Problems and solutions

First day — August 3, 1998

Problem 1. (20 points) Let V be a 10-dimensional real vector space and U_1 and U_2 two linear subspaces such that $U_1 \subseteq U_2$, $\dim U_1 = 3$ and $\dim U_2 = 6$. Let $\mathcal{E}$ be the set of all linear maps $T : V \to V$ which have U_1 and U_2 as invariant subspaces (i.e., $T(U_1) \subseteq U_1$ and $T(U_2) \subseteq U_2$). Calculate the dimension of $\mathcal{E}$ as a real vector space.

Solution First choose a basis $\{v_1, v_2, v_3\}$ of U_1. It is possible to extend this basis with vectors v_4, v_5 and v_6 to get a basis of U_2. In the same way we can extend a basis of U_2 with vectors $v_7, \ldots, v_{10}$ to get as basis of V.

Let $T \in \mathcal{E}$ be an endomorphism which has U_1 and U_2 as invariant subspaces. Then its matrix, relative to the basis $\{v_1, \ldots, v_{10}\}$ is of the form

$$\begin{bmatrix} * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \end{bmatrix}.$$

So the dimension of $\mathcal{E}$ is the number of stars above: $9 + 18 + 40 = 67$.

Problem 2. Prove that the following proposition holds for $n = 3$ (5 points) and $n = 5$ (7 points), and does not hold for $n = 4$ (8 points).

"For any permutation π_1 of $\{1, 2, \ldots, n\}$ different from the identity there is a permutation π_2 such that any permutation π can be obtained from π_1 and π_2 using only compositions (for example, $\pi = \pi_1 \circ \pi_1 \circ \pi_2 \circ \pi_1$)."

Solution Let S_n be the group of permutations of $\{1, 2, \ldots, n\}$.

1. When $n = 3$ the proposition is obvious: if $x = (12)$ we choose $y = (123)$; if $x = (123)$ we choose $y = (12)$.

2. $n = 4$. Let $x = (12)(34)$. Assume that there exists $y \in S_n$, such that $S_4 = \langle x, y \rangle$. Denote by K the invariant subgroup

   $$K = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}.$$

   By the fact that x and y generate the whole group S_4, it follows that the factor group S_4/K contains only powers of $\tilde{y} = yK$, i.e., S_4/K is cyclic. It is easy to see that this factor-group is not commutative (something more this group is not isomorphic to S_3).

3. $n = 5$

   (a) If $x = (12)$, then for y we can take $y = (12345)$.

   (b) If $x = (123)$, we set $y = (124)(35)$. Then $y^3 x y^3 = (125)$ and $y^4 = (124)$. Therefore (123), (124), (125) $\in \langle x, y \rangle$- the subgroup generated by x and y. From the fact that (123), (124), (125) generate the alternating subgroup A_5, it follows that $A_5 \subseteq \langle x, y \rangle$. Moreover y is an odd permutation, hence $\langle x, y \rangle = S_5$.

   (c) If $x = (123)(45)$, then as in b) we see that for y we can take the element (124).

   (d) If $x = (1234)$, we set $y = (12345)$. Then $(yx)^3 = (24) \in \langle x, y \rangle$, $x^2(24) = (13) \in \langle x, y \rangle$ and $y^2 = (13524) \in \langle x, y \rangle$. By the fact $(13) \in \langle x, y \rangle$ and $(13524) \in \langle x, y \rangle$, it follows that $\langle x, y \rangle = S_5$.

   (e) If $x = (12)(34)$, then for y we can take $y = (1354)$. Then $y^2 x = (125)$, $y^3 x = (124)(53)$ and by c $S_5 = \langle x, y \rangle$.

   (f) If $x = (12345)$, then it is clear that for y we can take the element $y = (12)$.

Problem 3. Let $f(x) = 2x(1 - x)$, $x \in \mathbb{R}$. Define

$$f_n = \underbrace{f \circ \ldots \circ f}_{n \text{ times}}.$$

a) (10 points) Find $\displaystyle\lim_{n \to \infty} \int_0^1 f_n(x)\, dx$.

b) (10 points) Compute $\int_0^1 f_n(x)\, dx$ for $n = 1, 2, \ldots$.

Solution.

a) Fix $x = x_0 \in (0,1)$. If we denote $x_n = f_n(x_0)$, $n = 1, 2, \dots$ it is easy to see that $x_1 \in (0, 1/2]$, $x_1 \leq f(x_1) \leq 1/2$ and $x_n \leq f(x_n) \leq 1/2$ (by induction). Then (x_n) is a bounded non-decreasing sequence and, since $x_{n+1} = 2x_n(1-x_n)$, the limit $l = \lim_{n\to\infty} x_n$ satisfies $l = 2l(1-l)$, which implies $l = 1/2$. Now the monotone convergence theorem implies that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1/2.$$

b) We prove by induction that

$$f_n(x) = \frac{1}{2} - 2^{2^n - 1}\left(x - \frac{1}{2}\right)^{2^n}$$

holds for $n = 1, 2, \dots$. For $n = 1$ this is true, since $f(x) = 2x(1-x) = \frac{1}{2} - 2(x - \frac{1}{2})^2$. If it holds for some $n = k$, then we have

$$f_{k+1}(x) = f_k(f(x)) = \frac{1}{2} - 2^{2^k - 1}\left(\frac{1}{2} - 2\left(x - \frac{1}{2}\right)^2\right)^{2^k} = \frac{1}{2} - 2^{2^k - 1}\left(-2(x - \frac{1}{2})^2\right)^{2^k} = \frac{1}{2} - 2^{2^{k+1}-1}\left(x - \frac{1}{2}\right)^{2^{k+1}},$$

which is our statement for $n = k+1$.

