Problems and solutions

First day — August 2, 1996

Problem 1. (10 points)
Let for $j = 0, \dots, n$, $a_j = a_0 + jd$, where a_0, d are fixed real numbers. Put

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_1 & a_0 & a_1 & \cdots & a_{n-1} \\ a_2 & a_1 & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_0 \end{pmatrix}.$$

Calculate $\det(A)$, where $\det(A)$ denotes the determinant of A.

Solution. Adding the first column of A to the last column we get that

$$\det(A) = (a_0 + a_n) \det \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & 1 \\ a_1 & a_0 & a_1 & \cdots & 1 \\ a_2 & a_1 & a_0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \cdots & 1 \end{pmatrix}.$$

Subtracting the n-th row of the above matrix from the $(n+1)$-st one, $(n-1)$-st from n-th, …, first from second we obtain that

$$\det(A) = (a_0 + a_n) \det \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & 1 \\ d & -d & -d & \cdots & 0 \\ d & d & -d & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d & d & d & \cdots & 0 \end{pmatrix}.$$

Hence,

$$\det(A) = (-1)^n (a_0 + a_n) \det \begin{pmatrix} d & -d & -d & \cdots & -d \\ d & d & -d & \cdots & -d \\ d & d & d & \cdots & -d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d & d & d & \cdots & d \end{pmatrix}.$$

Adding the last row of the above matrix to the other rows we have

$$\det(A) = (-1)^n (a_0 + a_n) \det \begin{pmatrix} 2d & 0 & 0 & \cdots & 0 \\ 2d & 2d & 0 & \cdots & 0 \\ 2d & 2d & 2d & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d & d & d & \cdots & d \end{pmatrix} = (-1)^n (a_0 + a_n) 2^{n-1} d^n.$$

Problem 2. (10 points)
Evaluate the definite integral

$$\int_{-\pi}^{\pi} \frac{\sin nx}{(1+2^x)\sin x}\, dx,$$

where n is a natural number.

Solution. We have

$$I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1+2^x)\sin x}\, dx$$
$$= \int_0^{\pi} \frac{\sin nx}{(1+2^x)\sin x}\, dx + \int_{-\pi}^{0} \frac{\sin nx}{(1+2^x)\sin x}\, dx.$$

In the second integral we make the change of variable $x = -x$ and obtain

$$I_n = \int_0^{\pi} \frac{\sin nx}{(1+2^x)\sin x}\, dx + \int_0^{\pi} \frac{\sin nx}{(1+2^{-x})\sin x}\, dx$$

$$= \int_0^\pi \frac{(1+2^x)\sin nx}{(1+2^x)\sin x}\,dx$$
$$= \int_0^\pi \frac{\sin nx}{\sin x}\,dx.$$

For $n \geq 2$ we have

$$I_n - I_{n-2} = \int_0^\pi \frac{\sin nx - \sin(n-2)x}{\sin x}\,dx$$
$$= 2\int_0^\pi \cos(n-1)x\,dx = 0.$$

The answer

$$I_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \pi & \text{if } n \text{ is odd.} \end{cases}$$

follows from the above formula and $I_0 = 0$, $I_1 = \pi$.

Problem 3. (15 points) The linear operator A on the vector space V is called an involution if $A^2 = E$ where E is the identity operator on V. Let $\dim V = n < \infty$.

(i) Prove that for every involution A on V there exists a basis of V consisting of eigenvectors of A.

(ii) Find the maximal number of distinct pairwise commuting involutions on V.

Solution.

(i) Let $B = \frac{1}{2}(A+E)$. Then

$$B^2 = \frac{1}{4}(A^2 + 2AE + E) = \frac{1}{4}(2AE + 2E) = \frac{1}{2}(A+E) = B.$$

Hence B is a projection. Thus there exists a basis of eigenvectors for B, and the matrix of B in this basis is of the form $\operatorname{diag}(1,\ldots,1,0,\ldots,0)$.

Since $A = 2B - E$ the eigenvalues of A are ± 1 only.

