Problems and solutions

First day — July 29, 1994

Problem 1. (13 points)

(a) Let A be a $n \times n$ ($n \geq 2$) symmetric invertible matrix with real positive elements. Show that $z_n \leq n^2 - 2n$ where z_n is the number of zero elements in A^{-1}.

(b) How many zero elements are there in the inverse of the $n \times n$ matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & 2 & \dots & 2 \\ 1 & 2 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & 2 & \dots & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 1 & 2 & \dots & \ddots \end{pmatrix}$$

Solution. Denote by a_{ij} and b_{ij} the elements of A and A^{-1} respectively. Then for $k \neq m$ we have $\sum_{i=0}^{n} a_{ki}b_{im} = 0$ and from the positivity of a_{ij} we conclude that at least one of $\{b_{im} : i = 1, 2, \dots, n\}$ is positive and at least one is negative. Hence we have at least two non-zero elements in every column of A^{-1}. This proves part (a). For part (b), all b_{ij} are zero except $b_{11} = 2$, $b_{nn} = (-1)^n$, $b_{ii+1} = b_{i+1i} = (-1)^i$ for $i = 1, 2, \dots, n-1$.

Problem 2. (13 points) Let $f \in C^1(a,b)$, $\lim\limits_{x \to a+} f(x) = +\infty$, $\lim\limits_{x \to b-} f(x) = -\infty$ and $f'(x) + f^2(x) \geq -1$ for $x \in (a,b)$. Prove that $b - a \geq \pi$ and give an example where $b - a = \pi$.

Solution. From the inequality we get $\frac{d}{dx}(\arctan f(x) + x) = \frac{1}{1+f^2(x)} \geq 0$ for $x \in (a,b)$. Thus $\arctan f(x) + x$ is non-decreasing in the interval and using the limits we get $\frac{\pi}{2} + a \leq -\frac{\pi}{2} + b$. Hence $b - a \geq \pi$. One has equality for $f(x) = \cot x$, $a = 0$, $b = \pi$.

Problem 3. (13 points) Given a set S of $2n-1$ ($n \in \mathbb{N}$) different irrational numbers. Prove that there are n different elements $x_1, x_2, \dots, x_n \in S$ such that for all non-negative rational numbers $a_1, a_2, \dots, a_n$ with $a_1 + a_2 + \dots + a_n > 0$ we have that $a_1x_1 + a_2x_2 + \dots + a_nx_n$ is an irrational number.

Solution. Let $\mathbb{I}$ be the set of irrational numbers, $\mathbb{Q}$ – the set of rational numbers, $\mathbb{Q}_+ = \mathbb{Q} \cap [0, \infty)$. We work by induction. For $n = 1$ the statement is trivial. Let it be true for $n - 1$. We start to prove it for n. From the induction argument, there are $n-1$ different elements $x_1, x_2, \dots, x_{n-1} \in S$ such that

$$a_1x_1 + a_2x_2 + \dots + a_{n-1}x_{n-1} \in \mathbb{I} \tag{1}$$

for all $a_1, a_2, \dots, a_n \in \mathbb{Q}^+$ with $a_1 + a_2 + \dots + a_{n-1} > 0$.

Denote the other elements of S by $x_n, x_{n+1}, \dots, x_{2n-1}$. Assume the statement is not true for n. Then for $k = 0, 1, \dots, n-1$ there are $r_k \in \mathbb{Q}$ such that

$$\sum_{i=1}^{n-1} b_{ik}x_i + c_kx_{n+k} = r_k \text{ for some } b_{ik}, c_k \in \mathbb{Q}^+, \sum_{i=1}^{n-1} b_{ik} + c_k > 0. \tag{2}$$

Also

$$\sum_{k=0}^{n-1} d_kx_{n+k} = R \text{ for some } d_k \in \mathbb{Q}^+, \sum_{k=0}^{n-1} d_k > 0, R \in \mathbb{Q}. \tag{3}$$

If in (2) $c_k = 0$ then (2) contradicts (1). Thus $c_k \neq 0$ and without loss of generality one may take $c_k = 1$. In (2) also $\sum_{i=1}^{n-1} b_{ik} > 0$ in view of $x_{n+k} \in \mathbb{I}$. Replacing (2) in (3) we get

$$\sum_{k=0}^{n-1} d_k \left(-\sum_{i=1}^{n-1} b_{ik}x_i + r_k \right) = R \text{ or } \sum_{i=1}^{n-1} \left(\sum_{k=0}^{n-1} d_kb_{ik} \right) x_i \in \mathbb{Q}, \tag{4}$$

which contradicts (1) because of the conditions on b's and d's.

Problem 4. (18 points) Let $\alpha \in \mathbb{R} \setminus \{0\}$ and suppose that F and G are linear maps (operators) from $\mathbb{R}^n$ into $\mathbb{R}^n$ satisfying $F \circ G - G \circ F = \alpha F$.

(a) Show that for all $k \in \mathbb{N}$ one has $F^k \circ G - G \circ F^k = \alpha k F^k$.

(b) Show that there exists $k \geq 1$ such that $F^k = 0$.

