13:4-Differentials

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Objectives

- Understand the concepts of increments and differentials.
- Extend the concept of differentiability to a function of two variables.
- Use a differential as an approximation

Increments and Differentials

- For $y = f(x)$, the differential of y was defined as $dy = f'(x)dx$.

- Similar terminology is used for a function of two variables, $z = f(x, y)$. That is, Δx and Δy are the **increments of x and y**, and the **increment of z** is

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).$$

Increment of z

Increments and Differentials

Definition of Total Differential

If $z = f(x, y)$ and Δx and Δy are increments of x and y, then the **differentials** of the independent variables x and y are

$$dx = \Delta x \quad \text{and} \quad dy = \Delta y$$

and the **total differential** of the dependent variable z is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x, y)\,dx + f_y(x, y)\,dy.$$

- This definition can be extended to a function of three or more variables. For instance, if $w = f(x, y, z, u)$, then $dx = \Delta x$, $dy = \Delta y$, $dz = \Delta z$, $du = \Delta u$, and the total differential of w is

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz + \frac{\partial w}{\partial u}du.$$

Example 1 – *Finding the Total Differential*

- Find the total differential for each function.

  a. $z = 2x \sin y - 3x^2y^2$ **b.** $w = x^2 + y^2 + z^2$

- Solution:

  a. The total differential dz for $z = 2x \sin y - 3x^2y^2$ is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \qquad \text{Total differential } dz$$

$$= (2 \sin y - 6xy^2)\, dx + (2x \cos y - 6x^2y)\, dy.$$

b. The total differential dw for $w = x^2 + y^2 + z^2$ is

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz \qquad \text{Total differential } dw$$

$$= 2x\, dx + 2y\, dy + 2z\, dz.$$

Differentiability

- For a *differentiable* function given by $y = f(x)$, you can use the differential $dy = f'(x)dx$ as an approximation (for small Δx) of the value $\Delta y = f(x + \Delta x) - f(x)$.
- When a similar approximation is possible for a function of two variables, the function is said to be **differentiable**. This is stated explicitly in the following definition.

Definition of Differentiability

A function f given by $z = f(x, y)$ is **differentiable** at (x_0, y_0) if Δz can be written in the form

$$\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where both ε_1 and $\varepsilon_2 \to 0$ as

$$(\Delta x, \Delta y) \to (0, 0).$$

The function f is **differentiable in a region** R if it is differentiable at each point in R.

Example 2 – Showing That a Function Is Differentiable

- Show that the function

  $$f(x, y) = x^2 + 3y$$

  is differentiable at every point in the plane.

- Solution:

  Letting $z = f(x, y)$, the increment of z at an arbitrary point (x, y) in the plane is

  $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$ Increment of z

  $$= (x + \Delta x)^2 + 3(y + \Delta y) - (x^2 + 3y)$$

  $$= x^2 + 2x\Delta x + (\Delta x^2) + 3y + 3\Delta y - x^2 - 3y$$

  $$= 2x\Delta x + (\Delta x)^2 + 3\Delta y$$

Example 2 – *Solution*

cont'd

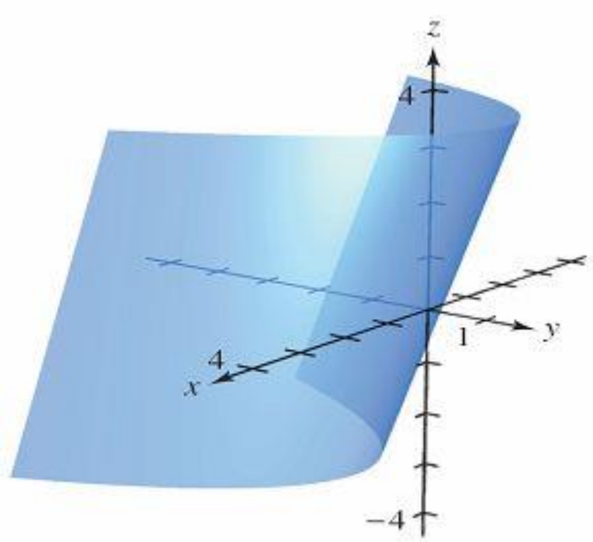
$$= 2x(\Delta x) + 3(\Delta y) + \Delta x(\Delta x) + 0(\Delta y)$$

$$= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

where $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = 0$.

✓ Because $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$, it follows that f is differentiable at every point in the plane.

The graph of f is shown in Figure.



