Second mock Olympiad for NUP team

July 2024

Problem 1. (10 points) Let $0 < a < b$. Prove that

$$\int_a^b (x^2+1)e^{-x^2}\,dx \geq e^{-a^2} - e^{-b^2}.$$

Problem 2. (10 points) Compute the sum of the series

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)} = \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{5\cdot 6\cdot 7\cdot 8} + \cdots.$$

Problem 3. (10 points) Define the sequence $x_1, x_2, \ldots$ inductively by $x_1 = \sqrt{5}$ and $x_{n+1} = x_n^2 - 2$ for each $n \geq 1$. Compute

$$\lim_{n\to\infty} \frac{x_1 \cdot x_2 \cdot x_3 \cdot \cdots \cdot x_n}{x_{n+1}}.$$

Problem 4. (10 points) Let a, b be two integers and suppose that n is a positive integer for which the set

$$\mathbb{Z} \setminus \{ax^n + by^n \mid x, y \in \mathbb{Z}\}$$

is finite. Prove that $n = 1$.

Problem 5. (10 points) Suppose that a, b, c are real numbers in the interval $[-1, 1]$ such that

$$1 + 2abc \geq a^2 + b^2 + c^2.$$

Prove that

$$1 + 2(abc)^n \geq a^{2n} + b^{2n} + c^{2n}$$

for all positive integers n.