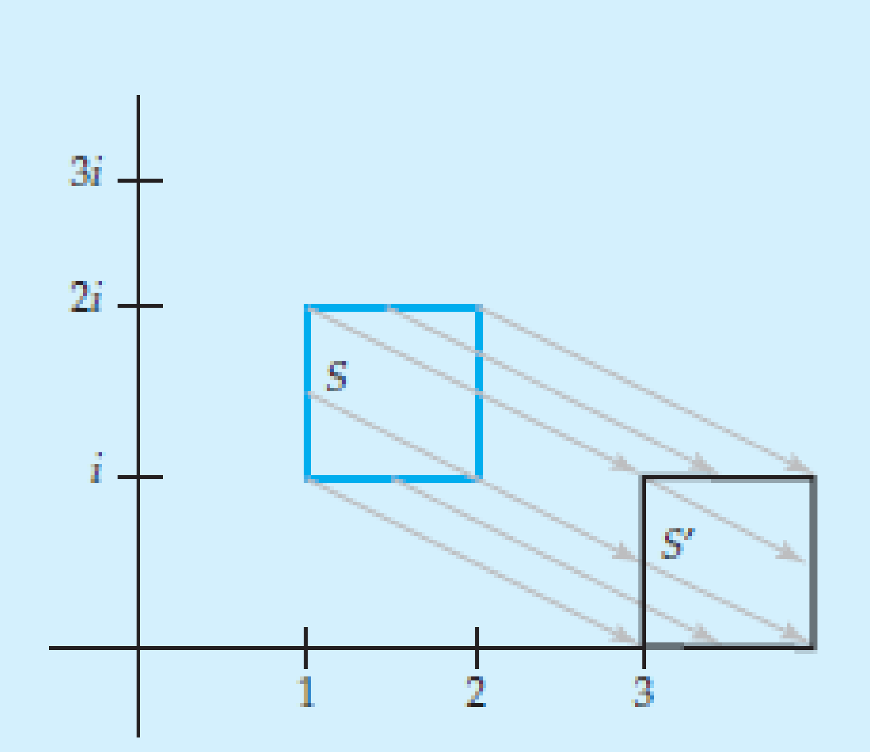


Complex Integration

ftp://10.220.20.26/*academic*/mce/maka

Definite Integral

$$\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a). \qquad (1)$$

Steps Leading to the Definition of the Definite Integral

1. *Let f be a function of a single variable x defined at all points in a closed interval $[a,\ b]$.*
2. *Let P be a partition:*

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

of $[a,\ b]$ into n subintervals $[x_{k-1},\ x_k]$ of length $\Delta x_k = x_k - x_{k-1}$. See Figure 5.1.

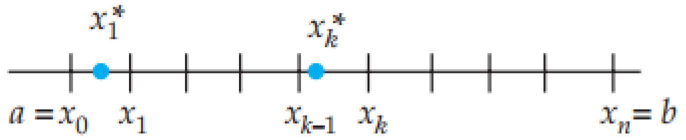


Figure 5.1 Partition of $[a,\ b]$ with x_k^* in each subinterval $[x_{k-1},\ x_k]$

Definite Integral

3. *Let $\|P\|$ be the **norm** of the partition P of $[a, b]$, that is, the length of the longest subinterval.*
4. *Choose a number x_k^* in each subinterval $[x_{k-1}, x_k]$* of $[a, b]$. *See Figure 5.1.*
5. *Form n products $f(x_k^*)\Delta x_k$, $k = 1, 2, \ldots, n$, and then sum these products:*

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k.$$

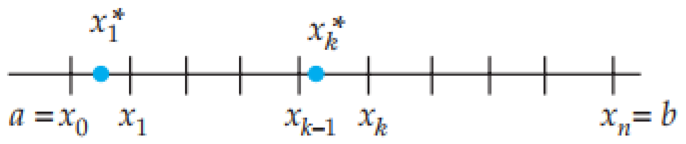


Figure 5.1 Partition of $[a, b]$ with x_k^* in each subinterval $[x_{k-1}, x_k]$

Definition 5.1 Definite Integral

The **definite integral** of f on $[a, b]$ is

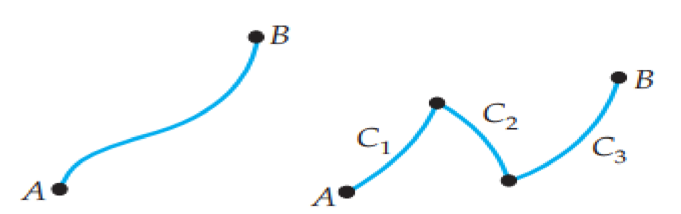
$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k. \tag{2}$$

