{draw} \le m) = \frac{m}{N}$ → for single draw.

Since draws are independant:

$$P(M \le m) = \left(\frac{m}{N}\right)^n \text{ for } n \text{ draw}$$

For $P(M=m)$ → Using the CDF

$$P(M=m) = P(M \le m) - P(M \le m-1)$$

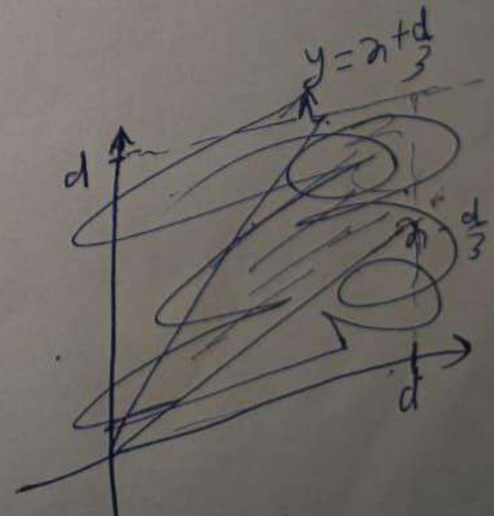
$$= \left(\frac{m}{N}\right)^n - \left(\frac{m-1}{N}\right)^n$$

$$E(M) = \sum_{m=1}^{N} m \cdot P(M=m) = \sum_{m=1}^{N} m\left[\left(\frac{m}{N}\right)^n - \left(\frac{m-1}{N}\right)^n\right]$$

(6) —d—

$P\left(|X-Y| < \frac{d}{3}\right)$

$-d < (X-Y) < \frac{d}{3}$



$(d^2 + \frac{d^2}{3})$

Q6. Let $U, V \sim \text{Unif}[0,1]$ independently
& define $D = |U-V|$

Let us derive CDF:

By def^n $F_D(r) = P(D \le r) = P(|U-V| \le r)$

In the unit square $[0,1]\times[0,1]$, this is the set of points within a diagonal band of width r:

$$\{(u,v) : |u-v| \le r\} = [0,1]^2 \setminus \{(u,v) : |u-v| > r\}$$

The area of the excluded set is two triangles each of side length $1-r$, so

$$P(|U-V| > r) = (1-r)^2$$

and thus $F_D(r) = 1-(1-r)^2 = 2r - r^2$, $0 \le r \le 1$

Differentiating gives the density,

$$F_D'(r) = 2 - 2r, \quad 0 < r < 1$$

thus, $E[D] = \int_0^1 r(2-2r)\,dr = 2\int_0^1 (r - r^2)\,dr$

$= 2\left[\frac{1}{2} - \frac{1}{3}\right] = 1/3$

By change of variables ($D_d = dD$)

$$P(D_d \le r) = P(D \le r/d) = 1 - \left(1 - \frac{r}{d}\right)^2$$

for $r = d/3$, $\boxed{\text{Ans} = 5/9}$

Q7. a) No return:

after the first spread, let someone other than O speaks there are $n+1$ possible receivers each time, and to avoid the speaker must choose among the the other n. Hence for the next $(n-1)$ tellings:

P (O not told
b) all 'n' distinct
we need that
is any of t
the $(n-1)$
P(all 'n' d

Q8. Let A_1,
bound
Using the Bo
Since for
we multipl

Q9. Given tw
a) If z_1
integer
b) From do
as F

P (0 not told again in r steps) $= \left(\frac{n}{n+1}\right)^{r-1}$

b) all 'r' distinct:

We need that at the first telling, the listener is any of the 'n' others; at the second one of the $(n-1)$ and so on, thus –

$$P(\text{all '}r\text{' distinct}) = \frac{n}{n+1} \times \frac{n-1}{n+1} \times \dots \frac{n-r+1}{n+1}$$

$$= \prod_{k=0}^{r-1} \frac{n-k}{n+1}$$

Q8. Let $A_1, \dots, A_n$ be events. We want an upper bound on $P(\cap_{i=1}^{n} A_i^c)$

Using the Bernoulli inequality,

Since for each i, $1 - P(A_i) \le e^{-P(A_i)}$,

we multiply:

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) = \prod_{i=1}^{n} [1 - P(A_i)] \le \prod_{i=1}^{n} e^{-P(A_i)}$$

$$= e^{-\sum_{i=1}^{n} P(A_i)}$$

Q9. Given two CDFs F & G, we define their convolution

$$H(z) = \int_{-\infty}^{\infty} F(z-y)\, dG(y)$$

& prove that H itself is a CDF.

a) If $z_1 < z_2$ then $\forall y$, $F(z_1 - y) \le F(z_2 - y)$

integrating gives $H(z_1) \le H(z_2)$ ($dG(y)$ is non negative)

b) From dominated convergence theorem, as F is right-continuous & $0 \le F \le 1$, thus so is H.

c) As $z \to -\infty$, $F(z-y) \to 0$ $\forall$ fixed y
so $H(z) \to 0$

as $z \to +\infty$, $F(z-y) \to 1$ so
$H(z) \to \int d(G(y)) = 1$

Hence H satisfies all properties of a CDF

Q10. Let $X \geq 0$ be a non-negative random variable with CDF $F(x) = P(X \leq x)$

then : $$X = \int_0^\infty \mathbb{I}_{\{X > t\}}\, dt$$

as an identity.

