Selection for IMC-2023, second olympiad

1. (10 points) Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that g is differentiable. Assume that
$$(f(0) - g'(0))\,(g'(1) - f(1)) > 0.$$
Show that there exists a point $c \in (0,1)$ such that $f(c) = g'(c)$.

2. (10 points) Let $C = \{4, 6, 8, 9, 10, \ldots\}$ be the set of composite positive integers. For each $n \in C$ let a_n be the smallest positive integer k such that $k!$ is divisible by n. Determine whether the following series converges:
$$\sum_{n\in C}\left(\frac{a_n}{n}\right)^n$$

3. (10 points) Let $x_1, \ldots, x_n$ be real numbers. For any set $I \subset \{1, 2, \ldots, n\}$ let $s(I) = \sum\limits_{i\in I} x_i$. Assume that the function $I \mapsto s(I)$ takes on at least 1.8^n values where I runs over all 2^n subsets of $\{1, 2, \ldots, n\}$. Prove that the number of sets $I \subset \{1, 2, \ldots, n\}$ for which $s(I) = 2019$ does not exceed 1.7^n.

4. (10 points) Let k be a positive integer. Find the smallest positive integer n for which there exist k nonzero vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$ such that for every pair i, j of indices with $|i - j| > 1$ the vectors v_i and v_j are orthogonal.

5. (10 points) Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers such that $a_0 = 0$ and
$$a_{n+1}^3 = a_n^2 - 8 \quad \text{for} \quad n = 0, 1, 2, \ldots$$
Prove that the following series is convergent:
$$\sum_{n=0}^{\infty} |a_{n+1} - a_n|$$

6. (10 points) Let $\Omega = \{(x, y, z) \in \mathbb{Z}^3 : y + 1 \geq x \geq y \geq z \geq 0\}$. A frog moves along the points of Ω by jumps of length 1. For every positive integer n, determine the number of paths the frog can take to reach (n, n, n) starting from $(0, 0, 0)$ in exactly $3n$ jumps.