Terminology

Suppose a curve C in the plane is parametrized by a set of equations $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, where $x(t)$ and $y(t)$ are continuous real functions. Let the **initial** and **terminal points** of C, that is, $(x(a), y(a))$ and $(x(b), y(b))$, be denoted by the symbols A and B, respectively. We say that:

(*i*) C is a **smooth curve** if x' and y' are continuous on the closed interval $[a, b]$ and not simultaneously zero on the open interval (a, b).

(*ii*) C is a **piecewise smooth curve** if it consists of a finite number of smooth curves $C_1, C_2, \ldots, C_n$ joined end to end, that is, the terminal point of one curve C_k coinciding with the initial point of the next curve C_{k+1}.



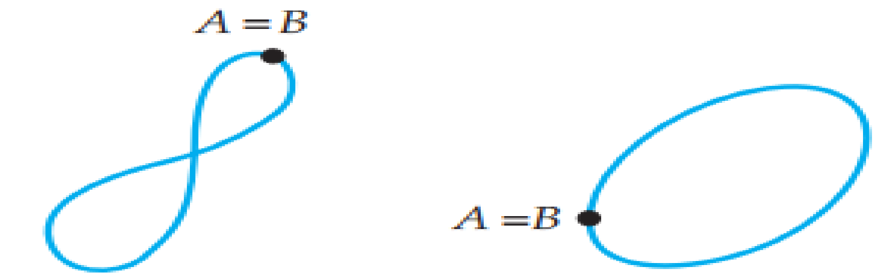
(a) Smooth curve and simple

(b) Piecewise smooth curve and simple

Terminology

Suppose a curve C in the plane is parametrized by a set of equations $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, where $x(t)$ and $y(t)$ are continuous real functions. Let the **initial** and **terminal points** of C, that is, $(x(a), y(a))$ and $(x(b), y(b))$, be denoted by the symbols A and B, respectively. We say that:

(*iii*) C is a **simple curve** if the curve C does not cross itself except possibly at $t = a$ and $t = b$.
(*iv*) C is a **closed curve** if $A = B$.
(*v*) C is a **simple closed curve** if the curve C does not cross itself and $A = B$; that is, C is simple and closed.



