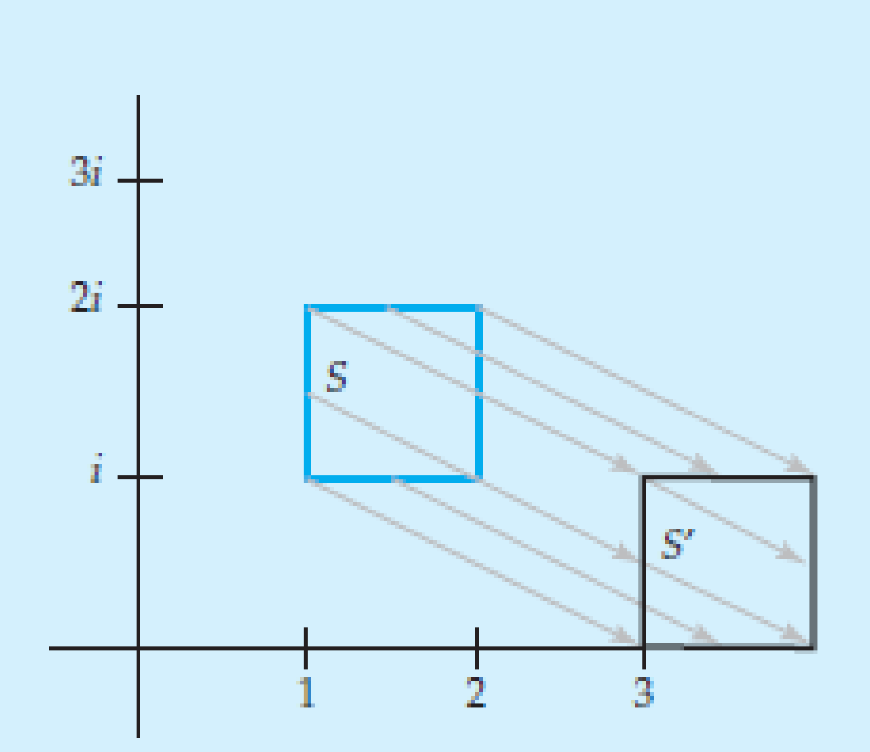

COMPLEX FUNCTION

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Complex Function

A complex function is a function f whose domain and range are subsets of the set C of complex numbers.

EXAMPLE:

The expression $z^2 - (2 + i)z$ can be evaluated at any complex number z and always yields a single complex number, and so $f(z) = z^2 - (2 + i)z$ defines a complex function. Values of f are found by using the arithmetic operations for complex numbers given in Section 1.1. For instance, at the points $z = i$ and $z = 1+i$

we have: $f(i) = (i)^2 - (2 + i)(i) = -1 - 2i + 1 = -2i$

And $f(1 + i) = (1 + i)^2 - (2 + i)(1 + i) = 2i - 1 - 3i = -1 - i$.

Complex Function

A symbol such as z,which can stand for any one of a set of complex number is called as Complex Variable.

EXAMPLE:

The expression $z^2 - (2 + i)z$ can be evaluated at any complex number z and always yields a single complex number.

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ELEMENTARY FUNCTIONS

1. **Polynomial Functions** are defined by

$$w = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = P(z) \tag{2.2}$$

where $a_0 \neq 0, a_1, \ldots, a_n$ are complex constants and n is a positive integer called the *degree* of the polynomial $P(z)$.

The transformation $w = az + b$ is called a *linear transformation*.

2. **Rational Algebraic Functions** are defined by

$$w = \frac{P(z)}{Q(z)} \tag{2.3}$$

where $P(z)$ and $Q(z)$ are polynomials. We sometimes call (2.3) a *rational transformation*. The special case $w = (az + b)/(cz + d)$ where $ad - bc \neq 0$ is often called a *bilinear* or *fractional linear transformation*.

3. **Exponential Functions** are defined by

$$w = e^z = e^{x+iy} = e^x(\cos y + i \sin y) \tag{2.4}$$

where e is the *natural base of logarithms*. If a is real and positive, we define

$$a^z = e^{z \ln a} \tag{2.5}$$

where $\ln a$ is the *natural logarithm of* a. This reduces to (4) if $a = e$.

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ELEMENTARY FUNCTIONS

Complex exponential functions have properties similar to those of real exponential functions. For example, $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$, $e^{z_1}/e^{z_2} = e^{z_1-z_2}$.

4. **Trigonometric Functions.** We define the trigonometric or circular functions $\sin z, \cos z$, etc., in terms of exponential functions as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sec z = \frac{1}{\cos z} = \frac{2}{e^{iz} + e^{-iz}}, \qquad \csc z = \frac{1}{\sin z} = \frac{2i}{e^{iz} - e^{-iz}}$$

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}, \qquad \cot z = \frac{\cos z}{\sin z} = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$$

Many of the properties familiar in the case of real trigonometric functions also hold for the complex trigonometric functions. For example, we have:

$$\sin^2 z + \cos^2 z = 1, \qquad 1 + \tan^2 z = \sec^2 z, \qquad 1 + \cot^2 z = \csc^2 z$$

$$\sin(-z) = -\sin z, \qquad \cos(-z) = \cos z, \qquad \tan(-z) = -\tan z$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\tan(z_1 \pm z_2) = \frac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \tan z_2}$$

