If I had spent my time learning quantum mechanics to make money, I would have my own little cafe by now.

1 Introduction

I, Abbas Abedini, want to take you inside the hydrogen atom.
Before learning this section, you need to know the Schrödinger equation.
Spherical coordinates and differential equations and three-dimensional modes
So let's go to the Schrödinger equation. I taught the Schrödinger equation in a book called Learning the Basics of Quantum Mechanics.
But since I didn't teach 3D, I have to explain it here. Of course, keep in mind that we have some difficult equations ahead of us, and since it's beyond my scope, I'll explain it briefly.
Here I will teach you how to get probability and I will also explain spin in hydrogen if I have the patience.
I'm trying to maintain the necessary standards here, unlike that introductory quantum book I wrote, some of which I wrote out of my own stupidity, but here I'm trying to act according to the standards.

2 Schrödinger in three dimensions

$$\hat{H}\psi(x) = E\psi(x)$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

Oh my child, but we don't need you.
But I can give you a chance to come and transform you into three dimensions or three dimensions that are in spherical coordinates.

$$h_1 = 1 \quad (forr) h_2 = r\sin\phi \quad (for\theta) h_3 = r \quad (for\phi)$$

Obtaining this part was also in mathematics and was obtained from geometry. Our main goal was not to obtain these values.

$$\nabla^2 f = \frac{1}{h_1\,h_2\,h_3}\left[\frac{\partial}{\partial q_1}\Big(\frac{h_2\,h_3}{h_1}\,\frac{\partial f}{\partial q_1}\Big) + \frac{\partial}{\partial q_2}\Big(\frac{h_3\,h_1}{h_2}\,\frac{\partial f}{\partial q_2}\Big) + \frac{\partial}{\partial q_3}\Big(\frac{h_1\,h_2}{h_3}\,\frac{\partial f}{\partial q_3}\Big)\right]$$

Oh, you didn't think Laplace would come??? If you don't know, go reread math or the first chapter of electromagnetism.

$$\nabla^2 f(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\,\frac{\partial f}{\partial r}\Big) \;+\; \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\Big(\sin\phi\,\frac{\partial f}{\partial\phi}\Big) \;+\; \frac{1}{r^2\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}\,.$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\,\frac{\partial\psi}{\partial\phi}\right)+\frac{1}{r^2\sin^2\phi}\frac{\partial^2\psi}{\partial\theta^2}\right]+V(r)\,\psi(r,\theta,\phi)=E\,\psi(r,\theta,\phi)$$

So it went into the Schrödinger equation.

$$-\frac{\hbar^2}{2m}\left[\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right)+\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial\psi}{\partial\phi}\right)+\frac{1}{\sin^2\phi}\frac{\partial^2\psi}{\partial\theta^2}\right]+r^2V(r)\psi=r^2E\psi$$

Multiplication : : :h$^2/2m$
$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left[\frac{2m}{\hbar^2}r^2(E-V(r))-\ell(\ell+1)\right]R=0$
$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+V_{eff}(r)\cdot u(r)=Eu(r)$

$$V_{eff}(r)=V(r)+\frac{\hbar^2}{2m}\cdot\frac{\ell(\ell+1)}{r^2}$$

Replace potential with the following phrase:

$$V(r)=-\frac{e^2}{4\pi\varepsilon_0\,r}$$

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We begin with the angular part of the Schrödinger equation in spherical coordinates:

$$\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial Y}{\partial\phi}\right)+\frac{1}{\sin^2\phi}\frac{\partial^2 Y}{\partial\theta^2}=-\ell(\ell+1)Y$$

Assume the separation of variables:

$$Y(\theta,\phi)=\Theta(\theta)\Phi(\phi)$$

Substituting into the equation and dividing by $\Theta(\theta)\Phi(\phi)$, we get:

$$\frac{1}{\Phi(\phi)}\cdot\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi}{d\phi}\right)+\frac{1}{\Theta(\theta)}\cdot\frac{1}{\sin^2\phi}\frac{d^2\Theta}{d\theta^2}=-\ell(\ell+1)$$

Since the left-hand side is a sum of two terms, one depending only on θ and one only on ϕ, we can set both equal to a constant, m^2, and write:

$$\frac{d^2\Theta(\theta)}{d\theta^2}+m^2\Theta(\theta)=0$$

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi(\phi)}{d\phi}\right)+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\phi}\right]\Phi(\phi)=0$$

$$P_\ell(x)=\frac{1}{2^\ell\,\ell!}\frac{d^\ell}{dx^\ell}\left(x^2-1\right)^\ell$$

$$P_\ell^m(x)=(1-x^2)^{\frac{m}{2}}\,\frac{d^m}{dx^m}P_\ell(x)$$

Probability Wave Function

In problems with spherical symmetry, the wavefunction separates as:

$$\Psi(r,\theta,\phi) = R(r) \times Y_\ell^m(\theta,\phi)$$

where $Y_\ell^m(\theta,\phi)$ are the **spherical harmonics** describing the angular dependence.