(c) Closed but not simple

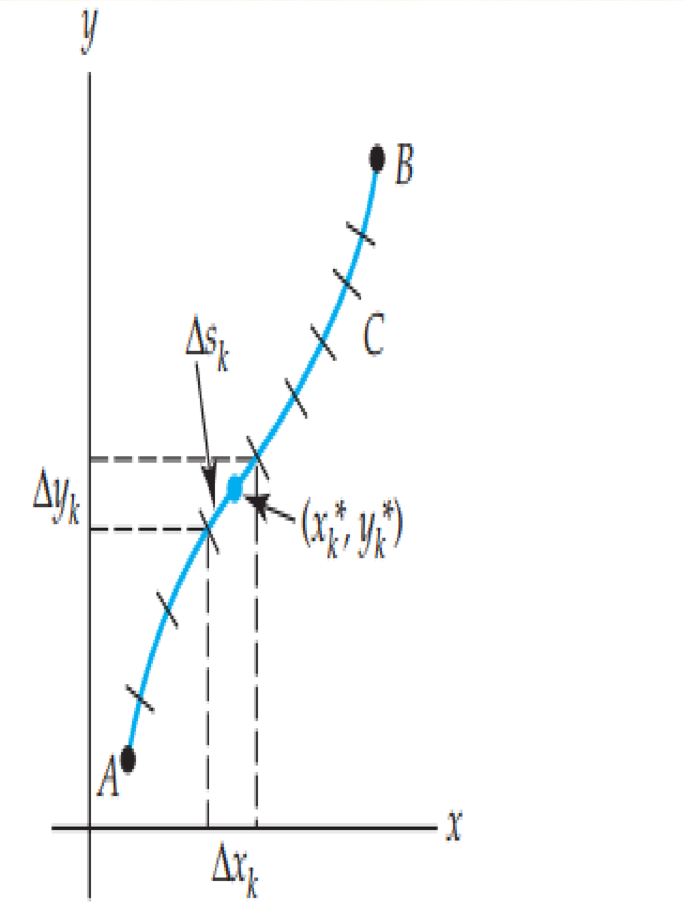
(d) Simple closed curve

Steps Leading to the Definition of Line Integrals

1. *Let G be a function of two real variables x and y defined at all points on a smooth curve C that lies in some region of the xy-plane. Let C be defined by the parametrization* $x = x(t), y = y(t)$, $a \leq t \leq b$.
2. *Let P be a partition of the parameter interval [a, b] into n subintervals* $[t_{k-1}, t_k]$ *of length* $\Delta t_k = t_k - t_{k-1}$:

   $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$

   The partition P induces a partition of the curve C into n subarcs of length Δsk. Let the projection of each subarc onto the x- and y-axes have lengths Δx_k *and* Δy_k, *respectively. See Figure 5.3.*



Steps Leading to the Definition of Line Integrals

3. Let $\|p\|$ be the norm of the partition P of $[a, b]$, that is, the length of the longest subinterval.
4. Choose a point (x_k^*, y_k^*) on each subarc of C. See Figure 5.3.
5. Form n products $G(x_k^*, y_k^*)\Delta x_k$, $G(x_k^*, y_k^*)\Delta y_k$, $G(x_k^*, y_k^*)\Delta s_k$, $k = 1, 2, \ldots, n$, and then sum these products

$$\sum_{k=1}^{n} G(x_k^*, yk^*)\Delta x_k$$

$$\sum_{k=1}^{n} G(x_k^*, yk^*)\Delta y_k$$

$$\sum_{k=1}^{n} G(x_k^*, yk^*)\Delta s_k$$

Line Integrals in the plane

(i) The **line integral of G along C with respect to x** is

$$\int_C G(x,y)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*,\, y_k^*)\,\Delta x_k. \tag{3}$$

(ii) The **line integral of G along C with respect to y** is

$$\int_C G(x,y)\,dy = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*,\, y_k^*)\,\Delta y_k. \tag{4}$$

(iii) The **line integral of G along C with respect to arc length s** is

$$\int_C G(x,y)\,ds = \lim_{\|P\|\to 0} \sum_{k=1}^{n} G(x_k^*,\, y_k^*)\,\Delta s_k. \tag{5}$$

Line Integral

Example 1: Evaluate (**a**) $\int_c xy^2 dx$, (**b**) $\int_c xy^2 dy$, and (**c**) $\int_c xy^2 ds$, where the path of integration C is the quarter circle defined by $x = 4\cos t, y = 4\sin t, 0 \le t \le \pi/2$.

Solution The path C of integration is shown in color in Figure 5.4. In each of the three given line integrals, x is replaced by $4\cos t$ and y is replaced by $4\sin t$.

(**a**) Since $dx = -4\sin t\,dt$, we have from (6):

$$\int_C xy^2 dx = \int_0^{\pi/2} (4\cos t)(4\sin t)^2(-4\sin t\,dt)$$

$$= -256\int_0^{\pi/2} \sin^3 t\cos t\,dt = -256\left[\frac{1}{4}\sin^4 t\right]_0^{\pi/2} = -64.$$

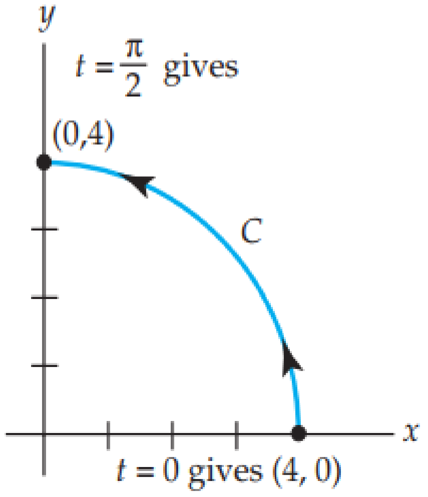


Figure 5.4 Path C of integration

Line Integral

Example 1: Evaluate (**a**) $\int_C xy^2 dx$, (**b**) $\int_C xy^2 dy$, and (**c**) $\int_C xy^2 ds$, where the path of integration C is the quarter circle defined by $x = 4\cos t, y = 4\sin t, 0 \le t \le \pi/2$.

