Ward_Abigail_HW5

Abigail Ward

7/17/2021

##question 1

Let $X_1, X_2, \ldots X_n$ be independent, identically distributed random variables with mean μ and variance σ^2

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$(X_i - \bar{X})^2 = X_i^2 - 2X_i\bar{X} + \bar{X}^2$$

$$E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = E\left[\sum_{i=1}^{n}(X_i^2 - 2X_i\bar{X} + \bar{X}^2)\right]$$

$$E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = E\left[\sum_{i=1}^{n} X_i^2 - \sum_{i=1}^{n} 2\,X_i\bar{X} + \sum_{i=1}^{n}\bar{X}^2\right]$$

$$E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = E\left[\sum_{i=1}^{n} X_i^2\right] - 2E\left[\sum_{i=1}^{n}\bar{X}\,X_i\right] + E\left[\sum_{i=1}^{n}\bar{X}^2\right]$$

##Question 2

$$Var[X_i] = E[X_i^2] - E[X_i]^2$$

$$Var[X_i] = \sigma^2$$

$$E[X_i] = \mu$$

$$\sigma^2 = E[X_i^2] - E[X_i]^2$$

$$\sigma^2 = E[X_i^2] - \mu^2$$

$$E[X_i^2] = \sigma^2 + \mu^2$$

If $\mu = 6$ and $\sigma^2 = 12$, then $ E [X_i^2]= 12+ 6^2$ so $ E [X_i^2]= 48$

```
f2<-function(x){x^2*dgamma(x,shape=3,scale=2)}
integrate(f2,0,Inf)$value

## [1] 48
```

Question 3

$$E[X_i^2] = \sigma^2 + \mu^2$$

$$E\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} E\,[Y_i]$$

$$E\left[\sum_{i=1}^{n} X_i^2\right] = E[X_i^2] = \sigma^2 + \mu^2$$

Question 4

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \mu$$

$$Var[X_i] = \frac{\sigma^2}{n}$$

$$E\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} E\,[Y_i]$$

$$E\left[\sum_{i=1}^{n} \bar{X}^2\right] = \sum_{i=1}^{n} E\,[\bar{X}^2]$$

$$E\left[\sum_{i=1}^{n} \bar{X}^2\right] = n[\mu^2 + \frac{\sigma^2}{n}]$$

$$E\left[\sum_{i=1}^{n} \bar{X}^2\right] = n\mu^2 + \sigma^2$$

Question 5

$$E\left[\sum_{i=1}^{n} \bar{X}\,X_i\right] = E\left[\bar{X}\sum_{i=1}^{n} X_i\right]$$

$$\sum_{i=1}^{n} X_i = n\bar{X}$$

$$E\left[\sum_{i=1}^{n} \bar{X}\,X_i\right] = E[n\bar{X}^2]$$

Question 6

$$E\left[\sum_{i=1}^{n} X_i^2\right] = n(\sigma^2 + \mu^2)$$

$$E\left[\sum_{i=1}^{n} \bar{X}^2\right] = n\left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$E\left[\sum_{i=1}^{n} \bar{X}\,X_i\right] = E[n\bar{X}^2] = n\left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$E\left[\sum_{i=1}^{n} X_i^2\right] - 2E\left[\sum_{i=1}^{n} \bar{X}\,X_i\right] + E\left[\sum_{i=1}^{n} \bar{X}^2\right] = n(\sigma^2 + \mu^2) - 2n\left(\frac{\sigma^2}{n} + \mu^2\right) + n\left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$E\left[\sum_{i=1}^{n} X_i^2\right] - 2E\left[\sum_{i=1}^{n} \bar{X}\,X_i\right] + E\left[\sum_{i=1}^{n} \bar{X}^2\right] = n\sigma^2 + n\mu^2 - 2\sigma^2 - 2n\mu^2 + \sigma^2 + n\mu^2$$

$$E\left[\sum_{i=1}^{n} X_i^2\right] - 2E\left[\sum_{i=1}^{n} \bar{X}\,X_i\right] + E\left[\sum_{i=1}^{n} \bar{X}^2\right] = (n-1)\sigma^2$$

Question 7

$$E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = \frac{1}{n-1}E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right]$$

$$\frac{1}{n-1}E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = \frac{1}{n-1}\left[E\left[\sum_{i=1}^{n} X_i^2\right] - 2E\left[\sum_{i=1}^{n} \bar{X}\,X_i\right] + E\left[\sum_{i=1}^{n} \bar{X}^2\right]\right]$$

$$\frac{1}{n-1}\left[E\left[\sum_{i=1}^{n} X_i^2\right] - 2E\left[\sum_{i=1}^{n} \bar{X}\,X_i\right] + E\left[\sum_{i=1}^{n} \bar{X}^2\right]\right] = \frac{1}{n-1}(n-1)\sigma^2$$

$$E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = \sigma^2$$

Question 8

Part 1

$$f_x(0) = \frac{1}{2}$$

$$f_x(1) = \frac{1}{2}$$

$$f_y(2) = \frac{1}{3}$$

$$f_y(3) = \frac{2}{3}$$

$f_{x,y}(0,2) = f_x(0) * f_y(2) = \frac{1}{2} * \frac{1}{3} = \frac{1}{6}$ for a

$f_{x,y}(0,3) = f_x(0) * f_y(3) = \frac{1}{2} * \frac{2}{3} = \frac{1}{3}$ for b and c

$f_{x,y}(1,2) = f_x(1) * f_y(2) = \frac{1}{2} * \frac{1}{3} = \frac{1}{6}$ for d

$f_{x,y}(1,3) = f_x(1) * f_y(3) = \frac{1}{2} * \frac{2}{3} = \frac{1}{3}$ for e and f

The random variables are independent because the probability of both events is the same as the probability of a single event occurring.

Part 2 $f_x(0) = \frac{1}{2}$

$f_x(1) = \frac{1}{2}$

$f_y(2) = \frac{1}{3}$

$f_y(3) = \frac{2}{3}$

$f_{x,y}(0,2) = f_x(0) * f_y(2) = \frac{1}{2} * \frac{1}{3} = \frac{1}{6}$ for a and d

$f_{x,y}(0,3) = f_x(0) * f_y(3) = \frac{1}{2} * \frac{2}{3} = \frac{1}{3}$ for e

$f_{x,y}(1,3) = f_x(1) * f_y(3) = \frac{1}{2} * \frac{2}{3} = \frac{1}{3}$ for b and c and f

The random variables are not independent because the probability of both events is not the same as the probability of a single event occurring.

##Question 9 weighted average $xa + (1-x)b$ with $x \in [0,1]$ minimizes the variance of $xa + (1-x)b$

$var[xa + (1-x)b] = var[xa] + var[(1-x)b]$

$= xvar[a] + (1-x)^2 var[b]$

$= x^2 v + (1-x)^2 w$

$= (w+v)x^2 - 2wx + w$

$$a = (w + v)$$

$$b = -2w$$

$$x = -\frac{b}{2a}$$

$$x = \frac{2w}{2(w + v)}$$

$$x = \frac{w}{(w + v)}$$