The angular probability density (probability to find the particle at angles θ,ϕ) is

$$P(\theta,\phi) = |Y_\ell^m(\theta,\phi)|^2$$

The spherical harmonics have the form:

$$Y_\ell^m(\theta,\phi) = N_{\ell m}\, P_\ell^m(\cos\theta)\, e^{im\phi}$$

$$N_{\ell m} = (-1)^m \sqrt{\frac{2\ell+1}{4\pi} \cdot \frac{(\ell-m)!}{(\ell+m)!}}$$

$$P(\theta,\phi) = |Y_\ell^m(\theta,\phi)|^2 = |N_{\ell m}|^2 \; |P_\ell^m(\cos\theta)|^2$$

Well, my beauty, we were able to obtain the probability of an angle, we need to go to Lagor.

$$R_{n\ell}(r) = N_{n\ell} \left(\frac{2r}{na_0}\right)^\ell e^{-r/(na_0)} L_{n-\ell-1}^{2\ell+1}\!\left(\frac{2r}{na_0}\right),$$

where a_0 is the Bohr radius, $L_k^\alpha(x)$ is the associated Laguerre polynomial of degree $k = n-\ell-1$ and parameter $\alpha = 2\ell+1$, and the normalization constant is

$$N_{n\ell} = \sqrt{\left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n\,(n+\ell)!}}\,.$$

Thus the full radial part reads

$$R_{n\ell}(r) = \sqrt{\left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n\,(n+\ell)!}} \left(\frac{2r}{na_0}\right)^\ell e^{-r/(na_0)} L_{n-\ell-1}^{2\ell+1}\!\left(\frac{2r}{na_0}\right).$$

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The full hydrogenic wavefunction in spherical coordinates is

$$\Psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r)\, Y_\ell^m(\theta,\phi)$$

where

$$R_{n\ell}(r)=\sqrt{\left(\frac{2}{na_0}\right)^3\frac{(n-\ell-1)!}{2n\,(n+\ell)!}}\;\left(\frac{2r}{na_0}\right)^{\ell}e^{-r/(na_0)}L_{n-\ell-1}^{2\ell+1}\!\left(\frac{2r}{na_0}\right)$$

and

$$Y_\ell^m(\theta,\phi)=(-1)^m\sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)\,e^{im\phi}\,.$$

Putting it all together:

$$\Psi_{n\ell m}(r,\theta,\phi)=\sqrt{\left(\frac{2}{na_0}\right)^3\frac{(n-\ell-1)!}{2n\,(n+\ell)!}}\left(\frac{2r}{na_0}\right)^{\ell}e^{-r/(na_0)}L_{n-\ell-1}^{2\ell+1}\!\left(\frac{2r}{na_0}\right)$$
$$(-1)^m\sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)\,e^{im\phi}.$$

This is the general wave function for radial and for angular we get
The Lagord equation, which gives a radius, the spherical Schrödinger equation, which we solved radially, had many equations, we just put them in.
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Ground State Wavefunction of Hydrogen ($n=1,\ \ell=0,\ m=0$)

The angular part is

$$Y_0^0(\theta,\phi)=\frac{1}{\sqrt{4\pi}}.$$

The radial part is

$$R_{10}(r)=2\,a_0^{-32}\;e^{-r/a_0},$$

where a_0 is the Bohr radius.

Hence the full normalized wavefunction is

$$\Psi_{100}(r,\theta,\phi)=R_{10}(r)\,Y_0^0(\theta,\phi)=\frac{1}{\sqrt{\pi\,a_0^3}}\;e^{-r/a_0}.$$

The probability density reads

$$|\Psi_{100}(r,\theta,\phi)|^2=\frac{1}{\pi\,a_0^3}\;e^{-2r/a_0}.$$