(**b**) Since $dy = 4\cos t\, dt$, we have from (7):

$$\int_C xy^2 dy = \int_0^{\pi/2} (4\cos t)\,(4\sin t)^2\,(4\cos t\, dt)$$
$$= 256\int_0^{\pi/2} \sin^2 t\cos^2 t\, dt$$
$$= 256\int_0^{\pi/2} \frac{1}{4}\sin^2 2t\, dt$$
$$= 64\int_0^{\pi/2} \frac{1}{2}(1-\cos 4t)\, dt = 32\left[t - \frac{1}{4}\sin 4t\right]_0^{\pi/2} = 16\pi.$$

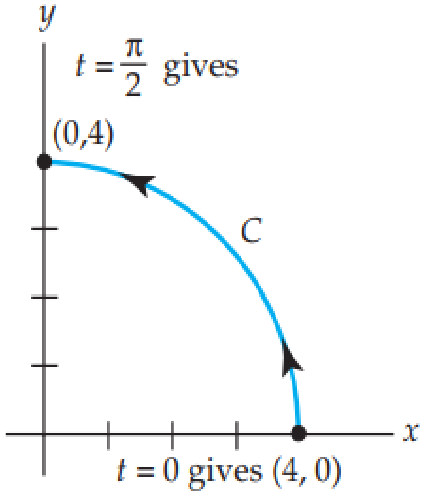


Figure 5.4 Path C of integration

Line Integral

Example 1: Evaluate (**a**) $\int_c \ xy^2dx$, (**b**) $\int_c \ xy^2dy$, and (**c**) $\int_c \ xy^2ds$, where the path of integration C is the quarter circle defined by $x = 4\cos t, y = 4\sin t, 0 \le t \le \pi/2$.

(c) Since $ds = \sqrt{16\left(\sin^2 t + \cos^2 t\right)}\,dt = 4\,dt$, it follows from (8):

$$\int_C xy^2 ds = \int_0^{\pi/2} (4\cos t)\,(4\sin t)^2\,(4\,dt)$$

$$= 256\int_0^{\pi/2} \sin^2 t\cos t\,dt = 256\left[\frac{1}{3}\sin^3 t\right]_0^{\pi/2} = \frac{256}{3}.$$

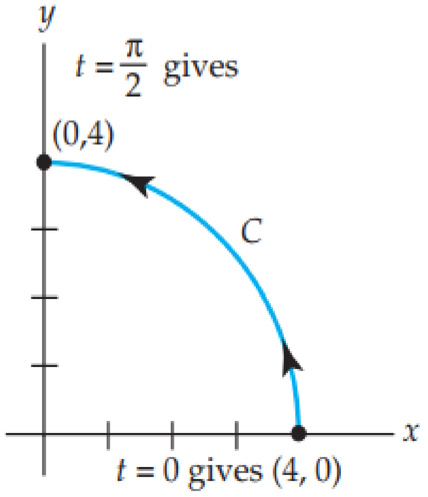


Figure 5.4 Path C of integration

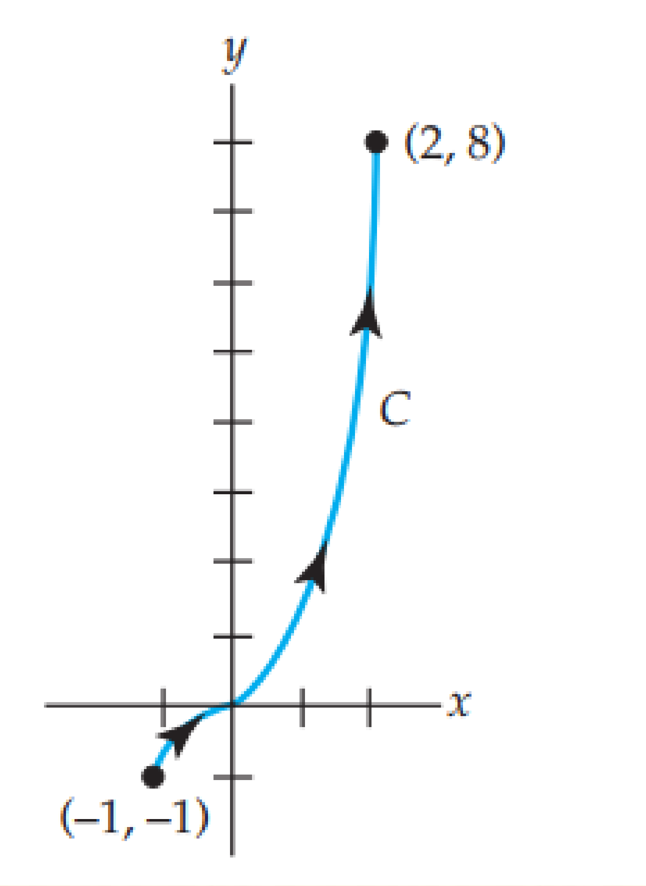
Line Integral

Example 2: Evaluate $\int_c \; xydx + x^2dy$, where the path of integration C is $y = x^3$, $-1 \le x \le 2$.

