A Simple Introduction to Learning Quantum Mechanics

By

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abbas abedini:Don't act like a quantum superposition in your personal life.

Proof of the Schrödinger equation

The goal is to proof the time-dependent Schrödinger equation in one spatial dimension: :

Since we do not know the exact location we are looking for the 'probability of presence'; and because the behavior of this probability is 'wave-like', and waves are described by energy, the Schrödinger equation must be written in terms of energy, not force.

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x,t)$$

where $\psi(x,t)$ is the wave function, m is the particle mass, $V(x)$ is the potential energy, and $\hbar$ is the reduced Planck constant.

We assume particles have wave-like behavior described by a wave function $\psi(x,t)$. A free particle can be represented by a plane wave:

$$\psi(x,t) = Ae^{i(kx-\omega t)}$$

where k is the wave number and ω is the angular frequency.

From quantum theory:

$$p = \hbar k, \quad E = \hbar\omega$$

where p is momentum and E is energy.

For a non-relativistic particle moving in potential $V(x)$, classical total energy is

$$E = \frac{p^2}{2m} + V(x)$$

To translate classical quantities into quantum mechanics, replace momentum and energy by operators acting on ψ:

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}, \quad \hat{E} = i\hbar\frac{\partial}{\partial t}$$

These definitions arise from the properties of plane waves, since:

$$\hat{p}\psi = p\psi, \quad \hat{E}\psi = E\psi$$

Constructing-the-Schrödinger-Equation

Substitute classical quantities with operators in the energy relation:

$$\hat{E}\psi = \left(\frac{\hat{p}^2}{2m} + V(x)\right)\psi$$

Using operator definitions:

$$i\hbar\frac{\partial}{\partial t}\psi = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi$$

which is the **time-dependent Schrödinger equation** in one dimension.

In quantum mechanics, classical physical quantities like momentum p and energy E are represented by operators acting on the wave function ψ.

Momentum

Consider a one-dimensional plane wave:

$$\psi(x) = e^{ikx}$$

Taking the spatial derivative of $\psi(x)$:

$$\frac{d}{dx}\psi(x) = ike^{ikx}$$

Multiplying both sides by $-i\hbar$, we get:

$$-i\hbar\frac{d}{dx}\psi(x) = \hbar k\psi(x)$$

Using the de Broglie relation, momentum is related to the wave number k as

$$p = \hbar k$$

Therefore, the momentum operator is defined as:

$$\hat{p} = -i\hbar\frac{d}{dx}$$

which acts on $\psi(x)$ to yield the momentum eigenvalue p.

$$\hat{k} = -i\frac{d}{dx}$$

Apply it to the wave function:

$$\hat{k}\psi(x) = -i\frac{d}{dx}e^{ikx} = -i(ike^{ikx}) = k\,e^{ikx}$$

We observed that it gave me the wave number, that expression in the presence of the wave function, of course this is for the plane

Energy

Similarly, consider the time-dependent part of the wave function:

$$\psi(t) = e^{-i\omega t}$$

Taking the time derivative:

$$\frac{\partial}{\partial t}\psi(t) = -i\omega e^{-i\omega t}$$

Multiplying by $i\hbar$:

$$i\hbar \frac{\partial}{\partial t} \psi(t) = \hbar\omega \psi(t)$$

Using the Planck-Einstein relation, energy is related to frequency ω as

$$E = \hbar\omega$$

Thus, the energy operator is defined as:

$$\hat{E} = i\hbar \frac{\partial}{\partial t}$$

which acts on $\psi(t)$ to yield the energy eigenvalue E.

The operators for momentum and energy are:

$$\hat{p} = -i\hbar \frac{d}{dx}, \quad \hat{E} = i\hbar \frac{\partial}{\partial t}$$

These operators allow us to write physical observables as differential operators acting on the wave function ψ, forming the basis for the Schrödinger equation.

Plane-Wave

Consider a free-particle plane wave in one dimension:

$$\psi(x,t) = A\, e^{i(kx - \omega t)}.$$

By de Broglie and Planck relations:

$$p = \hbar k, \qquad E = \hbar\omega.$$

Momentum

Apply a single spatial derivative:

$$\frac{\partial}{\partial x} \psi = ik\,\psi \implies -i\hbar \frac{\partial}{\partial x} \psi = \hbar k\,\psi = p\,\psi.$$

Hence the momentum operator is

$$\hat{p} = -\,i\hbar\, \frac{\partial}{\partial x}.$$

Kinetic-Energy

Classical kinetic energy: $E_{\text{kin}} = p^2 2m$. Replace p by $\hat{p}$:

$$\hat{T} = \frac{\hat{p}^2}{2m} = \frac{(-i\hbar)^2}{2m}\frac{\partial^2}{\partial x^2} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}.$$

Acting on our plane wave:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = -\frac{\hbar^2}{2m}(ik)^2\,\psi = \frac{\hbar^2 k^2}{2m}\,\psi = \frac{p^2}{2m}\,\psi = E_{\text{kin}}\,\psi.$$

Total-Energy

Apply a single time derivative:

$$\frac{\partial}{\partial t}\,\psi = -i\omega\,\psi \quad\Longrightarrow\quad i\hbar\,\frac{\partial}{\partial t}\,\psi = \hbar\omega\,\psi = E\,\psi.$$

Thus the Hamiltonian acting on a free particle is

$$\hat{H}\,\psi = i\hbar\,\frac{\partial\psi}{\partial t}.$$

Time-Dependent-Schrödinger-Equation

Including a potential $V(x)$, the full Hamiltonian is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x),$$

so the time-dependent Schrödinger equation reads

$$i\hbar\,\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\,\psi.$$

So we learned that the second derivative of position from the wave function gives us kinetic energy, and the first derivative of time from the wave function gives me total energy, and in this way, the Schrödinger equation is obtained.

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Newton's Second Law and the Schrödinger Equation

Abstract

The time-dependent Schrödinger equation, through the Ehrenfest theorem.

We work in one dimension. The wavefunction $\Psi(x,t)$ is normalized by

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = 1.$$

Operators:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x), \quad \hat{p} = -i\hbar x.$$

The time-dependent Schrödinger equation:

$$i\hbar \Psi t = \hat{H}\Psi.$$

Define expectation values: $\langle x \rangle = \int \Psi^*(x,t)\, x\, \Psi(x,t)\, dx$,
$\langle p \rangle = \int \Psi^*(x,t)\, \hat{p}\, \Psi(x,t)\, dx = -i\hbar \int \Psi^*\Psi x\, dx$. It's a number that tells you the average result you would get if you repeated a measurement many times.

Time-Derivative-of-$\langle x \rangle$

$$t\langle x \rangle = t \int \Psi^* \, x \, \Psi \, dx = \int \left(\Psi^* t \, x \, \Psi + \Psi^* \, x \, \Psi t \right) dx.$$

Using Schrödinger's equation and its conjugate:

$$\hat{H}\psi(\mathbf{r}) = E\psi(\mathbf{r}), \Psi t = \frac{1}{i\hbar} \, \hat{H}\Psi, \quad \Psi^* t = -\frac{1}{i\hbar} \, \hat{H}\Psi^*,$$

we get

$$t\langle x \rangle = \frac{1}{i\hbar} \int \Psi^* (\hat{H}x - x\hat{H})\Psi \, dx = \frac{1}{i\hbar} \langle [\hat{H}, x] \rangle.$$

Compute the commutator:

$$[\hat{H}, x] = \left[\frac{\hat{p}^2}{2m}, x \right] = \frac{1}{2m} (\hat{p}[\hat{p}, x] + [\hat{p}, x]\hat{p}) = -\frac{i\hbar}{m}\hat{p}.$$

$$t\langle x \rangle = \frac{1}{i\hbar} \left(-\frac{i\hbar}{m} \right) \langle \hat{p} \rangle = \frac{\langle p \rangle}{m}.$$

Time-Derivative-of $\langle p \rangle$

$$t\langle p \rangle = t \int \Psi^* \, \hat{p} \, \Psi \, dx = \int \left(\Psi^* t \, \hat{p} \, \Psi + \Psi^* \, \hat{p} \, \Psi t \right) dx.$$

Substitute time derivatives:

$$t\langle p \rangle = \frac{1}{i\hbar} \int \Psi^* (\hat{H}\hat{p} - \hat{p}\hat{H})\Psi \, dx = \frac{1}{i\hbar} \langle [\hat{H}, \hat{p}] \rangle.$$

Since $[\hat{p}, \hat{p}^2] = 0$ and $[\hat{p}, V(x)] = -i\hbar \, V'(x)$:

$$[\hat{H}, \hat{p}] = -i\hbar \, V'(x),$$

so

$$t\langle p \rangle = \frac{1}{i\hbar}(-i\hbar)\langle V'(x) \rangle = -\langle V'(x) \rangle.$$

Combining results:

$$m[2]t\langle x \rangle = t\langle p \rangle = -\langle V'(x) \rangle,$$

which is the quantum analogue of Newton's second law:

$$m\langle \ddot{x} \rangle = \langle F(x) \rangle.$$

Why the Wave Function Must Be Complex

Time-Dependent-Schrödinger-Equation

The time-dependent Schrödinger equation is given by:

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = \hat{H} \psi(x,t) \tag{1}$$

Suppose the wave function $\psi(x,t)$ is a purely **real-valued** function:

$$\psi(x,t) \in R \tag{2}$$

Then its time derivative is also real:

$$\frac{\partial}{\partial t} \psi(x,t) \in R \tag{3}$$

Now consider the left-hand side (LHS) of the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) \in C \tag{4}$$

This means the LHS is *purely imaginary* (since i multiplies a real function), while the right-hand side (RHS) is:

$$\hat{H} \psi(x,t) \in R \tag{5}$$

Contradiction

This leads to a contradiction:

$$LHS \in C \setminus R, \qquad RHS \in R$$

$$\Rightarrow LHS \neq RHS$$

Therefore, the Schrödinger equation cannot be satisfied if $\psi(x,t)$ is purely real.

Conclusion

To ensure both sides of the Schrödinger equation match, the wave function $\psi(x,t)$ must be complex-valued in general. This is why wave functions are typically written in the form:

$$\psi(x,t) = Ae^{i(kx-\omega t)}$$

Wave-Function-and-Sinusoidal-Form

We know that wave functions usually describe oscillatory behavior. A typical classical-looking wave may be written as:

$$\psi(x,t) = A\cos(kx - \omega t) \tag{6}$$

or

$$\psi(x,t) = A\sin(kx - \omega t) \tag{7}$$

These are **real-valued** functions and describe oscillations. However, in quantum mechanics, it is common to write the wave function using a **complex exponential**:

$$\psi(x,t) = Ae^{i(kx-\omega t)} \tag{8}$$

Using Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{9}$$

This means the complex exponential contains both sine and cosine components:

$$e^{i(kx-\omega t)} = \cos(kx - \omega t) + i\sin(kx - \omega t)$$

So we are not losing information—instead, we're using a more compact and powerful form.

Writing the wave function as a complex exponential provides many benefits:

- **Mathematical simplicity:** Differentiation and integration of exponentials is much easier than sine and cosine.

- **Phase information:** The phase of the wave is directly and clearly encoded in the exponential.

- **Interference calculations:** Adding and multiplying waves is easier using exponential notation.

- **Solving the Schrödinger Equation:** The time-dependent Schrödinger equation involves imaginary units, so a complex wave function is naturally compatible.

Simple-Example

If we write:

$$\psi(x,t) = Ae^{i(kx-\omega t)}$$

Then the time derivative becomes:

$$\frac{\partial \psi}{\partial t} = -i\omega Ae^{i(kx-\omega t)} = -i\omega\psi$$

This is a clean, compact result. But if we had used cosine or sine, the calculation would have been messier, involving chains of trigonometric identities.

In the end, all physical predictions are made using $|\psi|^2$, which is always a real number, so the use of complex numbers in the wave function does not affect measurable quantities.

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Plane-Wave-and-Non-Normalizability

Consider the plane wave function of a free particle:

$$\psi(x,t) = Ae^{i(kx-\omega t)}$$

Probability-Density

The probability density is given by:

$$|\psi(x,t)|^2 = \psi^*(x,t)\,\psi(x,t)$$

First compute the complex conjugate:

$$\psi^*(x,t) = A^* e^{-i(kx-\omega t)}$$

Assuming $A \in R$, then:

$$|\psi(x,t)|^2 = Ae^{-i(kx-\omega t)} \cdot Ae^{i(kx-\omega t)} = A^2$$

This is constant for all x and t.

Normalization-Attempt

To normalize the wave function, we must have:

$$\int_{-\infty}^{\infty} |\psi(x,t)|^2 \, dx = 1$$

But since $|\psi(x,t)|^2 = A^2$, we get:

$$\int_{-\infty}^{\infty} A^2 \, dx = A^2 \int_{-\infty}^{\infty} dx = A^2 \cdot \infty = \infty$$

This integral diverges, so the plane wave is **not normalizable**.

Conclusion

The plane wave is an idealization representing a particle with definite momentum $p = \hbar k$, but completely uncertain position. This is consistent with the Heisenberg uncertainty principle:

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

For a plane wave:

- $\Delta p = 0 \Rightarrow \Delta x = \infty$
- The particle is spread uniformly over all space

Entanglement

Imagine we have two particles: one with you, and one with your friend in another city.

Each particle can be in one of two states: **Left** or **Right**. We do not know exactly which one is in which state, but we know that:

They are always in opposite states!

Quantum-State-of-the-System

We cannot say:

$$First is Left, Second is Right \quad or \quad First is Right, Second is Left$$

Instead, the real quantum state is a superposition of both:

$$\Psi = \frac{1}{\sqrt{2}} \left(Left_1 Right_2 + Right_1 Left_2 \right)$$

This is called an **entangled state**.

Why-is-This-Special?

- If you look at only one particle (say, particle 1), you get 50% chance of Left and 50% of Right.

- But the moment you measure it, the state of particle 2 is instantly known!

- That means their states are "linked" even if they are far apart.

Entanglement means that the full wavefunction Ψ cannot be separated into:

$$\Psi \neq \psi_1 \cdot \psi_2$$

The information is stored in the whole system, not in each part separately.

Entanglement is one of the most fundamental and non-classical features of quantum mechanics. It shows how quantum systems can be deeply connected in a way that has no classical analogue.

Normalization

In quantum mechanics, the **wavefunction** $\psi(x,t)$ describes the *probability amplitude* of finding a particle at position x at time t. However, $\psi(x,t)$ itself is not a probability; rather, $|\psi(x,t)|^2$ gives the *probability density*. Therefore, to ensure that the total probability is unity, the wavefunction must be *normalized*.

A wavefunction $\psi(x,t)$ is said to be **normalized** if:

$$\int_{-\infty}^{+\infty} |\psi(x,t)|^2 \, dx = 1.$$

This condition guarantees that the particle is found *somewhere* in space with probability 1.

- In quantum mechanics, we only have probabilistic information about a particle's position.

- The probability density $|\psi(x,t)|^2$ must integrate to 1 over all space, since the particle must exist somewhere.

