Introduction to Functions of Several Variables

Md. Abul Kalam Azad
Assistant Professor
Mathematics, Department of MPE
IUT

Objectives

- Understand the notation for a function of several variables.
- Sketch the graph of a function of two variables.
- Sketch level curves for a function of two variables.
- Sketch level surfaces for a function of three variables.
- Use computer graphics to graph a function of two variables.

Functions of Several Variables

- We know only the functions of a single (independent) variable. Many familiar quantities, however, are functions of two or more variables. Here are three examples.

✓ The work done by a force, $W = FD$, is a function of two variables.

✓ The volume of a right circular cylinder, is a function of two variables.

✓ The volume of a rectangular solid, $V = lwh$, is a function of three variables.

Functions of Several Variables

- The notation for a function of two or more variables is similar to that for a function of a single variable. Here are two examples.

$$z = f\underbrace{(x, y)}_{\text{2 variables}} = x^2 + xy \qquad \text{Function of two variables}$$

- and

$$w = f\underbrace{(x, y, z)}_{\text{3 variables}} = x + 2y - 3z \qquad \text{Function of three variables}$$

Definition of a Function of Two Variables

Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is a **function of** x **and** y. The set D is the **domain** of f, and the corresponding set of values for $f(x, y)$ is the **range** of f. For the function

$$z = f(x, y)$$

x and y are called the **independent variables** and z is called the **dependent variable.**

Example 1:Domains of Functions of Several Variables

- Find the domain of each function.

a. $$f(x, y) = \frac{\sqrt{x^2 + y^2 - 9}}{x}$$

b. $$g(x, y, z) = \frac{x}{\sqrt{9 - x^2 - y^2 - z^2}}$$

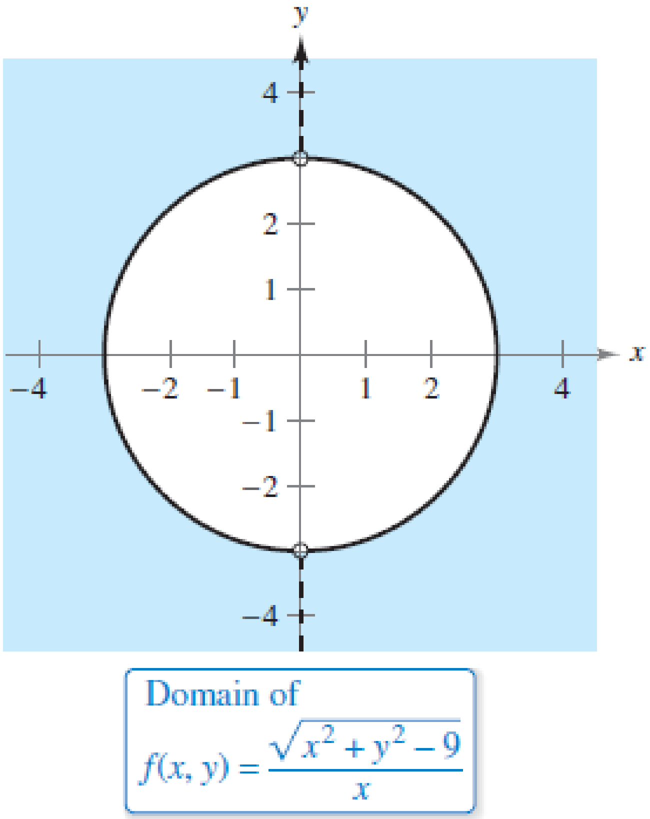
Solution:

a. The function *f* is defined for all points (*x*, *y*) such that

$x \neq 0$ and $x^2 + y^2 \geq 9.$

Example 1:Solution

- So, the domain is the set of all points lying on or outside the circle $x^2 + y^2 = 9$ *except* those points on the *y*-axis, as shown in following Figure.

