Subject:Commitator

$$[AB + CD, EF + GH] = [AB, EF] + [AB, GH] + [CD, EF] + [CD, GH]$$

$$(1)[AB - CD, EF - GH] = [AB, EF] - [AB, GH] - [CD, EF] + [CD, GH]$$

The top four sections of each opened section are:

$$(1)[AB, EF] = A[B, EF] + [A, EF]B$$

$$(1)[B, EF] = [B, E]F + E[B, D]$$

$$(1)[A, EF] = [A, E]F + E[A, D]$$

$$\boxed{(1)[AB, EF] = A[B, E]F + AE[B, F] + [A, E]FB + E[A, F]B}$$

The sixth mode is for one of the four modes.
Example:

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = \hat{i}(yp_z - zp_y) - \hat{j}(xp_z - zp_x) + \hat{k}(xp_y - yp_x)$$

use 1 , 2 , 3 , 4 and 5 This was only for one of the four episodes above.

$$L_z = xp_y - yp_x$$

$$\hat{L}_x = \hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y, \qquad \hat{L}_y = \hat{z}\,\hat{p}_x - \hat{x}\,\hat{p}_z$$

use (1)

$$[\hat{L}_x, \hat{L}_y] = i\hbar\,\hat{L}_z$$

$$[L_x, L_y]X \neq 0.$$

The empty commutator expresses the physical nature and algebraic structure of the operators and their relationship to each other.

Multiplying by X x shows that we are examining this algebraic relationship in a weighted position space or restricted to a particular point or interval.

$$L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right),$$

$$L_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right),$$

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

calculation:

$$[L_x, L_y]x = (L_xL_y - L_yL_x)x.$$

$$L_yx = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)x.$$

$$\frac{\partial x}{\partial x} = 1, \quad \frac{\partial x}{\partial z} = 0,$$

$$L_y x = -i\hbar(z \cdot 1 - x \cdot 0) = -i\hbar z.$$

$$L_x(-i\hbar z) = -i\hbar \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) (-i\hbar z).$$

$$\frac{\partial}{\partial z}(-i\hbar z) = -i\hbar, \quad \frac{\partial}{\partial y}(-i\hbar z) = 0,$$

$$L_x L_y x = -i\hbar \left(y(-i\hbar) - z \cdot 0 \right) = (-i)(-i)\hbar^2 y.$$

$$L_x L_y x = -\hbar^2 y.$$

$$L_x x = -i\hbar \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) x.$$

$$\frac{\partial x}{\partial z} = 0, \quad \frac{\partial x}{\partial y} = 0,$$

$$L_x x = 0.$$

$$L_y(L_x x) = L_y 0 = 0.$$

$$[L_x, L_y]x = L_x L_y x - L_y L_x x = -\hbar^2 y - 0 = -\hbar^2 y \neq 0.$$

-

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

$$[\hat{L}^2, \hat{L}_x] = [\hat{L}_x^2, \hat{L}_x] + [\hat{L}_y^2, \hat{L}_x] + [\hat{L}_z^2, \hat{L}_x]$$

$$[\hat{L}_x^2, \hat{L}_x] = 0$$

$$[A^2, B] = A[A, B] + [A, B]A$$
$$[AA, B]$$

with $A = \hat{L}_y$, $B = \hat{L}_x$:

$$[\hat{L}_y^2, \hat{L}_x] = \hat{L}_y[\hat{L}_y, \hat{L}_x] + [\hat{L}_y, \hat{L}_x]\hat{L}_y$$

$[\hat{L}_y, \hat{L}_x] = -i\hbar\,\hat{L}_z$:

$$[\hat{L}_y^2, \hat{L}_x] = -i\hbar(\hat{L}_y\hat{L}_z + \hat{L}_z\hat{L}_y)$$

$$[\hat{L}_z^2, \hat{L}_x] = \hat{L}_z[\hat{L}_z, \hat{L}_x] + [\hat{L}_z, \hat{L}_x]\hat{L}_z$$