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ELEMENTARY FUNCTIONS

5. Hyperbolic Functions are defined as follows:

$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^{-z}}{2}$$

$$\operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^z + e^{-z}}, \qquad \operatorname{csch} z = \frac{1}{\sinh z} = \frac{2}{e^z - e^{-z}}$$

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}}, \qquad \coth z = \frac{\cosh z}{\sinh z} = \frac{e^z + e^{-z}}{e^z - e^{-z}}$$

The following properties hold:

$$\cosh^2 z - \sinh^2 z = 1, \qquad 1 - \tanh^2 z = \operatorname{sech}^2 z, \qquad \coth^2 z - 1 = \operatorname{csch}^2 z$$

$$\sinh(-z) = -\sinh z, \qquad \cosh(-z) = \cosh z, \qquad \tanh(-z) = -\tanh z$$

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$$

$$\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$$

$$\tanh(z_1 \pm z_2) = \frac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$$

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ELEMENTARY FUNCTIONS

The following relations exist between the trigonometric or circular functions and the hyperbolic functions:

$$\sin iz = i\sinh z, \qquad \cos iz = \cosh z, \qquad \tan iz = i\tanh z$$
$$\sinh iz = i\sin z, \qquad \cosh iz = \cos z, \qquad \tanh iz = i\tan z$$

6. **Logarithmic Functions.** If $z = e^w$, then we write $w = \ln z$, called the *natural logarithm* of z. Thus the natural logarithmic function is the inverse of the exponential function and can be defined by

$$w = \ln z = \ln r + i(\theta + 2k\pi), \quad k = 0, \pm 1, \pm 2, \ldots$$

where $z = re^{i\theta} = re^{i(\theta+2k\pi)}$. Note that $\ln z$ is a multiple-valued (in this case, infinitely-many-valued) function. The *principal-value* or *principal branch* of $\ln z$ is sometimes defined as $\ln r + i\theta$ where $0 \le \theta < 2\pi$. However, any other interval of length 2π can be used, e.g., $-\pi < \theta \le \pi$, etc.

The logarithmic function can be defined for real bases other than e. Thus, if $z = a^w$, then $w = \log_a z$ where $a > 0$ and $a \ne 0, 1$. In this case, $z = e^{w \ln a}$ and so, $w = (\ln z)/(\ln a)$.

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ELEMENTARY FUNCTIONS

7. Inverse Trigonometric Functions. If $z = \sin w$, then $w = \sin^{-1} z$ is called the *inverse sine of* z or *arc sine of* z. Similarly, we define other inverse trigonometric or circular functions $\cos^{-1} z$, $\tan^{-1} z$, etc. These functions, which are multiple-valued, can be expressed in terms of natural logarithms as follows. In all cases, we omit an additive constant $2k\pi i$, $k = 0, \pm 1, \pm 2, \ldots$, in the logarithm:

$$\sin^{-1} z = \frac{1}{i}\ln\left(iz + \sqrt{1-z^2}\right), \qquad \csc^{-1} z = \frac{1}{i}\ln\left(\frac{i + \sqrt{z^2-1}}{z}\right)$$

$$\cos^{-1} z = \frac{1}{i}\ln\left(z + \sqrt{z^2-1}\right), \qquad \sec^{-1} z = \frac{1}{i}\ln\left(\frac{1 + \sqrt{1-z^2}}{z}\right)$$

$$\tan^{-1} z = \frac{1}{2i}\ln\left(\frac{1+iz}{1-iz}\right), \qquad \cot^{-1} z = \frac{1}{2i}\ln\left(\frac{z+i}{z-i}\right)$$

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ELEMENTARY FUNCTIONS

8. Inverse Hyperbolic Functions. If $z = \sinh w$, then $w = \sinh^{-1} z$ is called the *inverse hyperbolic sine of z*. Similarly, we define other inverse hyperbolic functions $\cosh^{-1} z$, $\tanh^{-1} z$, etc. These functions, which are multiple-valued, can be expressed in terms of natural logarithms as follows. In all cases, we omit an additive constant $2k\pi i$, $k = 0, \pm 1, \pm 2, \ldots$, in the logarithm:

$$\sinh^{-1} z = \ln\left(z + \sqrt{z^2+1}\right), \qquad \operatorname{csch}^{-1} z = \ln\left(\frac{1+\sqrt{z^2+1}}{z}\right)$$

$$\cosh^{-1} z = \ln\left(z + \sqrt{z^2-1}\right), \qquad \operatorname{sech}^{-1} z = \ln\left(\frac{1+\sqrt{1-z^2}}{z}\right)$$

$$\tanh^{-1} z = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right), \qquad \coth^{-1} z = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)$$

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BASIC DEFINATIONS

Distance: $|z - z_0|$ represents the distance between two points z and z_0.

Circle: $|z - z_0| = r$ represents a circle with centre at the point z_0 and radius r.

Interior of a circle: $|z - z_0| < r$ represents the interior of the circle.

Exterior of a circle: $|z - z_0| > r$ represents the exterior of the circle.

Annulus: The region between two concentric circles of radii r_1 and r_2 ($r_2 > r_1$) and centre at z_0 is known as the annulus region and is represented as

$$r_1 < |z - z_0| < r_2$$

Neighbourhood: The set of all points for which $|z - z_0| < r$ is known as the neighbourhood of z_0.