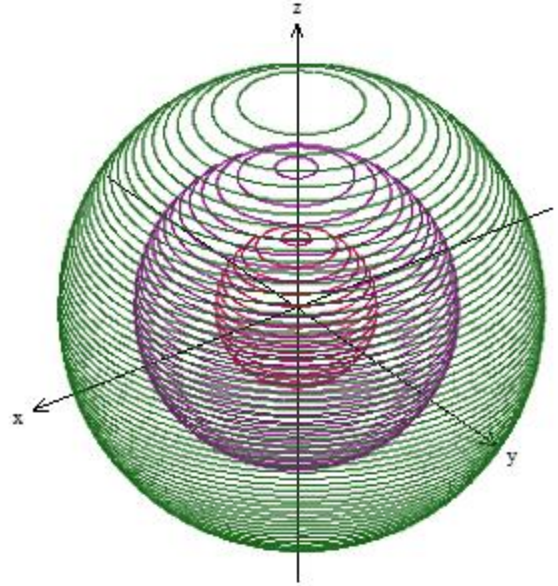


Example 1:Solution

b. The function g is defined for all points (x, y, z) such that

$$x^2 + y^2 + z^2 < 9.$$

Consequently, the domain is the set of all points (x, y, z) lying inside a sphere of radius 3 that is centered at the origin.



- Functions of several variables can be combined in the same ways as functions of single variables. For instance, you can form the sum, difference, product, and quotient of two functions of two variables as follows.

$$(f \pm g)(x, y) = f(x, y) \pm g(x, y) \quad \text{Sum or difference}$$

$$(fg)(x, y) = f(x, y)g(x, y) \quad \text{Product}$$

$$\frac{f}{g}(x, y) = \frac{f(x, y)}{g(x, y)}, \quad g(x, y) \neq 0 \quad \text{Quotient}$$

- You cannot form the composite of two functions of several variables. You can, however, form the **composite** function $(g \circ h)(x, y)$, where *g* is a function of a single variable and *h* is a function of two variables.

$$(g \circ h)(x, y) = g(h(x, y)) \quad \text{Composition}$$

Y'

- A function that can be written as a sum of functions of the form cx^my^n (where c is a real number m and n are nonnegative integers) is called a **polynomial function** of two variables.

✓ For instance, the functions

$$f(x, y) = x^2 + y^2 - 2xy + x + 2 \quad \text{and} \quad g(x, y) = 3xy^2 + x - 2$$

are polynomial functions of two variables.

- A **rational function** is the quotient of two polynomial functions.

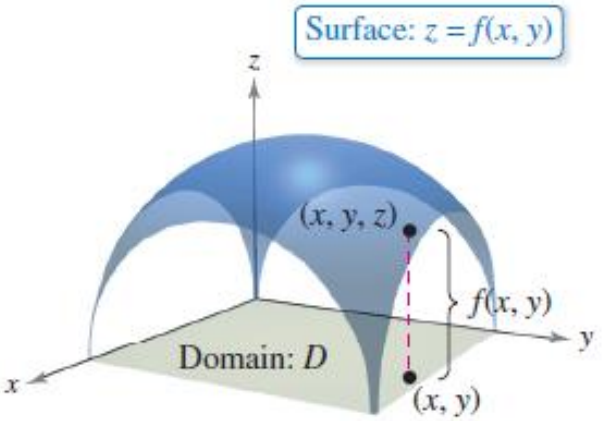
- Similar terminology is used for functions of more than two variables.

The Graph of a Function of Two Variables

The Graph of a Function of Two Variables

- The **graph** of a function f of two variables is the set of all points (x, y, z) for which $z = f(x, y)$ and (x, y) is in the domain of f.

- This graph can be interpreted geometrically as a *surface in space*. In Figure note that the graph of $z = f(x, y)$ is a surface whose projection onto the xy-plane is D, the domain of f.



The Graph of a Function of Two Variables

- To each point (x, y) in D there corresponds a point (x, y, z) on the surface, and, conversely, to each point (x, y, z) on the surface there corresponds a point (x, y) in D.

Example 2: *Describing the Graph of a Function of Two Variables*

- Consider the function given by

$$f(x, y) = \sqrt{16 - 4x^2 - y^2}.$$

a. Find the domain and range of the function.

b. Describe the graph of *f*.

