Problems and solutions

Second day — August 3, 1996

Problem 1. (10 points) Prove that if $f : [0,1] \to [0,1]$ is a continuous function, then the sequence of iterates $x_{n+1} = f(x_n)$ converges if and only if

$$\lim_{n\to\infty}(x_{n+1} - x_n) = 0.$$

Problem 2. (10 points) Let θ be a positive real number and let $\cosh t = \frac{e^t + e^{-t}}{2}$ denote the hyperbolic cosine. Show that if $k \in \mathbb{N}$ and both $\cosh k\theta$ and $\cosh(k+1)\theta$ are rational, then so is $\cosh\theta$.

Problem 3. (15 points) Let G be the subgroup of $GL_2(\mathbb{R})$, generated by A and B, where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Let H consist of those matrices $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ in G for which $a_{11} = a_{22} = 1$.

(a) Show that H is an abelian subgroup of G.

(b) Show that H is not finitely generated.

Remarks. $GL_2(\mathbb{R})$ denotes, as usual, the group (under matrix multiplication) of all 2×2 invertible matrices with real entries (elements). *Abelian* means commutative. A group is *finitely generated* if there are a finite number of elements of the group such that every other element of the group can be obtained from these elements using the group operation.

Problem 4. (20 points)

Let B be a bounded closed convex symmetric (with respect to the origin) set in $\mathbb{R}^2$ with boundary the curve Γ. Let B have the property that the ellipse of maximal area contained in B is the disc D of radius 1 centered at the origin with boundary the circle C. Prove that $A \cap \Gamma \neq \emptyset$ for any arc A of C of length $l(A) \geq \frac{\pi}{2}$.

Problem 5. (20 points)

(i) Prove that

$$\lim_{x\to+\infty} \sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} = \frac{1}{2}.$$

(ii) Prove that there is a positive constant c such that for every $x \in [1, \infty)$ we have

$$\left| \sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} - \frac{1}{2} \right| \leq \frac{c}{x}.$$

Problem 6. **(Carleman's inequality)** (25 points)

(i) Prove that for every sequence $\{a_n\}_{n=1}^{\infty}$, such that $a_n > 0$, $n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty} a_n < \infty$, we have

$$\sum_{n=1}^{\infty} (a_1 a_2 \ldots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n,$$

where e is the natural log base.

(ii) Prove that for every $\varepsilon > 0$ there exists a sequence $\{a_n\}_{n=1}^{\infty}$, such that $a_n > 0$, $n = 1, 2, \ldots$, $\sum_{n=1}^{\infty} a_n < \infty$ and

$$\sum_{n=1}^{\infty} (a_1 a_2 \ldots a_n)^{1/n} > (e - \varepsilon) \sum_{n=1}^{\infty} a_n.$$