Problems and solutions

Second day — August 3, 1996

Problem 1. (10 points) Prove that if $f : [0,1] \to [0,1]$ is a continuous function, then the sequence of iterates $x_{n+1} = f(x_n)$ converges if and only if

$$\lim_{n\to\infty} (x_{n+1} - x_n) = 0.$$

Solution. The "only if" part is obvious. Now suppose that $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$ and the sequence $\{x_n\}$ does not converge. Then there are two cluster points $K < L$. There must be points from the interval (K, L) in the sequence. There is an $x \in (K, L)$ such that $f(x) \neq x$. Put $\varepsilon = \frac{|f(x)-x|}{2} > 0$. Then from the continuity of the function f we get that for some $\delta > 0$ for all $y \in (x - \delta, x + \delta)$ it is $|f(y) - y| > \varepsilon$. On the other hand for n large enough it is $|x_{n+1} - x_n| < 2\delta$ and $|f(x_n) - x_n| = |x_{n+1} - x_n| < \varepsilon$. So the sequence cannot come into the interval $(x-\delta, x+\delta)$, but also cannot jump over this interval. Then all cluster points have to be at most $x - \delta$ (a contradiction with L being a cluster point), or at least $x + \delta$ (a contradiction with K being a cluster point).

Problem 2. (10 points) Let θ be a positive real number and let $\cosh t = \frac{e^t+e^{-t}}{2}$ denote the hyperbolic cosine. Show that if $k \in \mathbb{N}$ and both $\cosh k\theta$ and $\cosh(k+1)\theta$ are rational, then so is $\cosh\theta$.

Solution. First we show that

$$(1) \quad \text{If } \cosh t \text{ is rational and } m \in \mathbb{N}, \text{ then } \cosh mt \text{ is rational.}$$

Since $\cosh 0.t = \cosh 0 = 1 \in \mathbb{Q}$ and $\cosh 1.t = \cosh t \in \mathbb{Q}$, (1) follows inductively from

$$\cosh(m+1)t = 2\cosh t.\cosh mt - \cosh(m-1)t.$$

The statement of the problem is obvious for $k = 1$, so we consider $k \geq 2$. For any m we have

$$(2) \quad \cosh\theta = \cosh((m+1)\theta - m\theta) = \cosh(m+1)\theta.\cosh m\theta - \sinh(m+1)\theta.\sinh m\theta = \cosh(m+1)\theta.\cosh m\theta - \sqrt{\cosh^2(m+1)\theta - 1}.\sqrt{}$$

Set $\cosh k\theta = a$, $\cosh(k+1)\theta = b$, $a, b \in \mathbb{Q}$. Then (2) with $m = k$ gives

$$\cosh\theta = ab - \sqrt{a^2-1}\sqrt{b^2-1}$$

and then

$$(3) \quad (a^2-1)(b^2-1) = (ab - \cosh\theta)^2 = a^2b^2 - 2ab\cosh\theta + \cosh^2\theta.$$

Set $\cosh(k^2-1)\theta = A, \cosh k^2\theta = B$. From (1) with $m = k-1$ and $t = (k+1)\theta$ we have $A \in \mathbb{Q}$. From (1) with $m = k$ and $t = k\theta$ we have $B \in \mathbb{Q}$. Moreover $k^2 - 1 > k$ implies $A > a$ and $B > b$. Thus $AB > ab$. From (2) with $m = k^2 - 1$ we have

$$(4) \quad (A^2-1)(B^2-1) = (AB - \cosh\theta)^2 = A^2B^2 - 2AB\cosh\theta + \cosh^2\theta.$$

So after we cancel the $\cosh 2\theta$ from (3) and (4) we have a non-trivial linear equation in $\cosh\theta$ with rational coefficients.

Problem 3. (15 points) Let G be the subgroup of $GL_2(\mathbb{R})$, generated by A and B, where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Let H consist of those matrices $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ in G for which $a_{11} = a_{22} = 1$.

(a) Show that H is an abelian subgroup of G.

(b) Show that H is not finitely generated.

Remarks. $GL_2(\mathbb{R})$ denotes, as usual, the group (under matrix multiplication) of all 2×2 invertible matrices with real entries (elements). *Abelian* means commutative. A group is *finitely generated* if there are a finite number of elements of the group such that every other element of the group can be obtained from these elements using the group operation.

Solution.

(a) All of the matrices in G are of the form

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}.$$

So all of the matrices in H are of the form

$$M(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix},$$

so they commute. Since $M(x)^{-1} = M(-x)$, H is a subgroup of G.