(ii) Let $\{A_i : i \in I\}$ be a set of commuting diagonalizable operators on V, and let A_1 be one of these operators. Choose an eigenvalue λ of A_1 and denote $V_\lambda = \{v \in V : A_1 v = \lambda v\}$. Then V_λ is a subspace of V, and since $A_1 A_i = A_i A_1$ for each $i \in I$ we obtain that V_λ is invariant under each A_i. If $V_\lambda = V$ then A_1 is either E or $-E$, and we can start with another operator A_i. If $V_\lambda \neq V$ we proceed by induction on $\dim V$ in order to find a common eigenvector for all A_i. Therefore $\{A_i : i \in I\}$ are simultaneously diagonalizable.

If they are involutions then $|I| \leq 2^n$ since the diagonal entries may equal 1 or -1 only.

Problem 4. (15 points)
Let $a_1 = 1$, $a_n = \frac{1}{n}\sum_{k=1}^{n-1} a_k a_{n-k}$ for $n \geq 2$. Show that

(i) $\limsup_{n\to\infty} |a_n|^{1/n} < 2^{-1/2}$,

(ii) $\limsup_{n\to\infty} |a_n|^{1/n} \geq 2/3$.

Solution.

(i) We show by induction that

$$(*) \quad a_n \leq q^n \text{ for } n \geq 3,$$

where $q = 0.7$ and use that $0.7 < 2^{-1/2}$. One has $a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{3}$,

$$a_4 = \frac{11}{48}.$$

Therefore $(*)$ is true for $n = 3$ and $n = 4$. Assume $(*)$ is true for $n \leq N-1$ for some $N \geq 5$. Then

$$a_N = \frac{2}{N}a_{N-1} + \frac{1}{N}a_{N-2} + \frac{1}{N}\sum_{k=3}^{N-3} a_k a_{N-k} \leq \frac{2}{N}q^{N-1} + \frac{1}{N}q^{N-2} + \frac{N-5}{N}q^N \leq q^N$$

because $\frac{2}{q} + \frac{1}{q^2} \leq 5$.

(ii) We show by induction that

$$a_n \geq q^n \text{ for } n \geq 2,$$

where $q = \frac{2}{3}$. One has $a_2 = \frac{1}{2} > (\frac{2}{3})^2 = q^2$. Going by induction we have

$$a_N = \frac{2}{N} a_{N-1} + \frac{1}{N} \sum_{k=2}^{N-2} a_k a_{N-k} \geq \frac{2}{N} q^{N-1} + \frac{N-3}{N} q^N = q^N$$

because $\frac{2}{q} = 3$.

Problem 5. (25 points)

(i) Let a, b be real numbers such that $b \leq 0$ and $1 + ax + bx^2 \geq 0$ for every x in $[0, 1]$. Prove that

$$\lim_{n \to \infty} n \int_0^1 (1 + ax + bx^2)^n \, dx = \begin{cases} -\frac{1}{a} & \text{if } a < 0, \\ +\infty & \text{if } a \geq 0. \end{cases}$$

(ii) Let $f : [0, 1] \to [0, \infty)$ be a function with a continuous second derivative and let $f''(x) \leq 0$ for every x in $[0, 1]$. Suppose that $L = \lim_{n\to\infty} n \int_0^1 (f(x))^n \, dx$ exists and $0 < L < +\infty$. Prove that f' has a constant sign and $\min_{x\in[0,1]} |f'(x)| = L^{-1}$.

Solution.

(i) With a linear change of the variable (i) is equivalent to: (i') Let a, b, A be real numbers such that $b \leq 0$, $A > 0$ and $1 + ax + bx^2 > 0$ for every x in $[0, A]$. Denote $I_n = n \int_0^A (1 + ax + bx^2)^n \, dx$. Prove that

$$\lim_{n\to\infty} I_n = -\frac{1}{a} \text{ when } a < 0 \text{ and } \lim_{n\to\infty} I_n = +\infty \text{ when } a \geq 0.$$

Let $a < 0$. Set $f(x) = e^{ax} - (1 + ax + bx^2)$. Using that $f(0) = f'(0) = 0$ and $f''(x) = a^2 e^{ax} - 2b$ we get for $x > 0$ that

$$0 < e^{ax} - (1 + ax + bx^2) < cx^2$$

where $c = \frac{a^2}{2} - b$. Using the mean value theorem we get

$$0 < e^{anx} - (1 + ax + bx^2)^n < cx^2 n e^{a(n-1)x}.$$

Therefore

$$0 < n \int_0^A e^{anx} \, dx - n \int_0^A (1 + ax + bx^2)^n \, dx < cn^2 \int_0^A x^2 e^{a(n-1)x} \, dx.$$

Using that

$$n \int_0^A e^{anx} \, dx = \frac{e^{anA} - 1}{a} \to \frac{-1}{a} \quad \text{as } n \to \infty$$

and

$$\int_0^A x^2 e^{a(n-1)x} \, dx < \frac{1}{|a|^3 (n-1)^3} \int_0^\infty t^2 e^{-t} \, dt$$

we get (i') in the case $a < 0$.

