Proposed problems for the IMC 2023

Problem 1 (Alex Avdiushenko, Neapolis University Paphos, Cyprus). Find all non-constant functions that have a continuous second derivative and for which the equality $f(7x+1) = 49f(x)$ holds for all x.

Remark. A simple problem, suitable for the place of the first.

Solution.

Differentiating the equation twice, we get $f''(7x+1) = f''(x)$, or $f''(x) = f''\left(\frac{x-1}{7}\right)$ — 1 point.

Take an arbitrary $x = a$ and construct a sequence according to the rule $a_0 = a$, $a_{k+1} = \frac{a_k - 1}{7}$. Then the values of f'' at all points of this sequence are equal. The limit of this sequence is $-\frac{1}{6}$, since $\left|a_{k+1} + \frac{1}{6}\right| = \frac{1}{7}\left|a_k + \frac{1}{6}\right|$ — 5 points.

Due to continuity, the values of f'' at all points of this sequence are equal to $f''\left(-\frac{1}{6}\right)$, which means that $f''(x)$ is a constant — 2 points.

Then $f(x) = c_2x^2 + c_1x + c$. Substituting this expression into the original equation, we get a system of equations, from which we find $c_2 = 36c$, $c_1 = 12c$, and $f(x) = (6x+1)^2c$ — 2 points.

Problem 2 (Alex Avdiushenko, Neapolis University Paphos, Cyprus). Given the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$$

It is allowed to multiply or divide any row (column) element-wise by another row (respectively, column). Is it possible to obtain the following matrix B after several such operations?

$$B = \begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$$

Remark. A simple problem, suitable for the place of the first or may be second one.

Solution. Answer: No, it's not possible.

Let's represent all numbers in the matrix as powers of 2 (or any other number). Then the specified operations are reduced to adding or subtracting the corresponding exponents — 4 points.

Let's construct a matrix from the exponents:

$$C = \begin{pmatrix} 1 & \log_2 3 \\ 1 & 2 \end{pmatrix}$$

The determinant of this matrix does not change under the given operations — 4 points.

But it is not equal to 0 and is opposite to the determinant of the exponents of the original matrix — 2 points.

Problem 3 (Alex Avdiushenko, Neapolis University Paphos, Cyprus). Quadratic polynomials $p_1(x) = a_1x^2 + b_1x + c_1$ and $p_2(x) = a_2x^2 + b_2x + c_2$ with integer coefficients do not have common roots. For each $n \in \mathbb{N}$, let $d_n = \gcd(f(n)g(n))$. Prove that the sequence $\{d_n\}_{n=1}^{\infty}$ is bounded.

Remark. A more difficult problem, suitable for the place of the second or third.

Solution. Let $d_n = (f(n)g(n))$. Then $d_n | f(n)$ and $d_n | g(n)$ for any $n \in \mathbb{N}$, where $|$ denotes divisibility.

Therefore, $d_n | a_1p_2(n) - a_2p_1(n) = \gamma n + \beta$, where $\gamma = a_1b_2 - a_2b_1, \beta = a_1c_2 - a_2c_1$.

This means that $\gamma n \equiv -\beta \mod d_n$ — 4 points.

Consider $\gamma^2 p_i(n) \equiv 0 \mod d_n \Rightarrow a_i\beta^2 - b_i\gamma\beta + c_i\gamma^2 \equiv 0 \mod d_n$ — 4 points.

In the case of an unbounded sequence $\{d_n\}$, it follows from here that $a_i\beta^2 - b_i\gamma\beta + c_i\gamma^2 = 0,\ i = 1, 2$, which means $f(n)$ and $g(n)$ have a common root $x = -\frac{\beta}{\gamma}$ — 2 points.

Problem 4 (Alex Avdiushenko, Neapolis University Paphos, Cyprus). Let P and Q be square matrices such that $P^k = Q^l = 0$, where $k, l \in \mathbb{N}$, and 0 is the zero matrix. Prove that if $P \cdot Q = Q \cdot P$, then the matrix $I + \lambda P + \mu Q$ is invertible. Here, $\lambda, \mu \in \mathbb{R}$, I — the identity matrix.

Remark. A more difficult problem, suitable for the place of the second or third.

Solution. Let $S = \lambda P + \mu Q$. Then $S^{k+l-1} = \sum\limits_{i=0}^{k+l-1} \binom{k+l-1}{i} \lambda^i P^i \mu^{k+l-1-i} Q^{k+l-1-i} = 0$, since either the power of P is not less than k, or the power of Q is not less than l. Further, by expanding the brackets, it is checked that $(I+S)\left(I - S + S^2 - \cdots + (-1)^{k+l-2} S^{k+l-2}\right)$, i.e. the required matrix is invertible.