1 Helix equation fit

The points will lay on a helix, or (for our purposes) on a circle (we will neglect the z axis). The error ε_i of the measurement point i will be given by:w

$$\varepsilon_i = \frac{1}{2}\rho r_i^2 - (1+\rho d) r_i \sin(\phi_i - \phi_0) + \frac{1}{2}\rho d^2 + d \tag{1}$$

there r_i is the layer radius, ϕ_0 is the initial track's polar angle in the transverse plane, ϕ_i is the polar angle of each hit, ρ is the curvature of the track ($\rho = 1/R$, with R radius of curvature) and d is the distance of closest approach of the track to the z axis. The curvature ρ is related to the transverse momentum accoding to

$$\rho(\mathrm{m}) = \frac{0.3B}{p_T} \tag{2}$$

assuming a single-charged particle, with B measured in Tesla and p_T in GeV/c. Assuming the CMS magnet field of 3.8 T, this becomes

$$\rho(\mathrm{mm}) = \frac{1.14 \times 10^3}{p_T} \tag{3}$$

We have a set of measurement points, where the only error is baically on the y position of hit $y_i = d\sin(\phi - \phi_0)$. For momenta high enough, fitting the helix reduces to minimzing the χ^2 defined as follows

$$\chi^2 = \sum_{i,j} \varepsilon_j C_{i,j}^{-1} \varepsilon_i \tag{4}$$

with $C_{i,j}$ the correlation between the measurement points. The weight matrix W will be:

$$W_{k,l} = \sum_{i,j} \frac{\partial \varepsilon_i}{\partial \alpha_k} C_{i,j}^{-1} \frac{\partial \varepsilon_j}{\partial \alpha_l} \tag{5}$$

with $\alpha_1 = \rho$, $\alpha_2 = \phi$, $\alpha_3 = d$. The covariance matrix S is given by $S = W^{-1}$, and thus the measurement errors are:

$$\begin{aligned} \Delta\rho &= \sqrt{W_{1,1}^{-1}} \\ \Delta\phi &= \sqrt{W_{2,2}^{-1}} \\ \Delta d &= \sqrt{W_{3,3}^{-1}} \end{aligned}$$

which can be also written as

$$\begin{aligned} \sigma_\rho^2 &= S_{1,1} \\ \sigma_\phi^2 &= S_{2,2} \\ \sigma_d^2 &= S_{3,3} \\ \sigma_{\rho,\phi} &= S_{1,2} \\ &\ldots \end{aligned}$$

to evidentiate the covariances. The derivatives of (1) are:

$$\begin{aligned} \frac{\partial \varepsilon_i}{\partial \alpha_1} = \frac{\partial \varepsilon_i}{\partial \rho} &= \frac{1}{2} r_i^2 + d(d + y_i) \\ \frac{\partial \varepsilon_i}{\partial \alpha_2} = \frac{\partial \varepsilon_i}{\partial \phi} &= -x_i(1 + \rho d) \\ \frac{\partial \varepsilon_i}{\partial \alpha_3} = \frac{\partial \varepsilon_i}{\partial d} &= 1 + \rho(d - y_i) \end{aligned}$$

If we take into account that $d \ll r_i$, $d \ll 1/\rho$, $y_i \ll 1/\rho$, $r_i \simeq x_i$ we can approximate the previous set of equations as

$$\begin{aligned}
\frac{\partial \varepsilon_i}{\partial \alpha_1} = \frac{\partial \varepsilon_i}{\partial \rho} &= \frac{1}{2} r_i^2 \\
\frac{\partial \varepsilon_i}{\partial \alpha_2} = \frac{\partial \varepsilon_i}{\partial \phi} &= -r_i \\
\frac{\partial \varepsilon_i}{\partial \alpha_3} = \frac{\partial \varepsilon_i}{\partial d} &= 1
\end{aligned}$$

If we have M measurement points, the partial derivatives matrix D will be $M \times 3$ and defined by

$$D_{i,j} = \frac{\partial \varepsilon_i}{\partial \alpha_j} \tag{6}$$

and the 3×3 weight matrix W will be (in matrix notation) $W = D^T C^{-1} D$.

2 Error estimate

2.1 Track parameters

Given the layer radii $x_n = x_1, x_2, \ldots, x_N$ with scattering angles $\theta_1, \theta_2, \ldots, \theta_3$, then the deviation from the ideal path y_n is

$$y_n = \sum_{i=1}^{n-1} (x_n - x_i)\, \theta_i \tag{7}$$

The angles θ_i are distributed as a Gaussian, with r.m.s. such that

$$\left\langle \theta^2 \right\rangle = \left(\frac{13.6\,\mathrm{MeV}}{p} \right)^2 \frac{x}{X_0} \left[1 + 0.038 \log \left(\frac{x}{X_0} \right) \right]^2 \tag{8}$$

