Problems from IMC

First day — July 30, 2011

Problem 1. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. A point x is called a *shadow point* if there exists a point $y \in \mathbb{R}$ with $y > x$ such that $f(y) > f(x)$. Let $a < b$ be real numbers and suppose that

- all the points of the open interval $I = (a, b)$ are shadow points;
- a and b are not shadow points.

Prove that

1. $f(x) \leq f(b)$ for all $a < x < b$;
2. $f(a) = f(b)$.

Problem 2. (10 points) Does there exist a real 3×3 matrix A such that $\mathrm{tr}(A) = 0$ and $A^2 + A^t = I$? (Here, $\mathrm{tr}(A)$ denotes the trace of A, A^t is the transpose of A, and I is the identity matrix.)

Problem 3. (10 points) Let p be a prime number. Call a positive integer n *interesting* if

$$x^n - 1 = (x^p - x + 1)f(x) + pg(x)$$

for some polynomials f and g with integer coefficients.

1. Prove that the number $p^p - 1$ is interesting.
2. For which p is $p^p - 1$ the minimal interesting number?

Problem 4. (10 points) Let $A_1, A_2, \ldots, A_n$ be finite, nonempty sets. Define the function

$$f(t) = \sum_{k=1}^{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} (-1)^{k-1} t^{|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}|}.$$

Prove that f is nondecreasing on $[0, 1]$.

($|A|$ denotes the number of elements in A.)

Problem 5. (10 points) Let n be a positive integer and let V be a $(2n-1)$-dimensional vector space over the two-element field. Prove that for arbitrary vectors $v_1, \ldots, v_{4n-1} \in V$, there exists a sequence $1 \leq i_1 < \cdots < i_{2n} \leq 4n - 1$ of indices such that

$$v_{i_1} + \cdots + v_{i_{2n}} = 0.$$