13:5-Chain Rules for Functions of Several Variables

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Objectives

- Use the Chain Rules for functions of several variables.

- Find partial derivatives implicitly.

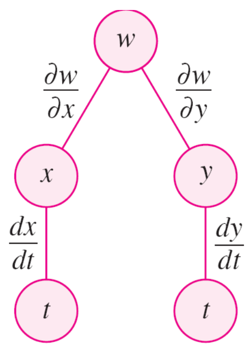
Chain Rules for Functions of Several Variables

THEOREM 13.6 Chain Rule: One Independent Variable

Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(t)$ and $y = h(t)$, where g and h are differentiable functions of t, then w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

The Chain Rule is shown schematically in Figure 13.39.



Chain Rule: one independent variable w is a function of x and y, which are each functions of t. This diagram represents the derivative of w with respect to t.

Figure 13.39

Example 1 – ***Chain Rule: One Independent Variable***

- Let $w = x^2y - y^2$, where $x = \sin t$ and $y = e^t$. Find dw/dt when $t = 0$.

- Solution:

  By the Chain Rule for one independent variable, you have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= 2xy(\cos t) + (x^2 - 2y)e^t$$

Example 1 – *Solution*

$$= 2(\sin t)(e^t)(\cos t) + (\sin^2 t - 2e^t)e^t$$

$$= 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}.$$

When $t = 0$, it follows that

$$\frac{dw}{dt} = -2.$$

Chain Rules for Functions of Several Variables

- The Chain Rule in Theorem 13.6 can provide alternative techniques for solving many problems in single-variable calculus. For instance, in Example 1, you could have used single-variable techniques to find *dw*/*dt* by first writing *w* as a function of *t*,

$$\begin{aligned} w &= x^2y - y^2 \\ &= (\sin t)^2(e^t) - (e^t)^2 \\ &= e^t \sin^2 t - e^{2t} \end{aligned}$$

and then $\dfrac{dw}{dt} = 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}$

Chain Rules for Functions of Several Variables

- The Chain Rule in Theorem 13.6 can be extended to any number of variables. For example, if each x_i is a differentiable function of a single variable t, then for

$$w = f(x_1, x_2, \ldots, x_n)$$

- you have

$$\frac{dw}{dt} = \frac{\partial w}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial w}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial w}{\partial x_n}\frac{dx_n}{dt}.$$

Chain Rules for Functions of Several Variables

THEOREM 13.7 Chain Rule: Two Independent Variables

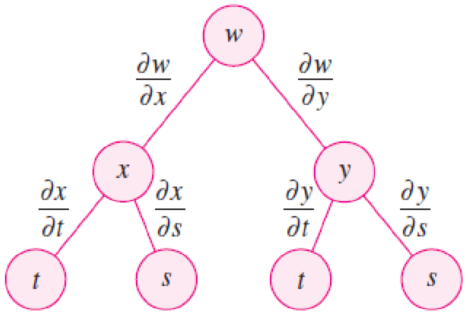
Let $w = f(x, y)$, where f is a differentiable function of x and y. If $x = g(s, t)$ and $y = h(s, t)$ such that the first partials $\partial x/\partial s$, $\partial x/\partial t$, $\partial y/\partial s$, and $\partial y/\partial t$ all exist, then $\partial w/\partial s$ and $\partial w/\partial t$ exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$

The Chain Rule is shown schematically in Figure 13.41.



Chain Rule: two independent variables

Figure 13.41

Example 4 – *The Chain Rule with Two Independent Variables*

- Use the Chain Rule to find $\partial w/\partial s$ and $\partial w/\partial t$ for $w = 2xy$ where $x = s^2 + t^2$ and $y = s/t$.

- Solution:

Using Theorem 13.7, you can hold t constant and differentiate with respect to s to obtain

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

$$= 2y(2s) + 2x\left(\frac{1}{t}\right)$$

Example 4 – *Solution* cont'd

$$= 2\left(\frac{s}{t}\right)(2s) + 2(s^2 + t^2)\left(\frac{1}{t}\right)$$ Substitute (s/t) for y and $s^2 + t^2$ for x.

$$= \frac{4s^2}{t} + \frac{2s^2 + 2t^2}{t}$$

$$= \frac{6s^2 + 2t^2}{t}.$$

Similarly, holding *s* constant gives

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

Example 4 – *Solution* cont'd

$$= 2y(2t) + 2x\left(\frac{-s}{t^2}\right)$$

$$= 2\left(\frac{s}{t}\right)(2t) + 2(s^2 + t^2)\left(\frac{-s}{t^2}\right)$$ Substitute (s/t) for y and $s^2 + t^2$ for x.

$$= 4s - \frac{2s^3 + 2st^2}{t^2}$$

$$= \frac{4st^2 - 2s^3 - 2st^2}{t^2}$$

$$= \frac{2st^2 - 2s^3}{t^2}.$$

Chain Rules for Functions of Several Variables

- The Chain Rule in Theorem 13.7 can also be extended to any number of variables. For example, if w is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ where each x_i is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$, then for $w = f(x_1, x_2, \ldots, x_n)$ you obtain the following.

$$\frac{\partial w}{\partial t_1} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdot\cdot\cdot + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_1}$$

$$\frac{\partial w}{\partial t_2} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdot\cdot\cdot + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_2}$$

$$\vdots$$

$$\frac{\partial w}{\partial t_m} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdot\cdot\cdot + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_m}$$

Implicit Partial Differentiation

- This section concludes with an application of the Chain Rule to determine the derivative of a function defined *implicitly*.

✓Let x and y be related by the equation $F(x, y) = 0$, where $y = f(x)$ is a differentiable function of x. To find dy/dx, you could use the techniques discussed in Section 2.5. You will see, however, that the Chain Rule provides a convenient alternative. Consider the function $w = F(x, y) = F(x, f(x))$.

- You can apply Theorem 13.6 to obtain

$$\frac{dw}{dx} = F_x(x, y)\frac{dx}{dx} + F_y(x, y)\frac{dy}{dx}.$$

Implicit Partial Differentiation

THEOREM 13.8 Chain Rule: Implicit Differentiation

If the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}, \quad F_y(x, y) \neq 0.$$

If the equation $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}, \quad F_z(x, y, z) \neq 0.$$

Example 6 – *Finding a Derivative Implicitly*

- Find dy/dx for $y^3 + y^2 - 5y - x^2 + 4 = 0$.
- Solution:

Begin by letting

$$F(x, y) = y^3 + y^2 - 5y - x^2 + 4.$$

Then

$$F_x(x, y) = -2x \quad \text{and} \quad F_y(x, y) = 3y^2 + 2y - 5.$$

- Using Theorem 13.8, you have

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = \frac{-(-2x)}{3y^2 + 2y - 5} = \frac{2x}{3y^2 + 2y - 5}.$$

Suggested Problems

Exercise 13.5:11,17,22,25,29.

Thanks a lot …