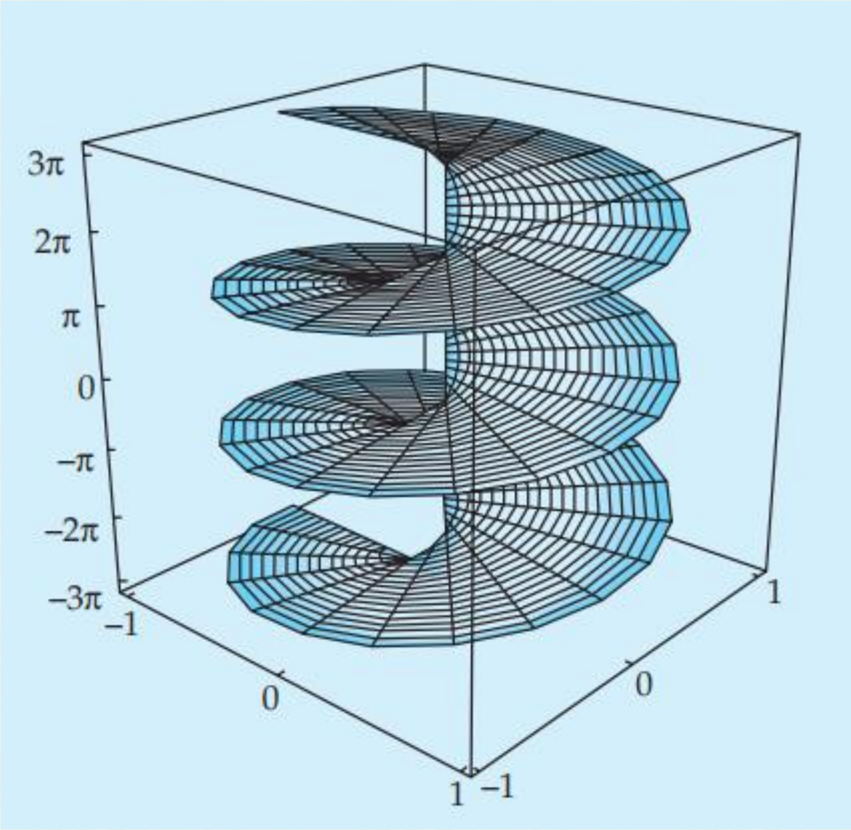
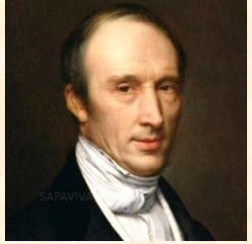
Complex Number System





Complex Number

A **complex number** is any number of the form $z = a + ib$ where a and b are real numbers and i is the imaginary unit.

The notations $a + ib$ and $a + bi$ are used interchangeably.

The real number a in $z = a + ib$ is called the **real part** of z; the real number b is called the **imaginary part** of z. The real and imaginary parts of a complex number z are abbreviated $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively. For example, if $z = 6i$ is a pure imaginary number.

Properties of Complex Number

- **Equality**

Complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are **equal**, $z_1 = z_2$, if $a_1 = a_2$ and $b_1 = b_2$.

In terms of the symbols $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, $z_1 = z_2$ if $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

- **Arithmetic Operations**

Complex numbers can be added, subtracted, multiplied, and divided. If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, these operations are defined as follows:

Addition: $z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$

Properties of Complex Number

- **Arithmetic Operations**

Complex numbers can be added, subtracted, multiplied, and divided. If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, these operations are defined as follows:

$$\begin{aligned}
&\textit{Subtraction:} && z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i(b_1 - b_2) \\
&\textit{Multiplication:} && z_1 \cdot z_2 = (a_1 + ib_1)(a_2 + ib_2) \\
& && = a_1 a_2 - b_1 b_2 + i(b_1 a_2 + a_1 b_2) \\
&\textit{Division:} && \frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2}, \ a_2 \neq 0, \text{ or } b_2 \neq 0 \\
& && \phantom{\frac{z_1}{z_2}} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\,\frac{b_1 a_2 - a_1 b_2}{a_2^2 + b_2^2}
\end{aligned}$$

Properties of Complex Number

- The familiar commutative, associative, and distributive laws hold for complex numbers:

$$\textit{Commutative laws:} \quad \begin{cases} z_1 + z_2 = z_2 + z_1 \\ z_1 z_2 = z_2 z_1 \end{cases}$$

$$\textit{Associative laws:} \quad \begin{cases} z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \\ z_1(z_2 z_3) = (z_1 z_2) z_3 \end{cases}$$

$$\textit{Distributive law:} \quad z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

- Conjugate

If z is a complex number, the number obtained by changing the sign of its imaginary part is called the **complex conjugate**, or simply **conjugate**, of z and is denoted by the symbol $\bar{z}$.

Properties of Complex Number

- Conjugate

  If z is a complex number, the number obtained by changing the sign of its imaginary part is called the **complex conjugate**, or simply **conjugate**, of z and is denoted by the symbol $\bar{z}$. In other words, if $z = a+ib$ then its conjugate is $\bar{z} = a - ib$. For example, if $z = 6 + 3i$, then $\bar{z} = 6 - 3i$; if $z = -5 - i$, then $\bar{z} = -5 + i$. If z is a real number, say, $z = 7$, then $\bar{z} = 7$.

Absolute Value

The *absolute value* or *modulus* of a complex number $a + bi$ is defined as $|a + bi| = \sqrt{a^2 + b^2}$.

EXAMPLE 1.1: $|-4 + 2i| = \sqrt{(-4)^2 + (2)^2} = \sqrt{20} = 2\sqrt{5}$.

If $z_1, z_2, z_3, \ldots, z_m$ are complex numbers, the following properties hold.

(1)	$\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$	or	$\lvert z_1 z_2 \cdots z_m\rvert = \lvert z_1\rvert\lvert z_2\rvert \cdots \lvert z_m\rvert$
(2)	$\left\lvert\dfrac{z_1}{z_2}\right\rvert = \dfrac{\lvert z_1\rvert}{\lvert z_2\rvert}$	if	$z_2 \neq 0$
(3)	$\lvert z_1 + z_2\rvert \leq \lvert z_1\rvert + \lvert z_2\rvert$	or	$\lvert z_1 + z_2 + \cdots + z_m\rvert \leq \lvert z_1\rvert + \lvert z_2\rvert + \cdots + \lvert z_m\rvert$
(4)	$\lvert z_1 \pm z_2\rvert \geq \lvert z_1\rvert - \lvert z_2\rvert$		

Roots of Complex Numbers

A number w is called an nth *root* of a complex number z if $w^n = z$, and we write $w = z^{1/n}$. From De Moivre's theorem we can show that if n is a positive integer,

$$\begin{aligned} z^{1/n} &= \{r(\cos\theta + i\sin\theta)\}^{1/n} \\ &= r^{1/n}\left\{\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right\} \quad k = 0, 1, 2, \ldots, n-1 \end{aligned} \tag{1.6}$$

from which it follows that there are n different values for $z^{1/n}$, i.e., n different nth roots of z, provided $z \neq 0$.

Graphical Representation of Complex Numbers

Complex Plane Because of the correspondence between a complex number $z = x + iy$ and one and only one point (x, y) in a coordinate plane, we shall use the terms *complex number* and *point* interchangeably. The coordinate plane illustrated in Figure 1.1 is called the **complex plane** or simply the **z-plane**. The horizontal or x-axis is called the **real axis** because each point on that axis represents a real number. The vertical or y-axis is called the **imaginary axis** because a point on that axis represents a pure imaginary number.

Graphical Representation of Complex Numbers

Vectors In other courses you have undoubtedly seen that the numbers in an ordered pair of real numbers can be interpreted as the components of a vector. Thus, a complex number $z = x + iy$ can also be viewed as a two-dimensional position **vector**, that is, a vector whose initial point is the origin and whose terminal point is the point (x, y). See Figure 1.2. This vector interpretation prompts us to define the length of the vector z as the distance $\sqrt{x^2 + y^2}$ from the origin to the point (x, y). This length is given a special name.