If an unnormalized wavefunction satisfies

$$\int_{-\infty}^{+\infty} |\psi(x,t)|^2 \, dx = A \neq 1,$$

we define the *normalized* wavefunction as

$$\psi_{norm}(x,t) = \frac{1}{\sqrt{A}}\,\psi(x,t),$$

so that

$$\int_{-\infty}^{+\infty} |\psi_{norm}(x,t)|^2\,dx = 1.$$

Example

Consider the trial wavefunction:

$$\psi(x) = A\,e^{-ax^2},$$

where $a > 0$ is a real constant. To determine A, impose normalization:

$$\int_{-\infty}^{+\infty} |\psi(x)|^2\,dx = \int_{-\infty}^{+\infty} |A|^2 e^{-2ax^2}\,dx = 1.$$

Using the Gaussian integral identity

$$\int_{-\infty}^{+\infty} e^{-bx^2}\,dx = \sqrt{\frac{\pi}{b}}, \quad b > 0,$$

we get

$$|A|^2 \int_{-\infty}^{+\infty} e^{-2ax^2}\,dx = |A|^2\sqrt{\frac{\pi}{2a}} = 1 \quad\Longrightarrow\quad |A|^2 = \sqrt{\frac{2a}{\pi}} \quad\Longrightarrow\quad A = \left(2a\pi\right)^{1/4}.$$

Hence, the fully normalized Gaussian wavefunction is

$$\psi(x) = \left(2a\pi\right)^{1/4} e^{-ax^2}.$$

$$\int_{-\infty}^{+\infty} \left|\left(2a\pi\right)^{1/4} e^{-ax^2}\right|^2 dx = \left(2a\pi\right)^{1/2}\int_{-\infty}^{+\infty} e^{-2ax^2}\,dx = \left(2a\pi\right)^{1/2}\sqrt{\frac{\pi}{2a}} = 1.$$

When-the-Normalization-Integral-is-Zero

$$\int_{-\infty}^{+\infty} |\psi(x,t)|^2\,dx = 0$$

$$|\psi(x,t)|^2 = 0 \quad \Rightarrow \quad \psi(x,t) = 0 \quad \forall x$$

$$\psi(x,t) = 0 \quad \Rightarrow \quad The\ particle\ does\ not\ exist$$

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Expectation Values of Position and Momentum in Quantum Mechanics

What-is-an-Expectation-Value?

In quantum mechanics, the **expectation value** of a physical observable represents the *average outcome* of measuring that quantity many times on identically prepared systems. Mathematically, for a wavefunction $\psi(x,t)$ and operator $\hat{Q}$:

$$\langle Q \rangle = \int_{-\infty}^{+\infty} \psi^*(x,t)\, \hat{Q}\, \psi(x,t)\, dx$$

The operator for position is simply multiplication by x, so:

$$\langle x \rangle = \int_{-\infty}^{+\infty} \psi^*(x,t)\, x\, \psi(x,t)\, dx$$

Physical-Interpretation

$\langle x \rangle$ gives the **average position** of the particle over many measurements. It is the center of the probability distribution $|\psi(x,t)|^2$. This does not mean the particle is located exactly at $\langle x \rangle$, but that this is the statistical mean.

Example

Suppose the wavefunction is significantly nonzero only between $x = 2$ and $x = 4$, and its peak is at $x = 3$. Then we expect:

$$\langle x \rangle \approx 3$$

This tells us that most measurements of position would give values close to $x = 3$.

The momentum operator in position space is:

$$\hat{p} = -i\hbar \frac{d}{dx}$$

Therefore, the expectation value of momentum is:

$$\langle p \rangle = \int_{-\infty}^{+\infty} \psi^*(x,t) \left(-i\hbar \frac{d}{dx} \right) \psi(x,t) \, dx$$

Note

If the wavefunction is symmetric and real (e.g., a stationary state centered at the origin), then:

$$\langle p \rangle = 0$$

because the particle has no preferred direction of motion.

- Expectation values provide the average outcomes of quantum measurements.
- $\langle x \rangle$ tells us where the particle is **on average**.
- $\langle p \rangle$ tells us the **average momentum** of the particle.

Why-is-the-Expectation-Value-of-Momentum-Zero-for-Symmetric-Wavepackets?

Expectation Value of Momentum

The expectation value of momentum is defined as:

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^*(x) \left(-i\hbar \frac{d}{dx} \right) \psi(x) \, dx$$

where $\psi(x)$ is the wavefunction of the particle.

Even (Symmetric) and Odd (Antisymmetric) Wavefunctions

A function $\psi(x)$ is called **even (symmetric)** if:

$$\psi(-x) = \psi(x)$$

and **odd (antisymmetric)** if:

$$\psi(-x) = -\psi(x).$$

For example:

– Even function: $\psi(x) = e^{-ax^2}$ because $e^{-a(-x)^2} = e^{-ax^2}$.
– Odd function: $\psi(x) = xe^{-ax^2}$ because $\psi(-x) = -xe^{-ax^2} = -\psi(x)$.

Why is $\langle p \rangle = 0$ for Even Wavefunctions?

If the wavefunction $\psi(x)$ is real and even, then the integrand in $\langle p \rangle$ is an odd function, which means:

$$f(-x) = -f(x).$$

Integrating an odd function over all space yields zero:

$$\int_{-\infty}^{\infty} f(x)dx = 0.$$

This means the average momentum $\langle p \rangle$ is zero.

Physically, this happens because the particle's probability distribution is symmetric about the origin — there is no preferred direction of motion.

A symmetric wavepacket can be thought of as a combination of two waves moving in opposite directions with momenta $+\hbar k$ and $-\hbar k$. These components cancel each other's average momentum, resulting in:

$$\langle p \rangle = 0.$$

In contrast, if the wavefunction is a simple plane wave like:

$$\psi(x) = e^{ikx},$$

then it has a definite momentum:

$$\langle p \rangle = \hbar k \neq 0.$$

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Mean Position and Standard Deviation in Quantum Mechanics

Suppose a particle has a normalized wavefunction $\psi(x)$ defined on some region (e.g., the length of a wall).

Mean Position

The expectation value (mean) of the position is given by:

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\psi(x)|^2 \, dx$$

If $\langle x \rangle = 0$, it only means that the probability distribution $|\psi(x)|^2$ is centered around $x = 0$. It does not tell us anything about how spread out the distribution is.

1 Expectation Value of Position in the Infinite Square Well

Consider a particle of mass m confined in a one-dimensional infinite square well of width L:

$$V(x) = \{\, 0 \,, 0 < x < L,$$

$\infty, otherwise.$ The normalized wavefunction for the ground state $(n = 1)$ is

$$\psi_1(x) = \sqrt{\frac{2}{L}}\,\sin\!\left(\frac{\pi x}{L}\right), \quad 0 < x < L.$$

The expectation value of the position operator is defined by

$$\langle x \rangle = \int_0^L \psi_1^*(x)\, x\, \psi_1(x)\, \mathrm{d}x = \int_0^L x\, |\psi_1(x)|^2\, \mathrm{d}x.$$

Since $\psi_1(x)$ is real,

$$\langle x \rangle = \frac{2}{L}\int_0^L x\,\sin^2(\pi x L)\, \mathrm{d}x.$$

Using the identity $\sin^2\theta = 1 - \cos(2\theta)2$ and evaluating the integrals one finds

$$\langle x \rangle = \frac{L}{2}.$$

Thus, in the ground state the particle's average position is at the center of the well.

2 Probability of Finding the Particle in a Given Region

For the same ground-state wavefunction $\psi_1(x)$, the probability of finding the particle between positions a and b is

$$P(a < x < b) = \int_a^b |\psi_1(x)|^2\, \mathrm{d}x = \frac{2}{L}\int_a^b \sin^2(\pi x L)\, \mathrm{d}x.$$

As a specific example, take $a = 0$ and $b = L/2$. Then

$$P(0 < x < L2) = \frac{2}{L}\int_0^{L/2} \sin^2(\pi x L)\, \mathrm{d}x = \frac{1}{L}\int_0^{L/2}[1 - \cos(2\pi x L)]\, \mathrm{d}x = \frac{1}{2}.$$

Therefore, in the ground state the probability of finding the particle in the left half of the well is 50%.

$$P(0 < x < L) = \int_0^L |\psi_1(x)|^2\, dx = \int_0^L \left(\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)\right)^2 dx = \frac{2}{L}\int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx$$

Using the identity:

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2},$$

we get

$$P(0 < x < L) = \frac{2}{L} \int_0^L \frac{1 - \cos\left(\frac{2\pi x}{L}\right)}{2} dx = \frac{1}{L} \int_0^L \left(1 - \cos\left(\frac{2\pi x}{L}\right)\right) dx$$

Calculate the integrals:

$$\int_0^L 1 \, dx = L, \quad \int_0^L \cos\left(\frac{2\pi x}{L}\right) dx = \left[\frac{L}{2\pi} \sin\left(\frac{2\pi x}{L}\right)\right]_0^L = 0,$$

Therefore,

$$P(0 < x < L) = \frac{1}{L}(L - 0) = 1.$$

The normalized wave function is chosen such that the sum of the probabilities of finding the particle in the entire possible region (from 0 to L) is equal to 1.

Standard Deviation of Position

The standard deviation of position (uncertainty in position) is:

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

Even if $\langle x \rangle = 0$, the value of Δx can be large, small, or even tend to zero depending on the shape of $|\psi(x)|^2$.

So I look at the wall of my room and I assume that there's a non-normalized wave function in it, meaning that the probability of a particle being there is the same all over my wall. Now when I calculate the expected value, it has nothing to do with the probability or the maximum probability of the particle being there. It tells me where the average measurement on the wall is. When I consider my wall to be one-dimensional, and if it's four centimeters, my expected value is two centimeters. We get the probability value using the wave function formula to the power of two and the absolute value.

The fact that $\langle x \rangle = 0$ does not imply that $\Delta x \to 0$ or $\Delta x \to \infty$. The mean and the standard deviation are independent quantities:

- $\langle x \rangle$ tells us the average outcome of position measurements.
- Δx tells us how spread out those measurements are.

Similarly, the uncertainty principle connects Δx and Δp via:

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

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Effect of Momentum Uncertainty on Energy in Quantum Mechanics

In quantum mechanics, we must distinguish between:

- The **energy eigenvalue** E_n when the system is in an energy eigenstate ψ_n, where

$$\hat{H}\psi_n = E_n\psi_n$$

- The **expectation value of energy** when the system is in a general state ψ, given by:

$$\langle E \rangle = \langle \psi | \hat{H} | \psi \rangle$$

Kinetic Energy and Momentum Uncertainty

For a free particle or in regions where the potential energy is constant, the Hamiltonian is:

$$\hat{H} = \frac{\hat{p}^2}{2m}$$

Then the expectation value of energy becomes:

$$\langle E \rangle = \left\langle \frac{\hat{p}^2}{2m} \right\rangle = \frac{\langle \hat{p}^2 \rangle}{2m}$$

Using the identity:

$$\langle \hat{p}^2 \rangle = (\Delta p)^2 + \langle p \rangle^2$$

we get:

$$\langle E \rangle = \frac{(\Delta p)^2 + \langle p \rangle^2}{2m}$$

This shows that an increase in the momentum uncertainty Δp increases the *expectation value* of the energy, but not necessarily the energy eigenvalue itself.

Conclusion

If the system is in a superposition of energy eigenstates, the expectation value $\langle E \rangle$ can increase as the spread in momentum Δp increases. However, if the system is in a single eigenstate of energy, then its energy is fixed and unaffected by Δp. $\langle E \rangle$ –infinity – alone does not guarantee that the particle has exited the well.

For complete exit, all components of the wave function must have energy greater than V0.

- Even (symmetric) wavefunctions imply no preferred direction, so average momentum is zero.
- This corresponds to a particle that, on average, is not moving in any particular direction.
- Wavefunctions that are not symmetric correspond to particles with a net average momentum. article amsmath physics tikz enumitem

Expectation Value of Momentum and Real vs. Complex Wavefunctions

[label=⇒]The expectation value of momentum in position space is given by:

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^*(x) \left(-i\hbar \frac{d}{dx} \right) \psi(x)\, dx$$

If the wavefunction is **real and symmetric**, like:

$$\psi(x) = A e^{-\frac{x^2}{2a^2}}$$

then the integrand becomes an **odd function**, and the result is:

$$\langle p \rangle = 0$$

If the wavefunction has a **complex phase**, such as:

$$\psi(x) = A e^{ikx} e^{-\frac{x^2}{2a^2}}$$

then:

$$\langle p \rangle = \hbar k$$

meaning the particle has average momentum in the direction of wave propagation. A wavefunction being **real** does **not** mean it is non-physical. For example, the ground state of the quantum harmonic oscillator is real and perfectly physical:

$$\psi_0(x) = A e^{-\frac{m\omega x^2}{2\hbar}}$$

However, **motion** and **non-zero momentum** require a complex wavefunction with phase, like e^{ikx}. summary:

Wavefunction Type	Physically Valid?	Mean Momentum	Motion
Real (no phase)	Yes	Zero	No
Complex with phase	Yes	Non-zero	Yes

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Real Wavefunctions in Ion Traps

[label=⇒]In modern quantum experiments, scientists trap single ions using electromagnetic or optical fields. This is done using:

* · Paul traps (radiofrequency traps)

- Penning traps (magnetic and electric fields)
- Optical traps (laser cooling and trapping)

* The goal is to isolate the ion from external disturbances and cool it down to its lowest quantum state (the ground state).
* In the ground state, the ion's wavefunction closely resembles that of a quantum harmonic oscillator in its lowest energy level:

$$\psi_0(x) = Ae^{-\frac{m\omega x^2}{2\hbar}}$$

* This wavefunction is:
    - **Real** (no imaginary or complex phase)
    - Physically meaningful and measurable
* In this state:
    - The average momentum is zero: $\langle p \rangle = 0$
    - But it still has non-zero kinetic energy due to quantum fluctuations
* Therefore, the wavefunction being **real and without phase** does not mean the particle is classical or "frozen", but rather it is in a symmetric, lowest-energy quantum state.

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Expectation Value Symmetry Table

Let $\psi(x)$ be the wave function, and $\hat{O}(x)$ be an operator. We consider the expectation value:

$$\langle \hat{O} \rangle = \int_{-a}^{a} \psi^*(x)\hat{O}(x)\psi(x)\,dx$$

The table below shows when this integral is zero based on parity (even/odd) and symmetry.

Wavefunction $\psi(x)$	Operator $\hat{O}(x)$	Symmetric Interval	Expectation Value $\langle$
Even	Even	Symmetric	It can be zero or non-ze
Even	Odd	Symmetric	$\langle \hat{O} \rangle = 0$
Odd	Even	Symmetric	$\langle \hat{O} \rangle = 0$
Odd	Odd	Symmetric	$\langle \hat{O} \rangle = 0$
Any	Any	Non-symmetric	It can be zero or non-ze

Note:

* A function $f(x)$ is **even** if $f(-x) = f(x)$.
* A function $f(x)$ is **odd** if $f(-x) = -f(x)$.
* The integral of an odd function over a symmetric interval $[-a, a]$ is zero.

Expectation Value of Kinetic Energy for a Gaussian Wave Packet

What Is a Gaussian Wave Packet?

In quantum mechanics, to describe a particle that is *localized* in space but still has some spread, we use a Gaussian wave packet:

$$\psi(x) = A \, \exp\!\left(-\frac{x^2}{4\sigma^2}\right) \exp(ik_0 x)$$

"The 'Gaussian wave packet' means a wave function that 'centrally' shows the particle statistically within a specific range.

* $A = (1/(2\pi\sigma^2))^{1/4}$ is the normalization constant,
* $\exp(-x^2/(4\sigma^2))$ is a bell-shaped envelope showing the probability is highest near $x = 0$ and decays away,
* σ controls the width: smaller σ means the packet is more tightly localized,
* k_0 is the central wave number, related to the average momentum $p = \hbar k_0$.

