Problems and solutions

Second day — July 30, 1994

Problem 1. (14 points) Let $f \in C^1[a,b]$, $f(a)=0$ and suppose that $\lambda \in \mathbb{R}, \lambda > 0$, is such that

$$|f'(x)| \le \lambda |f(x)|$$

for all $x \in [a,b]$. Is it true that $f(x)=0$ for all $x \in [a,b]$?

Solution. Assume that there is $y \in (a,b]$ such that $f(y) \ne 0$. Without loss of generality we have $f(y) > 0$. In view of the continuity of f there exists $c \in [a,y]$ such that $f(c)=0$ and $f(x) > 0$ for $x \in (c,y]$. For $x \in (c,y]$ we have $|f'(x)| \le \lambda f(x)$. This implies that the function $g(x) = \ln f(x) - \lambda x$ is not increasing in $(c,y]$ because of $g'(x) = \frac{f'(x)}{f(x)} - \lambda \le 0$. Thus $\ln f(x) - \lambda x \ge \ln f(y) - \lambda y$ and $f(x) \ge e^{\lambda x - \lambda y} f(y)$ for $x \in (c,y]$. Thus

$$0 = f(c) = f(c+0) \ge e^{\lambda c - \lambda y} f(y) > 0$$

— a contradiction. Hence one has $f(x)=0$ for all $x \in [a,b]$.

Problem 2. (14 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = (x^2-y^2)e^{-x^2-y^2}$.

a) Prove that f attains its minimum and its maximum.

b) Determine all points (x,y) such that $\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = 0$ and determine for which of them f has global or local minimum or maximum.

Solution. We have $f(1,0) = e^{-1}$, $f(0,1) = -e^{-1}$ and $te^{-t} \le 2e^{-2}$ for $t \ge 2$. Therefore $|f(x,y)| \le (x^2 + y^2)e^{-x^2-y^2} \le 2e^{-2} < e^{-1}$ for $(x,y) \notin M = \{(u,v) : u^2+v^2 \le 2\}$ and f cannot attain its minimum and its maximum outside M. Part a) follows from the compactness of M and the continuity of f. Let (x,y) be a point from part b). From $\frac{\partial f}{\partial x}(x,y) = 2x(1-x^2+y^2)e^{-x^2-y^2}$ we get

$$x(1-x^2+y^2) = 0 \tag{1}$$

Similarly

$$y(1+x^2-y^2) = 0 \tag{2}$$

All solutions (x,y) of the system (1), (2) are $(0,0)$, $(0,1)$, $(0,-1)$, $(1,0)$ and $(-1,0)$. One has $f(1,0) = f(-1,0) = e^{-1}$ and f has global maximum at the points $(1,0)$ and $(-1,0)$. One has $f(0,1) = f(0,-1) = -e^{-1}$ and f has global minimum at the points $(0,1)$ and $(0,-1)$. The point $(0,0)$ is not an extrema point because of $f(x,0) = x^2e^{-x^2} > 0$ if $x \ne 0$ and $f(y,0) = -y^2e^{-y^2} < 0$ if $y \ne 0$.

Problem 3. (14 points) Let f be a real-valued function with $n+1$ derivatives at each point of $\mathbb{R}$. Show that for each pair of real numbers $a, b, a < b$, such that

$$\ln\left(\frac{f(b)+f'(b)+\ldots+f^{(n)}(b)}{f(a)+f'(a)+\ldots+f^{(n)}(a)}\right) = b-a$$

there is a number c in the open interval (a,b) for which

$$f^{(n+1)}(c) = f(c).$$

Note that ln denotes the natural logarithm.

Solution. Set $g(x) = \left(f(x)+f'(x)+\ldots+f^{(n)}(x)\right)e^{-x}$. From the assumption one get $g(a)=g(b)$. Then there exists $c \in (a,b)$ such that $g'(c)=0$. Replacing in the last equality, $g'(x) = \left(f^{(n+1)}(x) - f(x)\right)e^{-x}$ we finish the proof.

Problem 4. (18 points) Let A be a $n \times n$ diagonal matrix with characteristic polynomial

$$(x-c_1)^{d_1}(x-c_2)^{d_2}\cdots(x-c_k)^{d_k},$$

where $c_1, c_2, \ldots, c_k$ are distinct (which means that c_1 appears d_1 times on the diagonal, c_2 appears d_2 times on the diagonal, etc. and $d_1+d_2+\cdots+d_k = n$). Let V be the space of all $n \times n$ matrices B such that $AB = BA$. Prove that the dimension of V is

$$d_1^2 + d_2^2 + \cdots + d_k^2.$$

Solution. Set $A = (a_{ij})_{i,j=1}^n$, $B = (b_{ij})_{i,j=1}^n$, $AB = (x_{ij})_{i,j=1}^n$ and $BA = (y_{ij})_{i,j=1}^n$. Then $x_{ij} = a_{ii}b_{ij}$ and $y_{ij} = a_{jj}b_{ij}$. Thus $AB = BA$ is equivalent to $(a_{ii} - a_{jj})b_{ij} = 0$ for $i, j = 1, 2, \ldots, n$. Therefore $b_{ij} = 0$ if $a_{ii} \ne a_{jj}$ and b_{ij} may be arbitrary if $a_{ii} = a_{jj}$. The number of indices (i,j) for which $a_{ii} = a_{jj} = c_m$ for some $m = 1, 2, \ldots, k$ is d_m^2. This gives the desired result.