Solution The curve C is illustrated in Figure 5.5 and is defined by the explicit function $y = x^3$. Hence we can use x as the parameter. Using the differential $dy = 3x^2dx$, we apply (9) and (10):

$$\int_C xy\,dx + x^2dy = \int_{-1}^{2} x\left(x^3\right)dx + x^2\left(3x^2dx\right)$$

$$= \int_{-1}^{2} 4x^4dx = \frac{4}{5}x^5\Big|_{-1}^{2} = \frac{132}{5}.$$



Line Integral

Self Study: 1. Evaluate $\oint_c \; xdx$, where C is the circle defined by $x = \cos t, y = \sin t, 0 \le t \le 2\pi$.

2. Evaluate $\oint_c \; y^2dx - x^2dy$, where C is the closed curve shown in Figure 5.6.

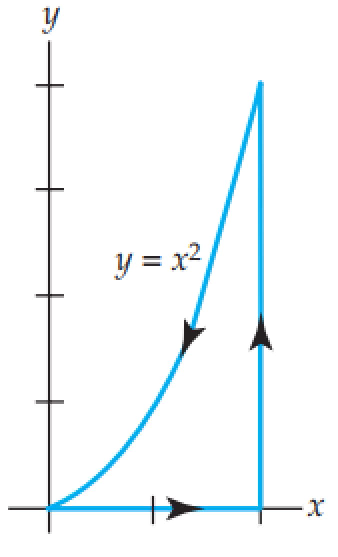


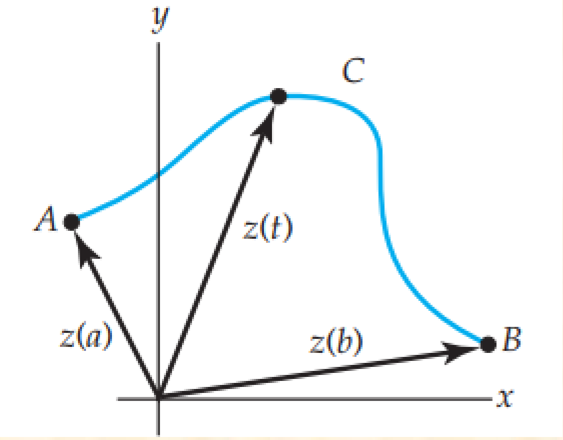
Figure 5.6 Piecewise smooth path of integration

Some Terminologies

Suppose the continuous real-valued functions $x = x(t), y = y(t)$, $a \leq t \leq b$, are parametric equations of a curve C in the complex plane. we can describe the points z on C by means of a complex-valued function of a real variable t called a **parametrization** of C:

$$z(t) = x(t) + iy(t), a \leq t \leq b. \qquad (1)$$

The point $z(a) = x(a) + iy(a)$ or $A = (x(a), y(a))$ is called the **initial point** of C and $z(b) = x(b)+iy(b)$ or $B = (x(b), y(b))$ is its **terminal point**.



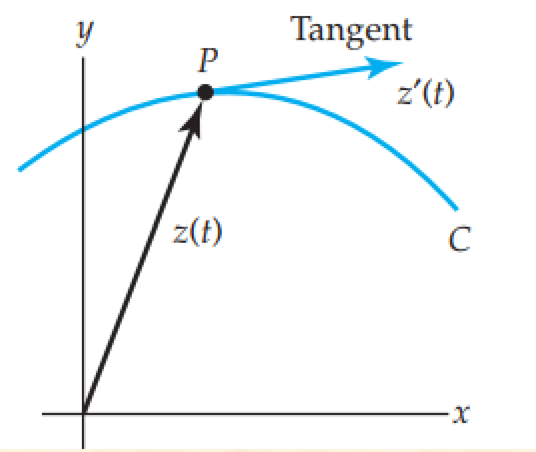
$z(t) = x(t) + iy(t)$ as a position vector

