13:3-Partial Derivatives

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Objectives

- Find and use partial derivatives of a function of two variables.
- Find and use partial derivatives of a function of three or more variables.
- Find higher-order partial derivatives of a function of two or three variables.

Partial Derivatives of a Function of Two Variables

- You can determine the rate of change of a function *f* with respect to one of its several independent variables.

- This process is called **partial differentiation,** and the result is referred to as the **partial derivative** of *f* with respect to the chosen independent variable.

Partial Derivatives of a Function of Two Variables

Definition of Partial Derivatives of a Function of Two Variables

If $z = f(x, y)$, then the **first partial derivatives** of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$ Partial derivative with respect to x

and

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$ Partial derivative with respect to y

provided the limits exist.

- This definition indicates that if $z = f(x, y)$, then to find f_x, you *consider y constant* and differentiate with respect to x.
- Similarly, to find f_y, you *consider x constant* and differentiate with respect to y.

Example 1 – Finding Partial Derivatives

- **a**. To find f_x for $f(x, y) = 3x - x^2y^2 + 2x^3y$, consider y to be constant and differentiate with respect to x.

$$f_x(x, y) = 3 - 2xy^2 + 6x^2y \qquad \text{Partial derivative with respect to } x$$

- To find f_y, consider x to be constant and differentiate with respect to y.

$$f_y(x, y) = -2x^2y + 2x^3 \qquad \text{Partial derivative with respect to } y$$

Example 1 – Finding Partial Derivatives cont’d

- **b**. To find f_x for $(x, y) = (\ln x)(\sin x^2y)$, consider y to be constant and differentiate with respect to x.

$$f_x(x, y) = (\ln x)(\cos x^2y)(2xy) + \frac{\sin x^2y}{x} \qquad \text{Partial derivative with respect to } x$$

- To find f_y, consider x to be constant and differentiate with respect to y.

$$f_y(x, y) = (\ln x)(\cos x^2y)(x^2) \qquad \text{Partial derivative with respect to } y$$

Partial Derivatives of a Function of Two Variables

Notation for First Partial Derivatives

For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x}$$

Partial derivative with respect to x

and

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}.$$

Partial derivative with respect to y

The first partials evaluated at the point (a, b) are denoted by

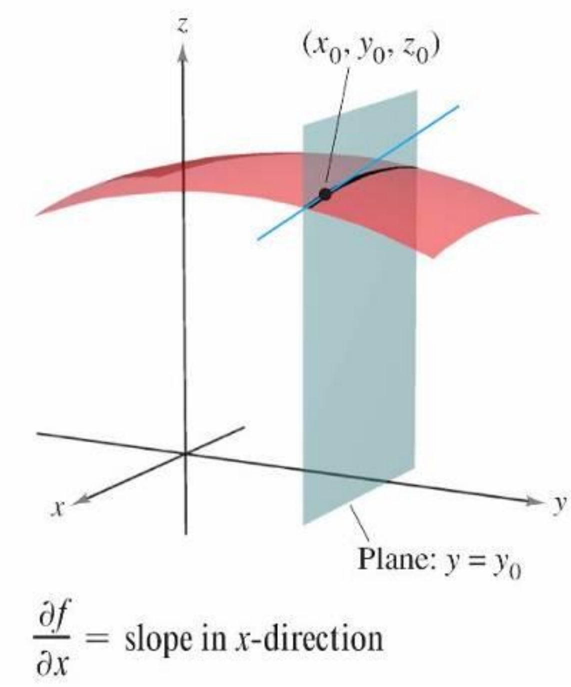
$$\left.\frac{\partial z}{\partial x}\right|_{(a,\, b)} = f_x(a, b)$$

and

$$\left.\frac{\partial z}{\partial y}\right|_{(a,\, b)} = f_y(a, b).$$

Partial Derivatives of a Function of Two Variables

- The partial derivatives of a function of two variables, $z = f(x, y)$, have a useful geometric interpretation.
- If $y = y_0$, then $z = f(x, y_0)$ represents the curve formed by intersecting the surface $z = f(x, y)$ with the plane $y = y_0$, as shown in Figure.



