# Review of Probability Theory and Linear Algebra

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# Outline

- Probability Theory

- Linear Algebra

# What is probability?

- Classical definition: $\mathbb{P}(A) = \dfrac{N_A}{N}$

...with $N$ mutually exclusive equally likely outcomes,
$N_A$ of which result in the occurrence of $A$. *Laplace, 1814*

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- Subjective probability: $\mathbb{P}(A)$ is a degree of belief.

  ...gives meaning to $\mathbb{P}(\text{“tomorrow will rain”})$.

# Key concepts: Sample space and events

- Sample space $\mathcal{X}$ = set of possible outcomes of a random experiment.

  Examples:
  - Tossing two coins: $\mathcal{X} = \{HH, TH, HT, TT\}$
  - Roulette: $\mathcal{X} = \{1, 2, ..., 36\}$
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- An event is a subset of $\mathcal{X}$

  Examples:
  - "exactly one H in 2-coin toss": $A = \{TH, HT\} \subset \{HH, TH, HT, TT\}$.
  - "odd number in the roulette": $B = \{1, 3, ..., 35\} \subset \{1, 2, ..., 36\}$.
  - "drawn a $\heartsuit$ card": $C = \{A\heartsuit, 2\heartsuit, ..., K\heartsuit\} \subset \{A\clubsuit, ..., K\diamondsuit\}$

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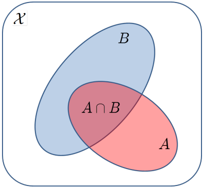
- From these axioms, many results can be derived. Examples:

  - $\mathbb{P}(\emptyset) = 0$
  - $C \subset D \;\Rightarrow\; \mathbb{P}(C) \leq \mathbb{P}(D)$
  - $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$



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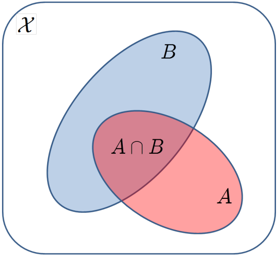

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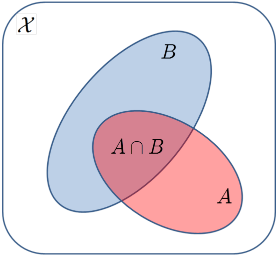


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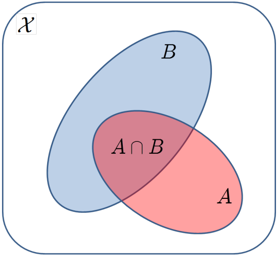


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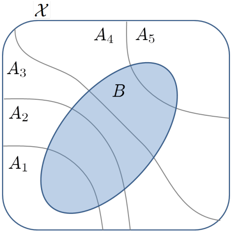
- Relationship with conditional probabilities:

$$A \perp\!\!\!\perp B \;\Leftrightarrow\; \mathbb{P}(A|B) = \mathbb{P}(A)$$

# Bayes Theorem

- Law of total probability: if $A_1, ..., A_n$ are a partition of $\mathcal{X}$

$$\mathbb{P}(B) = \sum_i \mathbb{P}(B|A_i)\mathbb{P}(A_i)$$




- Bayes' theorem: if $A_1, ..., A_n$ are a partition of $\mathcal{X}$

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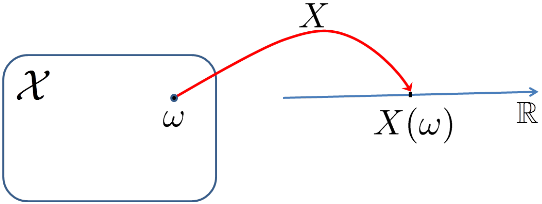
# Random Variables

- A (real) random variable (RV) is a function: $X : \mathcal{X} \to \mathbb{R}$



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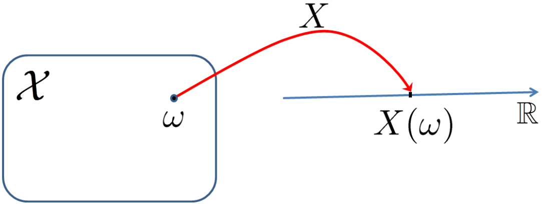