Differentiability

THEOREM 13.4 Sufficient Condition for Differentiability

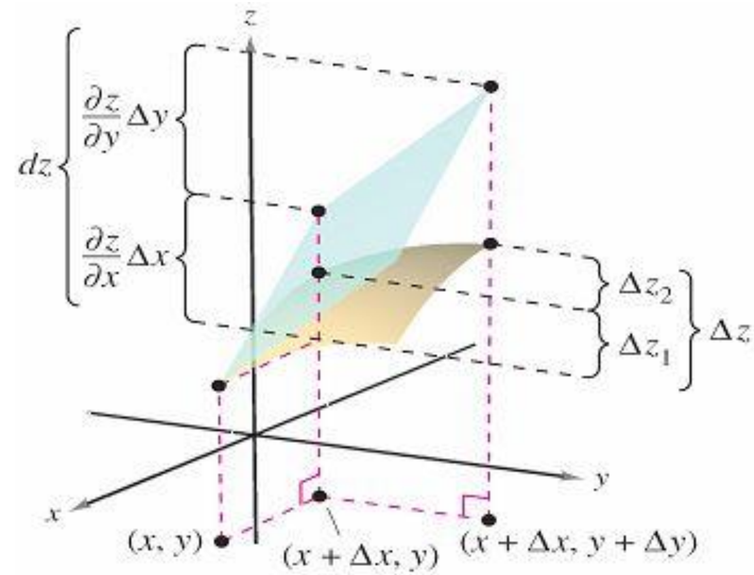
If f is a function of x and y, where f_x and f_y are continuous in an open region R, then f is differentiable on R.

Approximation by Differentials

- Theorem 13.4 tells you that you can choose $(x + \Delta x, y + \Delta y)$ close enough to (x, y) to make $\varepsilon_1 \Delta x$ and $\varepsilon_2 \Delta y$ insignificant. In other words, for small Δx and Δy, you can use the approximation

✓ $\Delta z \approx dz.$ Approximate change in z

✓ This approximation is illustrated graphically in Figure 13.35.



The exact change in z is Δz. This change can be approximated by the differential dz.

Approximation by Differentials

- The partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ can be interpreted as the slopes of the surface in the x- and y-directions.

- This means that

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

represents the change in height of a plane that is tangent to the surface at the point $(x, y, f(x, y))$.

- Because a plane in space is represented by a linear equation in the variables x, y, and z, the approximation of Δz by dz is called a **linear approximation**.

Example 3 – *Using a Differential as an Approximation*

- Use the differential dz to approximate the change in $z = \sqrt{4 - x^2 - y^2}$ as (x, y) moves from the point (1, 1) to the point (1.01, 0.97). Compare this approximation with the exact change in z.

- Solution:

  Letting $(x, y) = (1, 1)$ and $(x + \Delta x, y + \Delta y) = (1.01, 0.97)$ produces

  $dx = \Delta x = 0.01$ and $dy = \Delta y = -0.03$.

Example 3 – *Solution*

cont'd

- So, the change in z can be approximated by

$$\Delta z \approx dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$= \frac{-x}{\sqrt{4 - x^2 - y^2}}\Delta x + \frac{-y}{\sqrt{4 - x^2 - y^2}}\Delta y.$$

- When $x = 1$ and $y = 1$, you have

$$\Delta z \approx -\frac{1}{\sqrt{2}}(0.01) - \frac{1}{\sqrt{2}}(-0.03)$$

$$= \frac{0.02}{\sqrt{2}}$$

$$= \sqrt{2}(0.01) \approx 0.0141.$$

Example 3 – *Solution*

cont'd

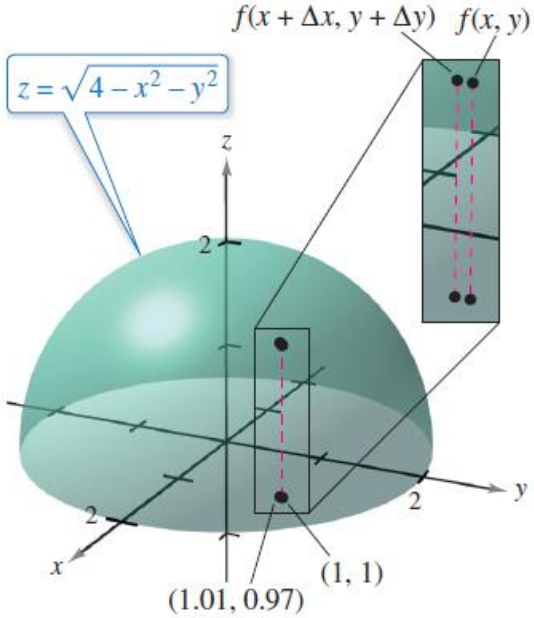
- In Figure, you can see that the exact change corresponds to the difference in the heights of two points on the surface of a hemisphere.

