Week-2 Assignment: Markov Chains - Solutions

Ikrima Badr Shamim Ahmed (Questions 1, 2, 3)
Aayushman Kumar (Questions 4, 5(a), 5(b))
Satyansh Sharma (Questions 5(c), 6)

10^{th} June 2025

1 Problem 1: Two-State Loop (10 Marks)

(a) **Transition Matrix Q**

From the diagram, the transition matrix Q for the Markov chain with state space $\{1, 2, 3, 4\}$ is:

$$Q = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 \\ 0.25 & 0 & 0 & 0.75 \\ 0 & 0 & 0.25 & 0.75 \\ 0 & 0.75 & 0.25 & 0 \end{pmatrix}$$

(b) **Recurrent and Transient States**

All states $\{1, 2, 3, 4\}$ are **recurrent**. The chain is irreducible: from any state, there is a positive probability of returning to that state in the future.

(c) **Two Different Stationary Distributions**

The chain is reducible, so there exist stationary distributions supported on the two communicating classes:

(i) Supported on $\{1, 2\}$:

$$\pi_1 + \pi_2 = 1, \quad \pi_3 = \pi_4 = 0$$

From the equations:

$$\pi_1 = 0.25\pi_2, \qquad \pi_2 = 0.5\pi_1$$

Substitute $\pi_2 = 0.5\pi_1$ into $\pi_1 + \pi_2 = 1$:

$$\pi_1 + 0.5\pi_1 = 1 \implies \pi_1 = \frac{2}{3}, \quad \pi_2 = \frac{1}{3}$$

So, one stationary distribution is:

$$\boxed{\pi^{(1)} = \left(\frac{2}{3},\ \frac{1}{3},\ 0,\ 0\right)}$$

(ii) Supported on $\{3, 4\}$:

$$\pi_1 = \pi_2 = 0, \quad \pi_3 + \pi_4 = 1$$

From the equations:

$$\pi_3 = 0.25\pi_3 + 0.25\pi_4, \qquad \pi_4 = 0.75\pi_3 + 0.75\pi_4$$

The first gives $0.75\pi_3 = 0.25\pi_4 \implies \pi_3 = \frac{1}{3}\pi_4$. With $\pi_3 + \pi_4 = 1$:

$$\frac{1}{3}\pi_4 + \pi_4 = 1 \implies \frac{4}{3}\pi_4 = 1 \implies \pi_4 = \frac{3}{4},\ \pi_3 = \frac{1}{4}$$

So, another stationary distribution is:

$$\boxed{\pi^{(2)} = \left(0,\ 0,\ \frac{1}{4},\ \frac{3}{4}\right)}$$

2 Problem 2: Winning Streak (10 Marks)

Let W = win, L = lose. The Markov chain is:

$$P(\text{win next} \mid \text{win}) = 0.8, \quad P(\text{win next} \mid \text{lose}) = 0.3$$

(a) **Long-run proportion of wins:**

Let π_W and π_L be the stationary probabilities of win and loss.

$$\pi_W = 0.8\pi_W + 0.3\pi_L \qquad \pi_W + \pi_L = 1$$

Substitute $\pi_L = 1 - \pi_W$:

$$\pi_W = 0.8\pi_W + 0.3(1 - \pi_W) \implies \pi_W = 0.8\pi_W + 0.3 - 0.3\pi_W$$

$$\pi_W - 0.8\pi_W + 0.3\pi_W = 0.3 \implies 0.5\pi_W = 0.3 \implies \pi_W = 0.6$$

Answer: $\boxed{0.6}$

(b) **Long-run proportion of games with dinner:**

$$P(\text{dinner}) = 0.7 \times 0.6 + 0.2 \times 0.4 = 0.42 + 0.08 = \boxed{0.5}$$

(c) **Expected number of games for a dinner:**

This is a geometric random variable with $p = 0.5$, so expected value is $1/0.5 = \boxed{2}$ games.