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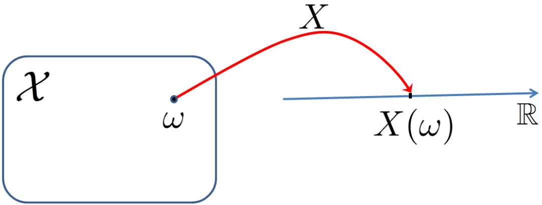


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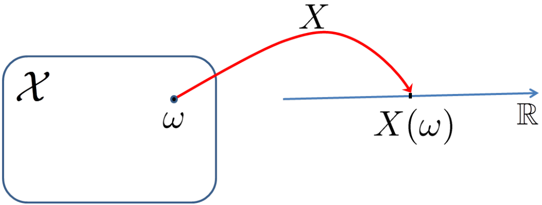


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  - Example: number of head in tossing two coins,
    $\mathcal{X} = \{HH, HT, TH, TT\}$,
    $X(HH) = 2$, $X(HT) = X(TH) = 1$, $X(TT) = 0$.
    Range of $X = \{0, 1, 2\}$.

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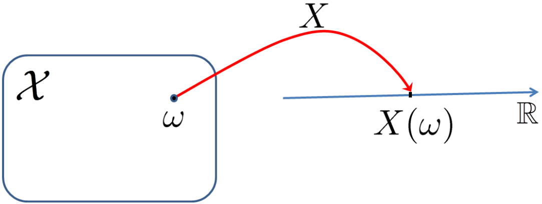


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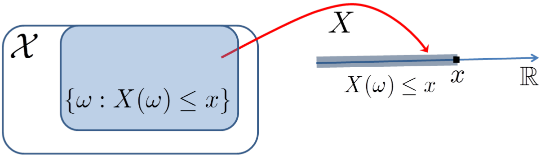
  - Example: distance traveled by a tossed coin; range of $X = \mathbb{R}_+$.

# Random Variables: Distribution Function

- Distribution function: $F_X(x) = \mathbb{P}(\{\omega \in \mathcal{X} : X(\omega) \leq x\})$



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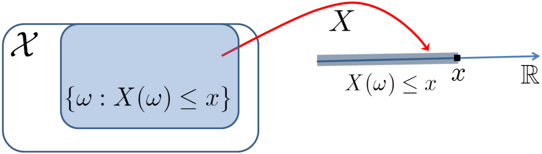


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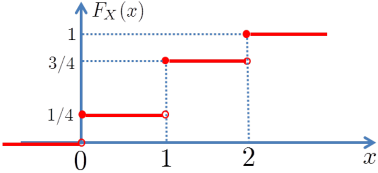


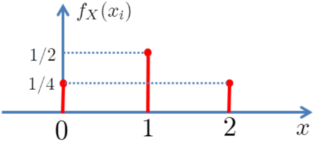

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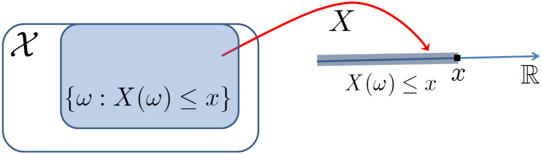


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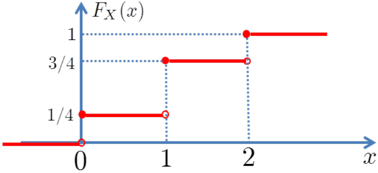


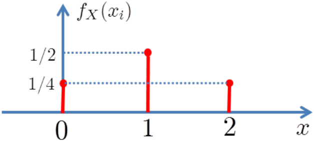


- Probability mass function (discrete RV): $f_X(x) = \mathbb{P}(X = x)$,

$$F_X(x) = \sum_{x_i \leq x} f_X(x_i).$$

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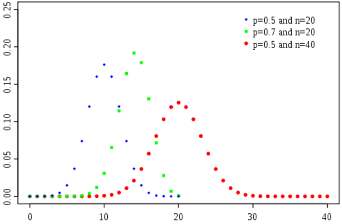
Binomial coefficients:

$$\binom{n}{x} = \frac{n!}{(n - x)! \, x!}$$



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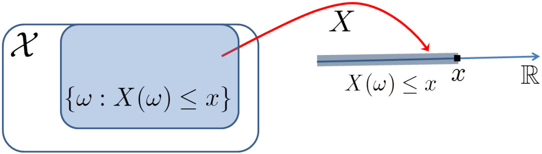