Problem 5. (18 points) Let $x_1, x_2, \ldots, x_k$ be vectors of m-dimensional Euclidean space, such that $x_1+x_2+\ldots+x_k = 0$. Show that there exists a permutation π of the integers $\{1, 2, \ldots, k\}$ such that

$$\left\|\sum_{i=1}^{n} x_{\pi(i)}\right\| \leq \left(\sum_{i=1}^{k} \|x_i\|^2\right)^{1/2}$$

for each $n = 1, 2, \ldots, k$.

Note that $\|\cdot\|$ denotes the Euclidean norm.

Solution. We define π inductively. Set $\pi(1) = 1$. Assume π is defined for $i = 1, \ldots, n$ and also

$$\left\|\sum_{i=1}^{n} x_{\pi(i)}\right\|^2 \leq \sum_{i=1}^{n} \|x_{\pi(i)}\|^2 \tag{1}$$

Note (1) is true for $n = 1$. We choose $\pi(n+1)$ in a way that (1) is fulfilled with $n+1$ instead of n. Set $y = \sum_{i=1}^{n} x_{\pi(i)}$ and $A = \{1, 2, \ldots, k\} \setminus \{\pi(i) : i = 1, 2, \ldots, n\}$. Assume that $(y, x_r) > 0$ for all $r \in A$. Then $(y, \sum_{r\in A} x_r) > 0$ and in view of $y + \sum_{r\in A} x_r = 0$ one gets $(y, y) > 0$, which is impossible. Therefore there is $r \in A$ such that

$$(y, x_r) \leq 0 \tag{2}$$

Put $\pi(n+1) = r$. Then using (2) and (1) we have

$$\left\|\sum_{i=1}^{n+1} x_{\pi(i)}\right\|^2 = \|y + x_r\|^2 = \|y\|^2 + 2(y, x_r) + \|x_r\|^2 \leq \|y\|^2 + \|x_r\|^2 \leq$$

$$\leq \sum_{i=1}^{n} \|x_{\pi(i)}\|^2 + \|x_r\|^2 = \sum_{i=1}^{n+1} \|x_{\pi(i)}\|^2,$$

which verifies (1) for $n + 1$. Thus we define π for every $n = 1, 2, \ldots, k$. Finally from (1) we get

$$\left\|\sum_{i=1}^{n} x_{\pi(i)}\right\|^2 \leq \sum_{i=1}^{n} \|x_{\pi(i)}\|^2 \leq \sum_{i=1}^{k} \|x_i\|^2.$$

Problem 6. (22 points) Find

$$\lim_{N\to\infty} \frac{\ln^2 N}{N} \sum_{k=2}^{N-2} \frac{1}{\ln k \cdot \ln(N-k)}.$$

Note that ln denotes the natural logarithm.

Solution. Obviously

$$A_N = \frac{\ln^2 N}{N} \sum_{k=2}^{N-2} \frac{1}{\ln k \cdot \ln(N-k)} \geq \frac{\ln^2 N}{N} \cdot \frac{N-3}{\ln^2 N} = 1 - \frac{3}{N} \tag{1}$$

Take $M, 2 \leq M < N/2$. Then using that $\frac{1}{\ln k \cdot \ln(N-k)}$ is decreasing in $[2, N/2]$ and the symmetry with respect to $N/2$ one get

$$\begin{aligned} A_N &= \frac{\ln^2 N}{N} \left\{ \sum_{k=2}^{M} + \sum_{k=N-M+1}^{N-2} \right\} \frac{1}{\ln k \cdot \ln(N-k)} \leq \\ &\leq \frac{\ln^2 N}{N} \left\{ \frac{2M-1}{\ln 2 \cdot \ln(N-2)} + \frac{N-2M-1}{\ln M \cdot \ln(N-M)} \right\} \leq \\ &\leq \frac{2 \cdot M \ln N}{\ln^2 N} + \frac{(1-2M/N) \ln N}{\ln M \cdot \ln(N-M)} + O\left(\frac{1}{\ln N}\right). \end{aligned}$$

Choose $M = \lfloor \frac{N}{\ln^2 N} \rfloor + 1$ to get

$$(2) \quad A_N < \left(1 - \frac{2}{\ln^2 N}\right) \frac{\ln N}{\ln N - 2\ln M} + O\left(\frac{1}{\ln N}\right) \leq 1 + O\left(\frac{\ln\ln N}{\ln N}\right).$$

Estimates (1) and (2) give

$$\lim_{N\to\infty} \frac{\ln^2 N}{N} \sum_{k=2}^{N-2} \frac{1}{\ln k \cdot \ln(N-k)} = 1.$$