Now taking Expectation & applying Fubini

$$E[X] = E\left[\int_0^\infty \mathbb{I}_{\{X>t\}}\, dt\right]$$

$$= \int_0^\infty E[\mathbb{I}_{\{X>t\}}]\, dt$$

$$= \int_0^\infty P(X>t)\, dt$$

$$= \int_0^\infty [1-F(t)]\, dt$$

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(11.) for $X \sim N(\mu, \sigma^2)$ when we complete the square in the Gaussian integral to obtain

$$M_X(u) = E[e^{uX}] = e^{u\mu + \frac{1}{2}u^2\sigma^2}$$

Using Jensen, since e^{ux} is convex,

$$\underline{\underline{E[e^{uX}] \geq e^{uE[X]}}}.$$

(12.) Programming Task:

Number of Dyck paths of length $2n$ is the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

To compute $C_n \bmod 10^9 + 7$ in $O(n^2)$ time one can use following python :—

```
MOD = 10**9 + 7
MAX_N = 200  # Since we need factorials up to 2n

# Precompute factorials and inverse factorials
fact = [1] * (MAX_N + 1)
inv_fact = [1] * (MAX_N + 1)

# Compute all factorials % MOD
for i in range(1, MAX_N + 1):
    fact[i] = fact[i - 1] * i % MOD

# Compute all inverse factorials using Fermat's little theorem
inv_fact[MAX_N] = pow(fact[MAX_N], MOD - 2, MOD)
for i in range(MAX_N - 1, 0, -1):
    inv_fact[i] = inv_fact[i + 1] * (i + 1) % MOD

def catalan_number(n):
    return fact[2 * n] * inv_fact[n] % MOD * inv_fact[n + 1] % MOD


# Compute and print Pn for n = 1 to 50
for n in range(1, 51):
    print(f"P_{n} = {catalan_number(n)}")
```

```
P_1 = 1
P_2 = 2
P_3 = 5
P_4 = 14
P_5 = 42
P_6 = 132
P_7 = 429
P_8 = 1430
P_9 = 4862
P_10 = 16796
P_11 = 58786
P_12 = 208012
P_13 = 742900
P_14 = 2674440
P_15 = 9694845
P_16 = 35357670
P_17 = 129644790
P_18 = 477638700
P_19 = 767263183
P_20 = 564120378
P_21 = 466266852
P_22 = 482563003
P_23 = 59611249
P_24 = 904138301
P_25 = 946367425
P_26 = 352943583
P_27 = 550429273
P_28 = 949904131
P_29 = 209635674
P_30 = 475387402
P_31 = 937414464
P_32 = 488309750
P_33 = 925890214
P_34 = 459122512
P_35 = 93302951
P_36 = 141865378
P_37 = 966114350
P_38 = 869670557
P_39 = 998231628
P_40 = 602941373
P_41 = 468488140
P_42 = 436489089
P_43 = 4616923
P_44 = 884518775
P_45 = 205311759
P_46 = 837590216
P_47 = 245662066
P_48 = 217873312
P_49 = 765348450
P_50 = 265470434
```

```
# Compute and print Pn for n = 51 to 100
for n in range(51, 101):
    print(f"P_{n} = {catalan_number(n)}")
```

```
P_51 = 185096680
P_52 = 323205961
P_53 = 812467623
P_54 = 961237645
P_55 = 63389378
P_56 = 931095477
P_57 = 938406495
P_58 = 709042248
P_59 = 165264749
P_60 = 202180493
P_61 = 143994823
P_62 = 895598835
P_63 = 467182928
P_64 = 887145589
P_65 = 467932736
P_66 = 337289196
P_67 = 848807734
P_68 = 364899808
P_69 = 628322100
P_70 = 685542858
P_71 = 185042843
P_72 = 889345934
P_73 = 458247558
P_74 = 316330417
P_75 = 319295576
P_76 = 44509913
P_77 = 251538890
P_78 = 88317157
P_79 = 171644840
P_80 = 747939002
P_81 = 619955577
P_82 = 784403821
P_83 = 724443566
P_84 = 881931175
P_85 = 861543437
P_86 = 697101768
P_87 = 740877392
P_88 = 452888603
P_89 = 48028493
P_90 = 826309900
P_91 = 773088937
P_92 = 590866122
P_93 = 155536848
P_94 = 391271379
P_95 = 853131050
P_96 = 988619170
P_97 = 608234667
P_98 = 577894130
P_99 = 676902861
P_100 = 558488487
```

Work Distribution

- Ikrima Badr Shamim Ahmed (230482): Questions 1 to 5
- Satyansh Sharma (230938): Questions 6 to 10
- Aayushman Kumar (230029): Questions 11 and 12