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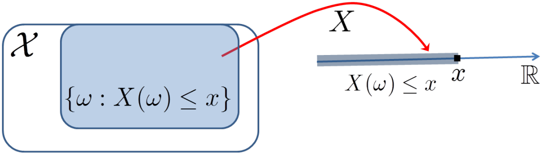
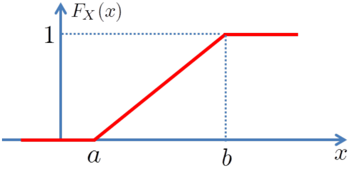
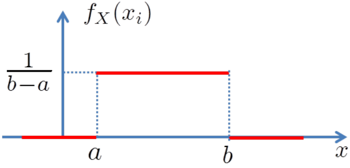



- Example: continuous RV with uniform distribution on $[a, b]$.



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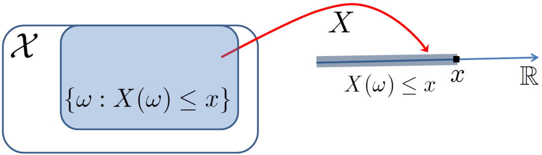


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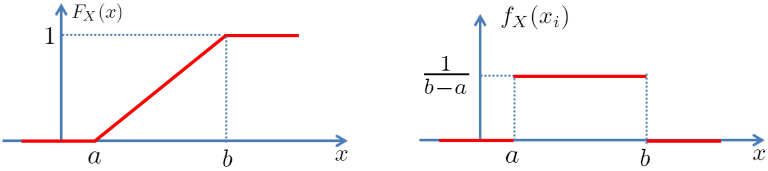


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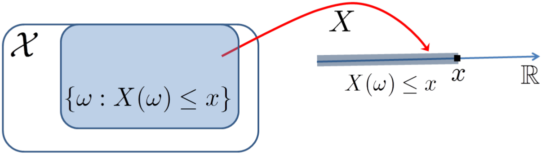


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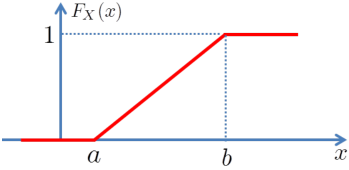


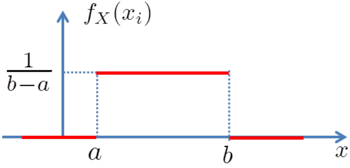


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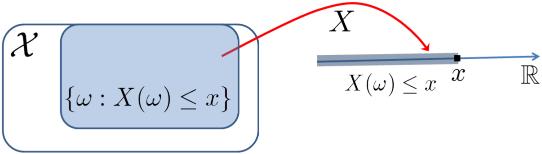


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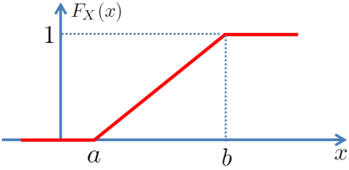


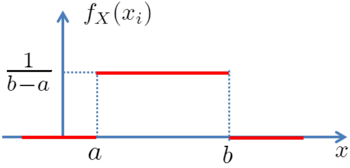


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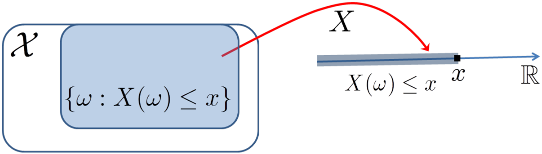


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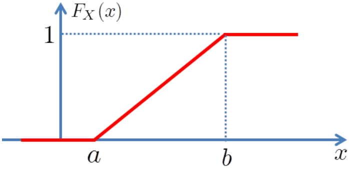