Kinetic Energy Operator

In quantum mechanics, kinetic energy becomes the operator

$$\hat{T} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}$$

The average kinetic energy is

$$\langle T \rangle = \int_{-\infty}^{\infty} \psi^*(x)\, \hat{T}\, \psi(x)\, dx = -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \psi^*(x)\, \frac{d^2\psi(x)}{dx^2}\, dx.$$

Derivatives

$$\frac{d\psi}{dx} = A\left(-\frac{x}{2\sigma^2} + ik_0\right) \exp\!\left(-\frac{x^2}{4\sigma^2}\right) e^{ik_0 x},$$

$$\frac{d^2\psi}{dx^2} = A\left[-\frac{1}{2\sigma^2} - \frac{x^2}{4\sigma^4} - i\frac{k_0 x}{\sigma^2} - k_0^2\right]\exp\left(-\frac{x^2}{4\sigma^2}\right)e^{ik_0 x}.$$

Using standard results:

$$\int_{-\infty}^{\infty} e^{-x^2/(2\sigma^2)}\,dx = \sqrt{2\pi\sigma^2}, \quad \int_{-\infty}^{\infty} x^2 e^{-x^2/(2\sigma^2)}\,dx = \sqrt{2\pi\sigma^2}\,(2\sigma^2), \quad \int_{-\infty}^{\infty} x\, e^{-x^2/(2\sigma^2)}\,dx = 0.$$

Combining everything yields

$$\langle T \rangle = \frac{\hbar^2 k_0^2}{2m} + \frac{\hbar^2}{8\,m\,\sigma^2}.$$

Physical Interpretation

* **First term:** $\frac{\hbar^2 k_0^2}{2m}$ is the "classical" kinetic energy from the average momentum $p = \hbar k_0$.
* **Second term:** $\frac{\hbar^2}{8m\sigma^2}$ is the *quantum correction*. A narrower packet (smaller σ) implies a larger momentum spread by the uncertainty principle, adding extra kinetic energy.

Example

For $\sigma = 1$ nm and $k_0 = 10^9$ m^{-1}, one finds

$$\langle T \rangle = \frac{\hbar^2 (10^9)^2}{2m} + \frac{\hbar^2}{8m(1 \text{ nm})^2} \approx \underbrace{3.8 \times 10^{-20}\,\text{J}}_{classical} + \underbrace{6.2 \times 10^{-21}\,\text{J}}_{quantum correction}.$$

If is large $\to$ the bell shape is broad $\to$ a particle can be found in a relatively wide area. If is small $\to$ the bell shape is sharp and narrow $\to$ a particle must almost certainly be around the center.

Help in understanding integrals and derivatives

Wave Function

Consider the normalized Gaussian wave packet:

$$\psi(x) = A \exp\left(-\frac{x^2}{4\sigma^2}\right) e^{ik_0 x}, \quad A = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4}.$$

$$\hat{T} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}.$$

$$\langle T \rangle = \int_{-\infty}^{\infty} \psi^*(x)\hat{T}\psi(x)\,dx = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} \psi^*(x)\frac{d^2\psi}{dx^2}dx.$$

First, calculate the first and second derivatives of $\psi(x)$:

$$\frac{d\psi}{dx} = A\left(-\frac{x}{2\sigma^2} + ik_0\right)\exp\left(-\frac{x^2}{4\sigma^2}\right)e^{ik_0 x},$$

$$\frac{d^2\psi}{dx^2} = A\left[-\frac{1}{2\sigma^2} - \frac{x^2}{4\sigma^4} - i\frac{k_0 x}{\sigma^2} - k_0^2\right]\exp\left(-\frac{x^2}{4\sigma^2}\right)e^{ik_0 x}.$$

$$\langle T \rangle = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\psi^*(x)\frac{d^2\psi}{dx^2}dx = -\frac{\hbar^2}{2m}A^2\int_{-\infty}^{\infty}e^{-\frac{x^2}{2\sigma^2}}e^{-ik_0 x}\left[-\frac{1}{2\sigma^2} - \frac{x^2}{4\sigma^4} - i\frac{k_0 x}{\sigma^2} - k_0^2\right]e^{ik_0 x}dx.$$

Simplify the exponentials:

$$e^{-ik_0 x}\cdot e^{ik_0 x} = 1,$$

so the integral reduces to

$$\langle T \rangle = -\frac{\hbar^2}{2m}A^2\int_{-\infty}^{\infty}e^{-\frac{x^2}{2\sigma^2}}\left[-\frac{1}{2\sigma^2} - \frac{x^2}{4\sigma^4} - i\frac{k_0 x}{\sigma^2} - k_0^2\right]dx.$$

Separating Terms

Break the integral into four parts:

$$\langle T \rangle = -\frac{\hbar^2}{2m} A^2 \left[-\frac{1}{2\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx - \frac{1}{4\sigma^4} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2\sigma^2}} dx - i\frac{k_0}{\sigma^2} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2\sigma^2}} dx / [\right.$$

- k$_0^2$ $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx$

Evaluating the Integrals

Recall the standard Gaussian integrals:

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx = \sqrt{2\pi\sigma^2},$$

$$\int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2\sigma^2}} dx = \sqrt{2\pi\sigma^2} \cdot 2\sigma^2,$$

$$\int_{-\infty}^{\infty} x e^{-\frac{x^2}{2\sigma^2}} dx = 0 \quad (because the integrand is odd).$$

$$\langle T \rangle = -\frac{\hbar^2}{2m} A^2 \left[-\frac{1}{2\sigma^2} \sqrt{2\pi\sigma^2} - \frac{1}{4\sigma^4} \cdot 2\sigma^2 \sqrt{2\pi\sigma^2} - 0 - k_0^2 \sqrt{2\pi\sigma^2} \right].$$

Simplify the terms inside the bracket:

$$-\frac{1}{2\sigma^2} \sqrt{2\pi\sigma^2} - \frac{1}{2\sigma^2} \sqrt{2\pi\sigma^2} - k_0^2 \sqrt{2\pi\sigma^2} = \left(-\frac{1}{\sigma^2} - k_0^2 \right) \sqrt{2\pi\sigma^2}.$$

Therefore,

$$\langle T \rangle = -\frac{\hbar^2}{2m} A^2 \left(-\frac{1}{\sigma^2} - k_0^2 \right) \sqrt{2\pi\sigma^2} = \frac{\hbar^2}{2m} A^2 \left(\frac{1}{\sigma^2} + k_0^2 \right) \sqrt{2\pi\sigma^2}.$$

Normalization

Recall that

$$A^2 = \left(\frac{1}{2\pi\sigma^2} \right)^{1/2} A^2 \sqrt{2\pi\sigma^2} = 1.$$

Thus,

$$\langle T \rangle = \frac{\hbar^2 k_0^2}{2m} + \frac{\hbar^2}{2m\sigma^2}.$$

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– Examples

1)Suppose the wavefunction of a particle at a given time is:

$$\psi(x) = \{ \sqrt{2}, if 0 \leq x \leq \frac{1}{2}$$

0, otherwise

We check whether the wavefunction is normalized:

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_0^{1/2} (\sqrt{2})^2 \, dx = \int_0^{1/2} 2 \, dx = 2 \times \frac{1}{2} = 1$$

So the wavefunction is properly normalized.

$$P = \int_0^{0.25} |\psi(x)|^2 \, dx = \int_0^{0.25} 2 \, dx = 2 \times 0.25 = 0.5$$

The probability of finding the particle between $x = 0$ and $x = 0.25$ is:

$$0.5 \quad or \quad 50\%$$

Examples

2)Suppose the wavefunction is defined as:

$$\psi(x) = \{\ A\ x\ if\ 0 \le x \le 1$$

0   otherwise

We impose the normalization condition:

$$\int_{-\infty}^{\infty} |\psi(x)|^2\, dx = 1 \Rightarrow \int_{0}^{1} |Ax|^2\, dx = 1$$

$$\int_{0}^{1} A^2 x^2\, dx = A^2 \int_{0}^{1} x^2\, dx = A^2 \cdot \frac{1}{3} = 1 \Rightarrow A^2 = 3 \Rightarrow A = \sqrt{3}$$

Thus, the normalized wavefunction is:

$$\psi(x) = \{\ \sqrt{3}x\ if\ 0 \le x \le 1$$

0   otherwise

$$P = \int_{0.5}^{1} |\psi(x)|^2\, dx = \int_{0.5}^{1} (\sqrt{3}x)^2\, dx = 3 \int_{0.5}^{1} x^2\, dx$$

$$\int_{0.5}^{1} x^2\, dx = \left[\frac{x^3}{3}\right]_{0.5}^{1} = \frac{1}{3} - \frac{0.125}{3} = \frac{0.875}{3}$$

$$P = 3 \cdot \frac{0.875}{3} = 0.875$$

$$P = 0.875 \quad or \quad 87.5\%$$

Examples

3)Expectation Value of Energy for a Free Particle with Gaussian Wavefunction
Let the wavefunction of a particle at time $t = 0$ be given by:

$$\psi(x) = Ae^{-ax^2}, \quad a > 0$$

Assume the particle is free, so the potential energy is zero:

$$V(x) = 0$$

Thus, the Hamiltonian operator is:

$$\hat{H} = -\frac{\hbar^2}{2m}[2]x$$

Expectation Value of Energy

The expectation value of the total mechanical energy is:

$$\langle E \rangle = \int_{-\infty}^{\infty} \psi^*(x)\hat{H}\psi(x)\,dx$$

Since $\psi(x)$ is real:

$$\langle E \rangle = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} \psi(x)[2]x\psi(x)\,dx$$

First derivative:

$$\psi x = -2axAe^{-ax^2}$$

Second derivative:

$$[2]\psi x = Ae^{-ax^2}(4a^2x^2 - 2a)$$

$$\langle E \rangle = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} Ae^{-ax^2} \cdot Ae^{-ax^2}(4a^2x^2 - 2a)\,dx$$

$$= -\frac{\hbar^2}{2m}A^2\int_{-\infty}^{\infty} e^{-2ax^2}(4a^2x^2 - 2a)\,dx$$

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$$\langle E \rangle = -\frac{\hbar^2}{2m} A^2 \left[4a^2 \int_{-\infty}^{\infty} x^2 e^{-2ax^2} dx - 2a \int_{-\infty}^{\infty} e^{-2ax^2} dx \right]$$

Use the Gaussian integrals:

$$\int_{-\infty}^{\infty} e^{-bx^2} dx = \sqrt{\frac{\pi}{b}}, \quad \int_{-\infty}^{\infty} x^2 e^{-bx^2} dx = \frac{1}{2b} \sqrt{\frac{\pi}{b}}$$

$b = 2a$

$$\int e^{-2ax^2} dx = \sqrt{\frac{\pi}{2a}}, \quad \int x^2 e^{-2ax^2} dx = \frac{1}{4a} \sqrt{\frac{\pi}{2a}}$$

$$\langle E \rangle = -\frac{\hbar^2}{2m} A^2 \left[4a^2 \cdot \frac{1}{4a} \sqrt{\frac{\pi}{2a}} - 2a \cdot \sqrt{\frac{\pi}{2a}} \right]$$

$$= -\frac{\hbar^2}{2m} A^2 \sqrt{\frac{\pi}{2a}} (a - 2a) = \frac{\hbar^2 a}{2m} A^2 \sqrt{\frac{\pi}{2a}}$$

Normalization of the Wavefunction

We require:

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1 \Rightarrow A^2 \int_{-\infty}^{\infty} e^{-2ax^2} dx = 1$$

$$A^2 \sqrt{\frac{\pi}{2a}} = 1 \Rightarrow A^2 = \sqrt{\frac{2a}{\pi}}$$

Substitute A^2 into the expectation value:

$$\langle E \rangle = \frac{\hbar^2 a}{2m}$$

This result shows that the energy depends on the parameter a, which controls the spread of the wavefunction. A larger a means a more localized wave packet in space, leading to a greater uncertainty in momentum and hence more kinetic energy.

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Compact Table: Effect of a on Quantum Properties

Quantity	$a \uparrow$	$a \downarrow$
$\psi(x)$ shape	Narrower	Wider
Δx	Decreases	Increases
Δp	Increases	Decreases
$\langle T \rangle$	Increases	Decreases

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Physical Interpretation of Kinetic Energy vs. Parameter a

Consider a Gaussian wavefunction:

$$\psi(x) = Ae^{-ax^2}, \quad a > 0$$

where a controls the spatial width of the wavefunction.

Effect of Increasing a:

* The wavefunction becomes more localized (narrower) in position space.

$$\Downarrow$$

* The position uncertainty Δx decreases.

$$\Downarrow$$

* By Heisenberg's uncertainty principle,

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2},$$

the momentum uncertainty Δp increases.

$$\Downarrow$$

* Since kinetic energy operator is

$$\hat{T} = \frac{\hat{p}^2}{2m},$$

a larger spread in momentum means higher expectation value of kinetic energy.

Summary:

$$a \uparrow \Delta x \downarrow, \quad \Delta p \uparrow, \quad \langle T \rangle \uparrow$$

$$a \downarrow \Delta x \uparrow, \quad \Delta p \downarrow, \quad \langle T \rangle \downarrow$$

//

In quantum mechanics, when a system's state is expressed as a superposition of energy eigenstates, we write the wavefunction as:

$$\psi(x, t) = \sum_n c_n \phi_n(x) \, e^{-iE_n t/\hbar}$$

where $\phi_n(x)$ are the energy eigenstates with eigenvalues E_n, and c_n are complex coefficients.

$$|c_n|^2$$

represents the **probability** of finding the particle in the energy eigenstate ϕ_n with energy E_n upon measurement.

These coefficients must satisfy the normalization condition:

$$\sum_n |c_n|^2 = 1$$

That means the sum of all probabilities equals 1.

$\langle E \rangle$ is the average energy obtained from repeatedly measuring a set of identical systems, taking into account the probabilities —cn—^{2}

$$\langle E \rangle = \sum_n |c_n|^2 \, E_n$$

which represents the average energy measured over many identical experiments. article amsmath amssymb

Example: Expected Energy in a Quantum Superposition

$$E_1 = 1\,eV, \quad E_2 = 2\,eV, \quad E_3 = 5\,eV$$

The wavefunction is given by a superposition of these eigenstates:

$$\psi = \frac{1}{\sqrt{2}}\phi_1 + \frac{1}{2}\phi_2 + \frac{1}{2}\phi_3$$

The coefficients are:

$$c_1 = \frac{1}{\sqrt{2}}, \quad c_2 = \frac{1}{2}, \quad c_3 = \frac{1}{2}$$

We compute the probabilities:

$$|c_1|^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

$$|c_2|^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}, \quad |c_3|^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

Now, the expected value of energy is:

$$\langle E \rangle = \sum_n |c_n|^2 E_n = \left(\frac{1}{2}\right)(1) + \left(\frac{1}{4}\right)(2) + \left(\frac{1}{4}\right)(5)$$

$$\langle E \rangle = \frac{1}{2} + \frac{1}{2} + \frac{5}{4} = 1 + 1.25 = 2.25 \, eV$$

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Consider a particle in a one-dimensional infinite square well of width a, with walls at $x = 0$ and $x = a$. The energy eigenstates and eigenvalues are

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right), \quad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}, \quad n = 1, 2, \ldots$$

wavefunction

At time $t = 0$, suppose the particle's wavefunction is uniformly spread across the well:

$$\psi(x, 0) = \{ \ \sqrt{\frac{1}{a}}, 0 \le x \le a,$$

$0, \quad$ elsewhere.

$-$

$-$ This is normalized because

$$\int_0^a |\psi(x, 0)|^2 \, dx \ = \ \int_0^a \frac{1}{a} \, dx \ = \ 1.$$

c_1

The coefficient for the first eigenstate is

$$c_1 = \int_0^a \phi_1^*(x) \, \psi(x, 0) \, dx = \sqrt{\frac{2}{a}} \, \sqrt{\frac{1}{a}} \int_0^a \sin\left(\frac{\pi x}{a}\right) dx.$$

Evaluate the integral:

$$\int_0^a \sin\left(\frac{\pi x}{a}\right) dx = \left[-\frac{a}{\pi} \cos\left(\frac{\pi x}{a}\right) \right]_0^a = \frac{2a}{\pi}.$$

Hence

$$c_1 = \sqrt{2a} \, \sqrt{1a} \, \frac{2a}{\pi} = \frac{2\sqrt{2}}{\pi}.$$

Probability of measuring E_1

The probability to find the energy E_1 is

$$P(E = E_1) = c_1^2 = \left(\frac{2\sqrt{2}}{\pi}\right)^2 = \frac{8}{\pi^2} \approx 0.81.$$

Thus there is about an 81% chance that an energy measurement yields the ground-state energy E_1. "'latex article amsmath

$$c_2 = \int_0^a \phi_2^*(x) \, \psi(x, 0) \, dx = \sqrt{\frac{2}{a}} \, \sqrt{\frac{1}{a}} \int_0^a \sin\left(\frac{2\pi x}{a}\right) dx = 0,$$

$$P(E = E_2) = c_2^2 = 0.$$

$$c_3 = \int_0^a \phi_3^*(x)\, \psi(x,0)\, dx = \sqrt{\frac{2}{a}}\, \sqrt{\frac{1}{a}} \int_0^a \sin\!\left(\frac{3\pi x}{a}\right) dx = \frac{2\sqrt{2}}{3\pi},$$

$$P(E = E_3) = c_3^2 = \left(\frac{2\sqrt{2}}{3\pi}\right)^2 = \frac{8}{9\pi^2} \approx 0.0901 \quad (9.01\%).$$

We place this equation for the rest; it is important because it has come up in the master's exam.