- Solution:

a. The domain *D* implied by the equation of *f* is the set of all points (x, y) such that

$$16 - 4x^2 - y^2 \geq 0.$$

So, *D* is the set of all points lying on or inside the ellipse

$$\frac{x^2}{4} + \frac{y^2}{16} = 1.$$

Ellipse in the *xy*-plane

Example 2 – *Solution*

- The range of f is all values $z = f(x, y)$ such that $0 \leq z \leq \sqrt{16}$, or

$$0 \leq z \leq 4. \qquad \text{Range of } f$$

b. A point (x, y, z) is on the graph of f if and only if

$$z = \sqrt{16 - 4x^2 - y^2}$$

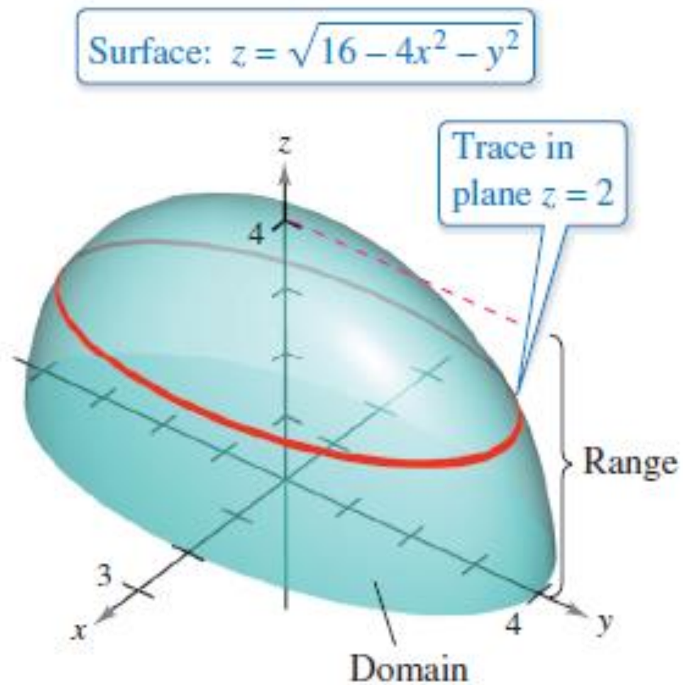
$$z^2 = 16 - 4x^2 - y^2$$

$$4x^2 + y^2 + z^2 = 16$$

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1, \qquad 0 \leq z \leq 4.$$

Example 2 – *Solution*

- You know that the graph of f is the upper half of an ellipsoid, as shown in below.



The graph of $f(x, y) = \sqrt{16 - 4x^2 - y^2}$ is the upper half of an ellipsoid.

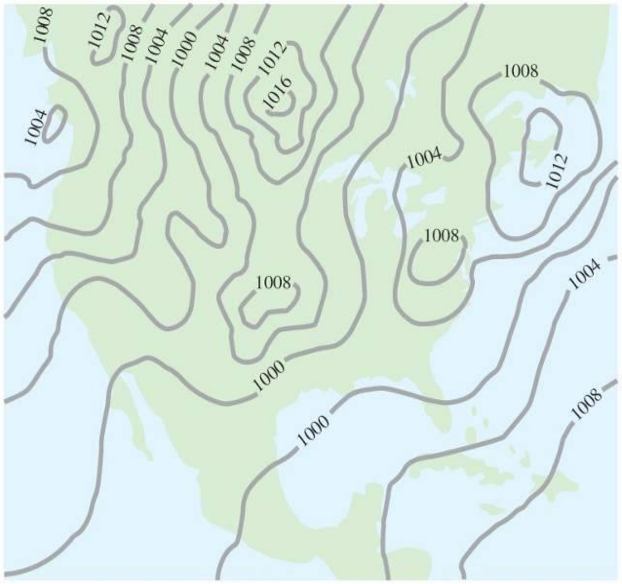
Level Curves

Level Curves

- A second way to visualize a function of two variables is to use a **scalar field** in which the scalar $z = f(x, y)$ is assigned to the point (x, y).

- A scalar field can be characterized by **level curves** (or **contour lines**) along which the value of $f(x, y)$ is constant.

