Comp Econ Homework 6

Date: 11th March

Due Date: 24th March

Analytical Exercises

In all of the following, let $\mathbf{X}$ be an element of $\mathcal{M}(n \times k)$ with linearly independent columns.

Exercise 1. Show that $\mathbf{X}'\mathbf{X}$ is invertible.[1]

Exercise 2. Let $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. Show that if $k = n$, then $\mathbf{P}$ is the identity. (Justify your steps.) Explain the geometric intuition in terms of the orthogonal projection theorem.

Exercise 3. Show that the projection of $\mathbf{y} \in \mathbb{R}^n$ onto $\operatorname{span}\{\mathbf{1}\}$ is the mean of the elements of $\mathbf{y}$. (Here $\operatorname{span}\{\mathbf{1}\}$ is the set of vectors in $\mathbb{R}^n$ that can be created as linear combinations of the elements (in fact element) of the singleton set consisting of a vector of ones.)

Exercise 4. It's well-known that for regressions with a constant term, the vector of residuals always sums to zero. To prove this, let $\mathbf{y} \in \mathbb{R}^n$ and let $\mathbf{X} \in \mathcal{M}(n \times k)$ have linearly independent columns. Let $S = \operatorname{span}(\mathbf{X})$, $\mathbf{P} = \operatorname{proj} S$ and $\mathbf{M} = \mathbf{I} - \mathbf{P}$. Let $\hat{\mathbf{u}} = \mathbf{M}\mathbf{y}$. Show that if $\mathbf{X}$ has a constant column (i.e., a vector of ones, or any other constant), then elements of $\hat{\mathbf{u}}$ sum to 0. If possible, give an argument based around orthogonal projection.

Exercise 5. Show that if S is a nonempty subset of $\mathbb{R}^n$, then $S \cap S^{\perp} = \{\mathbf{0}\}$.

Submit all of the preceding exercises as a PDF document. Please be sure to add your name at the top. For the file name please use the format

`firstname_lastname_analysis.pdf`

[1] Hint: Show that it is positive definite using the definition of the latter. What does this mean for the determinant?

Computational Exercises

Exercise 6. Complete the notebook you find here:

- https://github.com/jstac/quantecon_nyu_2016/tree/master/homework_assignments/hw_set6/ols_via_projection

When you submit, please add your name to the top and also change the file name to

`firstname_lastname_ols.ipynb`

The notebook in the previous exercise is in Python so you should probably go ahead and use it. For the next exercise you can use either Python or Julia.

Exercise 7. In this exercise, your job is to project $\mathbf{y}$ onto the column space of $\mathbf{X}$ in three different ways, where

$$\mathbf{y} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} \quad \text{and} \quad \mathbf{X} = \begin{pmatrix} 1 & 0 \\ 0 & -6 \\ 2 & 2 \end{pmatrix}$$

First use the ordinary expression for the projection. That is

$$\mathbf{P}\mathbf{y} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

Next, implement your own version of Gram–Schmidt, based of the algorithm given in the lecture. Use it to convert $\mathbf{X}$ to $\mathbf{U}$, a matrix with orthonormal columns and the same column space as $\mathbf{X}$. Now calculate the projection as $\mathbf{P}\mathbf{y} = \mathbf{U}\mathbf{U}'\mathbf{y}$. Third, use the same expression $\mathbf{P}\mathbf{y} = \mathbf{U}\mathbf{U}'\mathbf{y}$, but this time obtain $\mathbf{U}$ from the QR decomposition routine in either SciPy or Julia.

You should get the same vector for the projection in each case. Submit this exercise in a separate notebook, with file name

`firstname_lastname_projection.ipynb`