Boundary point: A point which does not lie in the interior or exterior of a region is known as a boundary point.

Open set: A set that does not contain its boundary points is known as an open set.

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BASIC DEFINATIONS

Closed set: A set that contains all its boundary points is known as a closed set.

Connected set: If any two points of the set can be joined by a polygonal line such that all the points of the line also belong to the set then the set is known as a connected set.

Domain: A set which is open and connected is known as a domain.

Bounded region: A region which can be enclosed in a circle of finite radius is known as a bounded region.

Compact region: A region that is closed and bounded is known as a compact region.

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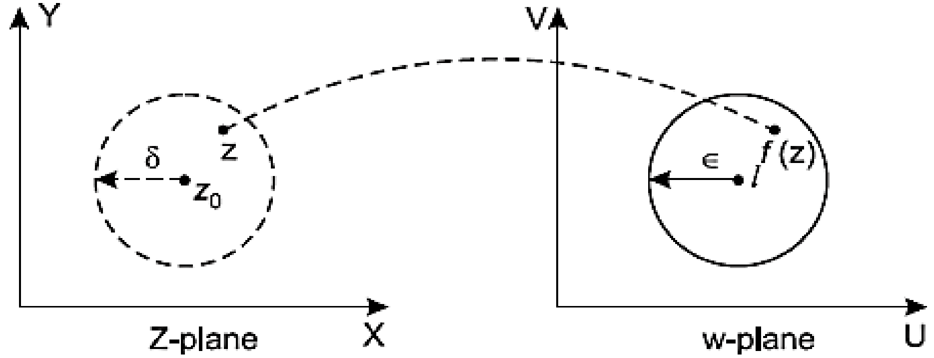
LIMIT OF FUNCTION OF COMPLEX VARIABLE

Let $f(z)$ be a single valued function defined at all points in some neighbourhood of point z_0. Then $f(z)$ is said to have the limit l as z approaches z_0 along any path if given an arbitrary real number $\in\ > 0$, however small there exists a real number $\delta > 0$, such that

$|f(z) - l| < \in$ whenever $0 < |z - z_0| < \delta$

i.e. for every $z \neq z_0$ in δ-disc (dotted) of z-plane, $f(z)$ has a value lying in the $\in$-disc of w-plane

In symbolic form, $\lim\limits_{z \to z_0} f(z) = l$



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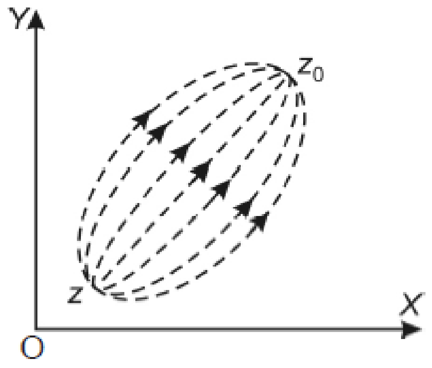
LIMIT OF FUNCTION OF COMPLEX VARIABLE

Note: (I) δ usually depends upon $\in$.

(II) $z \to z_0$ implies that z approaches z_0 along any path. The limits must be independent of the manner in which z approaches z_0

If we get two different limits as $z \to z_0$ along two different paths then limits does not exist.



Example:

(1)Find Following Limit, $\lim\limits_{z \to i} \frac{z^2+1}{z^6+1}$

As $z \to i, z^2 \to -1$

$$\therefore \lim_{z \to i} \frac{z^2+1}{z^6+1} = \lim_{z \to i} \frac{z^2+1}{(z^2+1)(z^4-z^2+1)}$$

$$= \lim_{z^2 \to -1} \frac{1}{(z^4-z^2+1)} = \frac{1}{3}$$

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LIMIT OF FUNCTION OF COMPLEX VARIABLE

Example (2): Show that $\lim_{z\to 0} \frac{z}{|z|}$ does not exist.

Solution. $\lim_{z\to 0} \frac{z}{|z|} = \lim_{\substack{x\to 0 \\ y\to 0}} \frac{x+iy}{\sqrt{x^2+y^2}}$

Let $y = mx$,

$$= \lim_{x\to 0} \frac{x+imx}{\sqrt{x^2+m^2x}} = \lim_{x\to 0} \frac{1+im}{\sqrt{1+m^2}} = \frac{1+mi}{\sqrt{1+m^2}}$$

The value of $\frac{1+mi}{\sqrt{1+m^2}}$ are different for different values of m.

Hence, limit of the function does not exist.

An Epsilon-Delta Proof of a Limit

EXAMPLE 3: Prove that $\lim_{z \to 1+i} (2+i)z = 1+3i$.