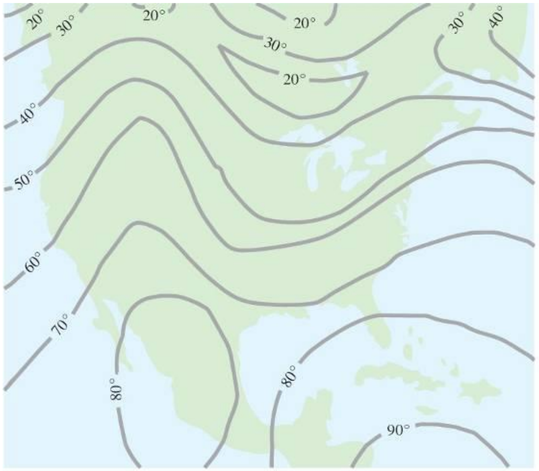
- For instance, the weather map in Figure shows level curves of equal pressure called **isobars.**



Level curves show the lines of equal pressure (isobars) measured in millibars.

Level Curves

- In weather maps for which the level curves represent points of equal temperature, the level curves are called **isotherms,** as shown in Figure.



Level curves show the lines of equal temperature (isotherms) measured in degrees Fahrenheit.

- Another common use of level curves is in representing electric potential fields.

- In this type of map, the level curves are called **equipotential lines.**

Level Curves

- Contour maps are commonly used to show regions on Earth's surface, with the level curves representing the height above sea level. This type of map is called a **topographic map.** For example, the mountain shown in Figure 13.7 is represented by the topographic map in Figure 13.8.

Figure 13.7

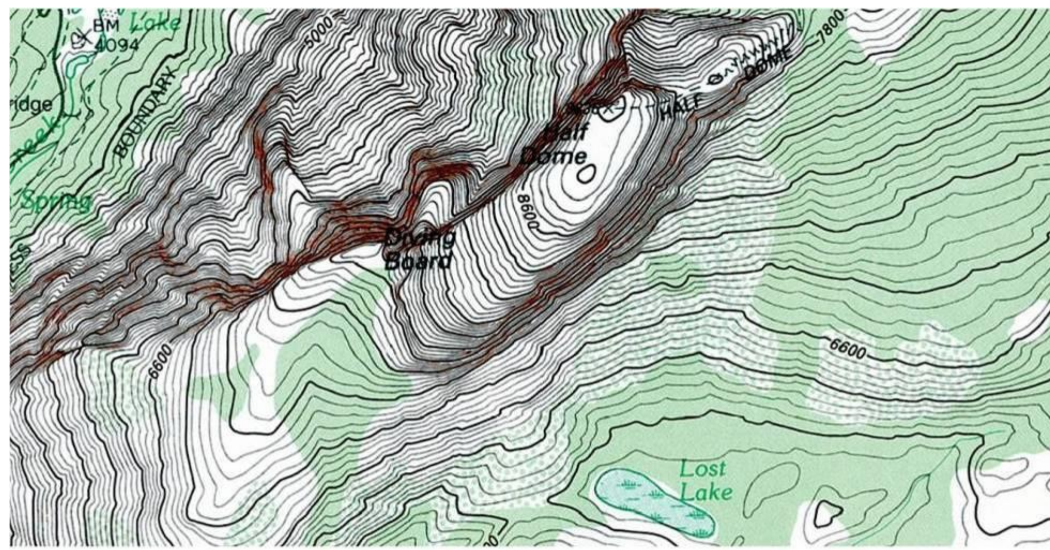


Figure 13.8

Level Curves

- A contour map depicts the variation of z with respect to x and y by the spacing between level curves.

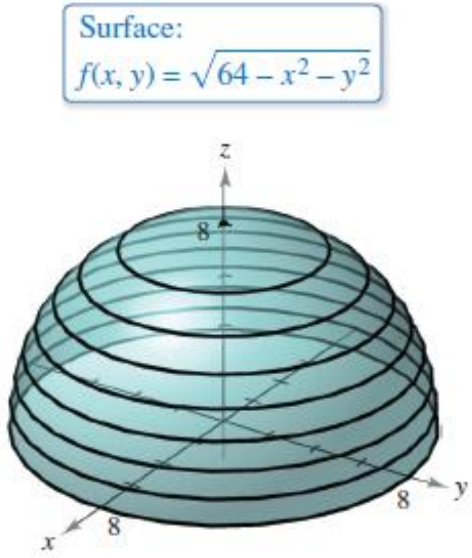
- Much space between level curves indicates that z is changing slowly, whereas little space indicates a rapid change in z.