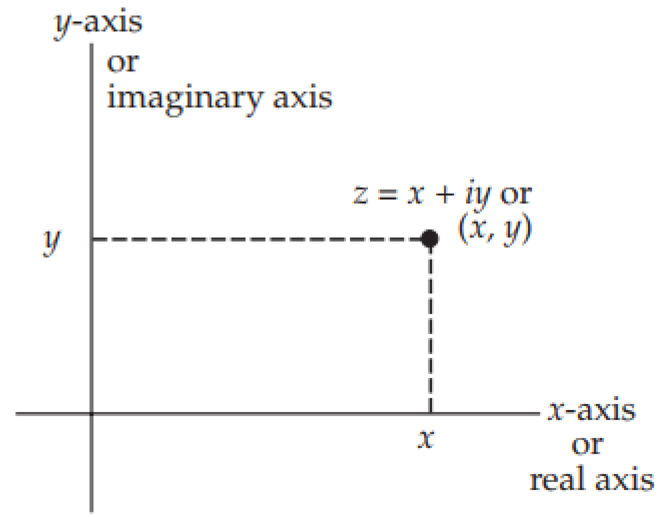


Figure 1.1 z-plane

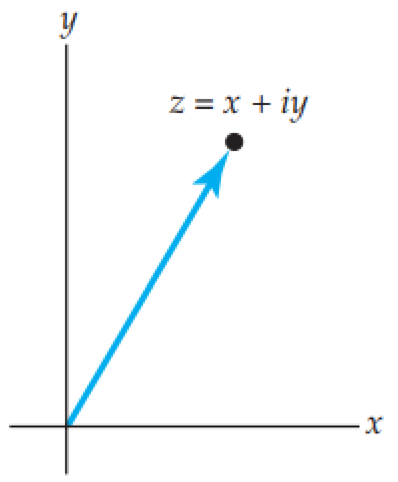


Figure 1.2 z as a position vector

Polar form of Complex Numbers

Let P be a point in the complex plane corresponding to the complex number (x, y) or $x + iy$. Then we see from Fig. 1-3 that

$$x = r\cos\theta, \qquad y = r\sin\theta$$

where $r = \sqrt{x^2 + y^2} = |x + iy|$ is called the *modulus* or *absolute value* of $z = x + iy$ [denoted by mod z or $|z|$]; and θ, called the *amplitude* or *argument* of $z = x + iy$ [denoted by arg z], is the angle that line OP makes with the positive x axis.

It follows that

$$z = x + iy = r(\cos\theta + i\sin\theta) \tag{1.1}$$

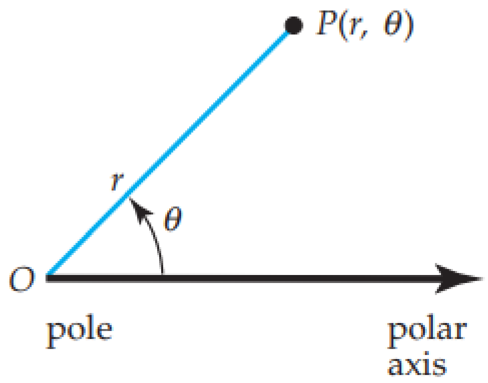


Figure 1.6 Polar coordinates

which is called the *polar form* of the complex number, and r and θ are called *polar coordinates*. It is sometimes convenient to write the abbreviation cis θ for $\cos\theta + i\sin\theta$.

For any complex number $z \neq 0$ there corresponds only one value of θ in $0 \leq \theta < 2\pi$. However, any other interval of length 2π, for example $-\pi < \theta \leq \pi$, can be used. Any particular choice, decided upon in advance, is called the *principal range*, and the value of θ is called its *principal value*.

De Moiver's Theorem

Let $z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1 z_2 = r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\} \tag{1.2}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\} \tag{1.3}$$

A generalization of (1.2) leads to

$$z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n\{\cos(\theta_1 + \theta_2 + \cdots + \theta_n) + i\sin(\theta_1 + \theta_2 + \cdots + \theta_n)\} \tag{1.4}$$

and if $z_1 = z_2 = \cdots = z_n = z$ this becomes

$$z^n = \{r(\cos\theta + i\sin\theta)\}^n = r^n(\cos n\theta + i\sin n\theta) \tag{1.5}$$

which is often called *De Moivre's theorem*.

Euler's Formula

By assuming that the infinite series expansion $e^x = 1 + x + (x^2/2!) + (x^3/3!) + \cdots$ of elementary calculus holds when $x = i\theta$, we can arrive at the result

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.7}$$

which is called *Euler's formula*. It is more convenient, however, simply to take (1.7) as a definition of $e^{i\theta}$. In general, we define

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y) \tag{1.8}$$

In the special case where $y = 0$ this reduces to e^x.

Note that in terms of (1.7) De Moivre's theorem reduces to $(e^{i\theta})^n = e^{in\theta}$.

n^{th} Roots of Complex Numbers

The solutions of the equation $z^n = 1$ where n is a positive integer are called the nth *roots of unity* and are given by

$$z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = e^{2k\pi i/n} \quad k = 0, 1, 2, \ldots, n-1 \tag{1.11}$$

If we let $\omega = \cos 2\pi/n + i\sin 2\pi/n = e^{2\pi i/n}$, the n roots are $1, \omega, \omega^2, \ldots, \omega^{n-1}$. Geometrically, they represent the n vertices of a regular polygon of n sides inscribed in a circle of radius one with center at the origin. This circle has the equation $|z| = 1$ and is often called the *unit circle*.

Perform each of the indicated operations.

$$\begin{aligned}(2-i)\{(-3+2i)(5-4i)\} &= (2-i)\{-15+12i+10i-8i^2\}\\ &= (2-i)(-7+22i) = -14+44i+7i-22i^2 = 8+51i\end{aligned}$$

$$(-1+2i)\{(7-5i)+(-3+4i)\} = (-1+2i)(4-i) = -4+i+8i-2i^2 = -2+9i$$

$$\frac{3-2i}{-1+i} = \frac{3-2i}{-1+i}\cdot\frac{-1-i}{-1-i} = \frac{-3-3i+2i+2i^2}{1-i^2} = \frac{-5-i}{2} = -\frac{5}{2}-\frac{1}{2}i$$

Example 1

Suppose $z_1 = 2 + i$, $z_2 = 3 - 2i$ and $z_3 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Evaluate each of the following.

Solution

(a) $$|3z_1 - 4z_2| = |3(2+i) - 4(3-2i)| = |6 + 3i - 12 + 8i|$$
$$= |-6 + 11i| = \sqrt{(-6)^2 + (11)^2} = \sqrt{157}$$

(b) $$z_1^3 - 3z_1^2 + 4z_1 - 8 = (2+i)^3 - 3(2+i)^2 + 4(2+i) - 8$$
$$= \{(2)^3 + 3(2)^2(i) + 3(2)(i)^2 + i^3\} - 3(4 + 4i + i^2) + 8 + 4i - 8$$
$$= 8 + 12i - 6 - i - 12 - 12i + 3 + 8 + 4i - 8 = -7 + 3i$$

(c) $$(\bar{z}_3)^4 = \left(\overline{-\frac{1}{2} + \frac{\sqrt{3}}{2}i}\right)^4 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^4 = \left[\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^2\right]^2$$
$$= \left[\frac{1}{4} + \frac{\sqrt{3}}{2}i + \frac{3}{4}i^2\right]^2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2 = \frac{1}{4} - \frac{\sqrt{3}}{2}i + \frac{3}{4}i^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

(d) $$\left|\frac{2z_2 + z_1 - 5 - i}{2z_1 - z_2 + 3 - i}\right|^2 = \left|\frac{2(3-2i) + (2+i) - 5 - i}{2(2+i) - (3-2i) + 3 - i}\right|^2$$
$$= \left|\frac{3 - 4i}{4 + 3i}\right|^2 = \frac{|3 - 4i|^2}{|4 + 3i|^2} = \frac{(\sqrt{(3)^2 + (-4)^2})^2}{(\sqrt{(4)^2 + (3)^2})^2} = 1$$

Example 2

Find real numbers x and y such that $3x + 2iy - ix + 5y = 7 + 5i$.

Solution

The given equation can be written as $3x + 5y + i(2y - x) = 7 + 5i$. Then equating real and imaginary parts, $3x + 5y = 7$, $2y - x = 5$. Solving simultaneously, $x = -1$, $y = 2$.

Example 3

Prove: (a) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, (b) $|z_1 z_2| = |z_1||z_2|$.

Solution

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then

(a) $$\overline{z_1 + z_2} = \overline{x_1 + iy_1 + x_2 + iy_2} = \overline{x_1 + x_2 + i(y_1 + y_2)}$$
$$= x_1 + x_2 - i(y_1 + y_2) = x_1 - iy_1 + x_2 - iy_2 = \overline{x_1 + iy_1} + \overline{x_2 + iy_2} = \bar{z}_1 + \bar{z}_2$$

(b) $$|z_1 z_2| = |(x_1 + iy_1)(x_2 + iy_2)| = |x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2)|$$
$$= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2} = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} = \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} = |z_1||z_2|$$

Example 4

Perform the indicated operations both analytically and graphically:

(a) $(3 + 4i) + (5 + 2i)$, (b) $(6 - 2i) - (2 - 5i)$, (c) $(-3 + 5i) + (4 + 2i) + (5 - 3i) + (-4 - 6i)$.

