# Problems

## Second day — July 30, 1994

**Problem 1.** (14 points) Let $f \in C^1[a,b]$, $f(a) = 0$ and suppose that $\lambda \in \mathbb{R}, \lambda > 0$, is such that

$$|f'(x)| \le \lambda |f(x)|$$

for all $x \in [a,b]$. Is it true that $f(x) = 0$ for all $x \in [a,b]$?

**Problem 2.** (14 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = (x^2 - y^2)e^{-x^2-y^2}$.

a) Prove that $f$ attains its minimum and its maximum.

b) Determine all points $(x,y)$ such that $\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y) = 0$ and determine for which of them $f$ has global or local minimum or maximum.

**Problem 3.** (14 points) Let $f$ be a real-valued function with $n+1$ derivatives at each point of $\mathbb{R}$. Show that for each pair of real numbers $a, b, a < b$, such that

$$\ln\left(\frac{f(b) + f'(b) + \ldots + f^{(n)}(b)}{f(a) + f'(a) + \ldots + f^{(n)}(a)}\right) = b - a$$

there is a number $c$ in the open interval $(a,b)$ for which

$$f^{(n+1)}(c) = f(c).$$

Note that ln denotes the natural logarithm.

**Problem 4.** (18 points) Let $A$ be a $n \times n$ diagonal matrix with characteristic polynomial

$$(x - c_1)^{d_1}(x - c_2)^{d_2} \cdots (x - c_k)^{d_k},$$

where $c_1, c_2, \ldots, c_k$ are distinct (which means that $c_1$ appears $d_1$ times on the diagonal, $c_2$ appears $d_2$ times on the diagonal, etc. and $d_1 + d_2 + \cdots + d_k = n$). Let $V$ be the space of all $n \times n$ matrices $B$ such that $AB = BA$. Prove that the dimension of $V$ is

$$d_1^2 + d_2^2 + \cdots + d_k^2.$$

**Problem 5.** (18 points) Let $x_1, x_2, \ldots, x_k$ be vectors of $m$-dimensional Euclidean space, such that $x_1 + x_2 + \ldots + x_k = 0$. Show that there exists a permutation $\pi$ of the integers $\{1, 2, \ldots, k\}$ such that

$$\left\| \sum_{i=1}^{n} x_{\pi(i)} \right\| \le \left( \sum_{i=1}^{k} \|x_i\|^2 \right)^{1/2}$$

for each $n = 1, 2, \ldots, k$.

Note that $\|\cdot\|$ denotes the Euclidean norm.

**Problem 6.** (22 points) Find

$$\lim_{N\to\infty} \frac{\ln^2 N}{N} \sum_{k=2}^{N-2} \frac{1}{\ln k \cdot \ln(N-k)}.$$

Note that ln denotes the natural logarithm.