Selection for IMC-2023, first olympiad

1. (10 points) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that f has infinitely many zeros, but there is no $x \in (a, b)$ with $f(x) = f'(x) = 0$.

   (a) Prove that $f(a)f(b) = 0$

   (b) Give an example of such a function on $[0, 1]$

2. (10 points) Let k and n be positive integers. A sequence $(A_1, ..., A_k)$ of $n \times n$ real matrices is preferred by Ivan the Confessor if $A_i^2 \neq 0$ for $1 \leq i \leq k$, but $A_i A_j = 0$ for $1 \leq i, j \leq k$ with $i \neq j$. Show that $k \leq n$ in all preferred sequences, and give an example of a preferred sequence with $k = n$ for each n.

3. (10 points) Let n be a positive integer. Also let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be real numbers such that $a_i + b_i > 0$ for $i = 1, 2, ..., n$. Prove that

$$\sum_{i=1}^{n} \frac{a_i b_i - b_i^2}{a_i + b_i} \leq \frac{\sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i - \left(\sum_{i=1}^{n} b_i\right)^2}{\sum_{i=1}^{n} (a_i + b_i)}$$

4. (10 points) For a positive integer x, denote its n-th decimal digit by $d_n(x)$, i.e. $d_n(x) \in \{0, 1, ..., 9\}$ and $x = \sum_{n=1}^{\infty} d_n(x) 10^{n-1}$. Suppose that for some sequence $\{a_n\}_{n=1}^{\infty}$, there are only finitely many zeros in the sequence $\{d_n(a_n)\}_{n=1}^{\infty}$. Prove that there are infinitely many positive integers that do not occur in the sequence $\{a_n\}_{n=1}^{\infty}$.

5. (10 points) Let $A = (a_{ij})_{i,j=1}^{n}$ be a symmetric $n \times n$ matrix with real entries, and let $\lambda_1, \lambda_2, ..., \lambda_n$ denote its eigenvalues. Show that

$$\sum_{1 \leq i < j \leq n} a_{ii} a_{jj} \geq \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j,$$

   and determine all matrices for which equality holds.

6. (10 points) Let $f(x) = \frac{\sin x}{x}$, for $x > 0$, and let n be a positive integer. Prove that $|f^{(n)}(x)| < \frac{1}{n+1}$, where $f^{(n)}$ denotes the n-th derivative of f.