Topics in Computational Economics

Lecture 8

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Today's Lecture

- Metric space
- Banach space
- Fixed point theorems
- Some other stuff

Comments:

- Please pick up your homework after class
- Class projects are due June 3

Motivation

Here's a typical example of why we need functional analysis

The **risk-neutral** price of a random payoff G_{t+1} at $t+1$ is

$$p_t = \beta \, \mathbb{E} \left[G_{t+1} \,|\, \mathscr{F}_t \right]$$

Here $\beta \in (0,1)$ is a discount factor

Now suppose we're pricing a claim to a dividend flow $\{D_t\}$

The same formula gives

$$p_t = \beta \, \mathbb{E} \left[D_{t+1} + p_{t+1} \,|\, \mathscr{F}_t \right]$$

Assume $D_t = d(X_t)$ for some **state process** $\{X_t\}$

Let $q(x, \cdot)$ be the density of X_{t+1} given $X_t = x$

Guess that $p_t = p(X_t)$ for some unknown function p

If p exists, it must satisfy

$$p(X_t) = \beta \, \mathbb{E} \left[d(X_{t+1}) + p(X_{t+1}) \,|\, \mathscr{F}_t \right]$$

or

$$p(x) = \beta \int [d(y) + p(y)] q(x, y) dy \qquad (x \in \mathbb{R})$$

But does such a function always exist? Is it unique? How to compute it?

To answer these questions we use functional analysis

Metric Space

Let M be any nonempty set

A function $\rho\colon M \times M \to \mathbb{R}$ is called a **metric** on M if

1. $\rho(x,y) \geqslant 0$ and $\rho(x,y) = 0 \iff x = y$
2. $\rho(x,y) = \rho(y,x)$
3. $\rho(x,y) \leqslant \rho(x,z) + \rho(z,y)$

for any $x,y,z \in M$

Together, (M,ρ) is called a **metric space**

Example. $\mathbb{R}^n$ with $\rho(x,y) = \|x-y\|$ is a metric space

Example. Consider the **discrete metric** on $\mathbb{R}^n$ given by

$$\rho(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Let's check it satisfies the triangle inequality

Pick any $x, y, z \in \mathbb{R}^n$

We claim that $\rho(x,y) \leqslant \rho(x,z) + \rho(z,y)$

If $x = y$, the bound is trivial, so suppose not

We need to show that $1 \leqslant \rho(x,z) + \rho(z,y)$

If not, then $\rho(x,z) + \rho(z,y) = 0$, so $x = z$ and $y = z$

Hence $x = y$ — a contradiction

Standard defs:

- The **ϵ-ball** around $x \in M$ is the set

$$B_\epsilon(x) := \{y \in M : \rho(x,y) < \epsilon\}$$

- $\{x_n\} \subset M$ **converges to $x \in M$** if

$$\forall\, \epsilon > 0,\ \exists\, N \in \mathbb{N} \text{ s.t. } n \geqslant N \implies x_n \in B_\epsilon(x)$$

- $x \in A \subset M$ is called **interior** to A if

$$\exists\, \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subset A$$

Standard defs:

- $G \subset M$ called **open** if every $x \in G$ is interior to G
- $F \subset M$ called **closed** if
$$\{x_n\} \subset F \text{ and } x_n \to x \in M \quad \Longrightarrow \quad x \in F$$
- $K \subset M$ called **compact** if every sequence in K has a subsequence converging to a point in K

Example. If $(M, \rho) = (\mathbb{R}^n, \| \cdot \|)$, then K is compact if and only if K is closed and bounded

A sequence $\{x_n\} \subset M$ is called **Cauchy** if

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } n, m \geqslant N \implies \rho(x_n, x_m) < \epsilon$$

Example. If $M = \mathbb{R}$, $\rho(x, y) = |x - y|$ and $x_n = 1/n$, then $\{x_n\}$ is Cauchy

A metric space (M, ρ) called **complete** if every Cauchy sequence in M converges to some point in M

Theorem Ordinary Euclidean space $(\mathbb{R}^n, \|\cdot\|)$ is complete

Ex. Show that if $M = (0, 1]$ and $\rho(x, y) = |x - y|$, then (M, ρ) is **not** complete