- Furthermore, to produce a good three-dimensional illusion in a contour map, it is important to choose c-values that are *evenly spaced*.

Example 3: *Sketching a Contour Map*

- The hemisphere $f(x, y) = \sqrt{64 - x^2 - y^2}$

is shown in Figure. Sketch a contour map of this surface using level curves corresponding to $c = 0, 1, 2, \ldots, 8$.



Hemisphere

Example 3 – *Solution*

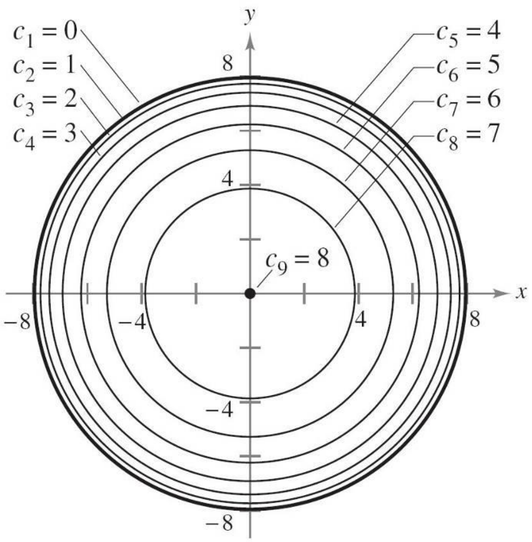
- For each value of c, the equation $f(x, y) = c$ is a circle (or point) in the xy-plane.

- For example, when $c_1 = 0$, the level curve is

$$x^2 + y^2 = 64 \qquad \text{Circle of radius 8}$$

which is a circle of radius 8.

Figure shows the nine level curves for the hemisphere.

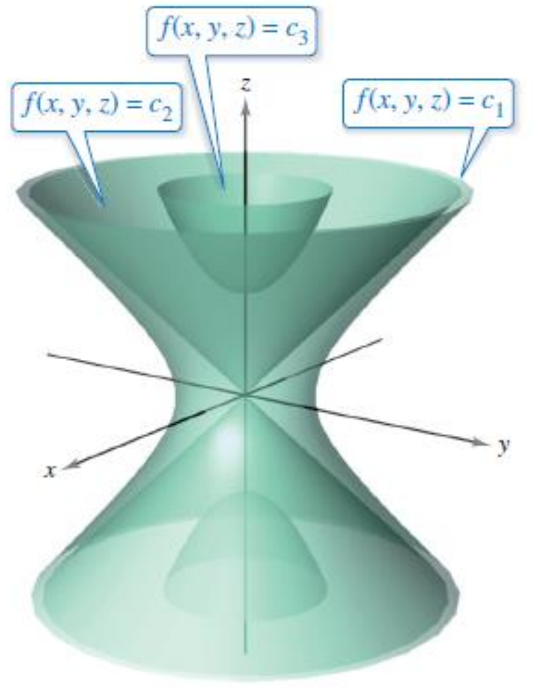


Contour map

Level Surfaces

- The concept of a level curve can be extended by one dimension to define a **level surface.**

- If f is a function of three variables and c is a constant, then the graph of the equation $f(x, y, z) = c$ is a **level surface** of f, as shown in Figure.



Level surfaces of f

Example 6 – *Level Surfaces*

- Describe the level surfaces of

$$f(x, y, z) = 4x^2 + y^2 + z^2.$$

- Solution:

Each level surface has an equation of the form

$$4x^2 + y^2 + z^2 = c. \qquad \text{Equation of level surface}$$

So, the level surfaces are ellipsoids (whose cross sections parallel to the *yz*-plane are circles).

As *c* increases, the radii of the circular cross sections increase according to the square root of *c*.