Solution According to Definition 2.8, $\lim_{z \to 1+i} (2+i)z = 1+3i$, if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|(2+i)z - (1+3i)| < \varepsilon$ whenever $0 < |z-(1+i)| < \delta$. Proving that the limit exists requires that we find an appropriate value of δ for a given value of ε. In other words, for a given value of ε we must find a positive number δ with the property that if $0 < |z-(1+i)| < \delta$, then $|(2+i)z - (1+3i)| < \varepsilon$. One way of finding δ is to "work backwards." The idea is to start with the inequality:

$$|(2+i)z - (1+3i)| < \varepsilon \tag{4}$$

An Epsilon-Delta Proof of a Limit

EXAMPLE 3: Prove that $\lim_{z \to 1+i} (2+i)z = 1+3i$.

and then use properties of complex numbers and the modulus to manipulate this inequality until it involves the expression $|z-(1+i)|$. Thus, a natural first step is to factor $(2+i)$ out of the left-hand side of (4):

$$|2+i| \cdot \left| z - \frac{1+3i}{2+i} \right| < \varepsilon. \tag{5}$$

Because $|2+i| = \sqrt{5}$ and $\dfrac{1+3i}{2+i} = 1+i$, (5) is equivalent to:

$$\sqrt{5} \cdot |z-(1+i)| < \varepsilon \quad \text{or} \quad |z-(1+i)| < \frac{\varepsilon}{\sqrt{5}}. \tag{6}$$

An Epsilon-Delta Proof of a Limit

EXAMPLE 3: Prove that $\lim_{z \to 1+i} (2+i)z = 1+3i$.

Thus, (6) indicates that we should take $\delta = \varepsilon/\sqrt{5}$. Keep in mind that the choice of δ is not unique. Our choice of $\delta = \varepsilon/\sqrt{5}$ is a result of the particular algebraic manipulations that we employed to obtain (6). Having found δ we now present the formal proof that $\lim_{z \to 1+i} (2+i)z = 1+3i$ that does not indicate how the choice of δ was made:

Given $\varepsilon > 0$, let $\delta = \varepsilon/\sqrt{5}$. If $0 < |z-(1+i)| < \delta$, then we have $|z-(1+i)| < \varepsilon/\sqrt{5}$. Multiplying both sides of the last inequality by $|2+i| = \sqrt{5}$ we obtain:

$$|2+i| \cdot |z-(1+i)| < \sqrt{5} \cdot \frac{\varepsilon}{\sqrt{5}} \quad \text{or} \quad |(2+i)z-(1+3i)| < \varepsilon.$$

Therefore, $|(2+i)z-(1+3i)| < \varepsilon$ whenever $0 < |z-(1+i)| < \delta$. So, according to Definition 2.8, we have proven that $\lim_{z \to 1+i} (2+i)z = 1+3i$.

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CONTINUITY

The function $f(z)$ of a complex variable z is said to be continuous at the point z_0 if for any given positive number $\in$, we can find a number δ such that $|f(z) - f(z_0)| < \in$ for all points z of the domain satisfying

$$|z - z_0| < \delta$$

$f(z)$ is said to be continuous at $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

CONTINUITY IN TERMS OF REAL & IMAGINARY NUMBER :-

If $w = f(z) = u(x, y) + iv(x, y)$ is continuous function at $z = z_0$ then $u(x, y)$ and $v(x, y)$ are separately continuous functions of x, y at (x_0, y_0) where $z_0 = x_0 + i\,y_0$.

Conversely, if $u(x, y)$ and $v(x, y)$ are continuous functions of x, y at (x_0, y_0) then $f(z)$ is continuous at $z = z_0$.

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CONTINUITY

Example 1

Show that the function f(z) defined by

$$f(z)=\begin{cases}\dfrac{\text{Re}(z)}{z} & , z\neq 0\\ 0 & , z=0\end{cases}$$ *is not continuous at* $z=0$

Solution. Here $f(z)=\dfrac{\text{Re}(z)}{z}$ when $z\neq 0$

$$\lim_{z\to 0}\frac{\text{Re}(z)}{z}=\lim_{\substack{x\to 0\\ y\to 0}}\frac{x}{x+iy}=\lim_{x\to 0}\left[\lim_{y\to 0}\frac{x}{x+iy}\right]=\lim_{x\to 0}\frac{x}{x}=1$$

Again $$\lim_{z\to 0}\frac{\text{Re}(z)}{z}=\lim_{y\to 0}\left[\lim_{x\to 0}\frac{x}{x+iy}\right]=0$$

As $z\to 0$, for two different paths limit have two different values. So, limit does not exist.
Hence $f(z)$ is not continuous at $z=0$ **Proved.**

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CONTINUITY

Example 2

Discuss the continuity of f(z) at the origin.

$$f(z) = \frac{\bar{z}}{z}, \quad z \neq 0$$
$$= 0, \quad z = 0$$

Solution

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\bar{z}}{z}$$
$$= \lim_{z\to 0} \frac{x - iy}{x + iy}$$

Let $z \to 0$ along the line $y = mx$.

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{x - imx}{x + imx}$$
$$= \lim_{x\to 0} \frac{1 - im}{1 + im}$$
$$= \frac{1 - im}{1 + im}$$

Since the limit depends on m, it takes different values along different paths. Thus, the limit does not exist. Hence, $f(z)$ is not continuous at the origin.

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CONTINUITY

Theorems on Continuity

THEOREM 2.2. Given $f(z)$ and $g(z)$ are continuous at $z = z_0$. Then so are the functions $f(z) + g(z)$, $f(z) - g(z)$, $f(z)g(z)$ and $f(z)/g(z)$, the last if $g(z_0) \neq 0$. Similar results hold for continuity in a region.