Solution

(a) *Analytically.* $(3 + 4i) + (5 + 2i) = 3 + 5 + 4i + 2i = 8 + 6i$

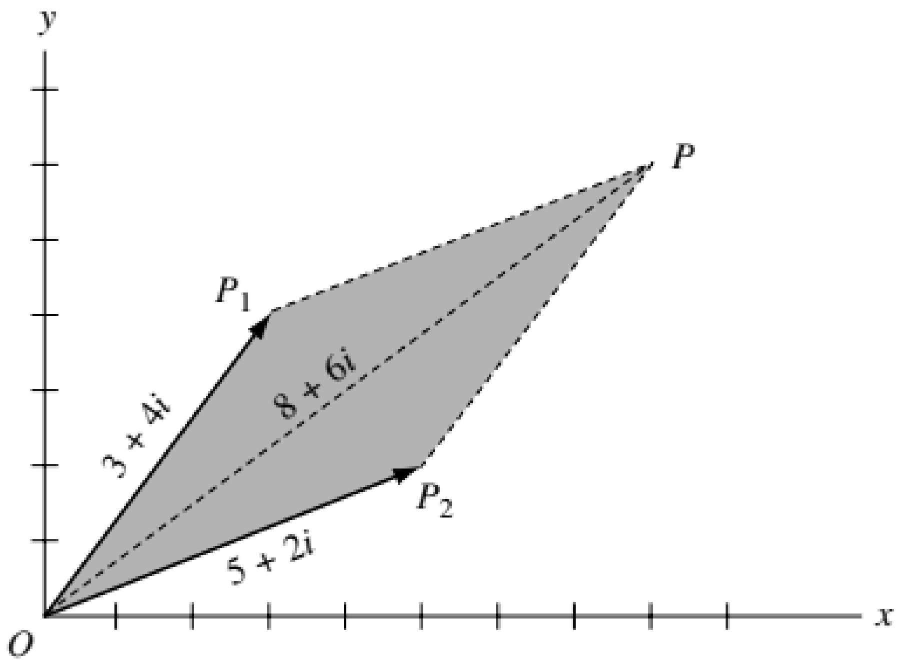


Fig. 1-7

Example 4

Graphically. Represent the two complex numbers by points P_1 and P_2, respectively, as in Fig. 1-7. Complete the parallelogram with OP_1 and OP_2 as adjacent sides. Point P represents the sum, $8 + 6i$, of the two given complex numbers. Note the similarity with the parallelogram law for addition of vectors OP_1 and OP_2 to obtain vector OP. For this reason it is often convenient to consider a complex number $a + bi$ as a vector having *components* a and b in the directions of the positive x and y axes, respectively.

(b) *Analytically.* $(6 - 2i) - (2 - 5i) = 6 - 2 - 2i + 5i = 4 + 3i$

Graphically. $(6 - 2i) - (2 - 5i) = 6 - 2i + (-2 + 5i)$. We now add $6 - 2i$ and $(-2 + 5i)$ as in part (a). The result is indicated by OP in Fig. 1-8.

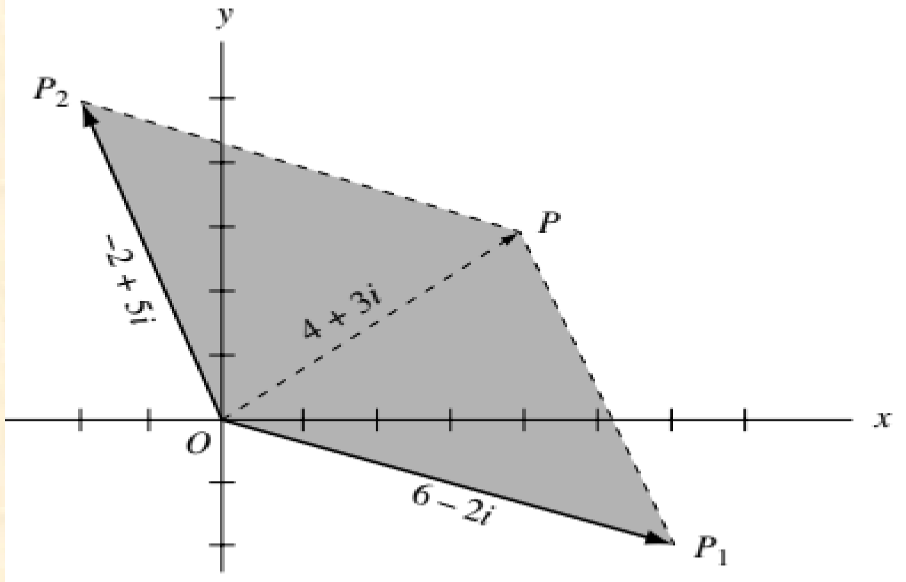


Fig. 1-8

Example 4

(c) *Analytically.*

$$(-3+5i)+(4+2i)+(5-3i)+(-4-6i)=(-3+4+5-4)+(5i+2i-3i-6i)=2-2i$$

Graphically. Represent the numbers to be added by z_1, z_2, z_3, z_4, respectively. These are shown graphically in Fig. 1-9. To find the required sum proceed as shown in Fig. 1-10. At the terminal point of vector z_1 construct vector z_2. At the terminal point of z_2 construct vector z_3, and at the terminal point of z_3 construct vector z_4. The required sum, sometimes called the *resultant*, is obtained by constructing the vector OP from the initial point of z_1 to the terminal point of z_4, i.e., $OP = z_1 + z_2 + z_3 + z_4 = 2 - 2i$.

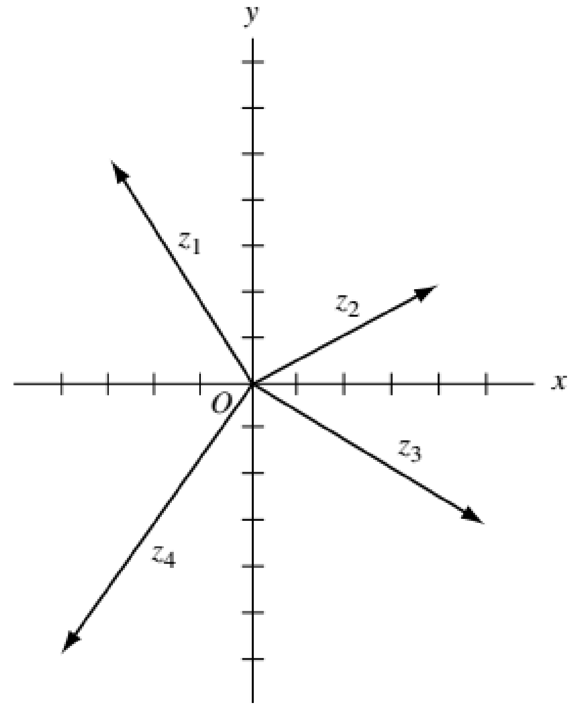


Fig. 1-9

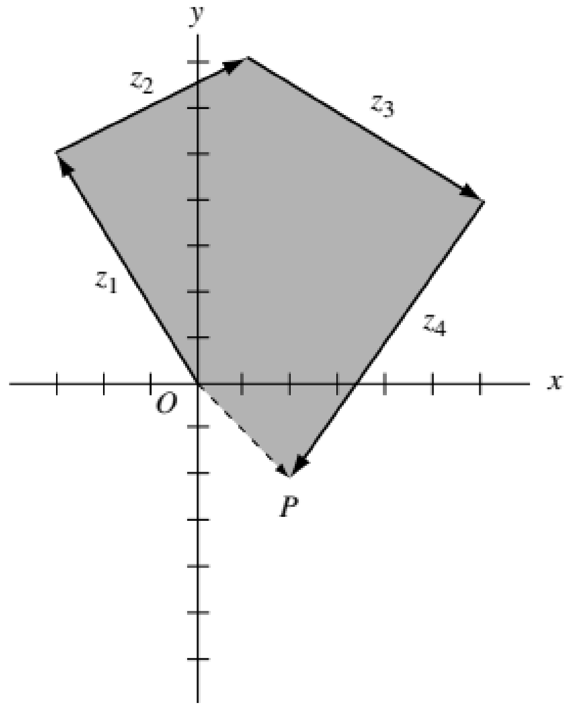


Fig. 1-10

Example 5

Suppose z_1 and z_2 are two given complex numbers (vectors) as in Fig. 1-11. Construct graphically

(a) $3z_1 - 2z_2$, (b) $\frac{1}{2}z_2 + \frac{5}{3}z_1$

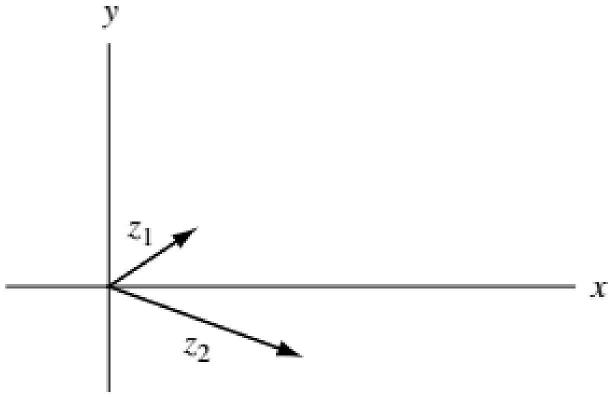


Fig. 1-11

Solution

(a) In Fig. 1-12, $OA = 3z_1$ is a vector having length 3 times vecter z_1 and the same direction. $OB = -2z_2$ is a vector having length 2 times vector z_2 and the opposite direction. Then vector $OC = OA + OB = 3z_1 - 2z_2$.

