Confidence interval:

def: I is a confidence interval of θ with level $100 \times (1-\alpha)\%$ if

$$P(\theta \in I) = 1-\alpha$$

Rk: I is a random interval

Construction on some examples

- $X_1, \ldots, X_n$ be n r.v with $\mathcal{N}(\mu, \sigma^2)$
  we assume that σ^2 is known!

$\rightarrow$ we want a confidence interval for μ.

an unbaised estimator of μ is $\overline{X}_n$

$$\overline{X}_n \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right) \quad (\text{true distribution})$$

$$\underbrace{\sqrt{n}\,\frac{\overline{X}_n - \mu}{\sigma}}_{T_n =} \sim \mathcal{N}(0, 1)$$

aim: to find $a(X_1, \dots, X_n), b(X_1, \dots, X_n)$ such that

$$P\big(a(X_1, \dots, X_n) \leq \mu \leq b(X_1, \dots, X_n)\big) = 1 - \alpha$$

If we find t_1 and t_2 such that

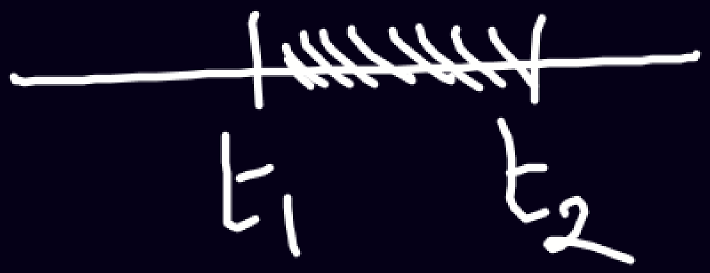
$$P(t_1 \le T_n \le t_2) = 1-\alpha$$

$$\Leftrightarrow P\left(t_1 \le \sqrt{n}\,\frac{\bar{X}_n - \mu}{\sigma} \le t_2\right) = 1-\alpha$$

$$\Leftrightarrow P\Big(\underbrace{\bar{X}_n - \frac{\sigma}{\sqrt{n}}\, t_2}_{a(X_1,\dots,X_n)} \le \mu \le \underbrace{\bar{X}_n - \frac{\sigma}{\sqrt{n}}\, t_1}_{b(X_1,\dots,X_n)}\Big) = 1-\alpha$$

How to define t_1 and t_2

$P(t_1 \le T_n \le t_2) = 1-\alpha$ with $T_n \sim \mathcal{N}(0;1)$



$\iff P(T_n \le t_1) + P(T_n \ge t_2) = \alpha$

$\iff P(T_n \le t_1) = \alpha_1 \quad P(T_n \ge t_2) = \alpha_2$

with $\alpha_1 + \alpha_2 = \alpha$

$\alpha_1 \ge 0, \ \alpha_2 \ge 0$

one choice to get some bilateral confidence interval is to take

$$\alpha_1 = \alpha_2 = \frac{\alpha}{2}$$

with this choice, $t_1 = -t_2$

$\Rightarrow$ to find t_2, we say that $P(\mathcal{N}(0;1) > t_2)$

$= 1 - P(\mathcal{N}(0;1) \leq t_2)$

$$\Rightarrow P(\mathcal{N}(0;1) \leq t_2) = 1 - \frac{\alpha}{2}$$

$\Rightarrow$ t_2 : quantile of order $1-\frac{\alpha}{2}$ associated to a $\mathcal{N}(0;1)$

Confidence interval :

$$\left[\overline{X}_n - \frac{\sigma}{\sqrt{n}} t_2 ; \overline{X}_n + \frac{\sigma}{\sqrt{n}} t_2\right]$$