Because the hydrogen atom is a two-particle atom, we previously considered Schrödinger as a single-particle equation for the electron only.
I try to consider the system as two particles and write the Schrödinger in two particle form.
We define the center of mass coordinate $\mathbf{R}$ and the relative coordinate $\mathbf{r}$ as:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

Expression for $\mathbf{r}_1$

$$\begin{aligned}(m_1 + m_2)\mathbf{R} &= m_1\mathbf{r}_1 + m_2\mathbf{r}_2 \\ &= m_1\mathbf{r}_1 + m_2(\mathbf{r}_1 - \mathbf{r}) \\ &= (m_1 + m_2)\mathbf{r}_1 - m_2\mathbf{r}\end{aligned}$$

Solving for $\mathbf{r}_1$:

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{m_1 + m_2}\mathbf{r}$$

Expression for $\mathbf{r}_2$

$$\begin{aligned}(m_1 + m_2)\mathbf{R} &= m_1(\mathbf{r}_2 + \mathbf{r}) + m_2\mathbf{r}_2 \\ &= (m_1 + m_2)\mathbf{r}_2 + m_1\mathbf{r}\end{aligned}$$

Solving for $\mathbf{r}_2$:

$$\mathbf{r}_2 = \mathbf{R} - \frac{m_1}{m_1 + m_2}\mathbf{r}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Then we can also write:

$$\mathbf{r}_1 = \mathbf{R} + \frac{\mu}{m_1}\mathbf{r}, \qquad \mathbf{r}_2 = \mathbf{R} - \frac{\mu}{m_2}\mathbf{r}$$

In fact, the big R has been replaced with the big X and the small R with the small X. Note that we did not mean one dimension, just notation. Otherwise, R is one of the three components XY and Z, and we wrote one component for notation.

Let $\mathbf{R} = (X, Y, Z)$, $\mathbf{r} = (x, y, z)$.

Gradient with respect to $\mathbf{r}_1$

$$(\nabla_1)_x = \frac{\partial}{\partial x_1} = \frac{\partial X}{\partial x_1}\frac{\partial}{\partial X} + \frac{\partial x}{\partial x_1}\frac{\partial}{\partial x} = \frac{m_1}{m_1+m_2}\frac{\partial}{\partial X} + \frac{\partial}{\partial x}$$

So the full

$$\nabla_1 = \frac{m_1}{m_1+m_2}\nabla_R + \nabla_r = \frac{\mu}{m_2}\nabla_R + \nabla_r$$

Gradient with respect to $\mathbf{r}_2$

$$(\nabla_2)_x = \frac{\partial}{\partial x_2} = \frac{\partial X}{\partial x_2}\frac{\partial}{\partial X} + \frac{\partial x}{\partial x_2}\frac{\partial}{\partial x} = \frac{m_2}{m_1+m_2}\frac{\partial}{\partial X} - \frac{\partial}{\partial x}$$

So the full gradient:

$$\nabla_2 = \frac{m_2}{m_1+m_2}\nabla_R - \nabla_r = \frac{\mu}{m_1}\nabla_R - \nabla_r$$

$$\boxed{\begin{aligned}\nabla_1 &= \frac{\mu}{m_2}\nabla_R + \nabla_r \\ \nabla_2 &= \frac{\mu}{m_1}\nabla_R - \nabla_r\end{aligned}}$$

These relations are useful when expressing kinetic energy operators or Laplacians in center-of-mass and relative coordinates. article amsmath

We calculate:

$$\nabla_1^2\psi = \nabla_1 \cdot (\nabla_1\psi) = \nabla_1 \cdot \left(\frac{\mu}{m_2}\nabla_R\psi + \nabla_r\psi\right)$$

Applying the divergence operator:

$$= \frac{\mu}{m_2}\nabla_R \cdot \left(\frac{\mu}{m_2}\nabla_R\psi + \nabla_r\psi\right) + \nabla_r \cdot \left(\frac{\mu}{m_2}\nabla_R\psi + \nabla_r\psi\right)$$

Expanding:

$$= \left(\frac{\mu}{m_2}\right)^2 \nabla_R^2\psi + 2\frac{\mu}{m_2}\left(\nabla_r \cdot \nabla_R\right)\psi + \nabla_r^2\psi$$

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The two-particle Hamiltonian acting on the wavefunction ψ is:

$$H\psi = -\frac{\hbar^2}{2m_1}\nabla_1^2\psi - \frac{\hbar^2}{2m_2}\nabla_2^2\psi + V(\mathbf{r}_1, \mathbf{r}_2)\psi$$

Using the transformations of $\nabla_1^2\psi$ and $\nabla_2^2\psi$:

$$
\begin{aligned}
H\psi = & -\frac{\hbar^2}{2m_1}\left[\left(\frac{\mu}{m_2}\right)^2\nabla_R^2\psi + 2\frac{\mu}{m_2}(\nabla_r\cdot\nabla_R)\psi + \nabla_r^2\psi\right] \\
& -\frac{\hbar^2}{2m_2}\left[\left(\frac{\mu}{m_1}\right)^2\nabla_R^2\psi - 2\frac{\mu}{m_1}(\nabla_r\cdot\nabla_R)\psi + \nabla_r^2\psi\right] + V(\mathbf{r})\psi
\end{aligned}
$$

Grouping terms gives:

$$
\begin{aligned}
H\psi = -\frac{\hbar^2}{2}\Bigg[& \left(\frac{\mu^2}{m_1 m_2^2} + \frac{\mu^2}{m_2 m_1^2}\right)\nabla_R^2\psi \\
& + \left(2\frac{\mu}{m_1 m_2} - 2\frac{\mu}{m_1 m_2}\right)(\nabla_r\cdot\nabla_R)\psi \\
& + \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\nabla_r^2\psi\Bigg] + V(\mathbf{r})\psi
\end{aligned}
$$

Since the cross terms cancel:

$$
2\frac{\mu}{m_1 m_2} - 2\frac{\mu}{m_1 m_2} = 0,
$$

and using:

$$
\frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1 + m_2}{m_1 m_2} = \frac{1}{\mu},
$$

and

$$
\frac{\mu^2}{m_1 m_2^2} + \frac{\mu^2}{m_2 m_1^2} = \frac{\mu^2(m_1 + m_2)}{m_1^2 m_2^2} = \frac{\mu}{m_1 m_2},
$$

we get

$$
H\psi = -\frac{\hbar^2}{2(m_1 + m_2)}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\mathbf{r})\psi = E\psi.
$$

—

Separation of variables:
Assume

$$
\psi(\mathbf{R}, \mathbf{r}) = \psi_R(\mathbf{R})\psi_r(\mathbf{r}),
$$

and divide the Schrödinger equation by $\psi_R\psi_r$:

$$
-\frac{\hbar^2}{2(m_1 + m_2)}\frac{1}{\psi_R}\nabla_R^2\psi_R - \frac{\hbar^2}{2\mu}\frac{1}{\psi_r}\nabla_r^2\psi_r + V(\mathbf{r}) = E.
$$

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