(b) A generator of H can only be of the form $M(x)$, where x is a binary rational, i.e., $x = \frac{p}{2^n}$ with integer p and non-negative integer n. In H it holds

$$M(x)M(y) = M(x+y)$$

and

$$M(x)M(y)^{-1} = M(x-y).$$

The matrices of the form $M\left(\frac{1}{2^n}\right)$ are in H for all $n \in \mathbb{N}$. With only finite number of generators all of them cannot be achieved.

Problem 4. (20 points)
Let B be a bounded closed convex symmetric (with respect to the origin) set in $\mathbb{R}^2$ with boundary the curve Γ. Let B have the property that the ellipse of maximal area contained in B is the disc D of radius 1 centered at the origin with boundary the circle C. Prove that $A \cap \Gamma \neq \emptyset$ for any arc A of C of length $l(A) \geq \frac{\pi}{2}$.

Solution. Assume the contrary – there is an arc $A \subseteq C$ with length $l(A) = \frac{\pi}{2}$ such that $A \cap B/\Gamma$. Without loss of generality we may assume that the ends of A are $M = (1/\sqrt{2}, 1/\sqrt{2})$, $N = (1/\sqrt{2}, -1/\sqrt{2})$. A is compact and Γ is closed. From $A \cap \Gamma = \emptyset$ we get $\delta > 0$ such that $\operatorname{dist}(x,y) > \delta$ for every $x \in A$, $y \in \Gamma$.

Given $\varepsilon > 0$ with E_ε we denote the ellipse with boundary: $\frac{x^2}{(1+\varepsilon)^2} + \frac{y^2}{b^2} = 1$, such that $M, N \in E_\varepsilon$. Since $M \in E_\varepsilon$ we get

$$b^2 = \frac{(1+\varepsilon)^2}{2(1+\varepsilon)^2 - 1}.$$

Then we have

$$\text{area} E_\varepsilon = \pi \frac{(1+\varepsilon)^2}{\sqrt{2(1+\varepsilon)^2 - 1}} > \pi = \text{area} D.$$

In view of the hypotheses, $E_\varepsilon \nsubseteq B$ for every $\varepsilon > 0$. Let $S = \{(x,y) \in \mathbb{R}^2 : |x| > |y|\}$. From $E_\varepsilon \subseteq S \subseteq D \subseteq B$ it follows that $E_\varepsilon \nsubseteq B \subseteq S$. Taking $\varepsilon < \delta$ we get that

$$\emptyset \neq E_\varepsilon \backslash B \subseteq E_\varepsilon \cap S \subseteq D_{1+\varepsilon} \cap S \subseteq B$$

– a contradiction (we use the notation $D_t = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq t^2\}$).

Remark. The ellipse with maximal area is well known as John's ellipse. Any coincidence with the President of the Jury is accidental.

Problem 5. (20 points)

(i) Prove that

$$\lim_{x \to +\infty} \sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} = \frac{1}{2}.$$

(ii) Prove that there is a positive constant c such that for every $x \in [1, \infty)$ we have

$$\left| \sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} - \frac{1}{2} \right| \leq \frac{c}{x}.$$

Solution.

(i) Set $f(t) = \frac{t}{(1+t^2)^2}$, $h = \frac{1}{\sqrt{x}}$. Then

$$\sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} = h \sum_{n=1}^{\infty} f(nh) \xrightarrow[h \to 0]{} \int_0^{\infty} f(t)dt = \frac{1}{2}.$$

The convergence holds since $h\sum_{n=1}^{\infty} f(nh)$ is a Riemann sum of the integral $\int_0^{\infty} f(t)dt$. There are no problems with the infinite domain because f is integrable and $f \to 0$ for $x \to \infty$ (thus $h\sum_{n=N}^{\infty} f(nh) \geq \int_{nN}^{\infty} f(t)dt \geq h\sum_{n=N+1}^{\infty} f(nh)$).

(ii) We have

$$(1) \quad \left|\sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} - \frac{1}{2}\right| = \left|\sum_{n=1}^{\infty} \left(hf(nh) - \int_{nh-\frac{h}{2}}^{nh+\frac{h}{2}} f(t)dt\right) - \int_0^{\frac{h}{2}} f(t)dt\right|$$
$$\leq \sum_{n=1}^{\infty} \left|hf(nh) - \int_{nh-\frac{h}{2}}^{nh+\frac{h}{2}} f(t)dt\right| + \int_0^{\frac{h}{2}} f(t)dt$$