THEOREM 2.3. Among the functions continuous in every finite region are (a) all polynomials, (b) e^z, (c) $\sin z$ and $\cos z$.

THEOREM 2.4. Suppose $w = f(z)$ is continuous at $z = z_0$ and $z = g(\zeta)$ is continuous at $\zeta = \zeta_0$. If $z_0 = g(\zeta_0)$, then the function $w = f[g(\zeta)]$, called a *function of a function* or *composite function*, is continuous at $\zeta = \zeta_0$. This is sometimes briefly stated as: A continuous function of a continuous function is continuous.

THEOREM 2.5. Suppose $f(z)$ is continuous in a closed and bounded region. Then it is bounded in the region; i.e., there exists a constant M such that $|f(z)| < M$ for all points z of the region.

THEOREM 2.6. If $f(z)$ is continuous in a region, then the real and imaginary parts of $f(z)$ are also continuous in the region.

Problems on Limit and Continuity

EXAMPLE 4: (a) Prove that $f(z) = z^2$ is continuous at $z = z_0$.

(b) Prove that $f(z) = \begin{cases} z^2 & z \neq z_0 \\ 0 & z = z_0 \end{cases}$, where $z_0 \neq 0$, is discontinuous at $z = z_0$.

Solution

(a) By Problem 2.23(a), $\lim_{z \to z_0} f(z) = f(z_0) = z_0^2$ and so $f(z)$ is continuous at $z = z_0$.

Another Method. We must show that given any $\epsilon > 0$, we can find $\delta > 0$ (depending on ϵ) such that $|f(z) - f(z_0)| = |z^2 - z_0^2| < \epsilon$ when $|z - z_0| < \delta$. The proof patterns that given in Problem 2.23(a).

(b) By Problem 2.23(b), $\lim_{z \to z_0} f(z) = z_0^2$, but $f(z_0) = 0$. Hence, $\lim_{z \to z_0} f(z) \neq f(z_0)$ and so $f(z)$ is discontinuous at $z = z_0$ if $z_0 \neq 0$.

If $z_0 = 0$, then $f(z) = 0$; and since $\lim_{z \to z_0} f(z) = 0 = f(0)$, we see that the function is continuous.

Problems on Limit and Continuity

EXAMPLE 5: Is the function $f(z) = \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$ continuous at $z = i$?

Solution

$f(i)$ does not exist, i.e., $f(x)$ is not defined at $z = i$. Thus $f(z)$ is not continuous at $z = i$.

By redefining $f(z)$ so that $f(i) = \lim_{z\to i} f(z) = 4 + 4i$ (see Problem 2.25), it becomes continuous at $z = i$. In such a case, we call $z = i$ a *removable discontinuity*.

EXAMPLE 6: Prove that $f(z) = z^2$ is continuous in the region $|z| \le 1$.

Solution

Let z_0 be any point in the region $|z| \le 1$. By Problem 2.23(a), $f(z)$ is continuous at z_0. Thus, $f(z)$ is continuous in the region since it is continuous at any point of the region.

Complex Functions as Mappings

Mappings A useful tool for the study of real functions in elementary calculus is the graph of the function. Recall that if $y = f(x)$ is a real-valued function of a real variable x, then the graph of f is defined to be the set of all points $(x, f(x))$ in the two-dimensional Cartesian plane. An analogous definition can be made for a complex function. However, if $w = f(z)$ is a complex function, then both z and w lie in a complex plane. It follows that the set of all points $(z, f(z))$ lies in four-dimensional space (two dimensions from the input z and two dimensions from the output w). Of course, a subset of four-dimensional space cannot be easily illustrated.

