Problems and solutions

Second Selection Test, May 2024

Problem 1. (10 points) For any integer $n \geq 2$ and two $n \times n$ matrices with real entries A, B that satisfy the equation

$$A^{-1} + B^{-1} = (A+B)^{-1},$$

prove that $\det(A) = \det(B)$. Does the same conclusion follow for matrices with complex entries?

Solution. Multiplying the equation by $(A+B)$ we get

$$I = (A+B)(A^{-1}+B^{-1}) = (A+B)(A+B)^{-1} = AA^{-1} + AB^{-1} + BA^{-1} + BB^{-1} = I + AB^{-1} + BA^{-1} + I = 0.$$

Let $X = AB^{-1}$, then $A = XB$ and $BA^{-1} = X^{-1}$, so we have $X + X^{-1} + I = 0$; multiplying by $(X-I)X$,

$$0 = (X-I)X \cdot (X + X^{-1} + I) = (X-I) \cdot (X^2 + X + I) = X^3 - I.$$

Hence,

$$X^3 = I \quad \text{so} \quad (\det X)^3 = \det(X^3) = \det I = 1 \quad \text{thus} \quad \det X = 1$$
$$\det A = \det(XB) = \det X \cdot \det B = \det B.$$

In case of complex matrices the statement is false. Let $\omega = \frac{1}{2}(-1 + i\sqrt{3})$. Obviously $\omega \notin \mathbb{R}$ and $\omega^3 = 1$, so $0 = 1 + \omega + \omega^2 = 1 + \omega + \overline{\omega}$.

Let $A = I$ and let B be a diagonal matrix with all entries along the diagonal equal to either ω or $\overline{\omega} = \omega^2$ such a way that $\det(B) \neq 1$ (if n is not divisible by 3 then one may set $B = \omega I$). Then $A^{-1} = I$, $B^{-1} = B$. Obviously $I + B + B = 0$ and

$$(A+B)^{-1} = (-B)^{-1} = -B = I + B = A^{-1} + B^{-1}.$$

By the choice of A and B, $\det A = 1 \neq \det B$.

Problem 2. (10 points) Let $f : [0; +\infty) \to \mathbb{R}$ be a continuous function such that $\lim_{x\to+\infty} f(x) = L$ exists (it may be finite or infinite). Prove that

$$\lim_{n\to\infty} \int_0^1 f(nx)\,dx = L.$$

Solution 1. *Case 1: L is finite.* Take an arbitrary $\epsilon > 0$. We construct a number $K \geq 0$ such that for every $n \geq K$ we have

$$\left| \int_0^1 f(nx)\,dx - L \right| < \epsilon.$$

Since $\lim_{x\to\infty} f(x) = L$, there exists a $K_1 \geq 0$ such that $|f(x) - L| < \frac{\epsilon}{2}$ for every $x \geq K_1$. Hence, for $n \geq K_1$ we have

$$\left| \int_0^1 f(nx)\,dx - L \right| = \left| \frac{1}{n} \int_0^n f(x)\,dx - L \right| = \frac{1}{n} \left| \int_0^n (f(x) - L)\,dx \right| \leq$$
$$\leq \frac{1}{n} \left(\int_0^{K_1} |f(x) - L|\,dx + \int_{K_1}^n |f(x) - L|\,dx \right) \leq \frac{1}{n} \left(\int_0^{K_1} |f(x) - L|\,dx + (n - K_1)\frac{\epsilon}{2} \right) =$$
$$= \frac{1}{n} \left(\int_0^{K_1} |f(x) - L|\,dx \right) + \frac{\epsilon}{2}$$

If $n \geq K_2 = \frac{2}{\epsilon} \int_0^{K_1} |f - L|\,dx$ then the first term is at most $\frac{\epsilon}{2}$. Then for $x \geq K := \max(K_1, K_2)$, we have

$$\left| \int_0^1 f(nx)\,dx - L \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Case 2: $L = +\infty$. Take an arbitrary real M; we need a $K \geq 0$ such that

$$\int_0^1 f(nx)\,dx > M$$

for every $x \geq K$. Since $\lim_{x\to\infty} f(x) = \infty$, there exists a $K_1 \geq 0$ such that $f(x) > M+1$ for every $x \geq K_1$. Hence, for $n \geq 2K_1$ we have

$$\int_0^1 f(nx)\,dx = \frac{1}{n}\int_0^n f(x)\,dx = \frac{1}{n}\left(\int_0^{K_1} f(x)\,dx + \int_{K_1}^n f(x)\,dx\right) =$$

$$= \frac{1}{n}\left(\int_0^{K_1} f(x)\,dx + \int_{K_1}^n (M+1)\,dx\right) = \frac{1}{n}\left(\int_0^{K_1} f\,dx + (n-K_1)(M+1)\right).$$

If $n \geq K_2 := \frac{1}{\int_0^{K_1} f - K_1(M+1)}$, then the first term is at most $-\frac{1}{2}$. For $x \geq K := \max(K_1, K_2)$, we have

$$\int_0^1 f(nx)\,dx > M.$$

Case 3: $L = -\infty$. We can repeat the steps in Case 2 for the function $-f$.