Using twice integration by parts one has

$$(2) \quad 2bg(a) - \int_{a-b}^{a+b} g(t)dt = -\frac{1}{2}\int_0^b (b-t)^2(g''(a+t) + g''(a-t))dt$$

for every $g \in C^2[a-b, a+b]$. Using $f(0) = 0$, $f \in C^2[0, h/2]$ one gets

$$(3) \quad \int_0^{h/2} f(t)dt = O(h^2).$$

From (1), (2) and (3) we get

$$\left|\sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} - \frac{1}{2}\right| \leq \sum_{n=1}^{\infty} h^2 \left|\int_{nh-\frac{h}{2}}^{nh+\frac{h}{2}} f''(t)dt\right| + O(h^2) =$$
$$= h^2 \int_{h/2}^{\infty} |f''(t)|dt + O(h^2) = O(h^2) = O(x^{-1}).$$

Problem 6. (Carleman's inequality) (25 points)
(i) Prove that for every sequence $\{a_n\}_{n=1}^{\infty}$, such that $a_n > 0$, $n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty} a_n < \infty$, we have

$$\sum_{n=1}^{\infty} (a_1 a_2 \ldots a_n)^{1/n} < e\sum_{n=1}^{\infty} a_n,$$

where e is the natural log base.
(ii) Prove that for every $\varepsilon > 0$ there exists a sequence $\{a_n\}_{n=1}^{\infty}$, such that $a_n > 0$, $n = 1, 2, \ldots$, $\sum_{n=1}^{\infty} a_n < \infty$ and

$$\sum_{n=1}^{\infty} (a_1 a_2 \ldots a_n)^{1/n} > (e - \varepsilon)\sum_{n=1}^{\infty} a_n.$$

Solution.

(i) Put for $n \in \mathbb{N}$

$$c_n = (n+1)^n / n^{n-1}. \tag{1}$$

Observe that $c_1 c_2 \cdot \ldots \cdot c_n = (n+1)^n$. Hence, for $n \in \mathbb{N}$,

$$(a_1 a_2 \cdot \ldots \cdot a_n)^{1/n} = (a_1 c_1 a_2 c_2 \cdot \ldots \cdot a_n c_n)^{1/n} / (n+1)$$
$$\leq (a_1 c_1 + \ldots + a_n c_n) / n(n+1).$$

Consequently,

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdot \ldots \cdot a_n)^{1/n} \leq \sum_{n=1}^{\infty} a_n c_n \left(\sum_{m=n}^{\infty} (m(m+1))^{-1}\right). \tag{2}$$

Since

$$\sum_{m=n}^{\infty}(m(m+1))^{-1}=\sum_{m=n}^{\infty}\left(\frac{1}{m}-\frac{1}{m+1}\right)=\frac{1}{n},$$

we have

$$\sum_{n=1}^{\infty}a_nc_n\left(\sum_{m=n}^{\infty}(m(m+1))^{-1}\right)=\sum_{n=1}^{\infty}a_nc_n/n=\sum_{n=1}^{\infty}a_n((n+1)/n)^n<e\sum_{n=1}^{\infty}a_n$$

(by (1)). Combining the last inequality with (2) we get the result.

(ii) Set $a_n=n^{-1}\ln(n+1)^{-n}$ for $n=1,2,...,N$ and $a_n=2^{-n}$ for $n>N$, where N will be chosen later. Then

$$(a_1\cdot\ldots\cdot a_n)^{1/n}=\frac{1}{n+1} \tag{3}$$

for $n\le N$. Let $K=K(\varepsilon)$ be such that

$$\left(\frac{n+1}{n}\right)^n>e-\frac{\varepsilon}{2}\text{ for }n>K. \tag{4}$$

Choose N from the condition

$$\sum_{n=1}^{K}a_n+\sum_{n=N+1}^{\infty}2^{-n}\le\frac{\varepsilon}{(2e-\varepsilon)(e-\varepsilon)}\sum_{n=K+1}^{N}\frac{1}{n}, \tag{5}$$

which is always possible because the harmonic series diverges. Using (3), (4) and (5) we have

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{K}a_n+\sum_{n=N+1}^{\infty}2^{-n}+\sum_{n=K+1}^{N}\frac{1}{n}\left(\frac{n}{n+1}\right)^n<$$

$$<\frac{\varepsilon}{(2e-\varepsilon)(e-\varepsilon)}\sum_{n=K+1}^{N}\frac{1}{n}+\left(e-\frac{\varepsilon}{2}\right)^{-1}\sum_{n=K+1}^{N}\frac{1}{n}=$$

$$=e-\varepsilon\sum_{n=K+1}^{N}\frac{1}{n}-e\sum_{n=K+1}^{N}\frac{1}{n}=e-\varepsilon-\sum_{n=1}^{\infty}(a_1\cdot\ldots\cdot a_n)^{1/n}.$$