Using $[\hat{L}_z, \hat{L}_x] = i\hbar\,\hat{L}_y$:

$$[\hat{L}_z^2, \hat{L}_x] = i\hbar(\hat{L}_z\hat{L}_y + \hat{L}_y\hat{L}_z)$$

$$-i\hbar(\hat{L}_y\hat{L}_z + \hat{L}_z\hat{L}_y) + i\hbar(\hat{L}_z\hat{L}_y + \hat{L}_y\hat{L}_z) = 0$$

$$\boxed{[\hat{L}^2, \hat{L}_x] = 0}$$

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y$$

$$\hat{L}_- = \hat{L}_x - i\hat{L}_y$$

$$\hat{L}_z \, |\ell, m\rangle = \hbar m \, |\ell, m\rangle$$
$$\hat{L}_\pm \, |\ell, m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m \pm 1)} \, |\ell, m \pm 1\rangle$$

$$-\ell \leq m \leq \ell$$

$$\hat{L}_+|\ell, \ell\rangle = 0, \quad \hat{L}_-|\ell, -\ell\rangle = 0$$

Consider the state $|\ell = 1, m = 0\rangle$.
Applying the raising operator $\hat{L}_+$:

$$\hat{L}_+|\ell = 1, m = 0\rangle = \hbar\sqrt{1 \cdot (1+1) - 0 \cdot (0+1)} \, |\ell = 1, m = 1\rangle = \hbar\sqrt{2} \, |\ell = 1, m = 1\rangle$$

Similarly, applying the lowering operator $\hat{L}_-$ on $|\ell = 1, m = 0\rangle$:

$$\hat{L}_-|\ell = 1, m = 0\rangle = \hbar\sqrt{1 \cdot (1+1) - 0 \cdot (0-1)} \, |\ell = 1, m = -1\rangle = \hbar\sqrt{2} \, |\ell = 1, m = -1\rangle$$

RULES

$$[A, B] = AB - BA.$$

$$[A, A] = 0.$$

$$[A, B] = -[B, A].$$

$$[A, c] = 0, \quad \text{scalar } c.$$

$$[A, cB] = c[A, B], \quad \text{scalar } c.$$

$$[A + B, C] = [A, C] + [B, C].$$

$$[A, BC] = [A, B]C + B[A, C].$$

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

$$[A, BC] = ABC - BCA$$

$$= ABC - BAC + BAC - BCA$$
$$= (AB - BA)C + B(AC - CA)$$

$$[AB, CD] = \Big(A[B, C] + [A, C]B\Big)D + C\Big(A[B, D] + [A, D]B\Big).$$

Important subtlety about operator factorization:

$$a^2 + b^2 = (a + ib)(a - ib).$$

$$L_x^2 + L_y^2 \neq (L_x + iL_y)(L_x - iL_y)$$

$$L_x^2 + L_y^2 \neq (L_x + iL_y)(L_x - iL_y).$$

Indeed,

$$(L_x + iL_y)(L_x - iL_y) = L_x^2 - iL_xL_y + iL_yL_x + L_y^2 = L_x^2 + L_y^2 - i(L_xL_y - L_yL_x)$$
$$= L_x^2 + L_y^2 - i[L_x, L_y] = L_x^2 + L_y^2 - i(i\hbar L_z) = L_x^2 + L_y^2 + \hbar L_z \neq L_x^2 + L_y^2.$$

$$[a,\, b + c] = [a, b] + [a, c].$$

$\hbar We show that [L^2, L_+] = 0.$

$$[L^2, L_+] = [L^2, L_x + iL_y] = [L^2, L_x] + i[L^2, L_y].$$

Since $L^2 = L_x^2 + L_y^2 + L_z^2$ commutes with each component L_x, L_y, L_z, we have