((((We say we broke the wave into pieces and named it. You calculate those. This is my viewpoint.))))))

article amsmath physics geometry a4paper, margin=1in

The Heisenberg Uncertainty Principle

– Abbas Abedini " You either know one or neither accurately, but you can't know both at the same time."

In quantum mechanics, the uncertainty principle states that it is impossible to simultaneously measure the exact position and momentum of a particle.

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2} \tag{10}$$

This principle is one of the fundamental concepts in quantum physics and arises from the wave-like nature of particles.

Robertson's Uncertainty Principle

In quantum mechanics, Robertson's uncertainty principle is a general form of the Heisenberg uncertainty principle, applicable to any two observables A and B:

$$\Delta A \cdot \Delta B \geq \frac{1}{2}\left|\langle[\hat{A}, \hat{B}]\rangle\right| \tag{11}$$

This inequality shows that if two observables do not commute, there is a fundamental limit to the precision with which they can be known simultaneously. For example, if $A = x$ and $B = p$, we have:

$$[\hat{x}, \hat{p}] = i\hbar \quad \Rightarrow \quad \Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

Non-Commuting Operators Cannot Be Simultaneously Measured

In quantum mechanics, if two observables A and B are represented by operators $\hat{A}$ and $\hat{B}$ such that:

$$[\hat{A}, \hat{B}] \neq 0$$

then they cannot be simultaneously measured with arbitrary precision.

Example:

$$[\hat{x}, \hat{p}] = i\hbar \quad \Rightarrow \quad \Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

Heisenberg Uncertainty Principle

The Heisenberg Uncertainty Principle states that it is impossible to precisely determine both the position and the momentum of a quantum particle at the same time. Mathematically, it is expressed as:

$$\Delta x \, \Delta p \geq \frac{\hbar}{2} \tag{12}$$

This implies that the more precisely we know the position (Δx is small), the less precisely we know the momentum (Δp is large), and vice versa.

Implication for Confined Particles

If a particle is confined in a very small region of space, then Δx is small, which means Δp is large. As a result, the expected kinetic energy increases:

$$E_{kin} = \frac{p^2}{2m} \tag{13}$$

If you want to 'trap' an electron very close to the nucleus, its kinetic energy becomes so high that its total energy becomes positive and it escapes because the nucleus itself is positive and they repel each other.

Where-did-the-tunneler-come-from-with-Abbas's-method,-which-is-the-principle-of-uncertainty?

For any two Hermitian operators A and B:

$$\Delta A \cdot \Delta B \geq \frac{1}{2} \left| [A, B] \right| \tag{14}$$

We apply this to the Hamiltonian H and the momentum operator p:

$$\Delta H \cdot \Delta p \geq \frac{1}{2} |[H,p]| \tag{15}$$

$$H = \frac{p^2}{2m} + V(x) \tag{16}$$

We compute: $[H, p] = \left[\frac{p^2}{2m}, p\right] + [V(x), p]$
$= 0 + [V(x), p]$ Since $[V(x), p] = -i\hbar Vx$, we get:

$$[H, p] = -i\hbar Vx \tag{17}$$

$$\Delta H \, \Delta p \ \geq \ \frac{\hbar}{2}\left|\langle\frac{dV}{dx}\rangle\right|.$$

$-$

$-$, if $\langle dV/dx \rangle$ is large (meaning the barrier is sharp and severe), then for the inequality to hold, either H or p must be large. If the energy of the particle is determined more precisely (i.e., if H is small), then p must be larger — meaning the fluctuation in momentum increases, and this indicates that the particle may have significant amounts of instantaneous energy.

Energy-Time Uncertainty Principle

If a quantum system remains in a specific state only for a very short time interval Δt, then the energy of that state cannot be precisely defined. In other words:

$$\Delta E \cdot \Delta t \frac{\hbar}{2}$$

This means that a smaller Δt implies a larger ΔE, representing the uncertainty in energy. In the future seasons, we will reach it. article amsmath physics bm

Stationary States and the Notion of "Rest" in Quantum Mechanics

In quantum mechanics, the concept of a particle being "at rest" is different from the classical picture. A particle in a stationary state is not necessarily motionless.

Definition of a Stationary State

A stationary state is a quantum state whose wavefunction has the form:

$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$$

where:

* $\psi(x)$ is the spatial part of the wavefunction,
* E is the energy eigenvalue,
* the probability density remains constant in time:

$$|\Psi(x,t)|^2 = |\psi(x)|^2$$

Comparison with Classical Rest

In classical physics, "at rest" means zero velocity $\rightarrow$ zero momentum $\rightarrow$ the particle is at rest.

In quantum physics, "at rest" means that the probability of finding a particle in different locations does not change over time.

That is, in the ground state of a potential well:

A particle may be found in a wide range of locations

But over time, the probability of being in these locations remains constant.

$$\langle x \rangle = 0, \quad \langle p \rangle = 0$$

but:

$$\Delta x \neq 0, \quad \Delta p \neq 0$$

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Mean Momentum in a Particle Confined in a Potential Well

Consider a particle confined in a potential well (e.g., an infinite potential well centered at $x = 0$).

Given Conditions:

$$\langle x \rangle \rightarrow 0, \quad \Delta x \rightarrow 0, \quad \Delta p \rightarrow \infty.$$

Question:

What is the mean momentum $\langle p \rangle$?

Answer:

The mean momentum is given by

$$\langle p \rangle = \int \psi^*(x) \left(-i\hbar \frac{d}{dx} \right) \psi(x)\, dx.$$

If the wavefunction $\psi(x)$ is **real-valued** (which is true for energy eigenstates in symmetric potential wells), then

$$\langle p \rangle = 0,$$

because the integrand is an odd function and the integral over symmetric limits vanishes.

Explanation:

* Energy eigenstates in an infinite or finite symmetric potential well are typically sine or cosine functions, which are real-valued.
* The particle has no preferred direction of motion inside the well.
* Therefore, the average momentum is zero.
* However, as $\Delta x \to 0$, the uncertainty in momentum Δp grows without bound due to the Heisenberg uncertainty principle:

$$\Delta x\, \Delta p \geq \frac{\hbar}{2}.$$

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Infinite Potential Well (1D)

Potential:
$$V(x) = \{\, 0 \ 0 < x < L$$

$\infty otherwise$

Time-independent Schrödinger equation:

$$[2]\psi(x)x + k^2 \psi(x) = 0, \quad k = \frac{\sqrt{2mE}}{\hbar}$$

Boundary conditions:

$$\psi(0) = 0, \quad \psi(L) = 0$$

Quantized solutions:

$$k_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, \dots$$

Wavefunctions:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L} \right)$$

Energy levels:

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$

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– The particle is confined in the well, while the potential outside the well is infinite, and the presence of the particle outside is zero. The particle is oscillating in the well; Abbas Abedini is stuck in the class.

Expansion Coefficients and Probabilities

A general wavefunction in an infinite potential well can be written as a linear combination of energy eigenstates:

$$\Psi(x, 0) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

The coefficients c_n are given by:

$$c_n = \int_0^L \psi_n^*(x) \Psi(x, 0)\, dx$$

The probability of finding the particle in the n-th energy level is:

$$p_n = |c_n|^2$$

Summary:

* $c_n \in C$: The wave function expansion coefficient is in the basis of the eigenstates. It may be complex (having a phase).
* $p_n \in [0, 1]$: Probability of measuring energy E_n
* $\sum_{n=1}^{\infty} |c_n|^2 = 1$: Normalization condition

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Orthogonality of Eigenfunctions in Infinite Potential Well

The eigenfunctions are:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

We evaluate the inner product:

$$\int_0^L \psi_n(x)\psi_m(x)\, dx = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

Using the identity:

$$\sin A \sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

we get:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{1}{2} \int_0^L \left[\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)\right] dx$$

Case 1: $n \neq m$

Both cosine integrals over a full period yield zero:

$$\Rightarrow \int_0^L \psi_n(x)\psi_m(x)\, dx = 0$$

Case 2: $n = m$

$$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{L}{2} \Rightarrow \int_0^L \psi_n(x)^2 dx = \frac{2}{L} \cdot \frac{L}{2} = 1$$

Final Result:

$$\int_0^L \psi_n(x)\psi_m(x)\, dx = \{ 1 \; n = m$$

0 n $\neq$ m article amsmath physics

Applications of the Case $m = n$ in Infinite Potential Well

The eigenfunctions are:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

We consider the inner product:

$$\int_0^L \psi_n(x)\psi_m(x)\, dx = \{ 1 \; n = m$$

0 n $\neq$ m

1. Normalization

To ensure the wavefunction is normalized:

$$\int_0^L |\psi_n(x)|^2 dx = 1 \quad only\, if\, n = m$$

2. Expansion Coefficient c_n

When expanding a general wavefunction:

$$\Psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

Then,

$$c_n = \int_0^L \psi_n(x)\Psi(x,0)\, dx$$

Only the term with $n = m$ contributes due to orthogonality.

3. Expectation Value of Energy

For a general state:
$$\langle E \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n$$

Cross-terms vanish because:
$$\langle \psi_n | \hat{H} | \psi_m \rangle = E_n \delta_{nm}$$

4. Orthogonality and Projection

The inner product simplifies many quantum calculations:
$$\langle \psi_n | \psi_m \rangle = \delta_{nm}$$

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Square Potential Well or finit

A one-dimensional square well of half-width a and depth $V_0 > 0$ is defined by
$$V(x) = \{ - V_0, |x| \leq a,$$

0, —x—¿a.

Behavior by energy range:

* **Bound states:**
$$E < 0.$$

Particle is confined inside the well. The wavefunction decays exponentially for $|x| > a$. Transmission $T = 0$.

* **Tunneling region:**
$$0 < E < V_0.$$

Particle can "leak" through the finite barrier. Transmission probability satisfies $0 < T < 1$.

* **Free propagation:**
$$E > V_0.$$

Particle has enough energy to go over the barrier. Transmission $T \approx 1$.

E	$State$	T
$E < 0$	$Bound(noescape)$	0
$0 < E < V_0$	$Tunneling(partial)$	$0 < T < 1$
$E > V_0$	$Free(fullpass)$	≈ 1

The right and left sides show a decrease, and the particle oscillates inside the well; here the particle has more energy than the well but is still negative, and outside the well, it is zero. article amsmath physics geometry margin=1in

Region	Description
Left side $|x < -a|$	Potential $V(x) = 0$; wavefunction decays exponentially
Inside the well $|x| < a$	Potential $V(x) = -V_0$; particle oscillates
Right side $|x > a|$	Potential $V(x) = 0$; wavefunction decays exponentially
Particle energy	$-V_0 < E < 0$: energy is greater than well bottom, but still negative

Table 1: Summary of particle behavior and potential in a finite square well

Probability Current Density in Quantum Mechanics

Definition

The **probability current density** $j(x,t)$ describes how probability flows through space. For a wave function $\psi(x,t)$, it is defined as:

$$j(x,t) = \frac{\hbar}{2mi} \left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right)$$

Alternatively, using the imaginary part:

$$j(x,t) = \frac{\hbar}{m} \, Im \left(\psi^* \frac{\partial \psi}{\partial x} \right)$$

Free Particle (Plane Wave
Let the wave function be:

$$\psi(x,t) = Ae^{i(kx-\omega t)}$$

Compute the spatial derivative:

$$\frac{\partial \psi}{\partial x} = ikAe^{i(kx-\omega t)}$$

Compute the complex conjugate:

$$\psi^*(x,t) = A^* e^{-i(kx-\omega t)}$$

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Now compute the product:

$$\psi^* \frac{\partial \psi}{\partial x} = A^* e^{-i(kx-\omega t)} \cdot ikA e^{i(kx-\omega t)} = ik|A|^2$$

So the probability current density is:

$$j = \frac{\hbar}{m} \cdot Im(ik|A|^2) = \frac{\hbar k}{m}|A|^2$$

– The probability density in quantum states is like the classical state where water flows through a pipe.

Interpretation

This result means:

* The particle is moving to the right (since $j > 0$ if $k > 0$).
* The current is proportional to the momentum $\hbar k$ and the probability density $|A|^2$.

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Transmission Coefficient T for a Finite Potential Barrier

Case 1: $E > V_0$ (Above the Barrier)

$$T = \frac{1}{1 + \frac{V_0^2 \sin^2(qa)}{4E(E-V_0)}} \tag{18}$$

where

$$q = \frac{\sqrt{2m(E - V_0)}}{\hbar}$$

Case 2: $E < V_0$ (Quantum Tunneling)

$$T = \frac{1}{1 + \frac{V_0^2 \sinh^2(\kappa a)}{4E(V_0-E)}} \tag{19}$$

where

$$\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

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Gamow–Geiger–Müller Experiment and Quantum Tunneling

In classical physics, a particle cannot escape a potential well if its total energy E is less than the potential barrier $V(x)$. However, in quantum mechanics, due to the wave-like nature of particles, there is a nonzero probability that a particle can **tunnel** through a barrier and appear on the other side. This phenomenon is known as **quantum tunneling**.

Gamow's Theory of Alpha Decay

In 1928, **George Gamow** used the concept of quantum tunneling to explain alpha decay in atomic nuclei. He modeled the nucleus as a potential well with a Coulomb barrier. The alpha particle inside the nucleus has energy less than the height of the barrier, yet it is observed to escape, which can only be explained through tunneling.

Hans Geiger and **Ernst Müller** developed a detector (Geiger–Müller tube) that could count alpha particles emitted from radioactive nuclei. This experimental setup provided direct evidence for alpha decay and, consequently, supported Gamow's theoretical prediction of quantum tunneling. article amsmath amssymb physics geometry margin=1in

Is the speed of light in tunneling faster than the speed of light in a vacuum???

(In these cases, physicists still don't understand the nature of time tunneling in a potential well.)

In classical physics, if we want to know how long it takes a particle to travel, we need to know the start and end times. But in quantum models, such as the "quantum clock" model, all we can observe and measure is the time difference (or travel time), not the absolute start time.

(There is no universal reference point for determining when the tunneling started.)

When the clock shows that the particle has only been inside the barrier for "say, 1 femtosecond," but because we don't know when that 1 femtosecond started, we can't tell the information from one point to another.

This is one of the main reasons why quantum tunneling remains consistent with special relativity, even though the "effective passage speed" apparently exceeds the speed of light.

In quantum mechanics, we can never say with certainty "when" a particle entered a particular region (e.g., a potential well). Not yesterday, not today, not in the future.

(I can only enter the well, see it, and calculate the time difference.)