Example 5

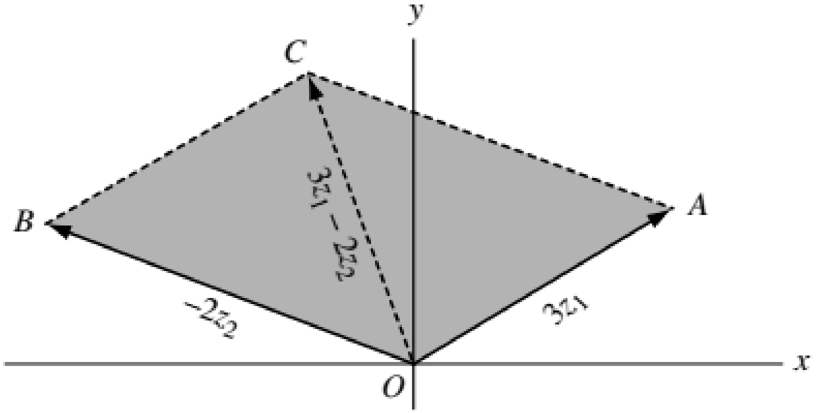


Fig. 1-12

(b) The required vector (complex number) is represented by *OP* in Fig. 1-13.

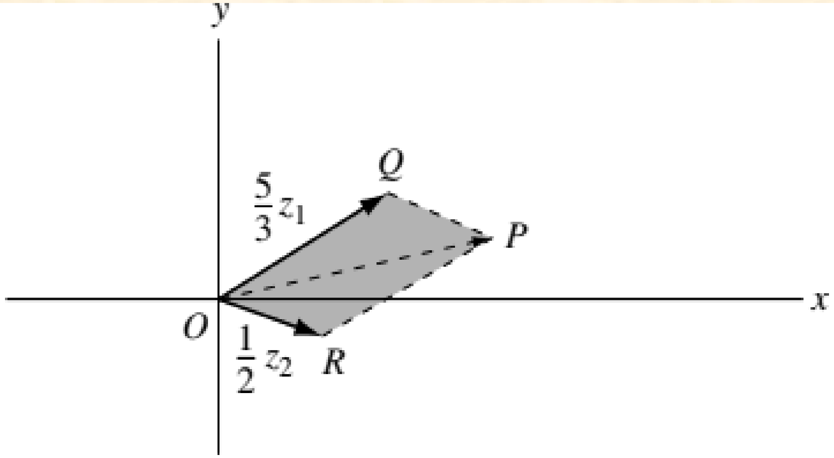


Fig. 1-13

Example 6

Prove (a) $|z_1 + z_2| \le |z_1| + |z_2|$, (b) $|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$, (c) $|z_1 - z_2| \ge |z_1| - |z_2|$ and give a graphical interpretation.

Solution

(a) *Analytically*. Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then we must show that

$$\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \le \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$$

Squaring both sides, this will be true if

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 \le x_1^2 + y_1^2 + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} + x_2^2 + y_2^2$$

i.e., if $$x_1x_2 + y_1y_2 \le \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

or if (squaring both sides again)

$$x_1^2x_2^2 + 2x_1x_2y_1y_2 + y_1^2y_2^2 \le x_1^2x_2^2 + x_1^2y_2^2 + y_1^2x_2^2 + y_1^2y_2^2$$

or $$2x_1x_2y_1y_2 \le x_1^2y_2^2 + y_1^2x_2^2$$

But this is equivalent to $(x_1y_2 - x_2y_1)^2 \ge 0$, which is true. Reversing the steps, which are reversible, proves the result.

Example 6: Solution

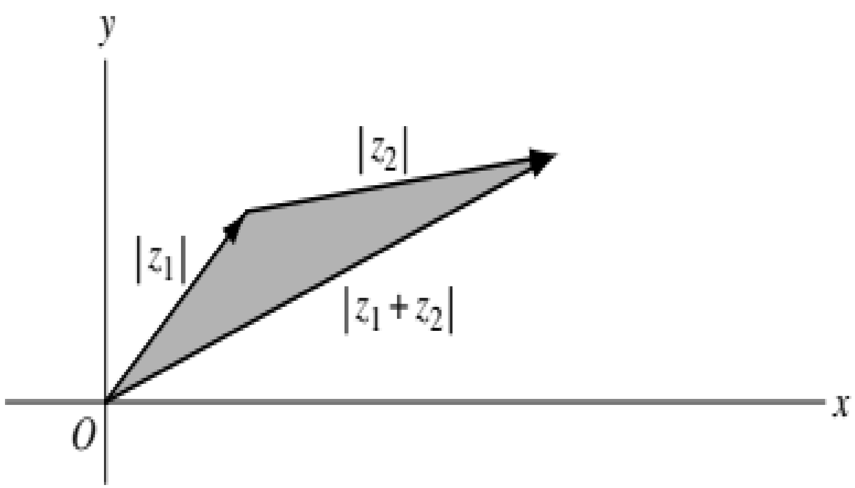


Fig. 1-14

Graphically. The result follows graphically from the fact that $|z_1|$, $|z_2|$, $|z_1 + z_2|$ represent the lengths of the sides of a triangle (see Fig. 1-14) and that the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side.

(b) *Analytically*. By part (a),

$$|z_1 + z_2 + z_3| = |z_1 + (z_2 + z_3)| \leq |z_1| + |z_2 + z_3| \leq |z_1| + |z_2| + |z_3|$$

Example 6

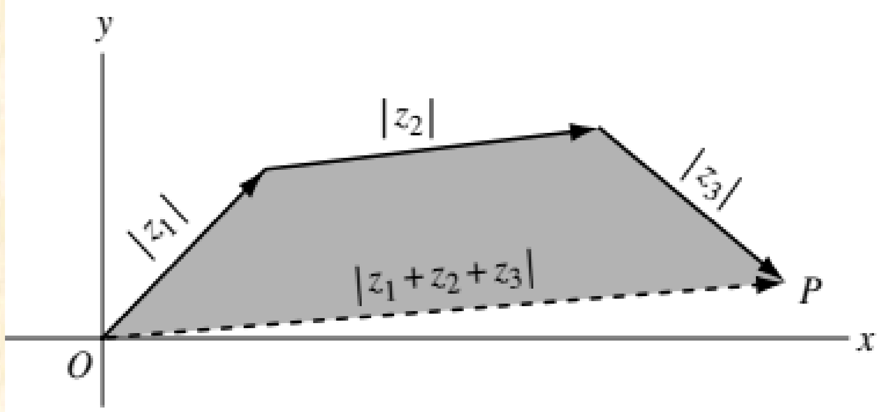


Fig. 1-15

Graphically. The result is a consequence of the geometric fact that, in a plane, a straight line is the shortest distance between two points O and P (see Fig. 1-15).

(c) *Analytically*. By part (a), $|z_1| = |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2|$. Then $|z_1 - z_2| \geq |z_1| - |z_2|$. An equivalent result obtained on replacing z_2 by $-z_2$ is $|z_1 + z_2| \geq |z_1| - |z_2|$.
Graphically. The result is equivalent to the statement that a side of a triangle has length greater than or equal to the difference in lengths of the other two sides.

Example 7

Let $A(1, -2)$, $B(-3, 4)$, $C(2, 2)$ be the three vertices of triangle ABC. Find the length of the median from C to the side AB.

Solution

The position vectors of A, B, and C are given by $z_1 = 1 - 2i$, $z_2 = -3 + 4i$ and $z_3 = 2 + 2i$, respectively. Then, from Fig. 1-19,

$$AC = z_3 - z_1 = 2 + 2i - (1 - 2i) = 1 + 4i$$

$$BC = z_3 - z_2 = 2 + 2i - (-3 + 4i) = 5 - 2i$$

$$AB = z_2 - z_1 = -3 + 4i - (1 - 2i) = -4 + 6i$$

$$AD = \tfrac{1}{2}AB = \tfrac{1}{2}(-4 + 6i) = -2 + 3i \quad \text{since } D \text{ is the midpoint of } AB.$$

$$AC + CD = AD \quad \text{or} \quad CD = AD - AC = -2 + 3i - (1 + 4i) = -3 - i.$$

Then the length of median CD is $|CD| = |-3 - i| = \sqrt{10}$.

