Problems

First day — August 3, 1998

Problem 1. (20 points) Let V be a 10-dimensional real vector space and U_1 and U_2 two linear subspaces such that $U_1 \subseteq U_2$, $\dim U_1 = 3$ and $\dim U_2 = 6$. Let $\mathcal{E}$ be the set of all linear maps $T : V \to V$ which have U_1 and U_2 as invariant subspaces (i.e., $T(U_1) \subseteq U_1$ and $T(U_2) \subseteq U_2$). Calculate the dimension of $\mathcal{E}$ as a real vector space.

Problem 2. Prove that the following proposition holds for $n = 3$ (5 points) and $n = 5$ (7 points), and does not hold for $n = 4$ (8 points).

"For any permutation π_1 of $\{1, 2, \ldots, n\}$ different from the identity there is a permutation π_2 such that any permutation π can be obtained from π_1 and π_2 using only compositions (for example, $\pi = \pi_1 \circ \pi_1 \circ \pi_2 \circ \pi_1$)."

Problem 3. Let $f(x) = 2x(1 - x)$, $x \in \mathbb{R}$. Define

$$f_n = \underbrace{f \circ \ldots \circ f}_{n \text{ times}}.$$

a) (10 points) Find $\displaystyle\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$.

b) (10 points) Compute $\int_0^1 f_n(x)\,dx$ for $n = 1, 2, \ldots$.

Problem 4. (20 points) The function $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and satisfies $f(0) = 2, f'(0) = -2$ and $f(1) = 1$. Prove that there exists a real number $\xi \in (0, 1)$ for which

$$f(\xi) \cdot f'(\xi) + f''(\xi) = 0.$$

Problem 5. Let P be an algebraic polynomial of degree n having only real zeros and real coefficients.

(a) (15 points) Prove that for every real x the following inequality holds:

$$(n-1)\left(P'(x)\right)^2 \geq nP(x)P''(x) \tag{1}$$

(b) (5 points) Examine the cases of equality.

Problem 6. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function with the property that for any x and y in the interval,

$$xf(y) + yf(x) \leq 1.$$

a) (15 points) Show that

$$\int_0^1 f(x)\,dx \leq \frac{\pi}{4}.$$

b) (5 points) Find a function, satisfying the condition, for which there is equality.