- This difference is given by

$\Delta z = f(1.01, 0.97) - f(1, 1)$

$$= \sqrt{4 - (1.01)^2 - (0.97)^2} - \sqrt{4 - 1^2 - 1^2}$$

$$\approx 0.0137.$$



As (x, y) moves from the point $(1, 1)$ to the point $(1.01, 0.97)$, the value of $f(x, y)$ changes by about 0.0137.

Approximation by Differentials

- A function of three variables $w = f(x, y, z)$ is **differentiable** at (x, y, z) provided that
  $\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$
  can be written in the form
  $\Delta w = f_x\Delta x + f_y\Delta y + f_z\Delta z + \varepsilon_1\Delta x + \varepsilon_2\Delta y + \varepsilon_3\Delta z$
  ✓where $\varepsilon_1, \varepsilon_2$, and $\varepsilon_3 \to 0$ as $(\Delta x, \Delta y, \Delta z) \to (0, 0, 0)$.

- With this definition of differentiability, Theorem 13.4 has the following extension for functions of three variables: If f is a function of x, y, and z, where f_x, f_y, and f_z are continuous in an open region R, then f is differentiable on R.

Approximation by Differentials

- As is true for a function of a single variable, when a function in two or more variables is differentiable at a point, it is also continuous there.

THEOREM 13.5 Differentiability Implies Continuity

If a function of x and y is differentiable at (x_0, y_0), then it is continuous at (x_0, y_0).

Example 5 – *A Function That Is Not Differentiable*

- For the function,

$$f(x, y) = \begin{cases} \dfrac{-3xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

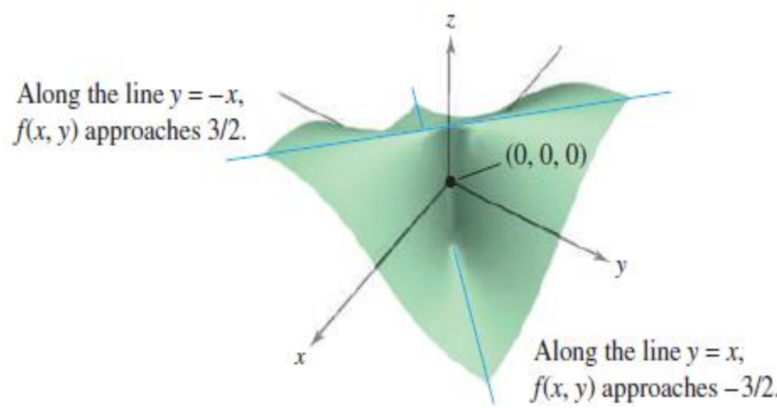
show that $f_x(0, 0)$ and $f_y(0, 0)$ both exist, but that f is not differentiable at (0, 0).

Example 5 – *Solution*

- You can show that f is not differentiable at (0, 0) by showing that it is not continuous at this point.

- To see that f is not continuous at (0, 0), look at the values of $f(x, y)$ along two different approaches to (0, 0), as shown in Figure.

$$f(x, y) = \begin{cases} \dfrac{-3xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$



Example 5 – *Solution* cont'd

- Along the line $y = x$, the limit is

$$\lim_{(x,\, x)\to(0,\, 0)} f(x, y) = \lim_{(x,\, x)\to(0,\, 0)} \frac{-3x^2}{2x^2}$$

whereas along $y = -x$ you have

$$\lim_{(x,\, -x)\to(0,\, 0)} f(x, y) = \lim_{(x,\, -x)\to(0,\, 0)} \frac{3x^2}{2x^2} = \frac{3}{2}.$$

- So, the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ does not exist, and you can conclude that f is not continuous at (0, 0).

- Therefore, by Theorem 13.5, you know that f is not differentiable at (0, 0).

Example 5 – *Solution* cont'd

- On the other hand, by the definition of the partial derivatives f_x and f_y you have

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0$$

and

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} = 0.$$

✓ So, the partial derivatives at (0, 0) exist.

Suggested Problems

Exercise 13.4:7,8,11,13,16,23,32.

Thanks a lot …