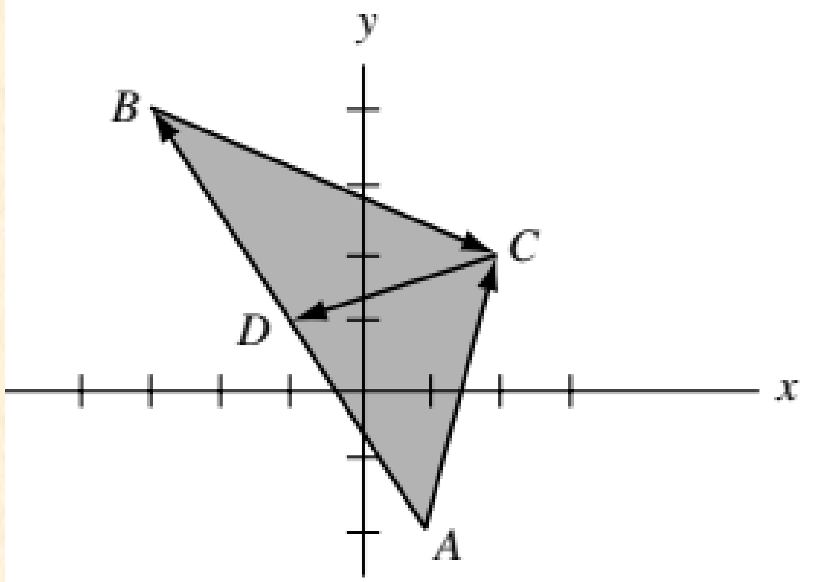


Fig. 1-19

Example 8

Find an equation for (a) a circle of radius 4 with center at $(-2, 1)$, (b) an ellipse with major axis of length 10 and foci at $(-3, 0)$ and $(3, 0)$.

Solution

(a) The center can be represented by the complex number $-2 + i$. If z is any point on the circle [Fig. 1-20], the distance from z to $-2 + i$ is

$$|z - (-2 + i)| = 4$$

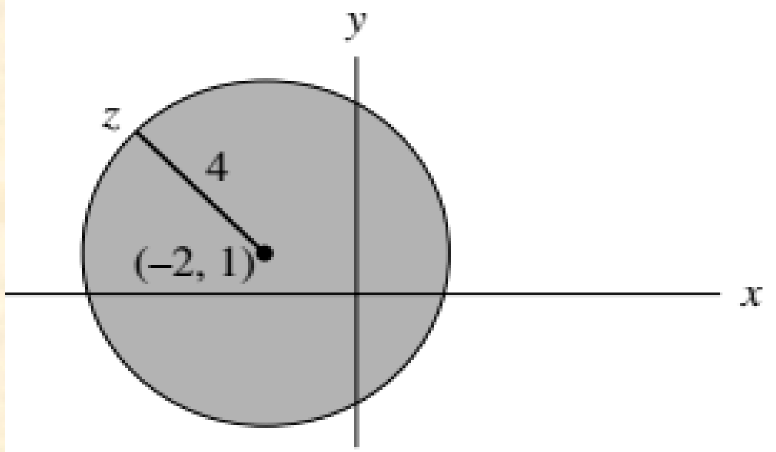


Fig. 1-20

Then $|z + 2 - i| = 4$ is the required equation. In rectangular form, this is given by

$$|(x + 2) + i(y - 1)| = 4, \quad \text{i.e., } (x + 2)^2 + (y - 1)^2 = 16$$

Example 8

(b) The sum of the distances from any point z on the ellipse [Fig. 1-21] to the foci must equal 10. Hence, the required equation is

$$|z+3| + |z-3| = 10$$

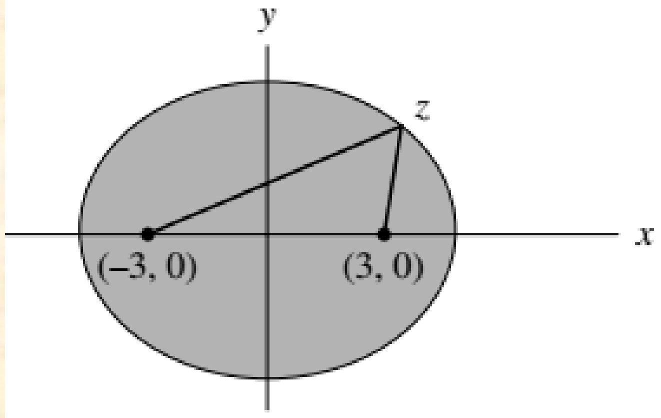


Fig. 1-21

In rectangular form, this reduces to $x^2/25 + y^2/16 = 1$

Example 9

Express each of the following complex numbers in polar form.

(a) $2+2\sqrt{3}i$, (b) $-5+5i$, (c) $-\sqrt{6}-\sqrt{2}i$, (d) $-3i$

Solution

(a) $2+2\sqrt{3}i$ [See Fig. 1-22.]

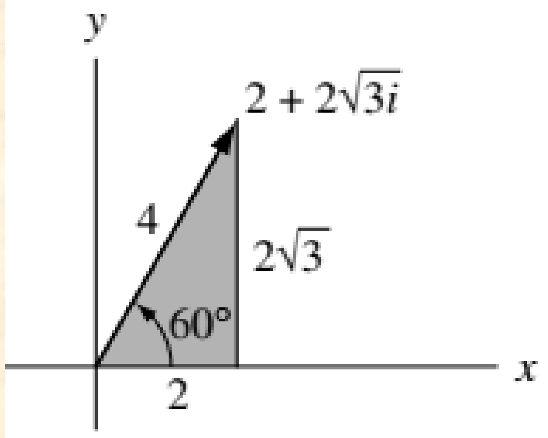


Fig. 1-22

Modulus or absolute value, $r = |2+2\sqrt{3}i| = \sqrt{4+12} = 4$.
Amplitude or argument, $\theta = \sin^{-1} 2\sqrt{3}/4 = \sin^{-1}\sqrt{3}/2 = 60° = \pi/3$ (radians).
Then

$$2+2\sqrt{3}i = r(\cos\theta + i\sin\theta) = 4(\cos 60° + i\sin 60°) = 4(\cos\pi/3 + i\sin\pi/3)$$

The result can also be written as $4 \text{ cis } \pi/3$ or, using Euler's formula, as $4e^{\pi i/3}$.

Example 9

(b) $-5 + 5i$ [See Fig. 1-23.]

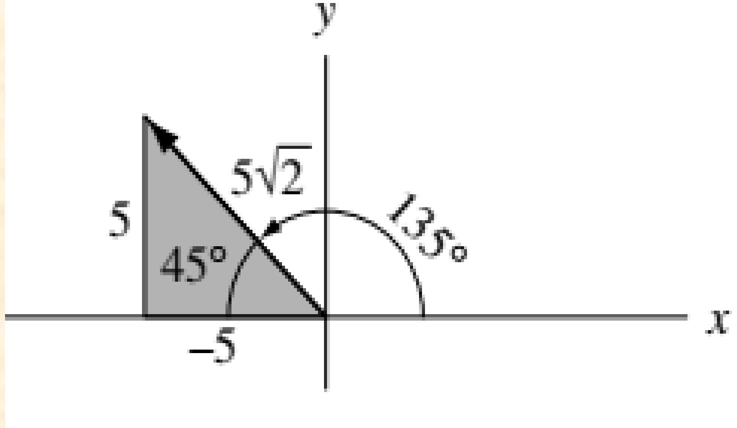


Fig. 1-23

$$r = |-5 + 5i| = \sqrt{25 + 25} = 5\sqrt{2}$$
$$\theta = 180° - 45° = 135° = 3\pi/4 \text{ (radians)}$$

$$-5 + 5i = 5\sqrt{2}(\cos 135° + i \sin 135°) = 5\sqrt{2} \text{ cis } 3\pi/4 = 5\sqrt{2}e^{3\pi i/4}$$

(c) $-\sqrt{6} - \sqrt{2}i$ [See Fig. 1-24.]