Using it we can compute the integral,

$$\int_0^1 f_n(x)\,dx = \left[\frac{x}{2} - \frac{2^{2^n - 1}}{2^n + 1}\left(x - \frac{1}{2}\right)^{2^n + 1}\right]_{x=0}^{x=1} = \frac{1}{2} - \frac{1}{2(2^n + 1)}.$$

Problem 4. (20 points) The function $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and satisfies $f(0) = 2, f'(0) = -2$ and $f(1) = 1$. Prove that there exists a real number $\xi \in (0,1)$ for which

$$f(\xi) \cdot f'(\xi) + f''(\xi) = 0.$$

Solution. Define the function

$$g(x) = \frac{1}{2} f^2(x) + f'(x).$$

Because $g(0) = 0$ and

$$f(x) \cdot f'(x) + f''(x) = g'(x),$$

it is enough to prove that there exists a real number $0 < \eta \leq 1$ for which $g(\eta) = 0$.

a) If f is never zero, let

$$h(x) = \frac{x}{2} - \frac{1}{f(x)}.$$

Because $h(0) = h(1) = -\frac{1}{2}$, there exists a real number $0 < \eta < 1$ for which $h'(\eta) = 0$. But $g = f^2 \cdot h'$, and we are done.

b) If f has at least one zero, let z_1 be the first one and z_2 be the last one. (The set of the zeros is closed.) By the conditions, $0 < z_1 \leq z_2 < 1$.

The function f is positive on the intervals $[0, z_1)$ and $(z_2, 1]$; this implies that $f'(z_1) \leq 0$ and $f'(z_2) \geq 0$. Then $g(z_1) = f'(z_1) \leq 0$ and $g(z_2) = f'(z_2) \geq 0$, and there exists a real number $\eta \in [z_1, z_2]$ for which $g(\eta) = 0$.

Remark. For the function $f(x) = \frac{2}{x+1}$ the conditions hold and $f \cdot f' + f''$ is constantly 0.

Problem 5. Let P be an algebraic polynomial of degree n having only real zeros and real coefficients.

(a) (15 points) Prove that for every real x the following inequality holds:

$$(n-1)\left(P'(x)\right)^2 \geq nP(x)P''(x) \tag{1}$$

(b) (5 points) Examine the cases of equality.

Solution. Observe that both sides of (1) are identically equal to zero if $n=1$. Suppose that $n>1$. Let $x_1,\ldots,x_n$ be the zeros of P. Clearly (1) is true when $x=x_i$, $i\in\{1,\ldots,n\}$, and equality is possible only if $P'(x_i)=0$, i.e., if x_i is a multiple zero of P. Now suppose that x is not a zero of P. Using the identities

$$\frac{P'(x)}{P(x)} = \sum_{i=1}^{n}\frac{1}{x-x_i},$$
$$\frac{P''(x)}{P(x)} = \sum_{1\le i<j\le n}\frac{2}{(x-x_i)(x-x_j)},$$

we find

$$(n-1)\left(\frac{P'(x)}{P(x)}\right)^2 - n\frac{P''(x)}{P(x)} = \sum_{i=1}^{n}\left(\frac{n-1}{(x-x_i)^2}\right)^2 - \sum_{1\le i<j\le n}\frac{2}{(x-x_i)(x-x_j)}.$$

But this last expression is simlply

$$\sum_{1\le i<j\le n}\left(\frac{1}{(x-x_i)}-\frac{1}{(x-x_j)}\right)^2,$$

and therefore is positive. The inequality is proved. In order that (1) holds with equality sign for every real x it is necessary that $x_1=x_2=\ldots=x_n$. A direct verification shows that indeed, if $P(x)=c(x-x_1)^n$, then (1) becomes an identity.

Problem 6. Let $f:[0,1]\to\mathbb{R}$ be a continuous function with the property that for any x and y in the interval,

$$xf(y)+yf(x)\le 1.$$

a) (15 points) Show that

$$\int_0^1 f(x)\,dx \le \frac{\pi}{4}.$$

b) (5 points) Find a function, satisfying the condition, for which there is equality.

Solution Observe that the integral is equal to

$$\int_0^{\frac{\pi}{2}} f(\sin\theta)\cos\theta\,d\theta$$

and to

$$\int_0^{\frac{\pi}{2}} f(\cos\theta)\sin\theta\,d\theta.$$

So, twice the integral is at most

$$\int_0^{\frac{\pi}{2}} 1\,d\theta = \frac{\pi}{2}.$$

Now let $f(x)=\sqrt{1-x^2}$. If $x=\sin\theta$ and $y=\sin\phi$ then

$$xf(y)+yf(x) = \sin\theta\cos\phi+\sin\phi\cos\theta = \sin(\theta+\phi)\le 1.$$