Let $a \geq 0$. Then for $n > \max\{A^{-2}, -b\} - 1$ we have

$$n \int_0^A (1 + ax + bx^2)^n \, dx > n \int_0^{\frac{1}{\sqrt{n+1}}} (1 + bx^2)^n \, dx$$

$$> n \cdot \frac{1}{\sqrt{n+1}} \cdot \left(1 + \frac{b}{n+1}\right)^n$$

$$> \frac{n}{\sqrt{n+1}} e^b \to \infty \quad \text{as } n \to \infty.$$

(i) is proved.

(ii) Denote $I_n = n\int_0^1 (f(x))^n\,dx$ and $M = \max\limits_{x\in[0,1]} f(x)$.

For $M < 1$ we have $I_n \le nM^n \to 0$, a contradiction.

If $M > 1$ since f is continuous there exists an interval $I \subset [0,1]$ with $|I| > 0$ such that $f(x) > 1$ for every $x \in I$. Then $I_n \ge n|I| \to \infty$, a contradiction. Hence $M = 1$. Now we prove that f' has a constant sign. Assume the opposite. Then $f'(x_0) = 0$ for some $x \in (0,1)$. Then $f(x_0) = M = 1$ because $f'' \le 0$. For $x_0 + h$ in $[0,1]$, $f(x_0+h) = 1 + \frac{h^2}{2}f''(\xi)$, for some ξ in (x_0, x_0+h). Let $m = \min_{x\in[0,1]} f''(x)$. So, $f(x_0+h) \ge 1 + \frac{h^2}{2}m$.

Let $\delta > 0$ be such that $1 + \frac{\delta^2}{2}m > 0$ and $x_0 + \delta < 1$. Then

$$I_n \ge n\int_{x_0}^{x_0+\delta} (f(x))^n\,dx \ge n\int_0^\delta \left(1 + \frac{m}{2}h^2\right)^n dh \to \infty \quad \text{as } n \to \infty$$

in view of (i') — a contradiction. Hence f is monotone and $M = f(0)$ or $M = f(1)$.

Let $M = f(0) = 1$. For h in $[0,1]$

$$1 + hf'(0) \ge f(h) \ge 1 + hf'(0) + \frac{m}{2}h^2,$$

where $f'(0) \ne 0$, because otherwise we get a contradiction as above. Since $f(0) = M$ the function f is decreasing and hence $f'(0) < 0$. Let $0 < A < 1$ be such that $1 + Af'(0) + \frac{m}{2}A^2 > 0$. Then

$$n\int_0^A (1 + hf'(0))^n\,dh \ge n\int_0^A (f(x))^n\,dx \ge n\int_0^A \left(1 + hf'(0) + \frac{m}{2}h^2\right)^n dh.$$

From (i') the first and the third integral tend to $-\frac{1}{f'(0)}$ as $n \to \infty$, hence so does the second. Also

$$n\int_A^1 (f(x))^n\,dx \le n(f(A))^n \to 0 \quad (f(A) < 1).$$

We get $L = -\frac{1}{f'(0)}$ in this case.

If $M = f(1)$ we get in a similar way $L = -\frac{1}{f'(1)}$.

Problem 6. (25 points) Upper content of a subset E of the plane $\mathbb{R}^2$ is defined as

$$C(E) = \inf\left\{\sum_{i=1}^n \operatorname{diam}(E_i)\right\}$$

where inf is taken over all finite families of sets $E_1, \ldots, E_n, n \in \mathbb{N}$, in $\mathbb{R}^2$ such that $E \subseteq \bigcup_{i=1}^n E_i$.

Lower content of E is defined as

$$K(E) = \sup\{\text{length}(L) : L \text{ is a closed line segment onto which } E \text{ can be contracted}\}.$$

Show that

(a) $C(L) = \text{length}(L)$ if L is a closed line segment;

(b) $C(E) \ge K(E)$;

(c) the equality in (b) needs not hold even if E is compact.