$$[L^2, L_x] = 0, \quad [L^2, L_y] = 0.$$

Therefore,

$$[L^2, L_+] = 0 + i(0) = 0.$$

$$[L^2, L_z] = [L_x^2 + L_y^2 + L_z^2,\, L_z] = [L_x^2, L_z] + [L_y^2, L_z] + [L_z^2, L_z]$$
$$= [L_xL_x, L_z] + [L_yL_y, L_z]$$
$$= L_x[L_x, L_z] + [L_x, L_z]L_x + L_y[L_y, L_z] + [L_y, L_z]L_y$$
$$= L_x(-i\hbar L_y) + (-i\hbar L_y)L_x + L_y(i\hbar L_x) + (i\hbar L_x)L_y$$
$$= \big(-i\hbar L_xL_y + i\hbar L_xL_y\big) + \big(-i\hbar L_yL_x + i\hbar L_yL_x\big) = 0.$$

- The eigenvalue and the eigenstet are

$$L^2|\alpha, \beta\rangle = \alpha|\alpha, \beta\rangle, \tag{1}$$

$$L_z|\alpha, \beta\rangle = \beta|\alpha, \beta\rangle, \tag{2}$$

$$[L_z, L_+] = L_zL_+ - L_+L_z = \hbar L_+, \tag{3}$$

$$L_zL_+ = L_+L_z + \hbar L_+. \tag{4}$$

$$\begin{aligned} L_zL_+|\alpha, \beta\rangle &= (L_+L_z + \hbar L_+)|\alpha, \beta\rangle && (5)\\ &= L_+L_z|\alpha, \beta\rangle + \hbar L_+|\alpha, \beta\rangle && (6)\\ &= L_+\beta|\alpha, \beta\rangle + \hbar L_+|\alpha, \beta\rangle && (7)\\ &= (\beta + \hbar)L_+|\alpha, \beta\rangle. && (8) \end{aligned}$$

$$L_z\left(L_+|\alpha, \beta\rangle\right) = (\beta + \hbar)\left(L_+|\alpha, \beta\rangle\right). \tag{9}$$

I move this angle along the z axis.

$$\left(L^2 - L_z^2\right)|\alpha, \beta\rangle = L^2|\alpha, \beta\rangle - L_z^2|\alpha, \beta\rangle = \alpha|\alpha, \beta\rangle - L_z\beta|\alpha, \beta\rangle = \alpha|\alpha, \beta\rangle - \beta^2|\alpha, \beta\rangle = \left(\alpha - \beta^2\right)|\alpha, \beta\rangle.$$

$$L_-L_+ = L^2 - L_z^2 - \hbar L_z,$$

applying to the eigenstate $|\alpha, \beta_{\max}\rangle$:

$$L_-L_+|\alpha, \beta_{\max}\rangle = 0,$$

$$\left(L^2 - L_z^2 - \hbar L_z\right)|\alpha, \beta_{\max}\rangle = 0,$$

$$L^2|\alpha, \beta_{\max}\rangle - L_z^2|\alpha, \beta_{\max}\rangle - \hbar L_z|\alpha, \beta_{\max}\rangle = 0.$$

$$L^2|\alpha, \beta_{\max}\rangle = \alpha|\alpha, \beta_{\max}\rangle,$$

$$L_z|\alpha, \beta_{\max}\rangle = \beta_{\max}|\alpha, \beta_{\max}\rangle,$$

$$\alpha|\alpha, \beta_{\max}\rangle - \beta_{\max}^2|\alpha, \beta_{\max}\rangle - \hbar\beta_{\max}|\alpha, \beta_{\max}\rangle = 0,$$

$$\left(\alpha - \beta_{\max}^2 - \hbar\beta_{\max}\right)|\alpha, \beta_{\max}\rangle = 0.$$

$$\boxed{\alpha = \beta_{\max}^2 + \hbar\beta_{\max}.}$$

$$\beta = \frac{n}{2}\hbar$$

This tells me that the minimum minus the maximum gives me two maximums and all my steps are N and are done at a distance of H.