Example 6 – *Solution*

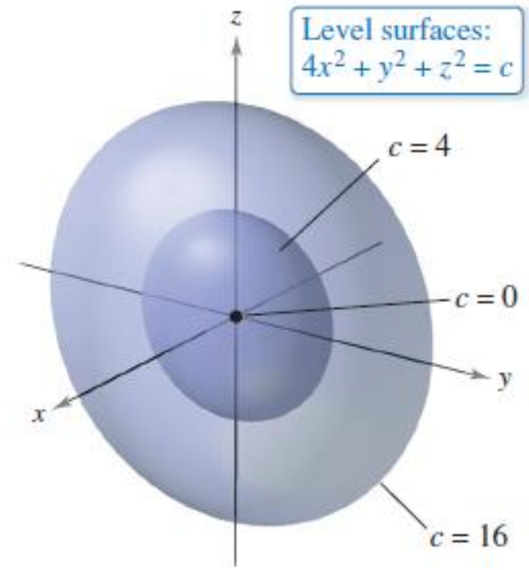
- For example, the level surfaces corresponding to the values c = 0, c = 4, and c = 16 are as follows.

$$4x^2 + y^2 + z^2 = 0 \qquad \text{Level surface for } c = 0 \text{ (single point)}$$

$$\frac{x^2}{1} + \frac{y^2}{4} + \frac{z^2}{4} = 1 \qquad \text{Level surface for } c = 4 \text{ (ellipsoid)}$$

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{16} = 1 \qquad \text{Level surface for } c = 16 \text{ (ellipsoid)}$$

- These level surfaces are shown in Figure.



Level Surfaces

- If the function represented the *temperature* at the point (x, y, z), then the level surfaces shown in previous Figure would be called **isothermal surfaces.**

Computer Graphics

- The problem of sketching the graph of a surface in space can be simplified by using a computer.

- Although there are several types of three-dimensional graphing utilities, most use some form of trace analysis to give the illusion of three dimensions.

- To use such a graphing utility, you usually need to enter the equation of the surface and the region in the *xy*-plane over which the surface is to be plotted. (You might also need to enter the number of traces to be taken.)

- For instance, to graph the surface

$$f(x, y) = (x^2 + y^2)e^{1-x^2-y^2}$$

- you might choose the following bounds for *x*, *y*, and *z*.

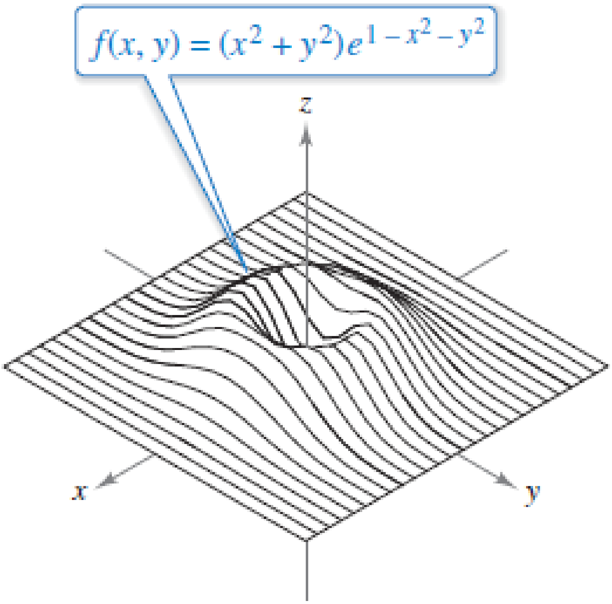
$$-3 \le x \le 3 \qquad \text{Bounds for } x$$

$$-3 \le y \le 3 \qquad \text{Bounds for } y$$

$$0 \le z \le 3 \qquad \text{Bounds for } z$$

- Figure shows a computer-generated graph of this surface using 26 traces taken parallel to the *yz*-plane.

- To heighten the three-dimensional effect, the program uses a "hidden line" routine.