Solution 2. Let $F(x) = \int_0^x f$. For $t > 0$ we have

$$\int_0^1 f(tx)\,dx = \frac{F(t)}{t}.$$

Since $\lim_{t\to\infty} t = \infty$ in the denominator and $\lim_{t\to\infty} F'(t) = \lim_{t\to\infty} f(t) = L$, L'Hospital's rule proves

$$\lim_{t\to\infty}\frac{F(t)}{t} = \lim_{t\to\infty}\frac{F'(t)}{1} = \lim_{t\to\infty} f(t) = L.$$

Then it follows that

$$\lim_{n\to\infty}\frac{F(n)}{n} = L.$$

Problem 3. (10 points) For a positive integer n, let $f(n)$ be the number obtained by writing n in binary and replacing every 0 with 1 and vice versa. For example, $n = 23$ is 10111 in binary, so $f(23)$ is 1000 in binary, therefore $f(23) = 8$. Prove that

$$\sum_{k=1}^n f(k) \leq \frac{n^2}{4}.$$

When does equality hold?

Solution. If r and k are positive integers with $2^{r-1} \leq k < 2^r$ then k has r binary digits, so $k + f(k) = 11\ldots1_2 = 2^r - 1$. Assume that $2^{s-1} - 1 \leq n \leq 2^s - 1$. Then

$$\frac{n(n+1)}{2} + \sum_{k=1}^n f(k) = \sum_{k=1}^n (k + f(k)) =$$

$$= \sum_{r=1}^{s-1}\sum_{k=2^{r-1}}^{2^r-1}(k+f(k)) + \sum_{k=2^{s-1}}^n (k+f(k)) =$$

$$= \sum_{r=1}^{s-1} 2^{2r-1}\cdot(2^r-1) + (n - 2^{s-1}+1)\cdot(2^s-1) =$$

$$= \sum_{r=1}^{s-1}\left(2^{2r-1} - \sum_{r=1}^{s-1} 2^r + 1\right) + (n-2^{s-1}+1)(2^s-1) =$$

$$= \frac{2}{3}(4^{s-1}-1) - (2^{s-1}-1) + (2^s-1)n - \frac{1}{3}4^s + 2^s - 2$$

$$= (2^s-1)n - \frac{1}{3}4^s + 2^s - \frac{2}{3}$$

and therefore

$$\frac{n^2}{4} - \sum_{k=1}^n f(k) = \frac{n^2}{4} - \left((2^s-1)n - \frac{1}{3}4^s + 2^s - \frac{2}{3} - \frac{n(n+1)}{2}\right) =$$

$$= \frac{3}{4}n^2 - (2^s - \frac{3}{2})n + \frac{1}{3}4^s - 2^s + \frac{2}{3} =$$
$$= \frac{3}{4}\left(n - \frac{2^{s+1}-2}{3}\right)\left(n - \frac{2^{s+1}-4}{3}\right).$$

Notice that the difference of the last two factors is less than 1, and one of them must be an integer: $\frac{2^{s+1}-2}{3}$ is integer if s is even, and $\frac{2^{s+1}-4}{3}$ is integer if s is odd. Therefore, either one of them is 0, resulting a zero product, or both factors have the same sign, so the product is strictly positive. This solves the problem and shows that equality occurs if $n = \frac{2^{s+1}-2}{3}$ (s is even) or $n = \frac{2^{s+1}-4}{3}$ (s is odd).

Problem 4. (10 points) For any positive integer m, denote by $P(m)$ the product of positive divisors of m (e.g. $P(6) = 36$). For every positive integer n define the sequence

$$a_1(n) = n, \quad a_{k+1}(n) = P(a_k(n)) \quad (k = 1, 2, \dots, 2024).$$

Determine whether for every set $S \subseteq \{1, 2, \dots, 2025\}$, there exists a positive integer n such that the following condition is satisfied:

For every k with $1 \le k \le 2025$, the number $a_k(n)$ is a perfect square if and only if $k \in S$.