Some Terminologies

Suppose, a curve C in the complex plane for a real variable t:

$z(t) = x(t) + iy(t),\ a \leq t \leq b.$ (1)

Suppose the derivative of (1) is $z'(t) = x'(t) + iy'(t)$. We say a curve C in the complex plane is **smooth** if $z'(t)$ is continuous and never zero in the interval $a \leq t \leq b$. As shown in the following Figure, since the vector $z'(t)$ is not zero at any point P on C, the vector $z'(t)$ is tangent to C at P. In other words, a smooth curve has a continuously turning tangent;



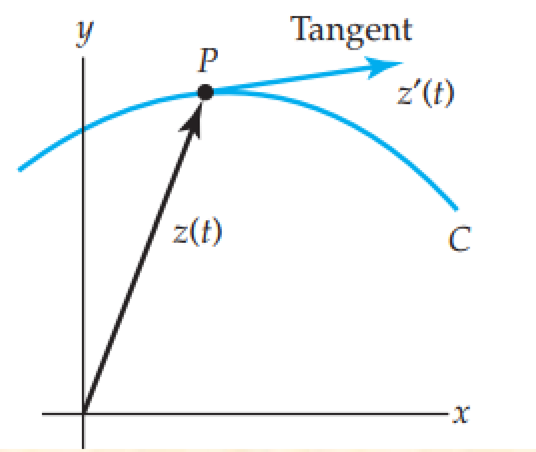
$Z'(t) = x'(t) + iy'(t)$ as a tangent vector

Some Terminologies

Suppose, a curve C in the complex plane for a real variable t:

$z(t) = x(t) + iy(t), a \leq t \leq b.$ (1)

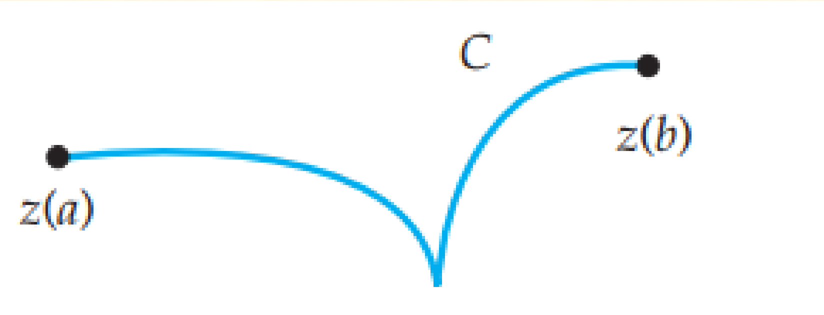
Suppose the derivative of (1) is $z'(t) = x'(t) + iy'(t)$. We say a curve C in the complex plane is **smooth** if $z'(t)$ is continuous and never zero in the interval $a \leq t \leq b$. As shown in the following Figure, since the vector $z'(t)$ is not zero at any point P on C, the vector $z'(t)$ is tangent to C at P. In other words, a smooth curve has a continuously turning tangent;



$Z'(t) = x'(t) + iy'(t)$ as a tangent vector

Some Terminologies

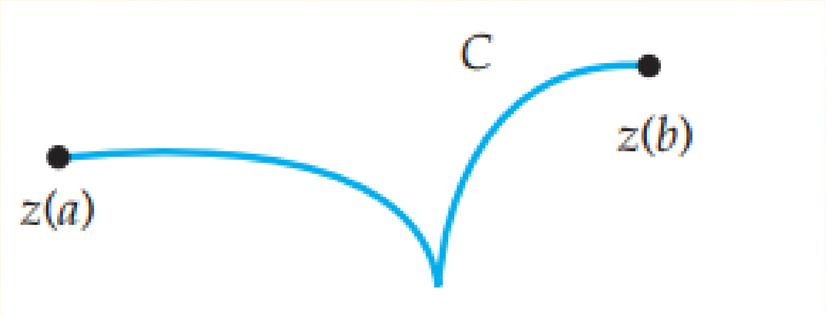
A **piecewise smooth curve** C has a continuously turning tangent, except possibly at the points where the component smooth curves $C_1, C_2, \ldots, C_n$ are joined together. a smooth curve can have no sharp corners or cusps. See the following Figure



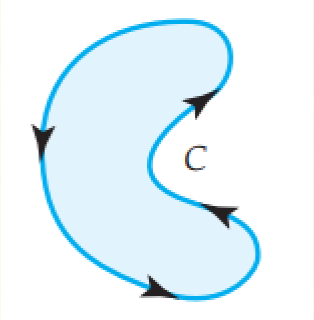
Curve C is not smooth since it has a cusp.