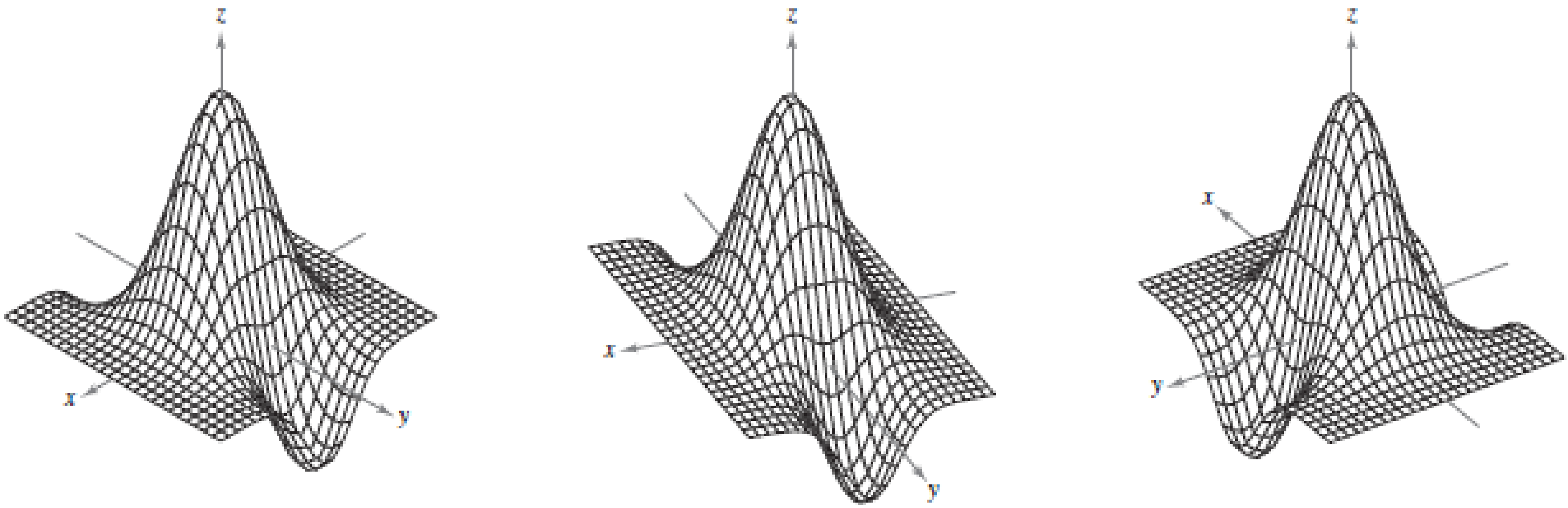


- That is, it begins by plotting the traces in the foreground (those corresponding to the largest x-values), and then, as each new trace is plotted, the program determines whether all or only part of the next trace should be shown.

- The graphs on the next slide show a variety of surfaces that were plotted by computer.

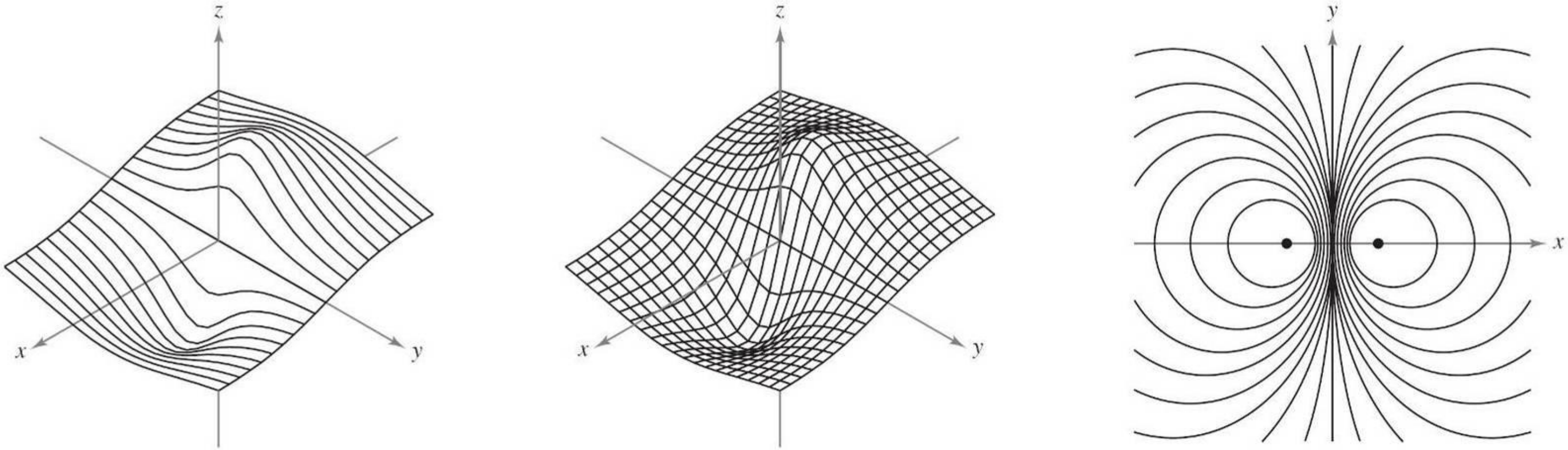
- If you have access to a computer drawing program, use it to reproduce these surfaces.