Partial Derivatives of a Function of Two Variables

- Therefore,

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

represents the slope of this curve at the point $(x_0, y_0, f(x_0, y_0))$.

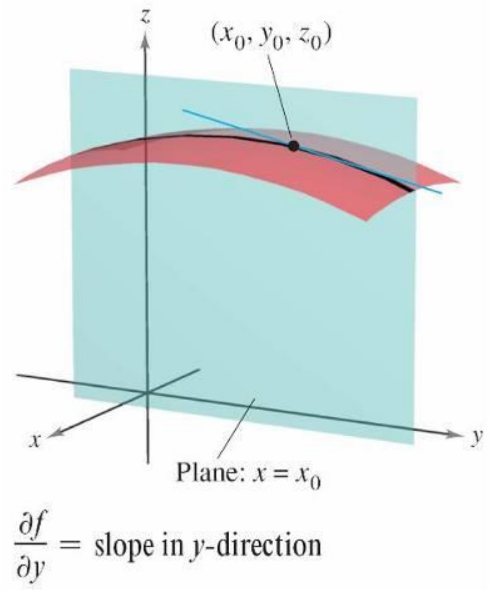
- Note that both the curve and the tangent line lie in the plane $y = y_0$.

Partial Derivatives of a Function of Two Variables

- Similarly,

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

represents the slope of the curve given by the intersection of $z = f(x, y)$ and the plane $x = x_0$ at $(x_0, y_0, f(x_0, y_0))$, as shown in Figure.



Partial Derivatives of a Function of Two Variables

- Informally, the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point (x_0, y_0, z_0) denote the **slopes of the surface in the *x*- and *y*-directions**, respectively.

Example 3 – Finding the Slopes of a Surface

- Find the slopes in the *x*-direction and in the *y*-direction of the surface

$$f(x, y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$$

at the point $\left(\frac{1}{2}, 1, 2\right)$.

- Solution:

The partial derivatives of *f* with respect to *x* and *y* are

$$f_x(x, y) = -x \qquad \text{and} \qquad f_y(x, y) = -2y.$$

Partial derivatives

Example 3 – Solution

cont'd

- So, in the *x*-direction, the slope is

$$f_x\left(\frac{1}{2}, 1\right) = -\frac{1}{2}$$

Figure 13.30

- and in the *y*-direction, the slope is

$$f_y\left(\frac{1}{2}, 1\right) = -2.$$

Figure 13.31

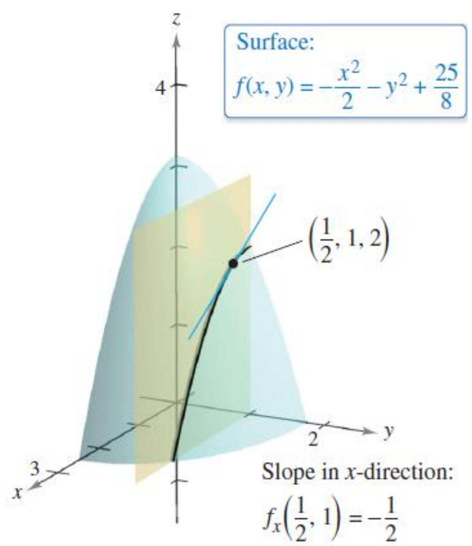


Figure 13.30

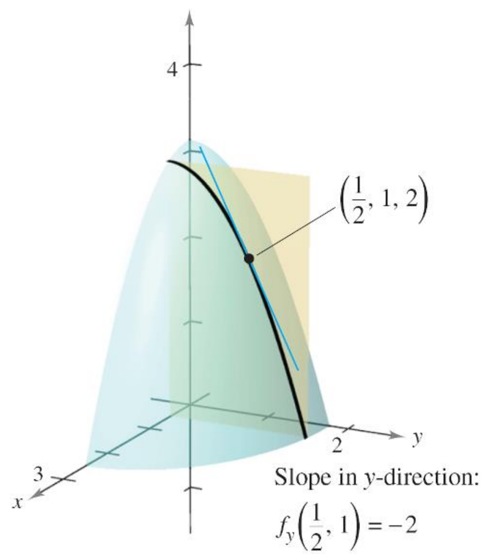


Figure 13.31

Partial Derivatives of a Function of Three or More Variables

- The concept of a partial derivative can be extended naturally to functions of three or more variables. For instance, if $w = f(x, y, z)$, then there are three partial derivatives, each of which is formed by holding two of the variables constant.