Solution. For a) using the assumptions we have

$$
\begin{aligned}
F^k \circ G - G \circ F^k &= \sum_{i=1}^{k} \left(F^{k-i+1} \circ G \circ F^{i-1} - F^{k-i} \circ G \circ F^i \right) \\
&= \sum_{i=1}^{k} F^{k-i} \circ (F \circ G - G \circ F) \circ F^{i-1} \\
&= \sum_{i=1}^{k} F^{k-i} \circ \alpha F \circ F^{i-1} = \alpha^k F^k.
\end{aligned}
$$

b) Consider the linear operator $L(F) = F \circ G - G \circ F$ acting over all $n \times n$ matrices F. It may have at most n^2 different eigenvalues. Assuming that $F^k \neq 0$ for every k we get that L has infinitely many different eigenvalues α^k in view of a) – a contradiction.

Problem 5. (18 points)

a) Let $f \in C[0,b]$, $g \in C(\mathbb{R})$ and let g be periodic with period b. Prove that

$$\lim_{n \to \infty} \int_0^b f(x) g(nx)\, dx = \frac{1}{b} \int_0^b f(x)\, dx \int_0^b g(x)\, dx.$$

b) Find

$$\lim_{n \to \infty} \int_0^{\pi} \frac{\sin x}{1 + 3 \cos 2nx}\, dx.$$

Solution. Set $\|g\|_1 = \int_0^b |g(x)|\, dx$ and

$$\omega(f,t) = \sup\{|f(x) - f(y)| : x, y \in [0,b], |x - y| \leq t\}.$$

In view of the uniform continuity of f we have $\omega(f,t) \to 0$ as $t \to 0$. Using the periodicity of g we get

$$
\begin{aligned}
\int_0^b f(x) g(nx)\, dx &= \sum_{k=1}^{n} \int_{\frac{b(k-1)}{n}}^{\frac{bk}{n}} f(x) g(nx)\, dx \\
&= \sum_{k=1}^{n} \int_{\frac{b(k-1)}{n}}^{\frac{bk}{n}} g(nx)\, dx + \sum_{k=1}^{n} \int_{\frac{b(k-1)}{n}}^{\frac{bk}{n}} \left(f(x) - f\left(\frac{bk}{n}\right) \right) g(nx)\, dx \\
&= \frac{1}{n} \sum_{k=1}^{n} \int_0^b g(x)\, dx + O(\omega(f, b/n) \|g\|_1)
\end{aligned}
$$

$$= \frac{1}{b} \sum_{k=1}^{n} \left(\int_{\frac{b(k-1)}{n}}^{\frac{bk}{n}} f(x)\, dx \right) \left(\int_0^b g(x)\, dx \right) + \frac{1}{b} \sum_{k=1}^{n} \left(\int_{\frac{b(k-1)}{n}}^{\frac{bk}{n}} - \int_{\frac{b(k-1)}{n}}^{\frac{bk}{n}} f(x)\, dx \right) \left(\int_0^b g(x)\, dx \right) + O(\omega(f, b/n) \|g\|_1)$$

$$= \frac{1}{b} \int_0^b f(x)\, dx \int_0^b g(x)\, dx + O(\omega(f, b/n) \|g\|_1).$$

This proves a). For b) we set $b = \pi$, $f(x) = \sin x$, $g(x) = (1 + 3 \cos 2x)^{-1}$. From a) and

$$\int_0^{\pi} \sin x\, dx = 2, \qquad \int_0^{\pi} (1 + 3 \cos 2x)^{-1}\, dx = \frac{\pi}{2}$$

we get

$$\lim_{n \to \infty} \int_0^{\pi} \frac{\sin x}{1 + 3 \cos 2nx}\, dx = 1.$$

Problem 6. (25 points) Let $f \in C^2[0,N]$ and $|f''(x)| < 1$, $f'(x) > 0$ for every $x \in [0,N]$. Let $0 \leq m_0 < m_1 < \ldots < m_k \leq N$ be integers such that $n_i = f(m_i)$ are also integers for $i = 0, 1, \ldots, k$. Denote $b_i = n_i - n_{i-1}$ and $a_i = m_i - m_{i-1}$ for $i = 1, 2, \ldots, k$.

a) Prove that

$$-1 < \frac{b_1}{a_1} < \frac{b_2}{a_2} < \ldots < \frac{b_k}{a_k} < 1.$$

b) Prove that for every choice of $\Lambda > 1$ there are no more than N/Λ indices j such that $a_j > \Lambda$.

c) Prove that $k \leq 3N^{2/3}$ (i.e. there are no more than $3N^{2/3}$ integer points on the curve $y = f(x)$, $x \in [0, N]$).

Solution.

a) For $i = 1, 2, \ldots, k$ we have

$$b_i = f(m_i) - f(m_{i-1}) = (m_i - m_{i-1})f'(x_i)$$

for some $x_i \in (m_{i-1}, m_i)$. Hence $\frac{b_i}{a_i} = f'(x_i)$ and so $-1 < \frac{b_i}{a_i} < 1$. From the convexity of f we have that f' is increasing and $\frac{b_i}{a_i} = f'(x_i) < f'(x_{i+1}) = \frac{b_{i+1}}{a_{i+1}}$ because of $x_i < m_i < x_{i+1}$.

b) Set $S_\Lambda = \{j \in \{0, 1, \ldots, k\} : a_j > \Lambda\}$. Then

$$N \geq m_k - m_0 = \sum_{i=1}^{k} a_i \geq \sum_{j \in S_\Lambda} a_j > \Lambda |S_\Lambda|$$

and hence $|S_\Lambda| < N/\Lambda$.

c) All different fractions in $(-1, 1)$ with denominators less or equal to Λ are no more than $2\Lambda^2$. Using b) we get $k < N/\Lambda + 2\Lambda^2$. Put $\Lambda = N^{1/3}$ in the above estimate and get $k < 3N^{2/3}$.