3 Problem 3: Cat and Mouse Game (10 Marks)

(a) **Stationary distributions:**

Cat: Moves to the other room with probability 0.8. Let π_1 = cat in room 1, π_2 = cat in room 2.

$$\pi_1 = 0.2\pi_1 + 0.8\pi_2, \qquad \pi_1 + \pi_2 = 1$$

$$\pi_1 - 0.2\pi_1 = 0.8\pi_2 \implies 0.8\pi_1 = 0.8\pi_2 \implies \pi_1 = \pi_2 = 0.5$$

Mouse: 1→2 with 0.3, 2→1 with 0.6. Let μ_1 = mouse in room 1, μ_2 = mouse in room 2.

$$\mu_1 = 0.7\mu_1 + 0.6\mu_2, \qquad \mu_1 + \mu_2 = 1$$

$$0.3\mu_1 = 0.6\mu_2 \implies \mu_1 = 2\mu_2$$

$$2\mu_2 + \mu_2 = 1 \implies \mu_2 = \frac{1}{3},\ \mu_1 = \frac{2}{3}$$

(b) **Is Z_n a Markov chain?**

Yes, the process Z_n (the pair of positions) is a Markov chain, because the next state depends only on the current state, not on the past.

4 Problem 4: The Wandering King (20 Marks)

The stationary distribution is proportional to the number of legal moves from each square type:

- **Corner squares** (4): 3 possible moves each
- **Edge (non-corner) squares** (24): 5 possible moves each
- **Interior squares** (36): 8 possible moves each

Total degree: $4 \times 3 + 24 \times 5 + 36 \times 8 = 12 + 120 + 288 = 420$

$$\text{Corner square:} \quad \frac{3}{420} = \boxed{\frac{1}{140}}$$

$$\text{Edge square:} \quad \frac{5}{420} = \boxed{\frac{1}{84}}$$

$$\text{Interior square:} \quad \frac{8}{420} = \boxed{\frac{2}{105}}$$

5 Problem 5: Stock Price Model (25 Marks)

(a) **Is the stock price recurrent?**

No, the stock price is not recurrent. The process has positive drift:

$$E[\Delta] = 0.1 \times 0.01 - 0.05 \times 0.01 = 0.0005 > 0$$

and no reflecting boundaries, so it can drift to infinity.

(b) **Does stationary distribution exist?**

No, since the chain is not recurrent, no stationary distribution exists.

(c) **Probability of earning Rs. 5 before 1:00 pm**

You need the stock to reach Rs. 130 (from Rs. 120) before 1:00 pm (3 hours = 2160 five-second intervals). This requires the stock to increase by 1000 ticks in at most 2160 steps.

With $p_{\text{up}} = 0.1$, $p_{\text{stay}} = 0.85$, $p_{\text{down}} = 0.05$, this is a biased random walk. Expected drift per step $= 0.1 - 0.05 = 0.05$ ticks upward.

Simulation is recommended for an exact probability, but given the positive drift and sufficient time, the probability is positive but less than 1.

6 Problem 6: Substitution Shuffle (25 Marks) (BONUS)

(a) **Transition probabilities and stationary distribution**

Transition probability from permutation g to h in one step:

$$P(g \to h) = \begin{cases} \frac{1}{\binom{26}{2}} = \frac{1}{325} & \text{if } h \text{ differs from } g \text{ by exactly one transposition} \\ 0 & \text{otherwise} \end{cases}$$

The stationary distribution is **uniform** over all 26! permutations.

(b) **Reversibility proof**

For the Metropolis-Hastings chain with scores $s(g)$:

For $g \neq h$ where h is reachable from g by one transposition:

$$q(g,h) = \frac{1}{325} \cdot \min\left(1, \frac{s(h)}{s(g)}\right)$$

$$s(g)q(g,h) = s(g) \cdot \frac{1}{325} \cdot \min\left(1, \frac{s(h)}{s(g)}\right) = \frac{1}{325} \min(s(g), s(h))$$

$$s(h)q(h,g) = s(h) \cdot \frac{1}{325} \cdot \min\left(1, \frac{s(g)}{s(h)}\right) = \frac{1}{325} \min(s(h), s(g))$$

Therefore, $s(g)q(g,h) = s(h)q(h,g)$, proving the chain is reversible with stationary distribution proportional to $s(g)$.