The correlation between two deviations y_n, y_m is (we will assume without loss of generality that $m \geq n$)

$$a_{n,m} \left\langle y_n y_m \right\rangle = \left\langle \sum_{i=1}^{m-1} (x_m - x_i)\, \theta_i \times \sum_{j=1}^{n-1} (x_n - x_j)\, \theta_j \right\rangle \tag{9}$$

Since the angles θ_i are uncorrelated, any term containing in $\langle \theta_i \theta_j \rangle$ with $i \neq j$ will be zero, thus

$$\begin{aligned}
a_{n,m} \left\langle y_n y_m \right\rangle &= \left\langle \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} (x_m - x_i)(x_n - x_j)\, \theta_i \theta_j \delta_{i,j} \right\rangle \\
&= \sum_{i=1}^{n-1} (x_m - x_i)(x_n - x_i) \left\langle \theta_i^2 \right\rangle
\end{aligned} \tag{10}$$

The measurement "error" depends both on the scattering of the real track with respect to the ideal case and also on the intrinsic measurement error σ_i, which depends approximately on the strip pitch p_i according to $\sigma_i = p_i/\sqrt{12}$, or $\sigma_i^2 = p_i^2/12$, thus the covariance matrix $b_{n,m}$ is

$$b_{n,m} = \begin{cases} \sum_{i=1}^{n-1} (x_m - x_i)(x_n - x_i) \left\langle \theta_i^2 \right\rangle & n < m \\ p_n^2/12 + \sum_{i=1}^{n-1} (x_n - x_i)^2 \left\langle \theta_i^2 \right\rangle & n = m \\ b_{m,n} & n > m \end{cases} \tag{11}$$

Let's suppose we have N hits, but in these N only M are measurement points and $N - M$ are hits on inactive surfaces. In this matrix $b_{n,m}$ is computed exactly in the same way, but the rows and columns corresponding to the inactive hits are removed. We thus start from a $N \times N$ square matrix of correlations $b_{n,m}$ and we end up with a $M \times M$ measurement point covariance matrix $C_{n,m}$, or in matrix notation C.

2.2 Extrapolation to the ECAL

We want to estimate the error in the extrapolation of the track to an external cylindrical surface, for example the inner surface of ECAL.

In the transverse plane this reduces to the interception of two circles: the given cylinder section of radius L and the track trajectory with radius $R = 1/\rho$. Given that typically $R \gg d$ and $L \gg d$ we will neglect d and consider the particle trajectory to be starting from the origin of the x, y plane (see Figure 1 below).

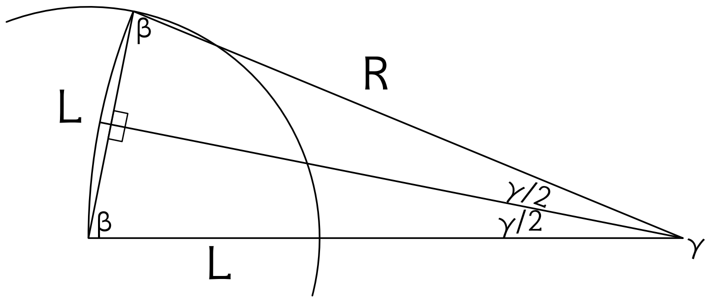

Figure 1: Target cylinder surface and particle trajectory in the x, y plane

To extrapolate the position on the target barrel the important parameter is φ:

$$\varphi = \phi + \beta \tag{12}$$

where ϕ is the trajectory angle at the origin already esitmated (plus a constant which is not influent in the error estimation) and β is shown in Figure 1.

From the figure one can easily get the relation

$$R\,\cos(\beta) = \frac{L}{2} \tag{13}$$

$$\beta = \arccos\left(\frac{L\rho}{2}\right) \tag{14}$$

where the relation $R = 1/\rho$ was used.

The error on φ can be estimated through

$$\sigma_\varphi^2 = \left(\frac{\partial\varphi}{\partial\rho}\right)^2 \sigma_\rho^2 + \left(\frac{\partial\varphi}{\partial\phi}\right)^2 \sigma_\phi^2 + 2\left(\frac{\partial\varphi}{\partial\rho}\right)\left(\frac{\partial\varphi}{\partial\phi}\right)\sigma_{\phi\rho} \tag{15}$$

given that

$$\frac{\partial\varphi}{\partial\phi} = 1 \tag{16}$$

and

$$\frac{\partial\varphi}{\partial\rho} = \frac{\partial\beta}{\partial\rho} = -\frac{L}{\sqrt{4 - L^2\rho^2}} \tag{17}$$

the error on φ can be written as

$$\sigma_\varphi^2 = \left(\frac{L^2}{4 - L^2\rho^2}\right)\sigma_\rho^2 + \sigma_\phi^2 - \left(\frac{2\,L}{\sqrt{4 - L^2\rho^2}}\right)\sigma_{\phi\rho} \tag{18}$$

the linear error in the $r - \varphi$ plane of the extrapolation Δs will be finally

$$\Delta s = L\sqrt{\sigma_\varphi^2} \tag{19}$$