Problems

First day — July 29, 1994

Problem 1. (13 points)

(a) Let A be a $n \times n$ ($n \geq 2$) symmetric invertible matrix with real positive elements. Show that $z_n \leq n^2 - 2n$ where z_n is the number of zero elements in A^{-1}.

(b) How many zero elements are there in the inverse of the $n \times n$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & 2 & \dots & 2 \\ 1 & 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & 2 & \dots & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 1 & 2 & \dots & \ddots \end{pmatrix}$$

Problem 2. (13 points) Let $f \in C^1(a,b)$, $\lim\limits_{x \to a+} f(x) = +\infty$, $\lim\limits_{x \to b-} f(x) = -\infty$ and $f'(x) + f^2(x) \geq -1$ for $x \in (a,b)$. Prove that $b - a \geq \pi$ and give an example where $b - a = \pi$.

Problem 3. (13 points) Given a set S of $2n-1$ ($n \in \mathbb{N}$) different irrational numbers. Prove that there are n different elements $x_1, x_2, \dots, x_n \in S$ such that for all non-negative rational numbers $a_1, a_2, \dots, a_n$ with $a_1 + a_2 + \dots + a_n > 0$ we have that $a_1x_1 + a_2x_2 + \dots + a_nx_n$ is an irrational number.

Problem 4. (18 points) Let $\alpha \in \mathbb{R} \setminus \{0\}$ and suppose that F and G are linear maps (operators) from $\mathbb{R}^n$ into $\mathbb{R}^n$ satisfying $F \circ G - G \circ F = \alpha F$.

(a) Show that for all $k \in \mathbb{N}$ one has $F^k \circ G - G \circ F^k = \alpha k F^k$.

(b) Show that there exists $k \geq 1$ such that $F^k = 0$.

Problem 5. (18 points)

a) Let $f \in C[0,b]$, $g \in C(\mathbb{R})$ and let g be periodic with period b. Prove that

$$\lim_{n \to \infty} \int_0^b f(x) g(nx)\, dx = \frac{1}{b} \int_0^b f(x)\, dx \int_0^b g(x)\, dx.$$

b) Find

$$\lim_{n \to \infty} \int_0^{\pi} \frac{\sin x}{1 + 3\cos 2nx}\, dx.$$

Problem 6. (25 points) Let $f \in C^2[0,N]$ and $|f''(x)| < 1$, $f'(x) > 0$ for every $x \in [0,N]$. Let $0 \leq m_0 < m_1 < \dots < m_k \leq N$ be integers such that $n_i = f(m_i)$ are also integers for $i = 0, 1, \dots, k$. Denote $b_i = n_i - n_{i-1}$ and $a_i = m_i - m_{i-1}$ for $i = 1, 2, \dots, k$.

a) Prove that

$$-1 < \frac{b_1}{a_1} < \frac{b_2}{a_2} < \dots < \frac{b_k}{a_k} < 1.$$

b) Prove that for every choice of $\Lambda > 1$ there are no more than N/Λ indices j such that $a_j > \Lambda$.

c) Prove that $k \leq 3N^{2/3}$ (i.e. there are no more than $3N^{2/3}$ integer points on the curve $y = f(x)$, $x \in [0,N]$).