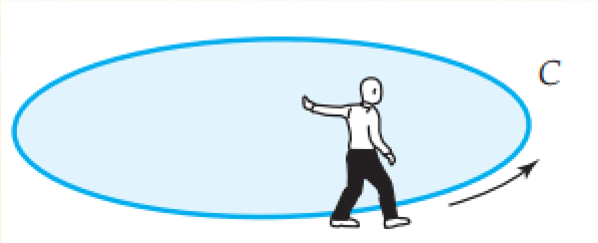
Some Terminologies

A curve C in the complex plane is said to be a **simple** if $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$, except possibly for $t = a$ and $t = b$. C is a **closed curve** if $z(a) = z(b)$. C is a **simple closed curve** if $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$ and $z(a) = z(b)$. In complex analysis, a piecewise smooth curve C is called a **contour** or **path**.



Simple Curve





Simple Closed Curve

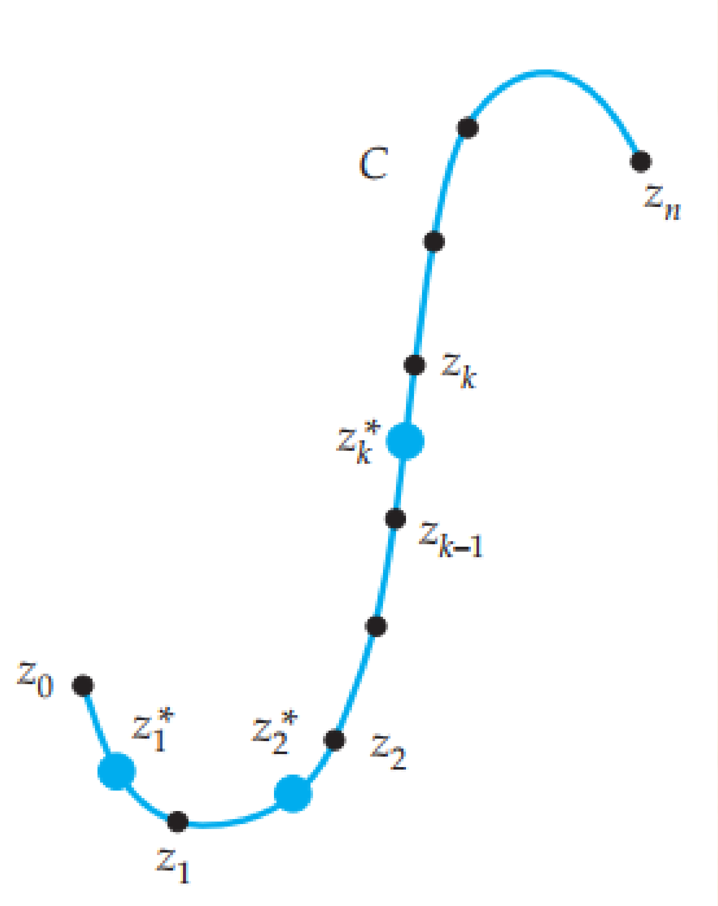
Complex Integration

The complex integration of *f* on *C*

$$\int_{c} f(z)dz = \lim_{\|p\|\to 0} \sum_{k=1}^{n} f(z_k^*)\Delta z_k^*$$

If C is a closed curve

$$\oint_{c} f(z)dz = \lim_{\|p\|\to 0} \sum_{k=1}^{n} f(z_k^*)\Delta z_k^*$$



Partition of curve *C* into *n* subarcs is induced by a partition *P* of the parameter interval [*a*, *b*].

Theorem: Evaluation of a Contour Integral

If f is continuous on a smooth curve C given by the parametrization $z(t) = x(t) + iy(t),\ a \le t \le b$, then

$$\int_C f(z)dz = \int f\big(z(t)\big)z'(t)\,dt$$

EXAMPLE 1: Evaluating a Contour Integral

Evaluate $\int_C \bar{z}\,dz$, where C is given by $x = 3t$, $y = t^2$, $-1 \le t \le 4$.