The papers have explained the matter very vaguely. In short, the phase velocity can exceed the speed of light, but that does not mean that information can also travel at the speed of light. Of course, today's experiments say so. I believe that information will travel faster than the speed of light in the future. Of course, we know that the phase velocity can travel faster than light. In quantum, if we make the well bigger or smaller, we will observe the same amount of tunneling time. That is, time is constant whether we increase or decrease the distance. This does not fit with classical theory because if the distance is bigger, the time to travel the distance should be longer. But in quantum mechanics, this is not the case and time will be the same. Of course, if we shine light on the glass and gradually increase its width, we will see that the transmission does not change to any extent and returns to its previous state (Faynman)

I believe that if we make the space smaller, the dispersion of the space will decrease by taking into account the uncertainty principle and we will witness the dispersion of momentum and as a result of this event, we will observe that the phase fluctuation will increase and as a result of the increase in the phase I will observe that the time will be less or in other words I increased the kinetic energy and the transit time has decreased and as a result we can say that the phase velocity has increased and we can say that we will have a constant time and we know that the probability current density is obtained from the Schrodinger equation and this relationship can be said that the probability current density has increased as a result of the decrease in the dispersion of the location of the probability current density and as a result this time in tunneling has decreased and my solution can be said that this can be said that I will have a constant time

Different methods There are formulas for getting the time it takes for a particle to pass through a well, but what caught my attention was the residence time formula, which doesn't give the time it takes for the particle to pass through, but rather the time it is there. I tried to come up with a new formula, but I had problems with dimensions and couldn't come up with a formula that had the time dimension and the necessary standards, but my formula was very similar to the residence time formula, or my way of thinking was similar to the residence time formula. Of course, if the location increases, the energy decreases, and if the energy decreases, the time must increase, but the time is the same because we reduced the interval, the probability decreases. Conversely, if you apply this to a different value, you will see that the time is the same on average. article amsmath physics

Derivation of the Probability Current from the Schrödinger Equation

We begin with the time-dependent Schrödinger equation in one dimension:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi \tag{20}$$

and its complex conjugate:

$$-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + V(x)\psi^* \tag{21}$$

The probability density is defined as:

$$\rho(x,t) = \psi^*(x,t)\psi(x,t)$$

Taking the time derivative:

$$\frac{\partial \rho}{\partial t} = \psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t} \tag{22}$$

Substitute from the Schrödinger equation and its conjugate:

$$\frac{\partial \psi}{\partial t} = \frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi\right)$$

$$\frac{\partial \psi^*}{\partial t} = -\frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + V\psi^*\right)$$

Now substitute into the time derivative of ρ:

$$\frac{\partial \rho}{\partial t} = \psi^* \cdot \frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi\right) + \psi \cdot \left(-\frac{1}{i\hbar}\right)\left(-\frac{\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + V\psi^*\right)$$

$$= -\hbar \frac{}{2mi\left(\psi^*\frac{\partial^2 \psi}{\partial x^2} - \psi\frac{\partial^2 \psi^*}{\partial x^2}\right)}$$

Now observe the following identity (product rule):

$$\frac{\partial}{\partial x}\left(\psi^*\frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x}\right) = \psi^*\frac{\partial^2 \psi}{\partial x^2} - \psi\frac{\partial^2 \psi^*}{\partial x^2}$$

Therefore:

$$\frac{\partial \rho}{\partial t} = -\frac{\hbar}{2mi}\frac{\partial}{\partial x}\left(\psi^*\frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x}\right)$$

Define the probability current density $j(x,t)$ as:

$$j(x,t) = \frac{\hbar}{2mi}\left(\psi^*\frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x}\right)$$

Then the continuity equation becomes:

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$

This is the continuity equation for probability, expressing conservation of probability in quantum mechanics. article amsmath physics

Calculation of Dwell Time for a Rectangular Potential Barrier

Definition of Dwell Time

Dwell time is defined as the average time a quantum particle spends inside a region (here, the potential barrier), regardless of whether it eventually transmits or reflects:

$$\tau_D = \frac{\int_{x_1}^{x_2} |\psi(x)|^2\, dx}{j_{in}}$$

Rectangular Potential Barrier

Let the potential be:
$$V(x) = \{\, 0 \quad x < 0$$

$V_0 \quad 0 \le x \le a$

$0 \text{ x } \text{¿ } a$

Assume $E < V_0$, so the particle undergoes tunneling. In the region $0 < x < a$, the wavefunction inside the barrier is:

$$\psi(x) = Ce^{-\kappa x} + De^{\kappa x}, \quad where \kappa = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

In the typical tunneling case, we consider only the decaying exponential:

$$|\psi(x)|^2 = |C|^2 e^{-2\kappa x}$$

$$\int_0^a |\psi(x)|^2 \, dx = |C|^2 \int_0^a e^{-2\kappa x} dx = |C|^2 \cdot \frac{1 - e^{-2\kappa a}}{2\kappa}$$

In the region $x < 0$, if the incident wave is $\psi(x) = Ce^{ikx}$, then the incoming probability current is:

$$j_{in} = \frac{\hbar k}{m}|A|^2, \quad where k = \sqrt{\frac{2mE}{\hbar^2}}$$

Putting everything together:

$$\tau_D = \frac{|C|^2 \cdot \frac{1-e^{-2\kappa a}}{2\kappa}}{\hbar k m |A|^2} = \left(\frac{m}{\hbar k}\right) \cdot \left(\frac{|C|^2}{|A|^2}\right) \cdot \left(\frac{1 - e^{-2\kappa a}}{2\kappa}\right)$$

The ratio $\frac{|c|^2}{|a|^2}$ can be determined from the continuity conditions at the boundaries of the potential. article amsmath physics

Transmission Coefficient for a Finite Square Potential Barrier

For a particle with energy $E < V_0$, tunneling through a finite square barrier of height V_0 and width a, the transmission coefficient is given by:

$$T = \frac{1}{1 + \left(\frac{V_0^2}{4E(V_0-E)}\right)\sinh^2(\kappa a)} \tag{23}$$

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Schrödinger Equation for the Finite Square Well

The potential is defined as:

$$V(x) = \{\, 0 \ for |x| > a$$

$-V_0 for |x| \leq a$

We solve the time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

This gives us three regions:

* **Region I:** $x < -a$
* left
* X is negative on the left, and placing it in the power of e reduces the exponent.

$$\frac{d^2\psi}{dx^2} - \kappa^2\psi = 0, \quad where \kappa = \sqrt{\frac{2m(-E)}{\hbar^2}}$$

$$\psi_I(x) = De^{\kappa x}$$

* **Region II:** $|x| \le a$
* inside

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0, \quad where k = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

$$\psi_{II}(x) = A\cos(kx) + B\sin(kx)$$

* **Region III:** $x > a$
* right
* X is positive on the left side and the placement in power e reduces the view.

$$\frac{d^2\psi}{dx^2} - \kappa^2\psi = 0$$

$$\psi_{III}(x) = Fe^{-\kappa x}$$

Quantization Conditions (Transcendental Equations

$$Even states: \quad k\tan(ka) = \kappa, \qquad Odd states: \quad k\cot(ka) = -\kappa.$$

$$Inside the well: k = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}, \qquad Outside the well: \kappa = \sqrt{\frac{2m(-E)}{\hbar^2}}$$

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In quantum mechanics, the discrete energy levels $n = 1, 2, 3, \ldots$ that arise in confined systems such as traps or finite potential wells correspond to specific stationary states of the system.

Each state n is associated with a specific wavefunction $\psi_n(x)$, which has a characteristic number of nodes (points where the wavefunction crosses zero). For example:

* For $n = 1$, the wavefunction $\psi_1(x)$ has no internal nodes and represents one "hump" (i.e., a single maximum or minimum).
* For $n = 2$, the wavefunction $\psi_2(x)$ has one node, and its shape resembles a sine wave with two "humps".
* Higher n values correspond to wavefunctions with more nodes and more complex oscillatory structure.

These wavefunctions form an orthonormal basis, and the probability of the system being in a particular state $\psi_n(x)$ is given by the square of the corresponding coefficient in the wavefunction's expansion:

$$\Psi(x, 0) = \sum_n c_n \psi_n(x), \quad with \quad |c_n|^2 = probability of being in state n$$

If $|c_1|^2 = 1$, it means the wavefunction is entirely in the ground state (first energy level), and the particle is in a stationary state:

$$\Psi(x, t) = \psi_1(x)e^{-iE_1t/\hbar}$$

In this case, the shape of the wavefunction remains constant over time (only the phase evolves), and the particle is considered to be "trapped" in the potential well in that specific energy state.

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Problem:

Given a trial wave function in an infinite potential well $0 \leq x \leq a$:

$$\psi(x) = \{ A\,x, 0 \leq x \leq \frac{a}{2}$$

A(a - x), a$\overline{_{2 < x \leq a}}$

0, otherwise

Find the normalization constant A and calculate the probability that this state corresponds to the ground state energy level.

Normalization condition:
$$\int_0^a |\psi(x)|^2 dx = 1$$

Calculate the integral:

$$\int_0^{a/2} (Ax)^2 dx + \int_{a/2}^a [A(a - x)]^2 dx = 1$$

Compute each integral:

$$\int_0^{a/2} A^2 x^2 dx = A^2 \left[\frac{x^3}{3}\right]_0^{a/2} = A^2 \frac{(a/2)^3}{3} = A^2 \frac{a^3}{24}$$

$$\int_{a/2}^a A^2 (a - x)^2 dx = A^2 \int_0^{a/2} y^2 dy = A^2 \frac{(a/2)^3}{3} = A^2 \frac{a^3}{24}$$

Sum:
$$A^2 \frac{a^3}{24} + A^2 \frac{a^3}{24} = A^2 \frac{a^3}{12} = 1$$

Solve for A:
$$A = \sqrt{\frac{12}{a^3}}$$

Ground state of infinite well:
$$\phi_1(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$$

Calculate:

$$c_1 = \int_0^a \phi_1(x)\psi(x)dx = \int_0^{a/2} \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) Ax\, dx + \int_{a/2}^a \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) A(a-x)\, dx$$

Substitute A:

$$c_1 = \sqrt{\frac{2}{a}}\sqrt{\frac{12}{a^3}} \left(\int_0^{a/2} x \sin\left(\frac{\pi x}{a}\right) dx + \int_{a/2}^a (a-x)\sin\left(\frac{\pi x}{a}\right) dx \right)$$

Simplify constants:

$$c_1 = \sqrt{\frac{24}{a^4}} (I_1 + I_2)$$

where

$$I_1 = \int_0^{a/2} x \sin\left(\frac{\pi x}{a}\right) dx, \quad I_2 = \int_{a/2}^a (a-x)\sin\left(\frac{\pi x}{a}\right) dx$$

Use integration by parts:

$$u = x, \quad dv = \sin\left(\frac{\pi x}{a}\right) dx$$

$$du = dx, \quad v = -\frac{a}{\pi}\cos\left(\frac{\pi x}{a}\right)$$

Then

$$I_1 = uv\Big|_0^{a/2} - \int_0^{a/2} v\, du = \left[-\frac{a}{\pi}x\cos\left(\frac{\pi x}{a}\right) \right]_0^{a/2} + \frac{a}{\pi}\int_0^{a/2} \cos\left(\frac{\pi x}{a}\right) dx$$

Calculate the boundary term:

$$-\frac{a}{\pi}\cdot\frac{a}{2}\cdot\cos\left(\frac{\pi}{2}\right) + 0 = 0$$

Calculate the integral:

$$\int_0^{a/2} \cos\left(\frac{\pi x}{a}\right) dx = \frac{a}{\pi}\sin\left(\frac{\pi x}{a}\right) \Big|_0^{a/2} = \frac{a}{\pi}$$

So,

$$I_1 = 0 + \frac{a}{\pi}\cdot\frac{a}{\pi} = \frac{a^2}{\pi^2}$$

Use substitution $y = a - x$:

$$I_2 = \int_0^{a/2} y\sin\left(\frac{\pi y}{a}\right) dy = I_1 = \frac{a^2}{\pi^2}$$

$$c_1 = \sqrt{\frac{24}{a^4}\left(\frac{a^2}{\pi^2} + \frac{a^2}{\pi^2}\right)} = \sqrt{\frac{24}{a^4}\frac{2a^2}{\pi^2}} = \frac{2\sqrt{24}}{\pi^2} = \frac{4\sqrt{6}}{\pi^2}$$

$$P_1 = |c_1|^2 = \left(\frac{4\sqrt{6}}{\pi^2}\right)^2 = \frac{16 \times 6}{\pi^4} = \frac{96}{\pi^4} \approx 0.986$$

—

The probability that the trial wave function corresponds to the ground state is approximately 0.986.

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Calculation for the Second Energy Level $n = 2$

The second energy eigenstate in the infinite potential well is:

$$\phi_2(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right)$$

The overlap coefficient:

$$c_2 = \int_0^a \psi(x)\phi_2(x)dx = \int_0^{a/2} Ax\sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right) dx + \int_{a/2}^a A(a-x)\sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right) dx$$

Substituting normalization constant:

$$c_2 = \sqrt{\frac{24}{a^4}} \left[\int_0^{a/2} x \sin\left(\frac{2\pi x}{a}\right) dx + \int_{a/2}^a (a - x) \sin\left(\frac{2\pi x}{a}\right) dx\right]$$

Change variable in the second integral $y = a - x$:

$$\int_{a/2}^a (a - x) \sin\left(\frac{2\pi x}{a}\right) dx = \int_0^{a/2} y \sin\left(2\pi - \frac{2\pi y}{a}\right) dy$$

Using $\sin(2\pi - \theta) = -\sin\theta$:

$$= -\int_0^{a/2} y \sin\left(\frac{2\pi y}{a}\right) dy$$

Thus,

$$c_2 = \sqrt{\frac{24}{a^4}} \left(\int_0^{a/2} x \sin\left(\frac{2\pi x}{a}\right) dx - \int_0^{a/2} y \sin\left(\frac{2\pi y}{a}\right) dy\right) = 0$$

Therefore, the probability of being in the second energy level is:

$$P_2 = |c_2|^2 = 0$$

The wave function here is nearly confined because it is close to one hundred percent.

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Even Parity State in Finite Potential Well

Consider the one-dimensional finite potential well:

$$V(x) = \{\, 0\,,\, |x| > a$$

- $V_0, |x| \le a$
where $V_0 > 0$.

The time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x).$$

Since the potential is symmetric, $V(x) = V(-x)$, the eigenfunctions $\psi(x)$ can be chosen to have definite parity: even or odd.

Inside the well $|x| \le a$

The potential is constant $-V_0$, so

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi \qquad \frac{d^2\psi}{dx^2} = -k^2\psi,$$

where

$$k = \sqrt{\frac{2m(E + V_0)}{\hbar^2}}.$$

The general solution inside is

$$\psi(x) = A\cos(kx) + B\sin(kx).$$

Outside the well $|x| > a$

The potential is zero:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \qquad \frac{d^2\psi}{dx^2} = \kappa^2\psi,$$

with

$$\kappa = \sqrt{\frac{-2mE}{\hbar^2}}.$$

Because $E < 0$, $\kappa > 0$, and the wavefunction must decay exponentially:

$$\psi(x) = Fe^{-\kappa x}, \quad x > a,$$

$$\psi(x) = Fe^{\kappa x}, \quad x < -a.$$

Even parity condition

For an even wavefunction, we set $B = 0$, so inside the well:

$$\psi(x) = A\cos(kx),$$

and outside the well:

$$\psi(x) = Fe^{-\kappa|x|}.$$

Boundary conditions at $x = a$

Continuity of the wavefunction:

$$A\cos(ka) = Fe^{-\kappa a}.$$

Continuity of the derivative:

$$-Ak\sin(ka) = -F\kappa e^{-\kappa a}.$$

Dividing the derivative condition by the wavefunction condition gives

$$\frac{-Ak\sin(ka)}{A\cos(ka)} = \frac{-F\kappa e^{-\kappa a}}{Fe^{-\kappa a}} - k\tan(ka) = -\kappa,$$

which simplifies to the transcendental equation

$$k\tan(ka) = \kappa.$$

This equation determines the allowed energy levels E for even parity bound states in the finite potential well.