Solution. We prove that the answer is yes; for every $S \subseteq \{1, 2, \dots, 2025\}$ there exists a suitable n. Specially, n can be a power of 2: $n = 2^{w_1}$ with some nonnegative integer w_1. Write $a_k(n) = 2^{w_k}$; then

$$2^{w_{k+1}} = a_{k+1}(n) = P(a_k(n)) = P(2^{w_k}) = 1 \cdot 2 \cdot 4 \cdot \ldots \cdot 2^{w_k} = 2^{\frac{w_k(w_k+1)}{2}},$$

so

$$w_{k+1} = \frac{w_k(w_k+1)}{2}.$$

The proof will be completed if we prove that for each choice of S there exists an initial value w_1 such that w_k is even if and only if $k \in S$.

Lemma. Suppose that the sequences $(b_1, b_2, \dots)$ and $(c_1, c_2, \dots)$ satisfy $b_{k+1} = \frac{b_k(b_k+1)}{2}$ and $c_{k+1} = c_k(c_k+1)$ for $k \ge 1$, and $c_1 = b_1 + 2^m$. Then for each $k = 1, \dots m$ we have $c_k \equiv b_k \pmod{2^m - k + 2}$.

As an immediate corollary, we have $b_k \equiv c_k \pmod 2$ for $1 \le k \le m$ and $b_{m+1} \equiv c_{m+1} + 1 \pmod 2$.

Proof. We prove the lemma by induction. For $k = 1$ we have $c_1 = b_1 + 2^m$ so the statement holds. Suppose the statement is true for some $k < m$, then for $k+1$ we have

$$c_{k+1} = \frac{c_k(c_k+1)}{2} = \frac{(b_k + 2^m - k + 1)(b_k + 2^m - k + 1 + 1)}{2}$$
$$= \frac{b_k^2 + 2^{m-k} + 2b_k + 2^{2m-2k} + 2 + b_k + 2^{m-k} + 1}{2}$$
$$= \frac{b_k(b_k+1) + 2^{m-k} + 2^{m-k+1}b_k + 2^{2m-2k+1}}{2}$$
$$\equiv b_k(b_k+1) + 2^{m-k} \pmod{2^{m-(k+1)+2}},$$

therefore $c_{k+1} \equiv b_{k+1} + 2^{m-(k+1)+1} \pmod{2^{m-(k+1)+2}}$.

Going back to the solution of the problem, for every $1 \le m \le 2025$ we construct inductively a sequence $(v_1, v_2, \dots)$ such that $v_{k+1} = \frac{v_k(v_k+1)}{2}$, and for every $1 \le k \le m$, v_k is even if and only if $k \in S$.

For $m = 1$ we can choose $v_1 = 0$ if $1 \in S$ or $v_1 = 1$ if $1 \notin S$. If we already have such a sequence $(v_1, v_2, \dots)$ for a positive integer m, we can choose either the same sequence or choose $v_1' = v_1 + 2^m$ and apply the same recurrence $v_{k+1}' = \frac{v_k'(v_k'+1)}{2}$. By the Lemma, we have $v_k \equiv v_k' \pmod 2$ for $k \le m$, but v_{m+1} and v_{m+1}' have opposite parities; hence, either the sequence (v_k) or the sequence (v_k') satisfies the condition for $m+1$.

Repeating this process for $m = 1, 2, \dots, 2025$, we obtain a suitable sequence (v_k).

Problem 5. (10 points) Determine whether or not there exist 15 integers $m_1, \dots, m_{15}$ such that

$$\sum_{k=1}^{15} m_k \cdot \arctan(k) = \arctan(16).$$

Solution We show that such integers $m_1, \ldots, m_{15}$ do not exist. Suppose that equation is satisfied by some integers $m_1, \ldots, m_{15}$. Then the argument of the complex number $z_1 = 1 + 16i$ coincides with the argument of the complex number

$$z_2 = (1+i)^{m_1}(1+2i)^{m_2}(1+3i)^{m_3} \cdots (1+15i)^{m_{15}}.$$

Therefore the ratio $R = \frac{z_2}{z_1}$ is real (and not zero). As $\mathrm{Re}(z_1) = 1$ and $\mathrm{Re}(z_2)$ is an integer, R is a nonzero integer.

By considering the squares of the absolute values of z_1 and z_2, we get

$$(1+16^2)R^2 = \prod_{k=1}^{15}(1+k^2)^{m_k}.$$

Notice that $p = 1 + 16^2 = 257$ is a prime (the fourth Fermat prime), which yields an easy contradiction through p-adic valuations: all prime factors in the right hand side are strictly below p (as $k < 16$ implies $1 + k^2 < p$). On the other hand, in the left hand side the prime p occurs with an odd exponent.