Exercise 1 (SVD and the reconstruction problem)

Let X be a $n \times p$ matrix. Show that a solution to the following optimization problem

$$\min_{u \in \mathbb{R}^n, v \in \mathbb{R}^p} J(u,v) \qquad \text{with} \qquad J(u,v) \;=\; \|X - uv^\top\|_F^2$$

with $\|v^\star\| = 1$, is given by

$$v^\star \qquad \text{and} \qquad u^\star = X\, v^\star,$$

where $v^\star$ is the normalized eigen vector associated with λ the largest eigen value of $X^\top X$. Furthermore, $\|u^\star\| = \sqrt{\lambda}$.

Exercise 2 (Iterated SVD)

The weighted SVD of order q problem is

$$\min_{U \in \mathbb{R}^{n \times q}, V \in \mathbb{R}^{p \times q}} \|X - UV^\top\|_W^2$$

where

$$\|M\|_W^2 = \sum_{i,j} W_{i,j} M_{i,j}^2. \tag{1}$$

We assume here that W is mask that is $W_{i,j} \in 0, 1$.

Let $\odot$ be the component-wise multiplication, the iterated SVD algorithm is given by

- Start by an initial factorization $U_0 V_0^\top$.
- Iterate T time:
  - Compute the completed matrix $R_t = W \odot X + (1 - W) \odot (U_t V_t^\top)$ where 1 is a matrix filled with 1 everywhere.
  - Use the SVD to obtain a rank q factorization of R_t by $U_{t+1} V_{t+1}^\top$
- Use the last factorization $U_T V_T^\top$.

We let $\mathcal{L}(U,V) = \|X - UV^\top\|_W^2$ and define

$$\mathcal{L}^+(U, V, U', V) = \|W \odot X + (1 - W) \odot U'V'^\top - UV^\top\|^2$$

1. Verify that $\mathcal{L}(U,V) \leq \mathcal{L}^+(U,V,U',V')$ and $\mathcal{L}(U,V) = \mathcal{L}^+(U,V,U,V)$
2. Prove that $\mathcal{L}(U_{t+1}, V_{t+1}) \leq \mathcal{L}(U_t, V_t)$. You may use the fact that the SVD can be used to minimize the problem

$$\min_{U \in \mathbb{R}^{n \times q}, V \in \mathbb{R}^{p \times q}} \|X - UV^\top\|^2.$$

Exercise 3 (Alternated least square)

The penalized weighted SVD of order q problem is

$$\min_{U \in \mathbb{R}^{n \times q}, V} \|X - UV^\top\|_W^2 + \lambda \|U\|^2 + \lambda \|V\|^2$$

with $\lambda \geq 0$ where

$$\|M\|_W^2 = \sum_{i,j} W_{i,j} M_{i,j}^2 \tag{2}$$

with $W_{i,j} \geq 0$.

To solve it using the penalized Alternating Least Square(ALS) the following sequence has to be implemented: initialize U and V with the SVD on the full matrix

loop

compute U that $\min_U \|X - UV^\top\|_W^2 + \lambda \|U\|^2$ with a fix V
compute V that $\min_V \|X - UV^\top\|_W^2 + \lambda \|V\|^2$ with a fix U

1. Show that
$$\min_U \|X - UV^\top\|_W^2 + \lambda\|U\|^2 \text{ with a fix } V$$
boils down to solving n least square problems of the form
$$\min_{u_i \in \mathbb{R}^q} \|x_i - Vu_i\|_{D_i}^2 + \lambda\|u_i\|^2, \qquad \text{for } i = 1, \ldots, n$$
with $x_i = X_{i,\cdot}$ a $\mathbb{R}^p$ vectors, $D_i = diag(W_{i,\cdot})$ the diagonal matrix with vector $W_{i,j}, j = 1, \ldots, p$ on the diagonal, $\|x\|_D^2 = \sum_j D_j x_j^2$, and whose solution is
$$u_i = (V^\top D_i V + \lambda I)^{-1} V^\top D_i x_i,$$
where I is the identity matrix of dimension q.

2. Show that
$$\min_V \|X - UV^\top\|_W^2 + \lambda\|V\|^2 \text{ with a fix } U$$
boils down at solving p least square problems of the form
$$\min_{v_j \in \mathbb{R}^q} \|x_j - Uv_j\|_{D_j}^2 + \lambda\|v_i\|^2, \qquad \text{for } j = 1, \ldots, p$$
with $x_j = X_{\cdot,j}$ a $\mathbb{R}^n$ vectors and $D_j = diag(W_{\cdot,j})$ the diagonal matrix with vector $W(i,j), i = 1, \ldots, n$ and whose solution is
$$v_j = (U^\top D_j U + \lambda I)^{-1} D^\top D_j x_j,$$
where I is the identity matrix of dimension q.