- That is, to define the partial derivative of w with respect to x, consider y and z to be constant and differentiate with respect to x.

- A similar process is used to find the derivatives of w with respect to y and with respect to z.

Partial Derivatives of a Function of Three or More Variables

$$\frac{\partial w}{\partial x} = f_x(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$$\frac{\partial w}{\partial y} = f_y(x, y, z) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

$$\frac{\partial w}{\partial z} = f_z(x, y, z) = \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

- In general, if $w = f(x_1, x_2, \ldots, x_n)$, then there are n partial derivatives denoted by

$$\frac{\partial w}{\partial x_k} = f_{x_k}(x_1, x_2, \ldots, x_n), \quad k = 1, 2, \ldots, n.$$

- To find the partial derivative with respect to one of the variables, hold the other variables constant and differentiate with respect to the given variable.

Example 6 – Finding Partial Derivatives

- **a.** To find the partial derivative of $f(x, y, z) = xy + yz^2 + xz$ with respect to z, consider x and y to be constant and obtain

$$\frac{\partial}{\partial z}[xy + yz^2 + xz] = 2yz + x.$$

- **b.** To find the partial derivative of $f(x, y, z) = z\sin(xy^2 + 2z)$ with respect to z, consider x and y to be constant. Then, using the Product Rule, you obtain

$$\begin{aligned}\frac{\partial}{\partial z}[z\sin(xy^2 + 2z)] &= (z)\frac{\partial}{\partial z}[\sin(xy^2 + 2z)] + \sin(xy^2 + 2z)\frac{\partial}{\partial z}[z] \\ &= (z)[\cos(xy^2 + 2z)](2) + \sin(xy^2 + 2z) \\ &= 2z\cos(xy^2 + 2z) + \sin(xy^2 + 2z).\end{aligned}$$

Example 6 – Finding Partial Derivatives

cont'd

- **c.** To find the partial derivative of

$$f(x, y, z, w) = \frac{x + y + z}{w}$$

with respect to w, consider x, y, and z to be constant obtain

$$\frac{\partial}{\partial w}\left[\frac{x + y + z}{w}\right] = -\frac{x + y + z}{w^2}.$$

Higher-Order Partial Derivatives

- The function $z = f(x, y)$ has the following second partial derivatives.
  - ✓ **1.** Differentiate twice with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}.$$

  - ✓ **2.** Differentiate twice with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$

  - ✓ **3.** Differentiate first with respect to x and then with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}.$$

Higher-Order Partial Derivatives

✓ **4.** Differentiate first with respect to *y* and then with respect to *x*:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

The third and fourth cases are called **mixed partial derivatives**.

Example 7 – Finding Second Partial Derivatives

- Find the second partial derivatives of $f(x, y) = 3xy^2 - 2y + 5x^2y^2$ and determine the value of $f_{xy}(-1, 2)$.

- Solution:

  Begin by finding the first partial derivatives with respect to x and y.

  $f_x(x, y) = 3y^2 + 10xy^2$ and $f_y(x, y) = 6xy - 2 + 10x^2y$

- Then, differentiate each of these with respect to x and y.

  $f_{xx}(x, y) = 10y^2$ and $f_{yy}(x, y) = 6x + 10x^2$

Example 7 – Solution

cont'd

$f_{xy}(x, y) = 6y + 20xy$ and $f_{yx}(x, y) = 6y + 20xy$

At (–1, 2), the value of f_{xy} is $f_{xy}(-1, 2) = 12 - 40 = -28$.

Suggested Problems

Exercise 13.3:16,28,37,43,52,55,62,68,88,92,71,129,128

Thanks a lot …