Complex Functions as Mappings

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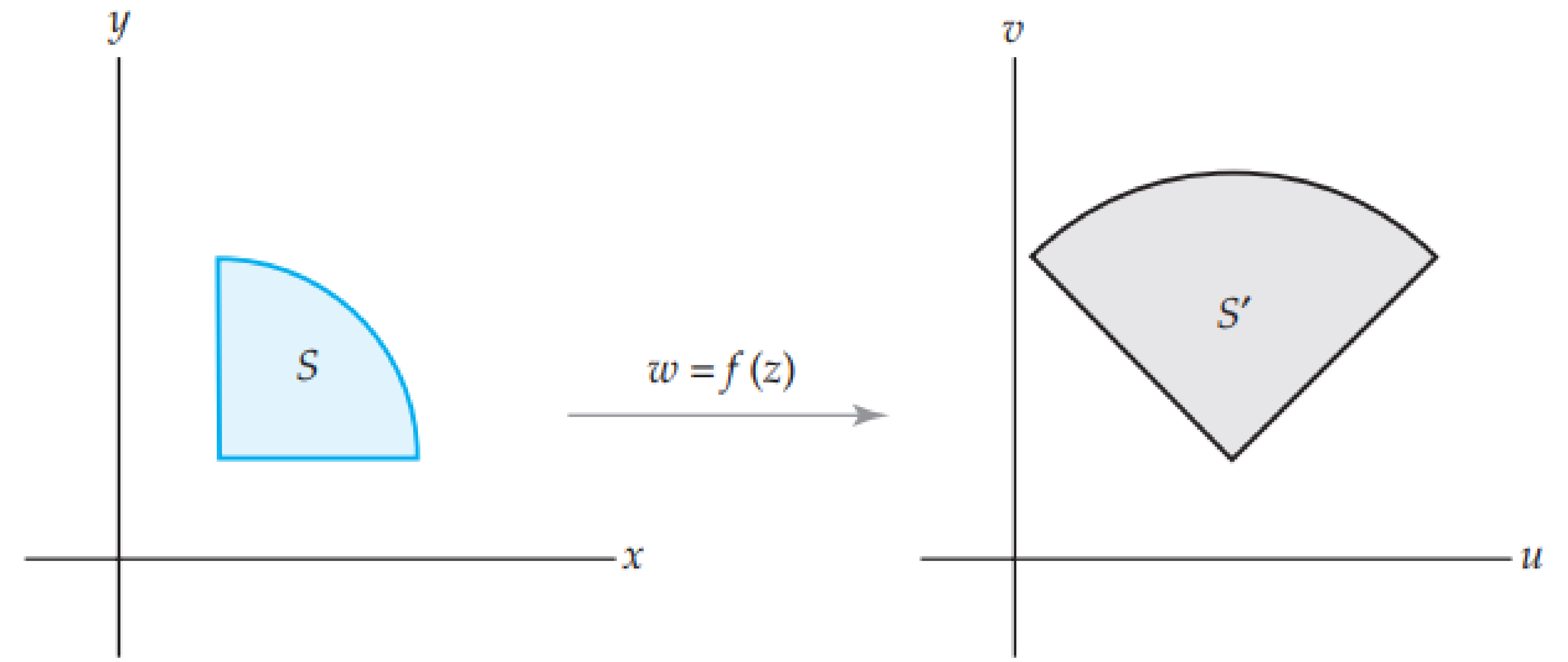
Complex Functions as Mappings



(a) The set S in the z-plane **(b)** The image of S in the w-plane

Complex Functions as Mappings

EXAMPLE 7: Let $w = f(z) = z^2$. Find the values of w that correspond to (a) $z = -2 + i$ and (b) $z = 1 - 3i$, and show how the correspondence can be represented graphically.

Solution

(a) $w = f(-2 + i) = (-2 + i)^2 = 4 - 4i + i^2 = 3 - 4i$

(b) $w = f(1 - 3i) = (1 - 3i)^2 = 1 - 6i + 9i^2 = -8 - 6i$

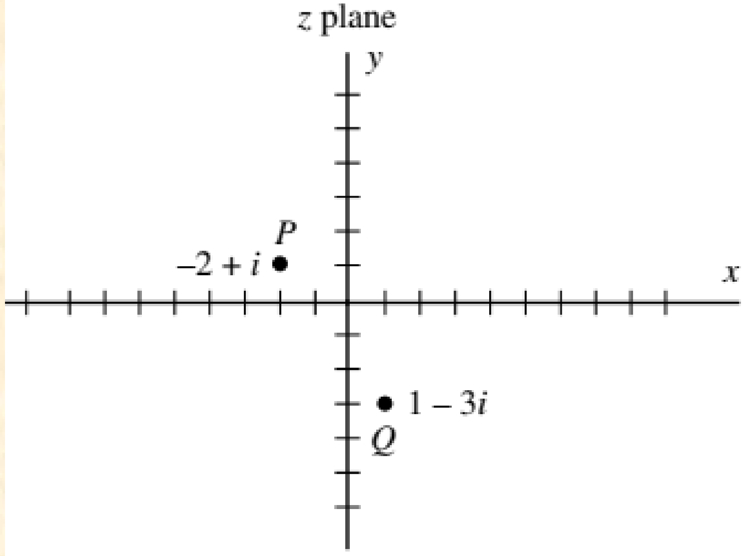


Fig. 2-6

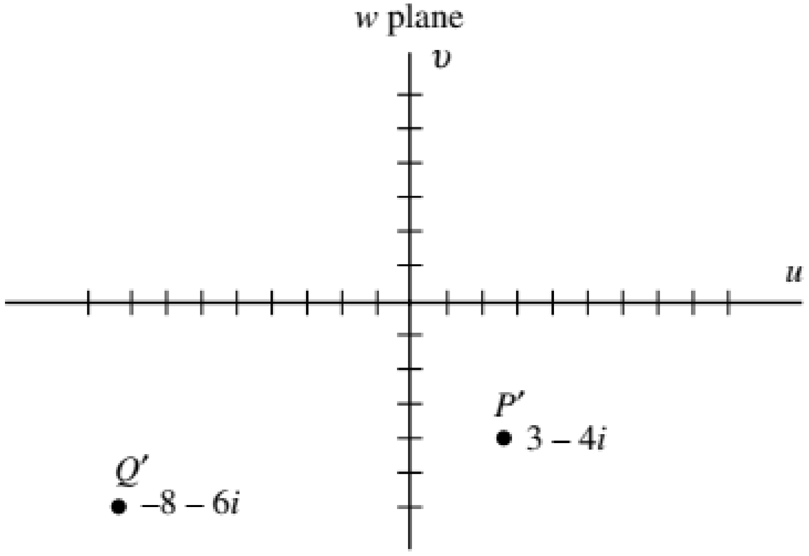


Fig. 2-7

Complex Functions as Mappings

EXAMPLE 8: A point P moves in a counterclockwise direction around a circle in the z plane having center at the origin and radius 1. If the mapping function is $w = z^3$, show that when P makes one complete revolution, the image P' of P in the w plane makes three complete revolutions in a counterclockwise direction on a circle having center at the origin and radius 1.