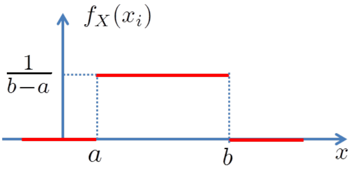


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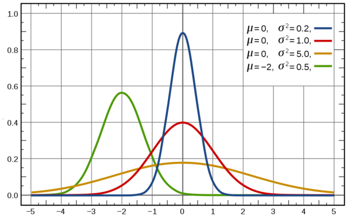
- Gaussian: $f_X(x) = \mathcal{N}(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\,\pi\,\sigma^2}} e^{-\frac{(x-\mu)^2}{2\,\sigma^2}}$



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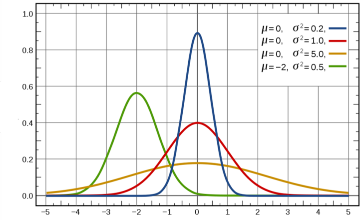


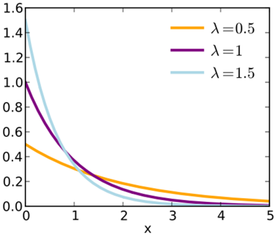


- Exponential: $f_X(x) = \text{Exp}(x; \lambda) = \begin{cases} \lambda e^{-\lambda x} & \Leftarrow \quad x \geq 0 \\ 0 & \Leftarrow \quad x < 0 \end{cases}$

# Expectation of Random Variables

- Expectation: $\mathbb{E}(X) = \begin{cases} \displaystyle\sum_i x_i f_X(x_i) & X \text{ discrete on } \{x_1, ...x_K\} \\ \displaystyle\int_{-\infty}^{\infty} x\, f_X(x)\, dx & X \text{ continuous} \end{cases}$

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- Linearity of expectation: $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$.

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- Probability as expectation of indicator, $\mathbf{1}_A(x) = \begin{cases} 1 & \Leftarrow & x \in A \\ 0 & \Leftarrow & x \notin A \end{cases}$

$$\mathbb{P}(X \in A) = \int_A f_X(x)\, dx = \int \mathbf{1}_A(x)\, f_X(x)\, dx = \mathbb{E}(\mathbf{1}_A(X))$$

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- Independence: $X \perp\!\!\!\perp Y \;\Leftrightarrow\; f_{X,Y}(x,y) = f_X(x)\,f_Y(y)$.

# Conditionals and Bayes' Theorem

- Conditional pmf (discrete RVs):
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- Also valid in the mixed case (*e.g.*, $X$ continuous, $Y$ discrete).

# Joint, Marginal, and Conditional Probabilities: An Example

- A pair of binary variables $X, Y \in \{0, 1\}$, with joint pmf:

| $f_{X,Y}(x,y)$ | $Y = 0$ | $Y = 1$ |
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| $X = 0$ | 1/5 | 2/5 |
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| $f_{X\|Y}(x\|y)$ | $Y=0$ | $Y=1$ |
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| $X=1$ | 1/3 | 3/7 |

| $f_{Y\|X}(y\|x)$ | $Y=0$ | $Y=1$ |
|---|---|---|
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# An Important Multivariate RV: Multinomial

- Multinomial: $X = (X_1, ..., X_K)$, $X_i \in \{0, ..., n\}$, such that $\sum_i X_i = n$,

$$f_X(x_1, ..., x_K) = \begin{cases} \binom{n}{x_1\ x_2\ \cdots\ x_K} p_1^{x_1}\, p_2^{x_2} \cdots p_k^{x_K} & \Leftarrow \quad \sum_i x_i = n \\ 0 & \Leftarrow \quad \sum_i x_i \neq n \end{cases}$$

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- Example: tossing $n$ independent fair dice, $p_1 = \cdots = p_6 = 1/6$.
  $x_i$ = number of outcomes with $i$ dots. Of course, $\sum_i x_i = n$.