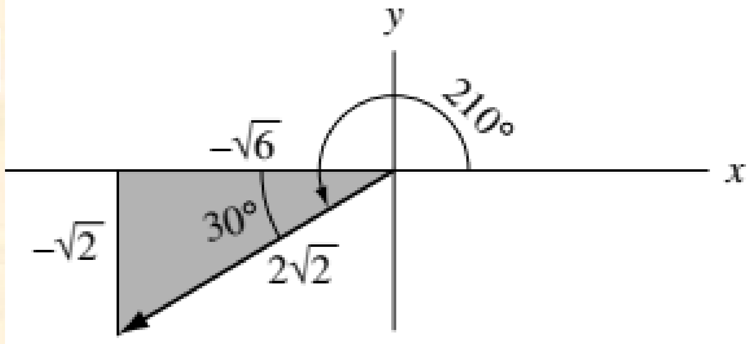


Fig. 1-24

Example 9

$$r = |-\sqrt{6} - \sqrt{2}i| = \sqrt{6+2} = 2\sqrt{2}$$
$$\theta = 180^\circ + 30^\circ = 210^\circ = 7\pi/6 \text{ (radians)}$$

Then $-\sqrt{6} - \sqrt{2}i = 2\sqrt{2}(\cos 210^\circ + i \sin 210^\circ) = 2\sqrt{2} \operatorname{cis} 7\pi/6 = 2\sqrt{2}e^{7\pi i/e}$

(d) $-3i$ [See Fig. 1-25.]

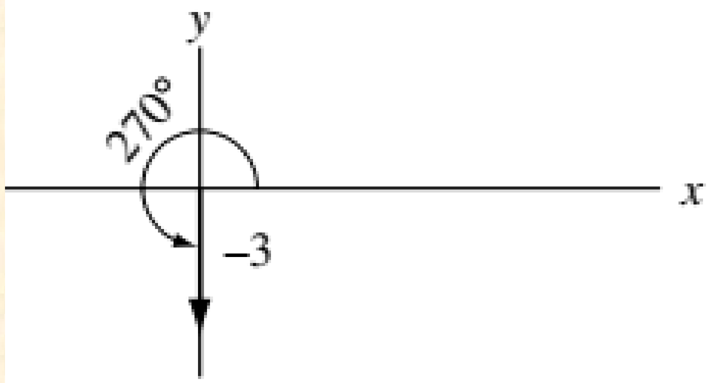


Fig. 1-25

$$r = |-3i| = |0 - 3i| = \sqrt{0+9} = 3$$
$$\theta = 270^\circ = 3\pi/2 \text{ (radians)}$$

Then $-3i = 3(\cos 3\pi/2 + i \sin 3\pi/2) = 3 \operatorname{cis} 3\pi/2 = 3e^{3\pi i/2}$

Example 10

Graph each of the following: (a) $6(\cos 240° + i \sin 240°)$, (b) $4e^{3\pi i/5}$, (c) $2e^{-\pi i/4}$.

Solution

(a) $6(\cos 240° + i \sin 240°) = 6 \text{ cis } 240° = 6 \text{ cis } 4\pi/3 = 6e^{4\pi i/3}$ can be represented graphically by OP in Fig. 1-26.

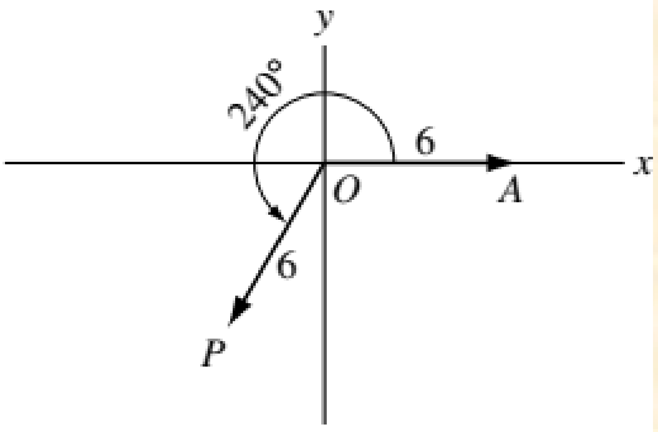


Fig. 1-26

If we start with vector OA, whose magnitude is 6 and whose direction is that of the positive x axis, we can obtain OP by rotating OA counterclockwise through an angle of 240°. In general, $re^{i\theta}$ is equivalent to a vector obtained by rotating a vector of magnitude r and direction that of the positive x axis, counterclockwise through an angle θ.

Example 10

(b) $$4e^{3\pi i/5} = 4(\cos 3\pi/5 + i\sin 3\pi/5) = 4(\cos 108^\circ + i\sin 108^\circ)$$
is represented by OP in Fig. 1-27.

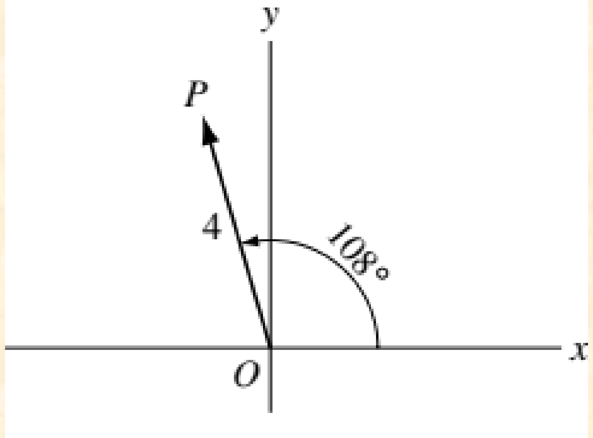


Fig. 1-27

(c) $$2e^{-\pi i/4} = 2\{\cos(-\pi/4) + i\sin(-\pi/4)\} = 2\{\cos(-45^\circ) + i\sin(-45^\circ)\}$$

This complex number can be represented by vector OP in Fig. 1-28. This vector can be obtained by starting with vector OA, whose magnitude is 2 and whose direction is that of the positive x axis, and rotating it counterclockwise through an angle of -45° (which is the same as rotating it *clockwise* through an angle of 45°).

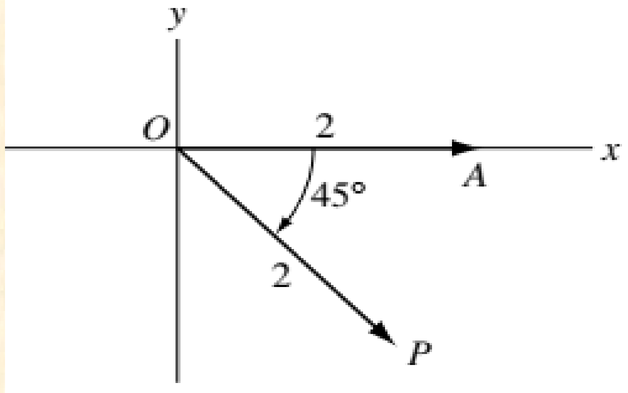


Fig. 1-28

Example 11

A man travels 12 miles northeast, 20 miles 30° west of north, and then 18 miles 60° south of west. Determine (a) analytically and (b) graphically how far and in what direction he is from his starting point.

Solution

(a) *Analytically*. Let O be the starting point (see Fig. 1-29). Then the successive displacements are represented by vectors OA, AB, and BC. The result of all three displacements is represented by the vector

$$OC = OA + AB + BC$$

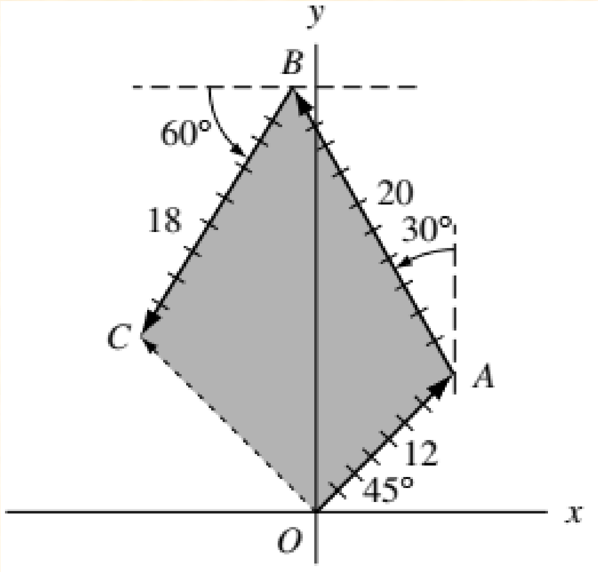


Fig. 1-29

Example 11

Now

$$OA = 12(\cos 45° + i \sin 45°) = 12e^{\pi i/4}$$

$$AB = 20\{\cos(90° + 30°) + i \sin(90° + 30°)\} = 20e^{2\pi i/3}$$

$$BC = 18\{\cos(180° + 60°) + i \sin(180° + 60°)\} = 18e^{4\pi i/3}$$

Then

$$\begin{aligned} OC &= 12e^{\pi i/4} + 20e^{2\pi i/3} + 18e^{4\pi i/3} \\ &= \{12\cos 45° + 20\cos 120° + 18\cos 240°\} + i\{12\sin 45° + 20\sin 120° + 18\sin 240°\} \\ &= \{(12)(\sqrt{2}/2) + (20)(-1/2) + (18)(-1/2)\} + i\{(12)(\sqrt{2}/2) + (20)(\sqrt{3}/2) + (18)(-\sqrt{3}/2)\} \\ &= (6\sqrt{2} - 19) + (6\sqrt{2} + \sqrt{3})i \end{aligned}$$

If $r(\cos\theta + i\sin\theta) = 6\sqrt{2} - 19 + (6\sqrt{2} + \sqrt{3})i$, then $r = \sqrt{(6\sqrt{2} - 19)^2 + (6\sqrt{2} + \sqrt{3})^2} = 14.7$ approximately, and $\theta = \cos^{-1}(6\sqrt{2} - 19)/r = \cos^{-1}(-.717) = 135°49'$ approximately.