As we increase the depth of the finite well, the constrained amount increases. The more we reduce the finite depth of the well, the amount of constraint decreases. article amsmath physics [margin=1in]geometry

Key Formulas for the Finite Square Well (Even States)

Definition of z_0:

$$z_0 = \frac{L}{2}\sqrt{\frac{2mV_0}{\hbar^2}}$$

Tangent function (even parity condition):

$$f_1(z) = \tan(z)$$

Definition of q_j:

$$z = \sqrt{\left(\frac{z_0}{z_j}\right)^2 - 1}$$

Final energy formula (even states):

$$E_n + V_0 = \frac{n^2 z^2 \hbar^2}{2m\left(\frac{L}{2}\right)^2}$$

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Physical Interpretation: Ground State Energy Approaches Zero

In a finite potential well, the energy of bound states is always **negative**, because the particle is confined inside the well and cannot move freely.

When we decrease the width (a) or depth (V_0) of the well, the following physical effects occur:

* The wavefunction has less space to oscillate.
* The particle's kinetic energy increases due to the uncertainty principle:

$$\Delta x \downarrow \quad \Rightarrow \quad \Delta p \uparrow \quad \Rightarrow \quad higher kinetic energy$$

* However, the total energy must remain negative for the particle to stay bound.

Therefore, the only way the particle can remain bound is if its energy gets closer to zero from below:

$$E_0 \to 0^-$$

That is, the ground state energy becomes less negative and approaches zero, but never becomes positive as long as the well still has some depth and width. article amsmath geometry margin=1in

In the above example that you solved

Suppose in our example the probability of the particle being in the ground state is approximately

$$|c_1|^2 \approx 98\%,$$

meaning there is a 90% chance that the particle occupies the first bound state.

This implies:

* The initial wavefunction is very close to the ground state wavefunction.
* When the ground state probability is very high, it usually indicates:
  · The ground state energy is relatively close to zero (less deeply bound).
  · The potential well is relatively **shallow** or **narrow**.
  · The particle is mostly confined to the lowest energy level.
* A shallow or narrow well supports fewer bound states, often just one or two.
* Consequently, the initial wavefunction overlaps mostly with the ground state because higher excited states are fewer or negligible.

Important note: If the potential well were very deep and wide, then the ground state probability would typically be lower, as excited states contribute more significantly.

$High|c_1|^2 Particle mostly in ground state Potential well is likely shallow or narrow$

In my opinion, it's better to say that in relation to energy.

In the past, we used graphics, but with this formula, we don't need graphics.

Thanks to Dr. Pravin Pathak for allowing us to use his article in our book. This is an excerpt from his article. Thanks to Dr. Abedini for allowing us to use his article in our book.

This equation is useful when we know the length and depth.

Consider a finite potential well with different effective masses inside and outside:

$$V(x) = \{ - V_0 for - L2 \leq x \leq L2$$

0 otherwise

$$\sigma = \frac{2m_o V_0 L^2}{\hbar^2}$$

The approximate ground state energy (even solution) is:

$$E_1 \approx \frac{\pi^2 \hbar^2}{2m_i L^2} \left(1 - \frac{2}{\sqrt{\sigma} + 2} \right)$$

What is zero mass (m0)?

Zero mass is usually referred to as the mass of a particle outside a potential well.

Zero mass is actually the effective mass in regions of zero potential or above.

Mass inside the well (miles) The mass inside is called the mass of the particle inside the potential region v0.

In textbooks, they are equated.

Mass Symbol	Location	Role
m_i	Inside well	Determines kinetic energy inside the well and energy levels
m_0	Outside well	Affects tunneling and boundary conditions

—

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Quantum Harmonic Oscillator

Potential and Hamiltonian

The potential energy is given by:

$$V(x) = \frac{1}{2}m\omega^2 x^2$$

The Hamiltonian operator is:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2x}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

Schrödinger Equation

We solve the time-independent Schrödinger equation:

$$\hat{H}\psi(x) = E\psi(x)$$

Energy Eigenvalues

The energy levels are quantized:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \quad n = 0, 1, 2, \ldots$$

The lowest possible energy (ground state) is:

$$E_0 = \frac{1}{2}\hbar\omega$$

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Derivation of Ladder Operators from the Schrödinger Equation

We start from the time-independent Schrödinger equation for the harmonic oscillator:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2\,\psi = E\,\psi.$$

Equivalently, in operator form,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\,\hat{x}^2.$$

Next, note that each squared operator can be written as a product:

$$\hat{p}^2 = \hat{p}\,\hat{p}, \qquad \hat{x}^2 = \hat{x}\,\hat{x}.$$

We now introduce the characteristic scales

$$x_0 = \sqrt{2\frac{\hbar}{m\omega}}, \quad p_0 = \sqrt{2\hbar m\omega},$$

and define dimensionless operators

$$X = \frac{\hat{x}}{x_0}, \qquad P = \frac{\hat{p}}{p_0}.$$

Then

$$\hat{x}^2 = x_0^2\, X^2, \quad \hat{p}^2 = p_0^2\, P^2,$$

and the Hamiltonian becomes

$$\hat{H} = \frac{p_0^2}{2m}\, P^2 + \frac{1}{2}\, m\omega^2\, x_0^2\, X^2 = \hbar\omega(P^2 + X^2).$$

Now form the complex combinations (annihilation/creation up to a factor):

$$X - iP, \qquad X + iP.$$

Because

$$P^2 + X^2 = (X - iP)(X + iP)$$

$$P^2 + X^2 = (X - iP)(X + iP)$$

Therefore

$$\hat{H} = \hbar\omega\left[(X - iP)(X + iP)\right.$$

If you multiply this, it creates both destructive and constructive effects.

Finally, define the true ladder operators

$$a = 1\sqrt{2}(X + iP), \qquad a^\dagger = 1\sqrt{2}(X - iP).$$

One checks

$$[a, a^\dagger] = 1,$$

and

$$\hat{H} = \hbar\omega\left(a^\dagger a + 12\right).$$

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Annihilation Operator and Transition Amplitude

We want to compute the following matrix element:

$$\langle 0|a|1\rangle$$

Action of the Annihilation Operator

The annihilation operator a acts on number states as:

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

Therefore:

$$a|1\rangle = \sqrt{1}\,|0\rangle = |0\rangle$$

Compute the Inner Product

Now apply the bra $\langle 0|$:

$$\langle 0|a|1\rangle = \langle 0|0\rangle = 1$$

"If a system is in the state $0\rangle$, then the probability of finding it in the same state $0\rangle$ is 100

Conclusion

The transition amplitude is:

$$\langle 0|a|1\rangle = 1$$

And the actual probability is the square of the modulus:

$$|\langle 0|a|1\rangle|^2 = |1|^2 = 1$$

This means the transition from $|1\rangle$ to $|0\rangle$ by applying the annihilation operator occurs with probability 1. Look a it's destructive, so the probability of going from one to zero is 100article amsmath

Example: Transition in the Quantum Harmonic Oscillator

Consider the annihilation operator a acting on the number state $|2\rangle$:

$$a|2\rangle = \sqrt{2}|1\rangle$$

Taking the inner product with $\langle 1|$:

$$\langle 1|a|2\rangle = \langle 1|\left(\sqrt{2}|1\rangle\right) = \sqrt{2}\langle 1|1\rangle = \sqrt{2}$$

Thus, the transition amplitude is:

$$\langle 1|a|2\rangle = \sqrt{2}$$

The squared modulus (often interpreted as transition strength) is:

$$|\langle 1|a|2\rangle|^2 = \left|\sqrt{2}\right|^2 = 2$$

More generally, for any number state $|n\rangle$, the annihilation operator acts as:

$$a|n\rangle = \sqrt{n}|n-1\rangle \quad \Rightarrow \quad \langle n-1|a|n\rangle = \sqrt{n}$$

– To find the probability of measuring $|1\rangle$ after applying a, we must normalize the new state:

$$|\psi\rangle = \frac{a|2\rangle}{\|a|2\rangle\|} = \frac{\sqrt{2}|1\rangle}{\sqrt{2}} = |1\rangle$$

– Now, the probability of finding the system in $|1\rangle$ after this operation is:

$$|\langle 1|\psi\rangle|^2 = |\langle 1|1\rangle|^2 = 1$$

Problem Statement

The potential energy operator for the one-dimensional harmonic oscillator is

$$V(x) = \frac{1}{2}m\omega^2 x^2.$$

We wish to compute its expectation value in the nth energy eigenstate $|n\rangle$:

$$\langle V\rangle_n = \left\langle n\left|\frac{1}{2}m\omega^2 x^2\right|n\right\rangle.$$

Recall the position operator in terms of the annihilation (a) and creation $(a^\dagger)$ operators:

$$\hat{x} = \sqrt{\frac{\hbar}{2\,m\,\omega}}\,(a + a^\dagger).$$

Hence,

$$x^2 = \frac{\hbar}{2\,m\,\omega}\,(a + a^\dagger)^2 = \frac{\hbar}{2\,m\,\omega}(a^2 + (a^\dagger)^2 + a\,a^\dagger + a^\dagger\,a).$$

For a number state $|n\rangle$:

$$a^2|n\rangle = \sqrt{n}\,\sqrt{n-1}\,|n-2\rangle, \quad (a^\dagger)^2|n\rangle = \sqrt{n+1}\,\sqrt{n+2}\,|n+2\rangle.$$

Since $\langle n|n-2\rangle = 0$ and $\langle n|n+2\rangle = 0$,

$$\langle n|a^2|n\rangle = 0, \qquad \langle n|(a^\dagger)^2|n\rangle = 0.$$

Use the commutation relation $[a, a^\dagger] = 1$, which gives

$$a\,a^\dagger = 1 + a^\dagger\,a \implies a\,a^\dagger + a^\dagger\,a = 1 + 2\,a^\dagger\,a.$$

Acting on $|n\rangle$:

$$a^\dagger\,a\,|n\rangle = n\,|n\rangle, \quad \langle n|n\rangle = 1,$$

thus

$$\langle n|a\,a^\dagger + a^\dagger\,a|n\rangle = \langle n|1 + 2\,a^\dagger\,a|n\rangle = 1 + 2n.$$

$$\langle n|x^2|n\rangle = \frac{\hbar}{2\,m\,\omega}\,\langle n|a^2 + (a^\dagger)^2 + a\,a^\dagger + a^\dagger\,a|n\rangle = \frac{\hbar}{2\,m\,\omega}\,(1 + 2n).$$

Finally,

$$\langle V\rangle_n = \frac{1}{2}m\omega^2\,\langle n|x^2|n\rangle = \frac{1}{2}m\omega^2 \times \frac{\hbar}{2\,m\,\omega}\,(1 + 2n) = \frac{\hbar\omega}{4}(2n + 1).$$

$$\langle V \rangle_n = \frac{\hbar\omega}{4}(2n+1) = \frac{\hbar\omega}{2}\left(n + 12\right).$$

This result is exactly half of the total energy $E_n = \hbar\omega\,(n+12)$. article amsmath
The orthogonality of quantum states implies:

$$\langle n|n-2 \rangle = 0$$

because the number states $|n\rangle$ form an orthonormal basis:

$$\langle m|n \rangle = \delta_{mn} = \{\, 1 \; if m = n$$

0 if m $\neq n$
Since $n \neq n-2$, we have:

$$\langle n|n-2 \rangle = 0$$

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Action of the Annihilation Operator

The annihilation (or lowering) operator a acts on the quantum harmonic oscillator number state $|n\rangle$ as follows:

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

This equation means that:

  * The operator a reduces the quantum number by one.
  * The coefficient $\sqrt{n}$ arises from the algebra of the ladder operators in the harmonic oscillator.

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Action of the Creation Operator

The creation (or raising) operator $a^\dagger$ acts on the quantum harmonic oscillator number state $|n\rangle$ as:

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

This equation means that:

  * The operator $a^\dagger$ increases the quantum number by one.
  * The coefficient $\sqrt{n+1}$ ensures the proper normalization of the state.

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Wave-Particle Duality and the Quantum Harmonic Oscillator

In classical physics, we separate the wave and the particle, but in quantum, the two are together. Whenever there is a particle, there is a wave, and whenever there is a wave, there is a particle. In the quantum dimension, the wave oscillates, which is why we study harmonic oscillations in it. article amsmath booktabs geometry margin=1in

Can a Particle Be Stationary in Quantum Mechanics?

Comparison: Classical vs Quantum View of a Stationary Particle

Classical Physics	Quantum Mechanics
Exactly known ($\Delta x = 0$)	Uncertain if momentum is known ($\Delta x \to \infty$)
Exactly zero ($\Delta p = 0$)	Uncertain if position is known ($\Delta p \to \infty$)
Both $\Delta x = 0$ and $\Delta p = 0$ possible	$\Delta x \cdot \Delta p \geq \hbar/2$
Not needed	Spread out over all space if $p = 0$
The particle is at rest at a known location	particle *cannot* be both at rest and localized

Explanation

In classical mechanics, the concept of a stationary particle is simple: it has zero velocity (or momentum), and its position is precisely known. There is no fundamental limitation on knowing both properties simultaneously.

However, in quantum mechanics, the **Heisenberg Uncertainty Principle** imposes a fundamental limit:

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

This means that the more precisely we know a particle's momentum, the less precisely we can know its position, and vice versa.

If a particle has a definite momentum (e.g., $p = 0$), then its wavefunction is completely delocalized in space—it is equally likely to be found anywhere. This is represented by a plane wave or a constant wavefunction:

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Why Only Discrete Energies Are Allowed in the Quantum Harmonic Oscillator

In the one-dimensional harmonic oscillator, the energy eigenvalues must satisfy

$$E_n = \left(n + 12\right)\hbar\omega, \quad n = 0, 1, 2, \ldots$$

Only for these values does the corresponding wavefunction remain normalizable.

Case 1: $E = 0.51\,\hbar\omega$ (just above E_0)
The solution of the Schrödinger equation oscillates or grows at large $|x|$, so

$$\lim_{|x|\to\infty} \psi(x) \neq 0 \quad \implies \quad \int_{-\infty}^{\infty} |\psi|^2 \, dx = \infty.$$

Thus the wavefunction is non-normalizable and $E = 0.51\,\hbar\omega$ is not allowed.

Case 2: $E = 0.49\,\hbar\omega$ (just below E_0)
Similarly, the solution diverges or fails to decay at infinity, so it cannot be normalized.

Physical meaning of diverging "tails":

* A normalizable wavefunction must satisfy $\lim_{|x|\to\infty} \psi(x) = 0$.
* Diverging or non-decaying tails mean $\int |\psi|^2 dx$ diverges.
* Only when $E = E_n$ do the series solutions terminate (Hermite polynomials), yielding Gaussian-damped wavefunctions that go to zero at infinity.

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Hilbert Space

A **Hilbert space** is a complete inner product space. It generalizes the notion of Euclidean space to (possibly) infinite-dimensional spaces and is essential in quantum mechanics. The states of a quantum system are represented as vectors in a Hilbert space. The wave function in quantum mechanics is considered as a vector in Hilbert space

Inner Product in a Hilbert Space

In a Hilbert space, the inner product between two vectors ψ and ϕ is denoted by

$$\psi\phi$$

and is a complex number. It satisfies the following properties:

Mathematical Properties

Conjugate Symmetry:
$$\psi\phi = \overline{\phi\psi}$$

Linearity in the Second Argument:

$$\psi a\phi_1 + b\phi_2 = a\psi\phi_1 + b\psi\phi_2$$

for all scalars $a, b \in C$ **Positive-Definiteness:**

$$\psi\psi \geq 0 \quad and \quad \psi\psi = 0 \iff \psi = 0$$

The **norm** (or length) of a vector is given by:

$$\|\psi\| = \sqrt{\psi\psi}$$

Physical Meaning in Quantum Mechanics

In quantum mechanics:

* The inner product $\psi\phi$ is the *probability amplitude* for the transition from state ϕ to ψ.

