Problems

First day — August 2, 1996

Problem 1. (10 points)
Let for $j = 0, \dots, n$, $a_j = a_0 + jd$, where a_0, d are fixed real numbers. Put

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_1 & a_0 & a_1 & \cdots & a_{n-1} \\ a_2 & a_1 & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_0 \end{pmatrix}.$$

Calculate $\det(A)$, where $\det(A)$ denotes the determinant of A.

Problem 2. (10 points)
Evaluate the definite integral

$$\int_{-\pi}^{\pi} \frac{\sin nx}{(1+2^x)\sin x}\, dx,$$

where n is a natural number.

Problem 3. (15 points) The linear operator A on the vector space V is called an involution if $A^2 = E$ where E is the identity operator on V. Let $\dim V = n < \infty$.

(i) Prove that for every involution A on V there exists a basis of V consisting of eigenvectors of A.

(ii) Find the maximal number of distinct pairwise commuting involutions on V.

Problem 4. (15 points)
Let $a_1 = 1$, $a_n = \frac{1}{n}\sum_{k=1}^{n-1} a_k a_{n-k}$ for $n \geq 2$. Show that

(i) $\limsup_{n\to\infty} |a_n|^{1/n} < 2^{-1/2}$,

(ii) $\limsup_{n\to\infty} |a_n|^{1/n} \geq 2/3$.

Problem 5. (25 points)

(i) Let a, b be real numbers such that $b \leq 0$ and $1 + ax + bx^2 \geq 0$ for every x in $[0,1]$. Prove that

$$\lim_{n\to\infty} n \int_0^1 (1 + ax + bx^2)^n \, dx = \begin{cases} -\frac{1}{a} & \text{if } a < 0, \\ +\infty & \text{if } a \geq 0. \end{cases}$$

(ii) Let $f : [0,1] \to [0,\infty)$ be a function with a continuous second derivative and let $f''(x) \leq 0$ for every x in $[0,1]$. Suppose that $L = \lim_{n\to\infty} n \int_0^1 (f(x))^n \, dx$ exists and $0 < L < +\infty$. Prove that f' has a constant sign and $\min_{x\in[0,1]} |f'(x)| = L^{-1}$.

Problem 6. (25 points) Upper content of a subset E of the plane $\mathbb{R}^2$ is defined as

$$C(E) = \inf \left\{ \sum_{i=1}^{n} \operatorname{diam}(E_i) \right\}$$

where inf is taken over all finite families of sets $E_1, \dots, E_n, n \in \mathbb{N}$, in $\mathbb{R}^2$ such that $E \subseteq \bigcup_{i=1}^n E_i$. Lower content of E is defined as

$$K(E) = \sup\{\text{length}(L) : L \text{ is a closed line segment onto which } E \text{ can be contracted}\}.$$

Show that

(a) $C(L) = \text{length}(L)$ if L is a closed line segment;

(b) $C(E) \geq K(E)$;

(c) the equality in (b) needs not hold even if E is compact.

Hint. If $E = T \cup T'$ where T is the triangle with vertices $(-2, 2), (2, 2)$ and $(0, 4)$, and T' is its reflexion about the x-axis, then $C(E) = 8 > K(E)$.

Remarks: All *distances* used in this problem are Euclidean. *Diameter* of a set E is $\operatorname{diam}(E) = \sup\{\operatorname{dist}(x, y) : x, y \in E\}$. *Contraction* of a set E to a set F is a mapping $f : E \to F$ such that $\operatorname{dist}(f(x), f(y)) \leq \operatorname{dist}(x, y)$ for all $x, y \in E$. A set E can be contracted onto a set F if there is a contraction f of E to F which is onto, i.e., such that $f(E) = F$. *Triangle* is defined as the union of the three segments joining its vertices, i.e., it does not contain the interior.