Solution

Let $z = re^{i\theta}$. Then, on the circle $|z| = 1$ [Fig. 2-8], $r = 1$ and $z = e^{i\theta}$. Hence, $w = z^3 = (e^{i\theta})^3 = e^{3i\theta}$. Letting (ρ, ϕ) denote polar coordinates in the w plane, we have $w = \rho e^{i\phi} = e^{3i\theta}$ so that $\rho = 1$, $\phi = 3\theta$.

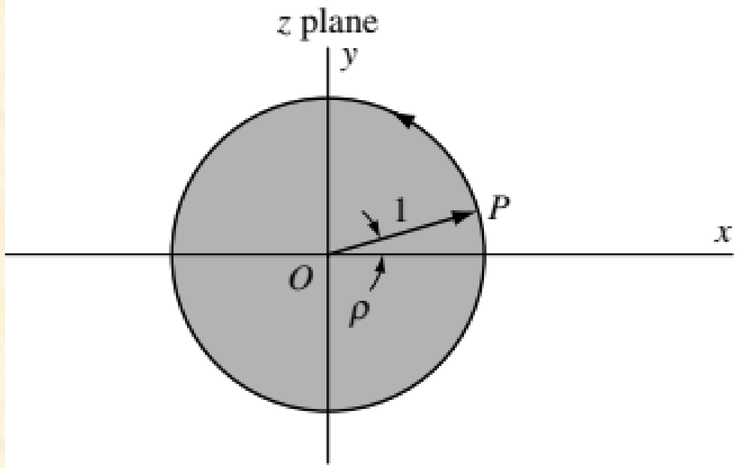


Fig. 2-8

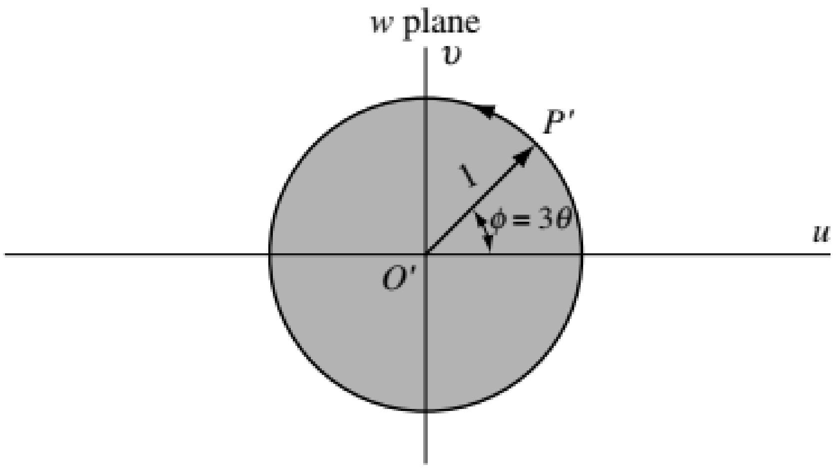


Fig. 2-9

Complex Functions as Mappings

EXAMPLE 8: A point P moves in a counterclockwise direction around a circle in the z plane having center at the origin and radius 1. If the mapping function is $w = z^3$, show that when P makes one complete revolution, the image P' of P in the w plane makes three complete revolutions in a counterclockwise direction on a circle having center at the origin and radius 1.

Solution

Let $z = re^{i\theta}$. Then, on the circle $|z| = 1$ [Fig. 2-8], $r = 1$ and $z = e^{i\theta}$. Hence, $w = z^3 = (e^{i\theta})^3 = e^{3i\theta}$. Letting (ρ, ϕ) denote polar coordinates in the w plane, we have $w = \rho e^{i\phi} = e^{3i\theta}$ so that $\rho = 1$, $\phi = 3\theta$.

Since $\rho = 1$, it follows that the image point P' moves on a circle in the w plane of radius 1 and center at the origin [Fig. 2-9]. Also, when P moves counterclockwise through an angle θ, P' moves counterclockwise through an angle 3θ. Thus, when P makes one complete revolution, P' makes three complete revolutions. In terms of vectors, it means that vector $O'P'$ is rotating three times as fast as vector OP.

Complex Functions as Mappings

EXAMPLE 9: Find the image of the vertical line $x = 1$ under the complex mapping $w = z2$ and represent the mapping graphically.

Solution Let C be the set of points on the vertical line $x = 1$ or, equivalently, the set of points $z = 1 + iy$ with $-\infty < y < \infty$. We proceed as in Example 1. From (1) of Section 2.1, the real and imaginary parts of $w = z^2$ are $u(x,\ y) = x^2 - y^2$ and $v(x,\ y) = 2xy$, respectively. For a point $z = 1 + iy$ in C, we have $u(1,\ y) = 1 - y^2$ and $v(1,\ y) = 2y$. This implies that the image of S is the set of points $w = u + iv$ satisfying the simultaneous equations:

$$u = 1 - y^2 \tag{3}$$

and

$$v = 2y \tag{4}$$

Complex Functions as Mappings

EXAMPLE 9: Find the image of the vertical line $x = 1$ under the complex mapping $w = z2$ and represent the mapping graphically.

