Comp Econ Homework 9

Date: 8th April

Due Date: 14th April

Submit the following in a single Jupyter notebook with the usual naming convention `firstname_lastname.ipynb`. Alternatively, if it's painful to write LaTeX in the notebook, then you can add a second file with the analytical solutions called `firstname_lastname_analysis.pdf`. Please make sure that you have your name at the top of each file, and that you put your file in the right folder in the homework GitHub repo.

As always, you are free to collaborate, and think through algorithms and proof solutions together. Just type up your own version at the end.

Exercise 1. Prove that for every $\mathbf{A} \in \mathcal{M}(n \times n)$ we have $\|\mathbf{A}\| = \sqrt{\rho(\mathbf{A}'\mathbf{A})}$. Hint: Use the Lagrange method for constrained optimization.

Exercise 2. Making use of Gelfand's formula, show that if $\rho(\mathbf{A}) < 1$, then $\sum_{k=0}^{\infty} \|\mathbf{A}^k\| < \infty$. In particular, show that under the stated condition there exists an $r < 1$ and a $C \in \mathbb{N}$ such that $\|\mathbf{A}^k\| \leq r^k C$ for all $k \in \mathbb{N}$.

Exercise 3. Show that if $\rho(\mathbf{A}) < 1$, then $\|\mathbf{A}^k\| \to 0$ as $k \to \infty$ *without* using Gelfand's formula. Assume that $\mathbf{A}$ is diagonalizable.

Exercise 4. Show that the set of nonnegative definite matrices is a closed subset of $(\mathcal{M}(n \times n), \|\cdot\|)$.

Exercise 5. Let $\mathbf{M}, \mathbf{A}$ be in $\mathcal{M}(n \times n)$ with $\rho(\mathbf{A}) < 1$. Let $\mathbf{X}^*$ be the unique solution to the Lyapunov equation

$$\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{A}' + \mathbf{M} \tag{1}$$

Show that

1. $\mathbf{M}$ symmetric $\implies$ $\mathbf{X}^*$ symmetric

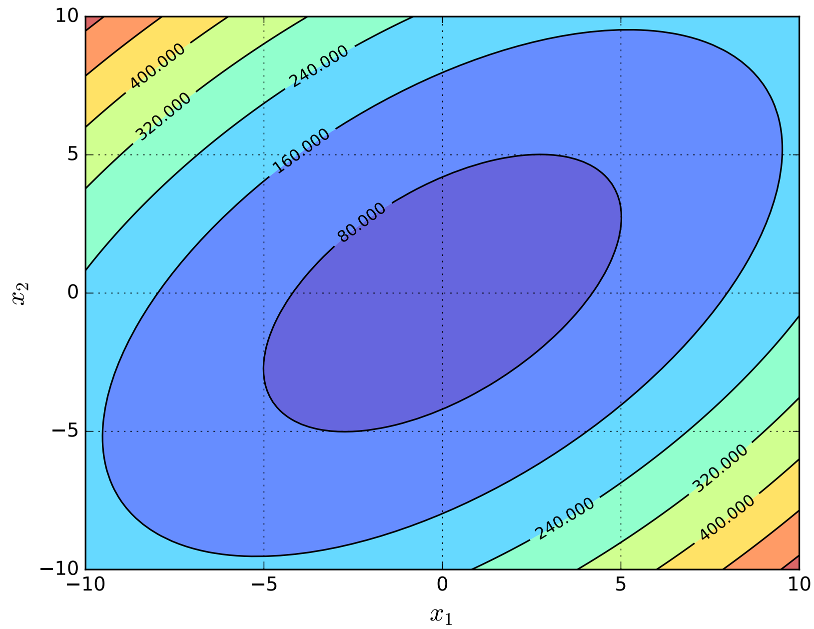

Figure 1: The equilibrium price function

2. **M** nonnegative definite $\implies$ $\mathbf{X}^*$ nonnegative definite
3. **M** positive definite $\implies$ $\mathbf{X}^*$ positive definite

Exercise 6. Write a function or class that takes the primitives of the linear quadratic asset pricing problem stated in the lectures and returns the price (or the matrix and scalar that define the price function). Your routine should test the stability condition and fail gracefully (i.e., with a useful message) when it doesn't hold.

In solving the Lyapunov equation, rather than using an existing routine, write your own function based on the iterative method suggested in the lectures. (You can test its performance by comparing output with the discrete Lyapunov solver in QuantEcon.py/QuantEcon.jl.)

Test your code in the setting of $\mathbb{R}^2$ with

$$\mathbf{A} = \begin{pmatrix} 0.8 & -0.1 \\ -0.1 & 0.8 \end{pmatrix}, \quad \mathbf{C} = \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $\beta = 0.9$. Compute the corresponding price function and plot it as a contour map. See if you can reproduce figure 1.