Thus, the man is 14.7 miles from his starting point in a direction $135°49' - 90° = 45°49'$ west of north.

(b) *Graphically*. Using a convenient unit of length such as PQ in Fig. 1-29, which represents 2 miles, and a protractor to measure angles, construct vectors OA, AB, and BC. Then, by determining the number of units in OC and the angle that OC makes with the y axis, we obtain the approximate results of (a).

Example 12

Suppose $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Prove:

(a) $z_1 z_2 = r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$, (b) $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$.

Solution

(a)
$$\begin{aligned} z_1 z_2 &= \{r_1(\cos\theta_1 + i\sin\theta_1)\}\{r_2(\cos\theta_2 + i\sin\theta_2)\} \\ &= r_1 r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\} \\ &= r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\} \end{aligned}$$

(b)
$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)} \\ &= \frac{r_1}{r_2}\left\{\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}\right\} \\ &= \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\} \end{aligned}$$

In terms of Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, the results state that if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$ and $z_1/z_2 = r_1 e^{i\theta_1}/r_2 e^{i\theta_2} = (r_1/r_2)e^{i(\theta_1-\theta_2)}$.

Proof of De Moiver's Theorem

Prove De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ where n is any positive integer.

Solution

We use the *principle of mathematical induction*. Assume that the result is true for the particular positive integer k, i.e., assume $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$. Then, multiplying both sides by $\cos\theta + i\sin\theta$, we find

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) = \cos(k+1)\theta + i\sin(k+1)\theta$$

by Problem 1.19. Thus, *if* the result is true for $n = k$, then it is also true for $n = k + 1$. But, since the result is clearly true for $n = 1$, it must also be true for $n = 1 + 1 = 2$ and $n = 2 + 1 = 3$, etc., and so must be true for all positive integers.

The result is equivalent to the statement $(e^{i\theta})^n = e^{ni\theta}$.

Example 12

Prove the identities: (a) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$;
(b) $(\sin 5\theta)/(\sin\theta) = 16\cos^4\theta - 12\cos^2\theta + 1$, if $\theta \neq 0, \pm\pi, \pm 2\pi, \ldots$.

Solution

We use the *binomial formula*

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n$$

where the coefficients

also denoted by $C(n,r)$ or ${}_nC_r$, are called the *binomial coefficients*. The number $n!$ or *factorial* n, is defined as the product $n(n-1)\cdots 3\cdot 2\cdot 1$ and we define $0! = 1$.

with $n = 5$, and the binomial formula,

$$\begin{aligned}\cos 5\theta + i\sin 5\theta &= (\cos\theta + i\sin\theta)^5 \\ &= \cos^5\theta + \binom{5}{1}(\cos^4\theta)(i\sin\theta) + \binom{5}{2}(\cos^3\theta)(i\sin\theta)^2 \\ &\quad + \binom{5}{3}(\cos^2\theta)(i\sin\theta)^3 + \binom{5}{4}(\cos\theta)(i\sin\theta)^4 + (i\sin\theta)^5\end{aligned}$$

Example 12

$$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta$$

$$- 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$$

$$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

$$+ i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$$

Hence

(a) $$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

$$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$$

$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

(b) $$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

or

$$\frac{\sin 5\theta}{\sin\theta} = 5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$= 5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$

$$= 16\cos^4\theta - 12\cos^2\theta + 1$$

provided $\sin\theta \neq 0$, i.e., $\theta \neq 0, \pm\pi, \pm 2\pi, \ldots$.

Example 13

Show that (a) $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$, (b) $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$.

Solution

We have

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad (1)$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \qquad (2)$$

(a) Adding (1) and (2),

$$e^{i\theta}+e^{-i\theta} = 2\cos\theta \quad \text{or} \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

(b) Subtracting (2) from (1),

$$e^{i\theta}-e^{-i\theta} = 2i\sin\theta \quad \text{or} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Example 14

Prove the identities (a) $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$, (b) $\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$.

Solution

(a) $$\sin^3\theta = \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^3 = \frac{(e^{i\theta}-e^{-i\theta})^3}{8i^3} = -\frac{1}{8i}\{(e^{i\theta})^3 - 3(e^{i\theta})^2(e^{-i\theta}) + 3(e^{i\theta})(e^{-i\theta})^2 - (e^{-i\theta})^3\}$$

$$= -\frac{1}{8i}(e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}) = \frac{3}{4}\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right) - \frac{1}{4}\left(\frac{e^{3i\theta}-e^{-3i\theta}}{2i}\right)$$

$$= \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

(b) $$\cos^4\theta = \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^4 = \frac{(e^{i\theta}+e^{-i\theta})^4}{16}$$

$$= \frac{1}{16}\{(e^{i\theta})^4 + 4(e^{i\theta})^3(e^{-i\theta}) + 6(e^{i\theta})^2(e^{-i\theta})^2 + 4(e^{i\theta})(e^{-i\theta})^3 + (e^{-i\theta})^4\}$$

$$= \frac{1}{16}(e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta}) = \frac{1}{8}\left(\frac{e^{4i\theta}+e^{-4i\theta}}{2}\right) + \frac{1}{2}\left(\frac{e^{2i\theta}+e^{-2i\theta}}{2}\right) + \frac{3}{8}$$

$$= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

Example 15

Evaluate each of the following.

(a) $[3(\cos 40^\circ + i\sin 40^\circ)][4(\cos 80^\circ + i\sin 80^\circ)]$, (b) $\dfrac{(2\ \text{cis}\ 15^\circ)^7}{(4\ \text{cis}\ 45^\circ)^3}$, (c) $\left(\dfrac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$

Solution

(a)
$$\begin{aligned}[3(\cos 40^\circ + i\sin 40^\circ)][4(\cos 80^\circ + i\sin 80^\circ)] &= 3\cdot 4[\cos(40^\circ+80^\circ) + i\sin(40^\circ+80^\circ)]\\ &= 12(\cos 120^\circ + i\sin 120^\circ)\\ &= 12\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = -6+6\sqrt{3}i\end{aligned}$$

(b)
$$\begin{aligned}\frac{(2\ \text{cis}\ 15^\circ)^7}{(4\ \text{cis}\ 45^\circ)^3} &= \frac{128\ \text{cis}\ 105^\circ}{64\ \text{cis}\ 135^\circ} = 2\ \text{cis}(105^\circ - 135^\circ)\\ &= 2[\cos(-30^\circ) + i\sin(-30^\circ)] = 2[\cos 30^\circ - i\sin 30^\circ] = \sqrt{3} - i\end{aligned}$$

(c)
$$\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10} = \left\{\frac{2\ \text{cis}(60^\circ)}{2\ \text{cis}(-60^\circ)}\right\}^{10} = (\text{cis}\ 120^\circ)^{10} = \text{cis}\ 1200^\circ = \text{cis}\ 120^\circ = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

Example 16

Find each of the indicated roots and locate them graphically.

(a) $(-1+i)^{1/3}$, (b) $(-2\sqrt{3}-2i)^{1/4}$

Solution

(a) $(-1+i)^{1/3}$

$$-1+i=\sqrt{2}\{\cos(3\pi/4+2k\pi)+i\sin(3\pi/4+2k\pi)\}$$

$$(-1+i)^{1/3}=2^{1/6}\left\{\cos\left(\frac{3\pi/4+2k\pi}{3}\right)+i\sin\left(\frac{3\pi/4+2k\pi}{3}\right)\right\}$$

If $k=0$, $z_1=2^{1/6}(\cos\pi/4+i\sin\pi/4)$.
If $k=1$, $z_2=2^{1/6}(\cos 11\pi/12+i\sin 11\pi/12)$.
If $k=2$, $z_3=2^{1/6}(\cos 19\pi/12+i\sin 19\pi/12)$.

Example 16

These are represented graphically in Fig. 1-32.