* The *probability* of measuring ψ when the system is in state ϕ is:

$$P = |\psi\phi|^2$$

Linearity of the Inner Product

The inner product between two quantum states shows how "aligned" they are. Let ψ, ϕ_1, ϕ_2 be vectors in a Hilbert space, and $a, b \in C$. The inner product is linear in the second argument:

$$\langle\psi|a\phi_1 + b\phi_2\rangle = a\langle\psi|\phi_1\rangle + b\langle\psi|\phi_2\rangle$$

In quantum mechanics, the inner product between two state vectors is written as:

$$\langle\phi|\psi\rangle$$

$$|\psi\rangle \leftrightarrow \psi(x), \quad |\phi\rangle \leftrightarrow \phi(x)$$

then their inner product is given by the integral:

$$\langle\phi|\psi\rangle = \int_{-\infty}^{+\infty} \phi^*(x)\,\psi(x)\,dx$$

The square of its magnitude gives the probability of transitioning from one state to another:

$$P = |\langle \phi | \psi \rangle|^2$$

If $\{|n\rangle\}$ is an orthonormal basis, then:

$$\langle m | n \rangle = \delta_{mn}$$

In quantum mechanics, the inner product between two state vectors is written as:

$$\langle \phi | \psi \rangle$$

This complex number represents the overlap between the two quantum states.
- If the inner product is equal to 1:

$$\langle \phi | \psi \rangle = 1$$

then the two states are identical (up to a global phase). They represent the **same physical state**.
- If the inner product is equal to 0:

$$\langle \phi | \psi \rangle = 0$$

then the two states are **orthogonal**, meaning they have no overlap and are **completely distinguishable**.
Consider two quantum states of a spin-$\frac{1}{2}$ particle:

$$|\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and $\quad |\phi\rangle = e^{i\pi/4} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{i\pi/4} \\ 0 \end{pmatrix}$

The inner product is:

$$\langle \phi | \psi \rangle = e^{-i\pi/4} \times 1 + 0 = e^{-i\pi/4}$$

with magnitude:

$$|\langle \phi | \psi \rangle| = 1$$

The two states $|\psi\rangle$ and $|\phi\rangle$ point in exactly the same direction in spin space, but one of them has a global phase factor $e^{i\pi/4}$.
This global phase does not affect any physical measurement (like spin measurement outcomes).
Therefore, physically, these two states are completely identical:

$$|\psi\rangle \equiv |\phi\rangle$$

showing they differ only by a global phase and represent the same physical state.

Consider:

$$|\alpha\rangle = (\,1\,)$$

0   and $-\beta\rangle = (\,1\,)$

0

The inner product is:

$$\langle\beta|\alpha\rangle = 1$$

with magnitude:

$$|\langle\beta|\alpha\rangle| = 1$$

Here, the states are exactly the same without any phase difference.

General Ket and Bra Vectors in a Complex Hilbert Space

A general *ket* vector in an n-dimensional complex Hilbert space $\mathcal{H}$ is written as

$$\psi = (\,\psi\,)_1$$

ψ_2

$\vdots$

$\psi_n, \quad \psi_i \in C.$

A general *bra* vector, which is the Hermitian conjugate of a ket, is written independently as

$$\phi = (\,\phi\,)_1^* \, \phi_2^* \cdots \phi_n^*, \quad \phi_i^* \in C.$$

The inner product between ψ and ϕ is then

$$\phi\psi = \sum_{i=1}^{n} \phi_i^* \, \psi_i.$$

Ket and Bra Vectors

Consider the two quantum states in a two-dimensional Hilbert space C^2:

$$\psi = 1$$

i , $\phi = \frac{1}{\sqrt{2}}1$

-1

The corresponding **bra vectors** are the Hermitian conjugates (conjugate transpose) of the kets:

$$\psi = (\psi)^\dagger = 1 - i, \quad \phi = (\phi)^\dagger = \frac{1}{\sqrt{2}}1 - 1$$

Full Dirac Notation Example in Real Hilbert Space

Let the quantum states be:

$$\psi = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \end{pmatrix}$$

2 , $\quad \phi = \frac{1}{5}\begin{pmatrix} 3 \end{pmatrix}$
4 .

Their corresponding bras are:

$$\psi = (\psi)^\dagger = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \end{pmatrix}2, \quad \phi = (\phi)^\dagger = \frac{1}{5}\begin{pmatrix} 3 \end{pmatrix}4.$$

Now compute the inner product $\phi\psi$:

$$\phi\psi = \phi\psi = \left(\frac{1}{5}\begin{pmatrix} 3 \end{pmatrix}4\right)\cdot\left(\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \end{pmatrix}\right)$$

$2 = \frac{1}{5\sqrt{5}}(3\cdot 1 + 4\cdot 2) = \frac{11}{5\sqrt{5}}.$

$$\phi\psi = \phi\psi = \frac{11}{5\sqrt{5}}$$

Another Example: Transition Probability Between Two States

Consider the following two normalized kets in C^2:

$$\psi = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \end{pmatrix}$$

$\sqrt{2},\qquad \phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \end{pmatrix}$
1 .

Their corresponding bras are

$$\psi = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \end{pmatrix}\sqrt{2}, \qquad \phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \end{pmatrix}1.$$

Compute the inner product:

$$\phi\psi = \phi\psi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \end{pmatrix}1\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \end{pmatrix}$$

$\sqrt{2} = 1\frac{}{\sqrt{6}\,(1+\sqrt{2}).}$

The probability to find the system in state ϕ when it is prepared in ψ is

$$P_{\psi\to\phi} = |\phi\psi|^2 = \frac{(1+\sqrt{2})^2}{6} = \frac{1+2\sqrt{2}+2}{6} = \frac{3+2\sqrt{2}}{6}.$$

$$P_{\psi\to\phi} = \frac{3+2\sqrt{2}}{6}$$

Spin-$\frac{1}{2}$ Measurement Example

Suppose an electron is in the state:

$$\psi = \frac{1}{\sqrt{2}}\uparrow_z + \frac{1}{\sqrt{2}}\downarrow_z$$

We want to find the probability that a measurement of spin along the x-axis yields $+\hbar/2$.

The eigenstate of spin along the x-axis corresponding to eigenvalue $+\hbar/2$ is:

$$\uparrow_x = \frac{1}{\sqrt{2}}\left(\uparrow_z + \downarrow_z\right)$$

The probability is given by:
$$P = \left|\uparrow_x |\psi\right|^2$$

Substitute the expressions:

$$\uparrow_x |\psi = \left\langle \frac{1}{\sqrt{2}}(\uparrow_z + \downarrow_z) \middle| \frac{1}{\sqrt{2}}(\uparrow_z + \downarrow_z) \right\rangle$$

Using linearity and orthonormality:

$$\uparrow_z \mid \uparrow_z = 1, \quad \downarrow_z \mid \downarrow_z = 1, \quad \uparrow_z \mid \downarrow_z = 0$$

Therefore:

$$\uparrow_x |\psi = \frac{1}{2}(1 + 0 + 0 + 1) = 1$$

Thus, the probability is:

$$P = \left|\uparrow_x |\psi\right|^2 = |1|^2 = 1$$

Conclusion: The probability of measuring spin-up along the x-axis is:

$$1$$

Spin-1/2 Wavefunction Example

The quantum state of a spin-1/2 particle can be represented as a two-component wavefunction (spinor):

Cauchy–Schwarz Inequality

Let $\vec{x}, \vec{y} \in R^n$. Then the **Cauchy–Schwarz inequality** states:

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \cdot \|\vec{y}\|$$

Where:

* $\vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i$ is the *dot product*
* $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ is the *length (norm)* of $\vec{x}$

Example

Let:
$$\vec{x} = (1, 2), \quad \vec{y} = (3, 4)$$

Then:
$$\vec{x} \cdot \vec{y} = 1 \cdot 3 + 2 \cdot 4 = 11$$
$$\|\vec{x}\| = \sqrt{1^2 + 2^2} = \sqrt{5}, \quad \|\vec{y}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$
$$\|\vec{x}\| \cdot \|\vec{y}\| = \sqrt{5} \cdot 5 \approx 11.18$$

Therefore:
$$|\vec{x} \cdot \vec{y}| = 11 \leq 11.18$$

The inequality holds.

Equality Condition

Equality holds **if and only if** $\vec{x}$ and $\vec{y}$ are *linearly dependent*, i.e., there exists a scalar $\lambda \in R$ such that:
$$\vec{x} = \lambda \vec{y}$$

Cauchy–Schwarz Inequality and Quantum Probability

In quantum mechanics, physical states are represented by vectors in a Hilbert space. Let $|\psi\rangle$ and $|\phi\rangle$ be two state vectors.

Transition Probability

The probability that a system initially in state $|\psi\rangle$ is found in state $|\phi\rangle$ upon measurement is given by:
$$P = |\langle \phi | \psi \rangle|^2$$

Cauchy–Schwarz Inequality

For any two vectors in a Hilbert space, the Cauchy–Schwarz inequality states:

$$|\langle \phi | \psi \rangle|^2 \leq \langle \psi | \psi \rangle \cdot \langle \phi | \phi \rangle$$

Normalization in Quantum Mechanics

Quantum states are typically normalized:

$$\langle\psi|\psi\rangle = 1, \quad \langle\phi|\phi\rangle = 1$$

Therefore, the inequality becomes:

$$|\langle\phi|\psi\rangle|^2 \leq 1$$

This guarantees:
$$0 \leq |\langle\phi|\psi\rangle|^2 \leq 1$$

Conclusion

The **Cauchy–Schwarz inequality** ensures that transition probabilities in quantum mechanics are always between 0 and 1, as expected for any probability.

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Observables and Hermitian Operators in Quantum Mechanics

In quantum mechanics, every observable quantity is represented by a **Hermitian operator**.

Key Properties of Hermitian Operators

* **Real eigenvalues**:
$$\hat{A}\phi = a\phi, \quad a \in R$$

* **Orthogonal eigenvectors**: eigenvectors corresponding to different eigenvalues are orthogonal.

* **Expectation value is real**:

$$\hat{A} = \psi\hat{A}\psi \in R$$

Examples of Observables and Their Operators

Position: $\hat{x}$

Momentum: $\hat{p} = -i\hbar\frac{d}{dx}$

Hamiltonian (Energy): $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$

Spin (z-component): $\hat{S}_z = \frac{\hbar}{2}\sigma_z$

Similarity	Description
Shared physical quantities	Both theories have observables like energy, position, momentum, etc.
Measurability	In both classical and quantum mechanics, observables are measurable.
Predictive role	Used in each theory to describe the state and predict system behavior.
Use of mathematics	(Classical: differential equations; Quantum: operators on Hilbert space)

Aspect	Classical Mechanics	Quantum Mechanics
Mathematical representation	Real-valued variable	Hermitian operator on Hilbert space
State description	Point in phase space	State vector (wavefunction)
Measurement outcome	Deterministic, exact	Probabilistic, one of the eigenvalues
Simultaneous measurement	Always possible	Only for commuting operators
Uncertainty principle	Absent	$\Delta x \, \Delta p \geq \hbar 2$
Wavefunction collapse	None	Occurs upon measurement
Time evolution	Newton's laws	Schrödinger equation

Conclusion

Any physical quantity that can be measured (observable) must be represented by a Hermitian operator, so that all possible outcomes of measurements are real numbers.

Yes, quantities in classical mechanics play the same role as "observables" in quantum mechanics. The difference is in their mathematical representation and behavior, not in their conceptual meaning. article amsmath physics bm

Physical Quantity (Observable)	Quantum Operator
Position x	$\hat{x}$
Momentum p	$\hat{p} = -i\hbar \dfrac{d}{dx}$
Energy E	$\hat{H}$ (Hamiltonian)
Spin around z	$\hat{S}_z$

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Spin in Quantum Mechanics

In quantum mechanics, **spin** is an intrinsic form of angular momentum carried by elementary particles, composite particles (hadrons), and atomic nuclei.

Key Properties

* Spin is a purely quantum mechanical property with no classical analogue.

* It is quantized and described by a quantum number s, where $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$
  * Each spin value determines $2s + 1$ possible spin projections.

Spin States

For a spin-$\frac{1}{2}$ particle (e.g., electron), the spin state is described in a two-dimensional Hilbert space:

$$\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Spin Operators

The spin operators are defined as:

$$\hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These are related to the Pauli matrices:

$$\hat{S}_i = \frac{\hbar}{2}\sigma_i, \quad i = x, y, z$$

Commutation Relations

The spin operators satisfy the angular momentum algebra:

$$[\hat{S}_i, \hat{S}_j] = i\hbar\varepsilon_{ijk}\hat{S}_k$$

$$\hat{S} = \frac{\hbar}{2}\sigma$$

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For example, the spin-z operator becomes:

$$\hat{S}_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Physical Quantity (Observable)	Quantum Operator
Position x	$\hat{x}$
Momentum p	$\hat{p} = -i\hbar \dfrac{d}{dx}$
Energy E	$\hat{H}$ (Hamiltonian)
Spin around z	$\hat{S}_z$

The eigenvalues of $\hat{S}_z$ are:

$$\hat{S}_z\uparrow = +\frac{\hbar}{2}\uparrow, \quad \hat{S}_z\downarrow = -\frac{\hbar}{2}\downarrow$$

These values represent the quantized spin angular momentum along the z-axis for a spin-$\frac{1}{2}$ particle like the electron.

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Definite States (Eigenstates) in Quantum Mechanics

In quantum mechanics, a **definite state** (or **eigenstate**) of an observable is a state in which the value of that observable is precisely known.

Mathematically, if $\hat{A}$ is an operator representing an observable and $|a\rangle$ is an eigenstate of $\hat{A}$, then:

$$\hat{A}|a\rangle = a|a\rangle,$$

where a is the eigenvalue (the definite measured value).

If the system is in the state $|a\rangle$, measuring the observable A will always give the value a.

eample suppoes we have a coin:

The definite tails state means that we know for sure that it is a tails coin.

But sometimes the coin is in a mixed state, for example, half lion and half lear . This means that we don't know which one will come up until we flip it and see. . A deterministic state is a state in which the response is certain when measured. **Example:**

The Hamiltonian operator $\hat{H}$ corresponds to the total energy. Its eigenstates satisfy:

$$\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle,$$

E_n are the energy eigenvalues, and $|\psi_n\rangle$ are the energy eigenstates. Example: Particle in a Box (Infinite Potential Well) Suppose a particle is in a very small, closed box, such as a very narrow room.

It can only move in this box and cannot get out.

The energy of the particle can only have certain values (not just any number).

These specific energy values are the eigenvalues of the Hamiltonian of this system.

A specific state of energy Each of these eigenvalues creates a specific state for the particle.

For example:

The first state has energy

and the waveform of the particle is a half-sine wave in the box.

The second state has energy

and the waveform is a more complete sine wave.

The third state is the same...

If the particle is in the first state, when you measure the energy, you will definitely get E1.

What does this mean? A deterministic state means that when the system is in that state, a measurement of the quantity of interest has a deterministic response.

In this example, the specific state of energy, i.e., the particle wave, is a specific form whose energy is precisely known. article amsmath physics bm

Discrete Spectra in Quantum Mechanics

In quantum mechanics, if an observable is represented by a Hermitian operator $\hat{A}$, the possible outcomes of measuring this observable are given by the **eigenvalues** of the operator.

If these eigenvalues are **discrete**, then we say the observable has a **discrete spectrum**.

Example 1: Particle in a Box (Infinite Potential Well)

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}, \quad n = 1, 2, 3, \ldots$$

Only specific energy values are allowed; the energy cannot take arbitrary values.

Example 2: Quantum Harmonic Oscillator

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots$$

The spectrum is discrete, and the energy levels are evenly spaced with gap $\hbar\omega$.

Example 3: Spin Component Along the z-axis

$$S_z = \pm\frac{\hbar}{2}$$

A spin-$\frac{1}{2}$ particle can only have two possible values for spin along the z-axis.

Conclusion: In a discrete spectrum, there are only specific allowed values for the observable, with no values in between. This contrasts with classical mechanics, where physical quantities can vary continuously.