Hint. If $E = T \cup T'$ where T is the triangle with vertices $(-2,2), (2,2)$ and $(0,4)$, and T' is its reflexion about the x-axis, then $C(E) = 8 > K(E)$.

Remarks: All *distances* used in this problem are Euclidean. *Diameter* of a set E is $\operatorname{diam}(E) = \sup\{\operatorname{dist}(x,y) : x, y \in E\}$. *Contraction* of a set E to a set F is a mapping $f : E \to F$ such that $\operatorname{dist}(f(x), f(y)) \le \operatorname{dist}(x,y)$ for all $x, y \in E$. A set E can be contracted onto a set F if there is a contraction f of E to F which is onto, i.e., such that $f(E) = F$. *Triangle* is defined as the union of the three segments joining its vertices, i.e., it does not contain the interior.

Solution.

(a) The choice $E_1 = L$ gives $C(L) \le \text{length}(L)$. If $E \subseteq \bigcup_{i=1}^n E_i$, then

$$\sum_{i=1}^n \text{diam}(E_i) \ge \text{length}(L).$$

By induction, $n = 1$ obvious, and assuming that E_{n+1} contains the end point a of L, define the segment $L_\varepsilon = \{x \in L : \text{dist}(x,a) > \text{diam}(E_{n+1}) + \varepsilon\}$ and use induction assumption to get

$$\sum_{i=1}^{n+1} \text{diam}(E_i) \ge \text{length}(L_\varepsilon) + \text{diam}(E_{n+1}) \ge \text{length}(L) - \varepsilon;$$

but $\varepsilon > 0$ is arbitrary.

(b) If f is a contraction of E onto L and $E \subseteq \bigcup_{i=1}^n E_i$, then $L \subseteq \bigcup_{i=1}^n f(E_i)$ and

$$\text{length}(L) \le \sum_{i=1}^n \text{diam}(f(E_i)) \le \sum_{i=1}^n \text{diam}(E_i).$$

(c1) Let $E = T \cup T'$ where T is the triangle with vertices $(-2,2),(2,2)$ and $(0,4)$, and T' is its reflexion about the x-axis. Suppose $E \subseteq \bigcup_{i=1}^n E_i$. If no set among E_i meets both T and T', then E_i may be partitioned into covers of segments $[(-2,2),(2,2)]$ and $[(-2,-2),(2,-2)]$, both of length 4, so

$$\sum_{i=1}^n \text{diam}(E_i) \ge 8.$$

If at least one set among E_i, say E_k, meets both T and T', choose $a \in E_k \cap T$ and $b \in E_k \cap T'$ and note that the sets $E_i' = E_i$ for $i \ne k$, $E_k' = E_k \cup [a,b]$ cover $T \cup T' \cup [a,b]$, which is a set of upper content at least 8, since its orthogonal projection onto y-axis is a segment of length 8. Since $\text{diam}(E_j) = \text{diam}(E_j')$, we get

$$\sum_{i=1}^n \text{diam}(E_i) \ge 8.$$

(c2) Let f be a contraction of E onto $L = [a',b']$. Choose $a = (a_1,a_2)$, $b = (b_1,b_2) \in E$ such that $f(a) = a'$ and $f(b) = b'$. Since $\text{length}(L) = \text{dist}(a',b') \le \text{dist}(a,b)$ and since the triangles have diameter only 4, we may assume that $a \in T$ and $b \in T'$. Observe that if $a_2 \le 3$ then a lies on one of the segments joining some of the points $(-2,2),(2,2),(-1,3),(1,3)$; since all these points have distances from vertices, and so from points, of T_2 at most $\sqrt{50}$, we get that $\text{length}(L) \le \text{dist}(a,b) \le \sqrt{50}$. Similarly, if $b_2 \ge -3$. Finally, if $a_2 > 3$ and $b_2 < -3$, we note that every vertex, and so every point of T is in the distance at most $\sqrt{10}$ for a and every vertex, and so every point, of T' is in the distance at most $\sqrt{10}$ of b. Since f is a contraction, the image of T lies in a segment containing a' of length at most $\sqrt{10}$ and the image of T' lies in a segment containing b' of length at most $\sqrt{10}$. Since the union of these two images is L, we get $\text{length}(L) \le 2\sqrt{10} \le \sqrt{50}$. Thus $K(E) \le \sqrt{50} < 8$.