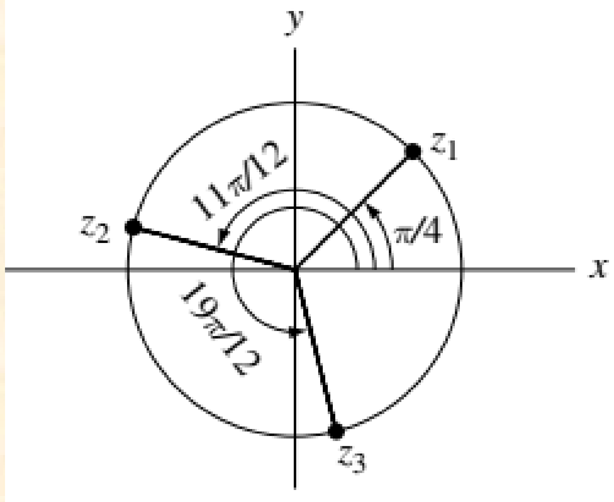


Fig. 1-32

(b) $(-2\sqrt{3} - 2i)^{1/4}$

$$-2\sqrt{3} - 2i = 4\{\cos(7\pi/6 + 2k\pi) + i\sin(7\pi/6 + 2k\pi)\}$$

$$(-2\sqrt{3} - 2i)^{1/4} = 4^{1/4}\left\{\cos\left(\frac{7\pi/6 + 2k\pi}{4}\right) + i\sin\left(\frac{7\pi/6 + 2k\pi}{4}\right)\right\}$$

If $k = 0$, $z_1 = \sqrt{2}(\cos 7\pi/24 + i\sin 7\pi/24)$.
If $k = 1$, $z_2 = \sqrt{2}(\cos 19\pi/24 + i\sin 19\pi/24)$.
If $k = 2$, $z_3 = \sqrt{2}(\cos 31\pi/24 + i\sin 31\pi/24)$.
If $k = 3$, $z_4 = \sqrt{2}(\cos 43\pi/24 + i\sin 43\pi/24)$.

Example 16

These are represented graphically in Fig. 1-33.

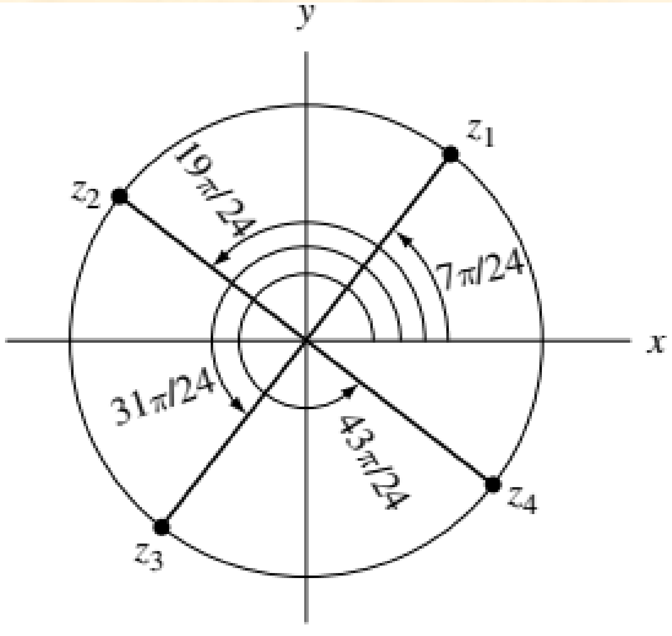


Fig. 1-33

Example 17

Find the square roots of $-15 - 8i$.

Solution

$$-15 - 8i = 17\{\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)\}$$

where $\cos\theta = -15/17$, $\sin\theta = -8/17$. Then the square roots of $-15 - 8i$ are

$$\sqrt{17}(\cos\theta/2 + i\sin\theta/2) \tag{1}$$

and

$$\sqrt{17}\{\cos(\theta/2 + \pi) + i\sin(\theta/2 + \pi)\} = -\sqrt{17}(\cos\theta/2 + i\sin\theta/2) \tag{2}$$

Now

$$\cos\theta/2 = \pm\sqrt{(1 + \cos\theta)/2} = \pm\sqrt{(1 - 15/17)/2} = \pm 1/\sqrt{17}$$

$$\sin\theta/2 = \pm\sqrt{(1 - \cos\theta)/2} = \pm\sqrt{(1 + 15/17)/2} = \pm 4/\sqrt{17}$$

Since θ is an angle in the third quadrant, $\theta/2$ is an angle in the second quadrant. Hence, $\cos\theta/2 = -1/\sqrt{17}$, $\sin\theta/2 = 4/\sqrt{17}$, and so from (1) and (2) the required square roots are $-1 + 4i$ and $1 - 4i$. As a check, note that $(-1 + 4i)^2 = (1 - 4i)^2 = -15 - 8i$.

Example 18

Solve the equation $z^2 + (2i - 3)z + 5 - i = 0$.

Solution

From Problem 1.31, $a = 1$, $b = 2i - 3$, $c = 5 - i$ and so the solutions are

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(2i-3) \pm \sqrt{(2i-3)^2 - 4(1)(5-i)}}{2(1)} = \frac{3 - 2i \pm \sqrt{-15-8i}}{2}$$

$$= \frac{3 - 2i \pm (1 - 4i)}{2} = 2 - 3i \quad \text{or} \quad 1 + i$$

using the fact that the square roots of $-15 - 8i$ are $\pm(1 - 4i)$ [see Problem 1.30]. These are found to satisfy the given equation.

Example 19

Find all the 5th roots of unity.

Solution

$z^5 = 1 = \cos 2k\pi + i \sin 2k\pi = e$ where $k = 0, \pm 1, \pm 2, \ldots$. Then

$$z = \cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5} = e^{2k\pi i/5}$$

where it is sufficient to use $k = 0, 1, 2, 3, 4$ since all other values of k lead to repetition.

Thus the roots are $1, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}$. If we call $e^{2\pi i/5} = \omega$, these can be denoted by $1, \omega, \omega^2, \omega^3, \omega^4$.

Example 20

Represent graphically the set of values of z for which (a) $\left|\frac{z-3}{z+3}\right| = 2$, (b) $\left|\frac{z-3}{z+3}\right| < 2$.

(a) The given equation is equivalent to $|z-3| = 2|z+3|$ or, if $z = x+iy$, $|x+iy-3| = 2|x+iy+3|$, i.e.,

$$\sqrt{(x-3)^2+y^2} = 2\sqrt{(x+3)^2+y^2}$$

Squaring and simplifying, this becomes

$$x^2+y^2+10x+9 = 0 \quad \text{or} \quad (x+5)^2+y^2 = 16$$

i.e., $|z+5| = 4$, a circle of radius 4 with center at $(-5, 0)$ as shown in Fig. 1-36.

Geometrically, any point P on this circle is such that the distance from P to point $B(3, 0)$ is twice the distance from P to point $A(-3, 0)$.

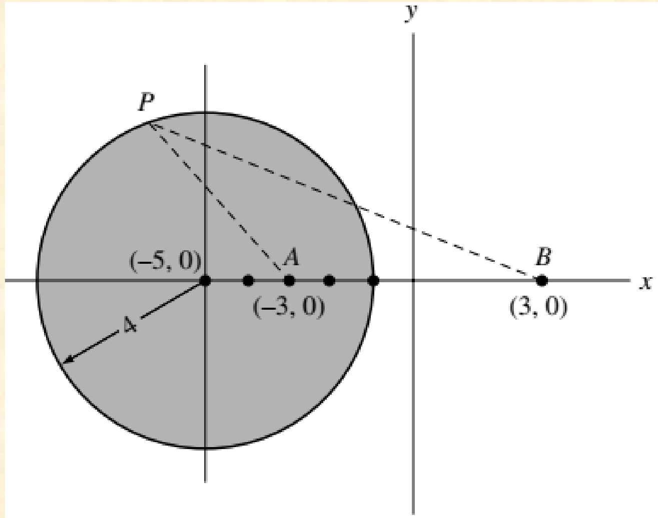


Fig. 1-36

(b) The given inequality is equivalent to $|z-3| < 2|z+3|$ or $\sqrt{(x-3)^2+y^2} < 2\sqrt{(x+3)^2+y^2}$. Squaring and simplifying, this becomes $x^2+y^2+10x+9 > 0$ or $(x+5)^2+y^2 > 16$, i.e., $|z+5| > 4$.

The required set thus consists of all points external to the circle of Fig. 1-36.

Example 21

Solve $z^2(1 - z^2) = 16$.

Method 1. The equation can be written $z^4 - z^2 + 16 = 0$, i.e., $z^4 + 8z^2 + 16 - 9z^2 = 0$, $(z^2 + 4)^2 - 9z^2 = 0$ or $(z^2 + 4 + 3z)(z^2 + 4 - 3z) = 0$. Then, the required solutions are the solutions of $z^2 + 3z + 4 = 0$ and $z^2 - 3z + 4 = 0$, or

$$-\frac{3}{2} \pm \frac{\sqrt{7}}{2}i \quad \text{and} \quad \frac{3}{2} \pm \frac{\sqrt{7}}{2}i$$

Thanks a lot …