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– The Dirac Delta Function and Its Physical Implications

Physical Quantity	Modeled As	Description
Point mass	$\rho(x) = m\delta(x - x_0)$	All mass at one point
Point charge	$\rho(x) = q\delta(x - x_0)$	Charge at one point
Point energy source	$u(x) = E\delta(x - x_0)$	Energy localized at a point
Impulse force	$F(t) = I\delta(t - t_0)$	Force applied at an instant

The Dirac delta function is used to mathematically represent any physical quantity that is concentrated at a point in space (or time).

– Proof that the Dirac Delta Potential Well Has Exactly One Bound State

We consider the one-dimensional potential

$$V(x) = -\alpha\,\delta(x), \quad \alpha > 0.$$

1. Schrödinger Equation and Ansatz

For a bound state with energy $E < 0$, set

$$E = -\frac{\hbar^2 k^2}{2m}, \quad k > 0.$$

The time-independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\,\delta(x)\,\psi(x) = E\,\psi(x).$$

Away from $x = 0$, the delta term vanishes and we have

$$\psi''(x) = k^2\,\psi(x),$$

whose general solution is

$$\psi(x) = \{\, A\ e^{kx}, x < 0,$$

$B\,e^{-kx}, x > 0.$

2. Continuity and Derivative-Jump Conditions

(i) Continuity at $x = 0$.

$$\psi(0^-) = \psi(0^+) \quad \Longrightarrow \quad A = B \equiv C.$$

(ii) Integrate across $x = 0$. Integrate the Schrödinger equation from $-\varepsilon$ to $+\varepsilon$:

$$\int_{-\varepsilon}^{+\varepsilon} \left(-\hbar^2 2m \psi''(x) \right) dx - \alpha \int_{-\varepsilon}^{+\varepsilon} \delta(x) \psi(x) \, dx = \int_{-\varepsilon}^{+\varepsilon} E \psi(x) \, dx.$$

As $\varepsilon \to 0$, the right-hand side vanishes and we get

$$-\frac{\hbar^2}{2m} [\psi'(0^+) - \psi'(0^-)] - \alpha \psi(0) = 0,$$

or

– Note that the zeros that are negative and positive indicate that they are decreasing exponentially in positive x or negative x. For example, in positive x, the exponential function should put a negative next to x.

$$\psi'(0^+) - \psi'(0^-) \;=\; -\frac{2m\alpha}{\hbar^2} \psi(0).$$

$I_1 = \int_{-\varepsilon}^{+\varepsilon} \left(-\hbar^2 2m \, \psi''(x) \right) dx = -\frac{\hbar^2}{2m} \left[\psi'(x) \right]_{x=-\varepsilon}^{x=+\varepsilon} = -\frac{\hbar^2}{2m} (\psi'(+\varepsilon) - \psi'(-\varepsilon)) \; \varepsilon \to 0 \; - \frac{\hbar^2}{2m} (\psi'(0^+) - \psi'(0^-)),$

$I_2 = \int_{-\varepsilon}^{+\varepsilon} (-\alpha \, \delta(x) \, \psi(x)) \, dx = -\alpha \, \psi(0),$

$I_3 = \int_{-\varepsilon}^{+\varepsilon} E \, \psi(x) \, dx \approx E \psi(0) \, (2\varepsilon) \; \varepsilon \to 0 \; 0.$

Combining $I_1 + I_2 = I_3$ in the $\varepsilon \to 0$ limit gives

$$-\frac{\hbar^2}{2m} [\psi'(0^+) - \psi'(0^-)] \; - \; \alpha \psi(0) = 0 \quad \Longrightarrow \quad \psi'(0^+) - \psi'(0^-) = -\frac{2m\alpha}{\hbar^2} \psi(0).$$

3. Evaluating the Derivative Jump

With $A = B = C$, the wavefunction is

$$\psi(x) = \{ C \; e^{kx}, x < 0,$$

$C\,e^{-kx}, x > 0$. Hence

$$\psi'(0^-) = k \, C, \qquad \psi'(0^+) = -k \, C, \qquad \psi(0) = C.$$

Substitute into the jump condition:

$$(-k \, C) - (k \, C) = -2k \, C \;=\; -\frac{2m\alpha}{\hbar^2} C \quad \Longrightarrow \quad k = \frac{m\alpha}{\hbar^2}.$$

4. Uniqueness of the Bound State

The matching conditions yield the single algebraic equation

$$k = \frac{m\alpha}{\hbar^2},$$

which has exactly one positive solution for k. Therefore the bound-state energy is unique:

$$E = -\frac{\hbar^2 k^2}{2m} = -\frac{m \alpha^2}{2\hbar^2}.$$

No other bound-state energies are possible. The delta well is infinitely narrow, but its depth is not infinitely large. It can only hold one state of negative energy. If you want to have two bound states, you need more space or greater depth. There is a sharp peak at x=0. This means: The highest probability of finding a particle is right at the center of the well. article amsmath,amssymb

5. Conclusion

Because the continuity and derivative-jump conditions produce exactly one condition on k with a unique positive solution, the Dirac delta potential well supports exactly one bound state.

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Wave Packet Dynamics in Quantum Mechanics

In quantum mechanics, a **wave packet** represents a localized particle by superposing many different energy eigenstates. Unlike a single stationary state, which corresponds to a specific energy and does not change its shape over time, a wave packet is a *time-dependent* combination of these states.

Motion of the Packet

A wave packet moves through space, and its position evolves with time. This motion reflects the particle-like behavior of the system. As time progresses, the packet may **shift in position** and also **spread out**, a phenomenon known as *dispersion*. Dispersion occurs because the constituent energy states in the packet move at different phase velocities.

Physical Interpretation

This time evolution means the system does not remain in a single quantum state. Instead, the wave packet changes its shape continuously, reflecting the uncertainty in both position and momentum. The packet's center may represent the most probable location of the particle, but there is always a spread due to the probabilistic nature of quantum mechanics.

Conclusion

The moving wave packet is a powerful tool in quantum theory to understand how quantum particles behave like classical ones in motion, while still respecting quantum uncertainty and superposition. It shows us that position and energy are not simultaneously definite and that the system explores many "quasi-states" as time flows.

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Superposition in Quantum Mechanics

In quantum mechanics, a particle can exist in **multiple states at once**. This is called the **superposition principle**.

If a system can be in the state ψ_1 and also in the state ψ_2, then it can also be in a combination of both:

$$\Psi = c_1\psi_1 + c_2\psi_2$$

Here:

* c_1 and c_2 are complex numbers (called amplitudes),
* The total probability must be 1, so:

$$|c_1|^2 + |c_2|^2 = 1$$

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Incompatible Measurements and Quantum Superposition

In quantum mechanics, measuring one property of a system can disturb or change other properties. This is because some observables do not commute and cannot have definite values simultaneously.

Example: Color and Texture

Suppose a quantum particle has two observable properties:

* **Color:** White or Black
* **Texture:** Soft or Hard

These properties are represented by quantum states:

$$White, \quad Black, \quad Soft, \quad Hard$$

Now, assume you perform a measurement and find the particle in the "White" state:

$$\psi = White$$

In terms of the texture basis, this state may be a superposition:

$$White = aSoft + bHard$$

After this, if you measure the **texture**, the system will collapse into either:

$$Soft \quad (with probability |a|^2), \quad or \quad Hard \quad (with probability |b|^2)$$

Now the system is no longer in the "White" state! If you measure the color again, it could collapse to:

$$White \quad or \quad Black$$

depending on the new state after the previous measurement.

Conclusion

In quantum mechanics:

* ∗ Measurement changes the state of the system.
* ∗ Some properties (observables) are **incompatible** and cannot be precisely known at the same time.
* ∗ This leads to fundamental uncertainty and behavior not seen in classical systems.

....

In the case of superposition in the potential well, we solved and obtained that two states are allowed in the well. Now, for simplicity, we assume that the probability of both is the same. So by solving this at time t zero, we notice where the maximum value is. And in a possible time when they are not the same, and with the obtained probabilities, we can obtain the maximum value of the probability in the well. In a situation where the value of the state is a number, such as the Dirac delta well, well, when this well has one state, we no longer have superposition and there is only one state, the maximum value of the probability density can be calculated with the method I will provide and the maximum value can be obtained. article amsmath amssymb

Finding the Maximum of the Probability Density for a Single Bound State

For a single bound state with wavefunction $\psi(x)$, the probability density is

$$P(x) = |\psi(x)|^2.$$

To locate its maxima:

1. Compute the derivative:

$$\frac{dP}{dx} = \frac{d}{dx}(\psi^*(x)\,\psi(x)) = 2\,\Re[\psi^*(x)\,\psi'(x)].$$

2. Set $dPdx = 0$. This yields

$$\psi^*(x)\,\psi'(x) = 0,$$

which splits into $\psi(x) = 0 \quad (nodes, where P(x) = 0)$,
$\psi'(x) = 0 \quad (critical points : candidates for maxima or minima)$.

3. Perform a second-derivative test, $d^2Pdx^2 < 0$, to identify true maxima.

Example: Infinite Square Well

For the nth eigenstate in a one-dimensional infinite well of width L,

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad P_n(x) = \frac{2}{L} \sin^2\left(\frac{n\pi x}{L}\right).$$

* Derivative:
$$\frac{dP_n}{dx} = \frac{4n\pi}{L^2} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) = 0.$$

* Solutions: $\sin(n\pi x L) = 0 \implies x = m\frac{L}{n}, \quad m = 0, 1, \ldots, n \quad (nodes),$
  $\cos(n\pi x L) = 0 \implies x = \frac{(2k+1)L}{2n}, \quad k = 0, 1, \ldots, n-1 \quad (maxima).$

Thus the maxima of $P_n(x)$ occur at

$$x_k = \frac{(2k+1)L}{2n}, \quad k = 0, 1, \ldots, n-1.$$

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Example: Equal Superposition of Two States in Infinite Square Well

Let the two lowest eigenstates of the infinite square well be:

$$\psi_1(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right), \qquad \psi_2(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right)$$

Consider the equal superposition:

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\psi_1(x)e^{-iE_1t/\hbar} + \frac{1}{\sqrt{2}}\psi_2(x)e^{-iE_2t/\hbar}$$

Then the probability density is: $P(x,t) = -\Psi(x,t)|^2$
$$= 1\frac{}{2|\psi_1(x)|^2 + \frac{1}{2}|\psi_2(x)|^2 + \Re\left[\psi_1^*(x)\psi_2(x)e^{-i(E_2-E_1)t/\hbar}\right]}$$
$$= 1\frac{}{L\left[\sin^2\left(\frac{\pi x}{L}\right) + \sin^2\left(\frac{2\pi x}{L}\right)\right] + \frac{2}{L}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{(E_2-E_1)t}{\hbar}\right)}$$

This final term shows interference and time-dependent modulation due to the superposition:

$$P(x,t) = \frac{1}{L}\left[\sin^2\left(\frac{\pi x}{L}\right) + \sin^2\left(\frac{2\pi x}{L}\right) + 2\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{(E_2-E_1)t}{\hbar}\right)\right]$$

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Superposition of Two Energy States in an Infinite Square Well

Assume a particle in a one-dimensional infinite potential well of width L. The energy eigenstates are:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots$$

Now, consider a superposition of the first two states:

$$\Psi(x, t) = c_1\psi_1(x)e^{-iE_1t/\hbar} + c_2\psi_2(x)e^{-iE_2t/\hbar}$$

Let's consider equal amplitudes:

$$c_1 = c_2 = \frac{1}{\sqrt{2}}$$

So,

$$\Psi(x, t) = \frac{1}{\sqrt{2}}\left[\psi_1(x)e^{-iE_1t/\hbar} + \psi_2(x)e^{-iE_2t/\hbar}\right]$$

The probability density is:

$$|\Psi(x, t)|^2 = \Psi^*(x, t)\Psi(x, t)$$

Expanding this gives:

$$|\Psi(x, t)|^2 = \frac{1}{2}\left[\psi_1^2(x) + \psi_2^2(x) + 2\psi_1(x)\psi_2(x)\cos\left(\frac{(E_2 - E_1)t}{\hbar}\right)\right]$$

This interference term causes **time-dependent oscillations** in the probability density.

—

Numerical Example

Let the width of the well be $L = 1$. and t=0 ;The wavefunctions are:

$$\psi_1(x) = \sqrt{2}\sin(\pi x), \quad \psi_2(x) = \sqrt{2}\sin(2\pi x)$$

Then:

$$\Psi(x, 0) = \frac{1}{\sqrt{2}}\left[\sqrt{2}\sin(\pi x) + \sqrt{2}\sin(2\pi x)\right] = \sin(\pi x) + \sin(2\pi x)$$

So,

$$|\Psi(x, 0)|^2 = [\sin(\pi x) + \sin(2\pi x)]^2 = \sin^2(\pi x) + \sin^2(2\pi x) + 2\sin(\pi x)\sin(2\pi x)$$

Using the identity:

$$2\sin a \sin b = \cos(a-b) - \cos(a+b)$$

We get:

$$|\Psi(x,0)|^2 = \sin^2(\pi x) + \sin^2(2\pi x) + \cos(\pi x) - \cos(3\pi x)$$

You can now plot this to find the location of **maximum probability** in the well.

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Wavefunction at $t = 0$

We consider a superposition of the first and second energy eigenstates with unequal probability amplitudes:

$$\Psi(x,0) = \sqrt{0.8} \cdot \psi_1(x) + \sqrt{0.2} \cdot \psi_2(x)$$

Where the eigenstates for a particle in an infinite potential well of width $L = 1$ are:

$$\psi_1(x) = \sqrt{2}\sin(\pi x), \qquad \psi_2(x) = \sqrt{2}\sin(2\pi x)$$

Probability Density

The square of the modulus of the wavefunction is:

$$—\Psi(x,0)|^2 = \left(\sqrt{0.8} \cdot \psi_1(x) + \sqrt{0.2} \cdot \psi_2(x)\right)^2$$
$$= 0.8 \cdot \psi_1^2(x) + 0.2 \cdot \psi_2^2(x) + 2 \cdot \sqrt{0.8 \cdot 0.2} \cdot \psi_1(x)\psi_2(x)$$
$$= 0.8 \cdot \left(2\sin^2(\pi x)\right) + 0.2 \cdot \left(2\sin^2(2\pi x)\right) + 2 \cdot \sqrt{0.8 \cdot 0.2} \cdot \left(\sqrt{2}\sin(\pi x)\right)\left(\sqrt{2}\sin(2\pi x)\right)$$
$$= 1.6\,\sin^2(\pi x) + 0.4\sin^2(2\pi x) + 0.8\sin(\pi x)\sin(2\pi x)$$

$$|\Psi(x,0)|^2 = 1.6\sin^2(\pi x) + 0.4\sin^2(2\pi x) + 0.8\sin(\pi x)\sin(2\pi x)$$

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The destruction of the view that consciousness causes collapse, from Abbas's perspective

I think the blanket I have spread on my bed as I write this is my awareness that it is soft but does not collapse, but when I run my hand over it, it collapses, meaning that I have measured it and that the consciousness and awareness of the human and the cat do not cause the collapse of physical states. The same thing is raised in the Schrödinger cat hypothesis, in which the cat is not a physical measurer and cannot cause it to collapse. If consciousness causes

states to collapse, then God is also something that is aware of everything in the universe, so Winger is wrong. In the double slit, I am aware that a photon will pass through the double slit, even if it is a photon, and provided that the measuring device is not there, I was aware that this will cause interference. He might say to himself that this is a sign of interference in the experiment of a measurement. You're right, but this measurement is before the two photons interfered, and this interference has already created a wave state and interference. I might say to myself, I've measured the softness of a blanket before, for example, as a child, I knew about the softness of a blanket, and what I measured before and what affected my consciousness has no effect on the experiment. The reason is that for the first time, humans didn't know what the outcome of the experiment would be, and when they did the experiment, they measured it, and the outcome of the experiment was the same. So the past measurement that made me aware of the experiment has no effect on the current experiment. And if God did the experiment and has perfect knowledge of the world before and after, then it has no effect on the outcome of the experiment. In my opinion, Winger was wrong, and I put together this view, which is a combination of other views.