Computer Graphics



Three different views of the graph of $f(x, y) = (2 - y^2 + x^2)e^{1-x^2-(y^2/4)}$

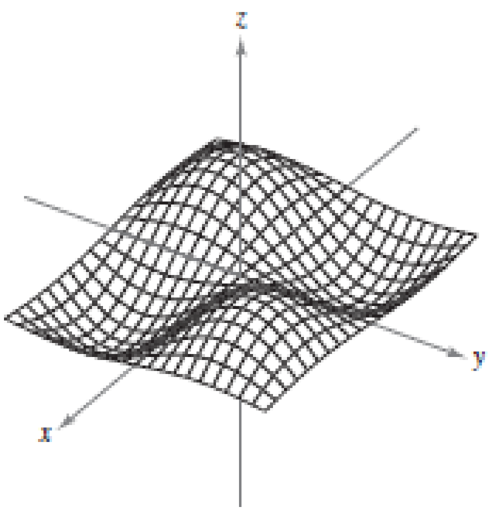
Computer Graphics



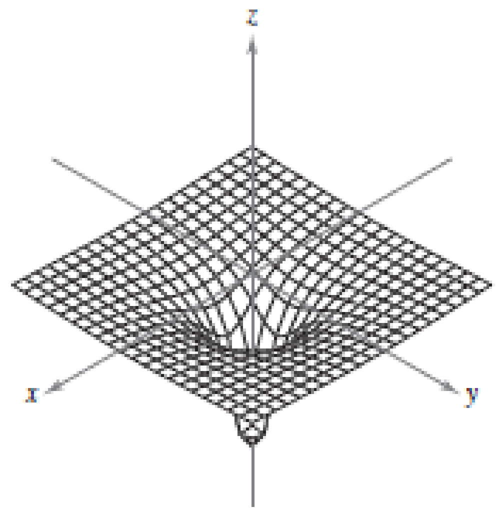
Single traces
Double traces
Level curves

Traces and level curves of the graph of $f(x, y) = \dfrac{-4x}{x^2 + y^2 + 1}$

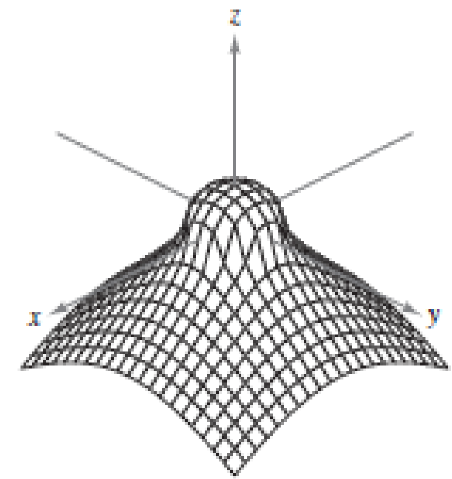
Computer Graphics



$f(x, y) = \sin x \sin y$



$$f(x, y) = -\frac{1}{\sqrt{x^2 + y^2}}$$



$$f(x, y) = \frac{1 - x^2 - y^2}{\sqrt{\left|1 - x^2 - y^2\right|}}$$

Suggested Problems

Exercise: 13.1-9,10,14,27,28,43,44,54,57

Thanks a lot …