Solution From (1) a parametrization of the contour C is $z(t) = 3t + it^2$. Therefore, with the identification $f(z) = \bar{z}$ we have $f(z(t)) = \overline{3t + it^2} = 3t - it^2$. Also, $z'(t) = 3 + 2it$, and so by (11) the integral is

$$\int_C \bar{z}\,dz = \int_{-1}^{4} (3t - it^2)(3 + 2it)\,dt = \int_{-1}^{4} \left[2t^3 + 9t + 3t^2 i\right] dt.$$

Theorem: Evaluation of a Contour Integral

EXAMPLE 1: Evaluating a Contour Integral

Now in view of (4), the last integral is the same as

$$\int_C \bar{z}\,dz = \int_{-1}^{4} (2t^3 + 9t)\,dt + i\int_{-1}^{4} 3t^2\,dt$$

$$= \left(\frac{1}{2}t^4 + \frac{9}{2}t^2\right)\Bigg|_{-1}^{4} + it^3\Bigg|_{-1}^{4} = 195 + 65i.$$

Theorem: Evaluation of a Contour Integral

EXAMPLE 2: Evaluating a Contour Integral

Evaluate $\oint_C \frac{1}{z}\,dz$, where C is the circle $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$.

Solution In this case $z(t) = \cos t + i \sin t = e^{it}$, $z'(t) = ie^{it}$, and $f(z(t)) = \frac{1}{z(t)} = e^{-it}$. Hence,

$$\oint_C \frac{1}{z}\,dz = \int_0^{2\pi} (e^{-it})\, ie^{it}\,dt = i\int_0^{2\pi} dt = 2\pi i.$$

Theorem: Properties of Contour Integrals

Suppose the functions *f* and *g* are continuous in a domain *D*, and *C* is a smooth curve lying entirely in *D*. Then

(i) $\int_C kf(z)\,dz = k\int_C f(z)\,dz$, k a complex constant.

(ii) $\int_C [f(z)+g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$.

(iii) $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$, where C consists of the smooth curves C_1 and C_2 joined end to end.

(iv) $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$, where $-C$ denotes the curve having the opposite orientation of C.

Theorem: Evaluation of a Contour Integral

EXAMPLE 3: *C* Is a Piecewise Smooth Curve

Evaluate $\int_C (x^2 + iy^2)\, dz$, where C is the contour shown in Figure 5.20.

Solution In view of Theorem 5.2(*iii*) we write

$$\int_C (x^2 + iy^2)\, dz = \int_{C_1} (x^2 + iy^2)\, dz + \int_{C_2} (x^2 + iy^2)\, dz.$$

Since the curve C_1 is defined by $y = x$, it makes sense to use x as a parameter. Therefore, $z(x) = x + ix,\ z'(x) = 1 + i, \quad f(z) = x^2 + iy^2,\ f(z(x)) = x^2 + ix^2,$ and

$$\int_{C_1} (x^2 + iy^2)\, dz = \int_0^1 \overbrace{(x^2 + ix^2)}^{(1+i)x^2}(1 + i)\, dx$$

$$= (1 + i)^2 \int_0^1 x^2 dx = \frac{(1 + i)^2}{3} = \frac{2}{3}i. \qquad (12)$$

Theorem: Evaluation of a Contour Integral

EXAMPLE 3: *C* Is a Piecewise Smooth Curve

The curve C_2 is defined by $x = 1$, $1 \le y \le 2$. If we use y as a parameter, then $z(y) = 1 + iy$, $z'(y) = i$, $f(z(y)) = 1 + iy^2$, and

$$\int_{C_2} (x^2 + iy^2)\,dz = \int_1^2 (1 + iy^2)i\,dy = -\int_1^2 y^2\,dy + i\int_1^2 dy = -\frac{7}{3} + i. \quad (13)$$

Combining (10) and (13) gives $\int_C (x^2 + iy^2)\,dz = \frac{2}{3}i + (-\frac{7}{3} + i) = -\frac{7}{3} + \frac{5}{3}i.$

Evaluation of a Contour Integral

Self Study: Evaluate the contour integral of $\int_c f(z)dz$ using the parametric representations for C, where $f(z) = \dfrac{z^2 - 1}{z}$

and the curve C is

(a) the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$;

(b) the semicircle $z = 2e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;

(c) the circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$;

Thanks a lot …