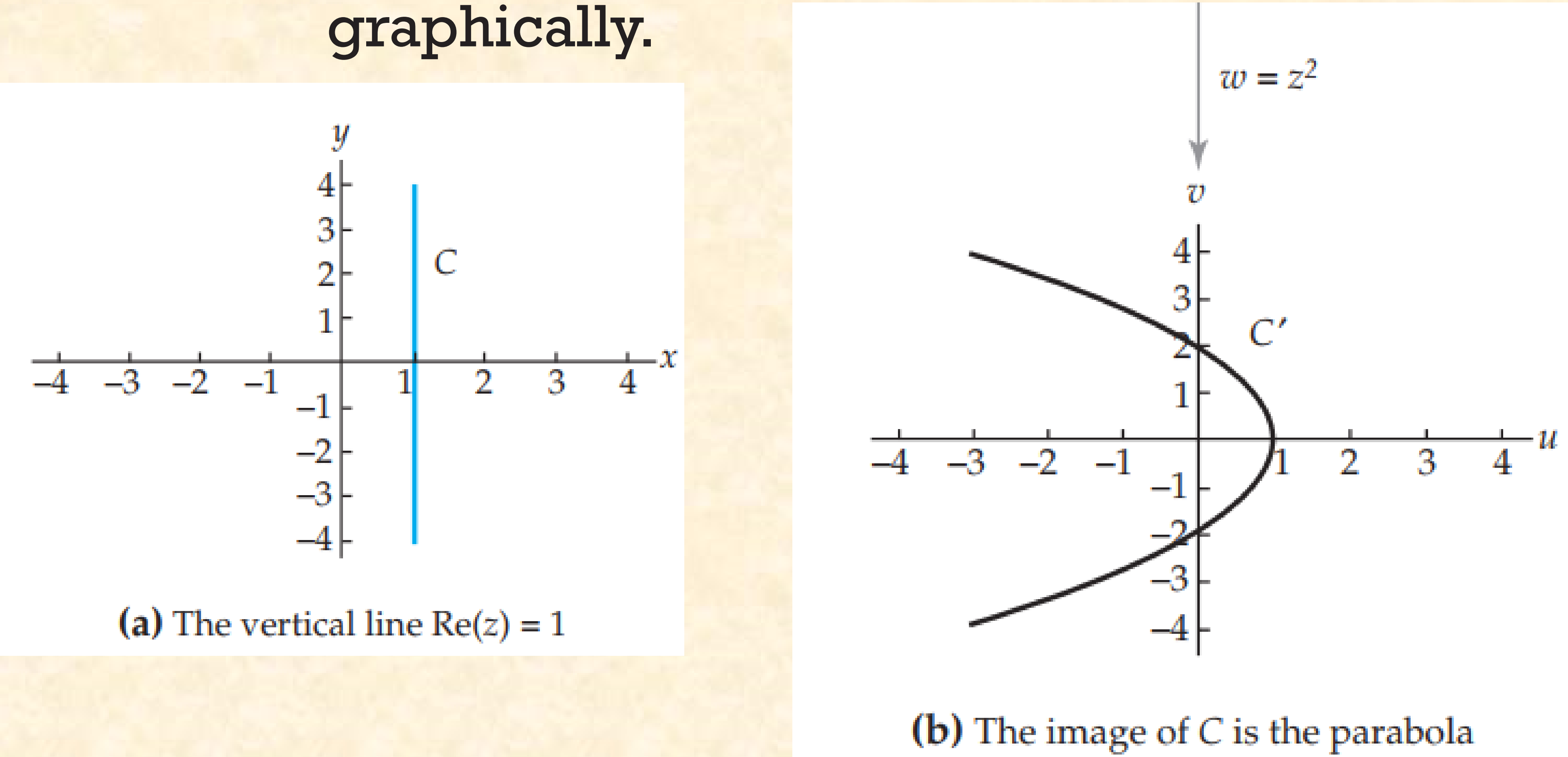
for $-\infty < y < \infty$. Equations (3) and (4) are *parametric equations* in the real parameter y, and they define a curve in the w-plane. We can find a Cartesian equation in u and v for this curve by eliminating the parameter y. In order to do so, we solve (4) for y and then substitute this expression into (3):

$$u = 1 - \left(\frac{v}{2}\right)^2 = 1 - \frac{v^2}{4}. \tag{5}$$

Since y can take on any real value and since $v = 2y$, it follows that v can take on any real value in (5). Consequently, C'—the image of C—is a parabola in the w-plane with vertex at (1,0) and u-intercepts at $(0, \pm 2)$. See Figure 2.3(b). In conclusion, we have shown that the vertical line $x = 1$ shown in color in Figure 2.3(a) is mapped onto the parabola $u = 1 - \frac{1}{4}v^2$ shown in black in Figure 2.3(b) by the complex mapping $w = z^2$.

Complex Functions as Mappings

EXAMPLE 9: Find the image of the vertical line $x = 1$ under the complex mapping $w = z2$ and represent the mapping graphically.



(a) The vertical line Re(z) = 1

(b) The image of C is the parabola $u = 1 - \frac{1}{4}v^2$

Thanks a lot …