Second Olympiad for NUP team selection

May 2024

Problem 1. (10 points) For any integer $n \geq 2$ and two $n \times n$ matrices with real entries A, B that satisfy the equation
$$A^{-1} + B^{-1} = (A+B)^{-1},$$
prove that $\det(A) = \det(B)$. Does the same conclusion follow for matrices with complex entries?

Problem 2. (10 points) Let $f : [0; +\infty) \to \mathbb{R}$ be a continuous function such that $\lim_{x\to+\infty} f(x) = L$ exists (it may be finite or infinite). Prove that
$$\lim_{n\to\infty} \int_0^1 f(nx)\,dx = L.$$

Problem 3. (10 points) For a positive integer n, let $f(n)$ be the number obtained by writing n in binary and replacing every 0 with 1 and vice versa. For example, $n = 23$ is 10111 in binary, so $f(23)$ is 1000 in binary, therefore $f(23) = 8$. Prove that
$$\sum_{k=1}^{n} f(k) \leq \frac{n^2}{4}.$$
When does equality hold?

Problem 4. (10 points) For any positive integer m, denote by $P(m)$ the product of positive divisors of m (e.g. $P(6) = 36$). For every positive integer n define the sequence
$$a_1(n) = n, \quad a_{k+1}(n) = P(a_k(n)) \quad (k = 1, 2, \ldots, 2024).$$
Determine whether for every set $S \subseteq \{1, 2, \ldots, 2025\}$, there exists a positive integer n such that the following condition is satisfied:

For every k with $1 \leq k \leq 2025$, the number $a_k(n)$ is a perfect square if and only if $k \in S$.

Problem 5. (10 points) Determine whether or not there exist 15 integers $m_1, \ldots, m_{15}$ such that
$$\sum_{k=1}^{15} m_k \cdot \arctan(k) = \arctan(16).$$