1 Helix equation fit

The points will lay on a helix, or (for our purposes) on a circle (we will neglect the z axis). The error ε_i of the measurement point i will be given by:w

$$\varepsilon_i = \frac{1}{2}\rho r_i^2 - (1+\rho d) r_i \sin(\phi_i - \phi_0) + \frac{1}{2}\rho d^2 + d \tag{1}$$

there r_i is the layer radius, ϕ_0 is the initial track's polar angle in the transverse plane, ϕ_i is the polar angle of each hit, ρ is the curvature of the track ($\rho = 1/R$, with R radius of curvature) and d is the distance of closest approach of the track to the z axis. The curvature ρ is related to the transverse momentum accoding to

$$\rho(\mathrm{m}) = \frac{0.3B}{p_T} \tag{2}$$

assuming a single-charged particle, with B measured in Tesla and p_T in GeV/c. Assuming the CMS magnet field of 3.8 T, this becomes

$$\rho(\mathrm{mm}) = \frac{1.14 \times 10^3}{p_T} \tag{3}$$

We have a set of measurement points, where the only error is baically on the y position of hit $y_i = d\sin(\phi - \phi_0)$. For momenta high enough, fitting the helix reduces to minimzing the χ^2 defined as follows

$$\chi^2 = \sum_{i,j} \varepsilon_j C^{-1}_{i,j} \varepsilon_i \tag{4}$$

with $C_{i,j}$ the correlation between the measurement points. The weight matrix W will be:

$$W_{k,l} = \sum_{i,j} \frac{\partial \varepsilon_i}{\partial \alpha_k} C^{-1}_{i,j} \frac{\partial \varepsilon_j}{\partial \alpha_l} \tag{5}$$

with $\alpha_1 = \rho$, $\alpha_2 = \phi$, $\alpha_3 = d$. The covariance matrix S is given by $S = W^{-1}$, and thus the measurement errors are:

$$\begin{aligned} \Delta\rho &= \sqrt{W^{-1}_{1,1}} \\ \Delta\phi &= \sqrt{W^{-1}_{2,2}} \\ \Delta d &= \sqrt{W^{-1}_{3,3}} \end{aligned}$$

which can be also written as

$$\begin{aligned} \sigma^2_\rho &= S_{1,1} \\ \sigma^2_\phi &= S_{2,2} \\ \sigma^2_d &= S_{3,3} \\ \sigma_{\rho,\phi} &= S_{1,2} \\ &\ldots \end{aligned}$$

to evidentiate the covariances. The derivatives of (1) are:

$$\begin{aligned} \frac{\partial \varepsilon_i}{\partial \alpha_1} = \frac{\partial \varepsilon_i}{\partial \rho} &= \frac{1}{2} r_i^2 + d(d + y_i) \\ \frac{\partial \varepsilon_i}{\partial \alpha_2} = \frac{\partial \varepsilon_i}{\partial \phi} &= -x_i(1 + \rho d) \\ \frac{\partial \varepsilon_i}{\partial \alpha_3} = \frac{\partial \varepsilon_i}{\partial d} &= 1 + \rho(d - y_i) \end{aligned}$$

If we take into account that $d \ll r_i$, $d \ll 1/\rho$, $y_i \ll 1/\rho$, $r_i \simeq x_i$ we can approximate the previous set of equations as

$$\begin{aligned}
\frac{\partial \varepsilon_i}{\partial \alpha_1} = \frac{\partial \varepsilon_i}{\partial \rho} &= \frac{1}{2} r_i^2 \\
\frac{\partial \varepsilon_i}{\partial \alpha_2} = \frac{\partial \varepsilon_i}{\partial \phi} &= -r_i \\
\frac{\partial \varepsilon_i}{\partial \alpha_3} = \frac{\partial \varepsilon_i}{\partial d} &= 1
\end{aligned}$$

If we have M measurement points, the partial derivatives matrix D will be $M \times 3$ and defined by

$$D_{i,j} = \frac{\partial \varepsilon_i}{\partial \alpha_j} \tag{6}$$

and the 3×3 weight matrix W will be (in matrix notation) $W = D^T C^{-1} D$.

2 Error estimate

2.1 Track parameters

Given the layer radii $x_n = x_1, x_2, \ldots, x_N$ with scattering angles $\theta_1, \theta_2, \ldots, \theta_3$, then the deviation from the ideal path y_n is

$$y_n = \sum_{i=1}^{n-1} (x_n - x_i)\, \theta_i \tag{7}$$

The angles θ_i are distributed as a Gaussian, with r.m.s. such that

$$\left\langle \theta^2 \right\rangle = \left(\frac{13.6\,\mathrm{MeV}}{p} \right)^2 \frac{x}{X_0} \left[1 + 0.038 \log \left(\frac{x}{X_0} \right) \right]^2 \tag{8}$$

The correlation between two deviations y_n, y_m is (we will assume without loss of generality that $m \geq n$)

$$a_{n,m} \left\langle y_n y_m \right\rangle = \left\langle \sum_{i=1}^{m-1} (x_m - x_i)\, \theta_i \times \sum_{j=1}^{n-1} (x_n - x_j)\, \theta_j \right\rangle \tag{9}$$

Since the angles θ_i are uncorrelated, any term containing in $\langle \theta_i \theta_j \rangle$ with $i \neq j$ will be zero, thus

$$\begin{aligned}
a_{n,m} \left\langle y_n y_m \right\rangle &= \left\langle \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} (x_m - x_i)(x_n - x_j)\, \theta_i \theta_j \delta_{i,j} \right\rangle \\
&= \sum_{i=1}^{n-1} (x_m - x_i)(x_n - x_i) \left\langle \theta_i^2 \right\rangle
\end{aligned} \tag{10}$$

The measurement "error" depends both on the scattering of the real track with respect to the ideal case and also on the intrinsic measurement error σ_i, which depends approximately on the strip pitch p_i according to $\sigma_i = p_i/\sqrt{12}$, or $\sigma_i^2 = p_i^2/12$, thus the covariance matrix $b_{n,m}$ is

$$b_{n,m} = \begin{cases} \sum_{i=1}^{n-1} (x_m - x_i)(x_n - x_i) \left\langle \theta_i^2 \right\rangle & n < m \\ p_n^2/12 + \sum_{i=1}^{n-1} (x_n - x_i)^2 \left\langle \theta_i^2 \right\rangle & n = m \\ b_{m,n} & n > m \end{cases} \tag{11}$$

Let's suppose we have N hits, but in these N only M are measurement points and $N - M$ are hits on inactive surfaces. In this matrix $b_{n,m}$ is computed exactly in the same way, but the rows and columns corresponding to the inactive hits are removed. We thus start from a $N \times N$ square matrix of correlations $b_{n,m}$ and we end up with a $M \times M$ measurement point covariance matrix $C_{n,m}$, or in matrix notation C.