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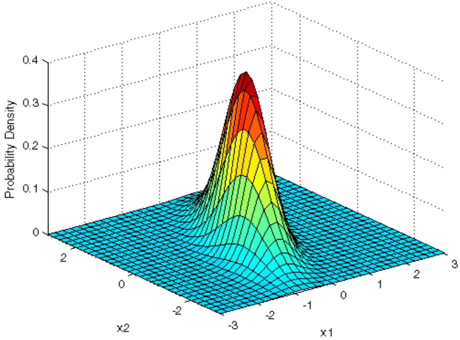

# Statistical Inference 101 (A Lot More Tomorrow)

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- Posterior mean (PM) criterion (for continuous RVs):

$$\widehat{x}_{\text{PM}} = \mathbb{E}(X|Y=y) = \int x\, f_{X|Y}(x|y)\, dx$$

# Statistical Inference: Example

- Observed $n$ i.i.d. (independent identically distributed) Bernoulli RVs: $Y = (Y_1, ..., Y_n)$, with $Y_i \in \{0, 1\}$.
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- Example: $n = 10$, observed $y = (1, 1, 1, 0, 1, 0, 0, 1, 1, 1)$, $\widehat{x}_{\mathrm{ML}} = 7/10$.

# Statistical Inference: Example (Continuation)

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- Likelihood: $f_{Y|X}\big((y_1, ..., y_n)|x\big) = \prod_{i=1}^{n} x^{y_i}(1-x)^{1-y_i} = x^{\sum_i y_i}(1-x)^{n-\sum_i y_i}$

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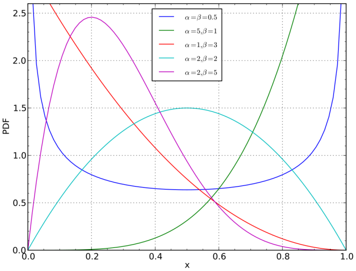

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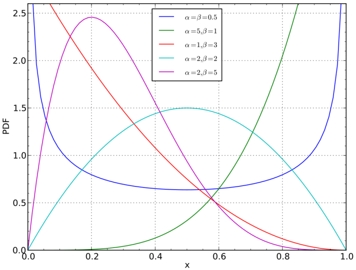

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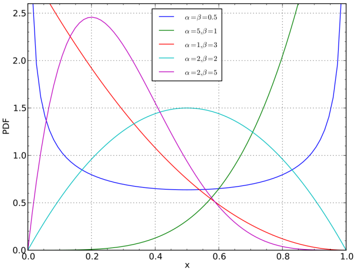
  - Example: $\alpha = 2$, $\beta = 5$, $n = 10$,
    $y = (1,1,1,0,1,0,0,1,1,1)$,

    $$\widehat{x}_{\text{MAP}} = \frac{8}{15} \quad \left(\text{recall } \widehat{x}_{\text{ML}} = 7/10\right)$$



# Agenda

- ~~Probability Theory~~ ✓
- Linear Algebra

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- It can be solved as $x = A^{-1}b$ (if $A^{-1}$ exists).

# Notation: Matrices and Vectors

- $A \in \mathbb{R}^{m \times n}$ is a matrix with $m$ rows and $n$ columns.

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- Transpose of sum: $(A+B)^T = A^T + B^T$.

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# Special Matrices

- The identity matrix $I \in \mathbb{R}^{n \times n}$ is a square matrix such that

$$I_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

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- There are many algorithms to compute $A^{-1}$; general case, computational cost $O(n^3)$.

# Quadratic Forms and Positive (Semi-)Definite Matrices

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# Quadratic Forms and Positive (Semi-)Definite Matrices

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$$x^T A x = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j}\, x_i\, x_j \;\in\; \mathbb{R}$$

  is called a quadratic form.

- A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive semi-definite (PSD) if, for any $x \in \mathbb{R}^n$, $x^T A x \geq 0$.

- A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive definite (PD) if, for any $x \in \mathbb{R}^n$, $(x \neq 0) \Rightarrow x^T A x > 0$.

- Matrix $A \in \mathbb{R}^{n \times n}$ is PSD $\Leftrightarrow$ all $\lambda_i(A) \geq 0$.

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# Agenda

- ~~Probability Theory~~ ✓
- ~~Linear Algebra~~ ✓

Enjoy LxMLS!