Essential Facts

Let (M, ρ) be any metric space

1. A set G is open in M if and only if G^c is closed in M
2. Every singleton set in M is closed
3. Arbitrary unions and finite intersections of open sets in M are open in M
4. Arbitrary intersections and finite unions of closed sets in M are closed in M
5. If (M, ρ) is complete and $C \subset M$ is closed, then (C, ρ) is itself a complete metric space

Continuous Functions

Let (M,σ) and (Y,τ) be two metric spaces and let $f\colon M \to Y$

We call f **continuous at** $x \in M$ if

$$x_n \to x \text{ in } (M,\sigma) \quad \Longrightarrow \quad f(x_n) \to f(x) \text{ in } (Y,\tau)$$

We call f **continuous on** M if f is continuous at all $x \in M$

Theorem. $f\colon M \to Y$ is continuous on M if and only if

$$G \text{ open in } (Y,\tau) \quad \Longrightarrow \quad f^{-1}(G) \text{ open in } (M,\sigma)$$

Theorem. If f is continuous on M and $K \subset M$ is compact, then the image set $f(K)$ is compact on Y

Let S be any set and let $\mathbb{R}^S :=$ the set of all functions $f \colon S \to \mathbb{R}$

Example. If $b\mathbb{R}^S :=$ all bounded $f \in \mathbb{R}^S$ and

$$d_\infty(f,g) := \sup_{x \in S} |f(x) - g(x)|$$

then $(b\mathbb{R}^S, d_\infty)$ is a metric space

Ex. Prove it

Fact. $(b\mathbb{R}^S, d_\infty)$ is complete

Fact. $cb\mathbb{R}^S :=$ all continuous $f \in b\mathbb{R}^S$ is closed in $(b\mathbb{R}^S, d_\infty)$

Hence $(cb\mathbb{R}^S, d_\infty)$ is complete

Example.

Let $\mathscr{C}$:= all continuously differentiable $f\colon [-1,1] \to \mathbb{R}$

As before let

$$d_\infty(f,g) := \sup_{x \in S} |f(x) - g(x)|$$

The set $\mathscr{C}$ is **not** a closed subset of $(b\mathbb{R}^S, d_\infty)$

Ex.

- Show that $d_\infty(f_n, f) \to 0$ as $n \to \infty$ when

$$f_n(x) := (x^2 + 1/n)^{1/2} \quad \text{and} \quad f(x) := |x|$$

- Conclude that $(\mathscr{C}, d_\infty)$ is not closed

Example. Let S be any countable set, let $p \geqslant 1$ and let

$$\ell_p(S) := \left\{ \text{all } f \in \mathbb{R}^S \text{ with } \sum_{x \in S} |f(x)|^p < \infty \right\}$$

Let

$$d_p(f,g) := \|f-g\|_p := \left\{ \sum_{x \in S} |f(x) - g(x)|^p \right\}^{1/p}$$

Theorem.

1. d_p is a metric on $\ell_p(S)$
2. $(\ell_p(S), d_p)$ is complete

Proof: See Cheney, section 8.7

Note that $(\ell_p(S), d_p)$ is an extension of ordinary Euclidean space

Indeed, suppose $S = \{1, \ldots, n\}$ and let $p = 2$

For $f, g \in \ell_2(S)$, we have

$$d_2(f,g) = \left\{ \sum_{i \in S} |f(i) - g(i)|^2 \right\}^{1/2}$$

$$= \left\{ \sum_{i=1}^{n} |f_i - g_i|^2 \right\}^{1/2} \qquad (f_i := f(i),\ g_i := g(i))$$

This is the Euclidean distance between **vectors** (f_i) and (g_i)

Closed subset of $\ell_p(S)$ include

1. the positive cone $\{f \in \ell_p(S) : f \geqslant 0\}$
2. the sphere $\{f \in \ell_p(S) : \|f\|_p \leqslant 1\}$

- Remark: $f \geqslant 0$ means $f(s) \geqslant 0$, for all $s \in S$

Ex. (Proof of 1) Show that for any $\{f_n\} \subset \ell_p(S)$, we have

$$d_p(f_n, f) \to 0 \implies f_n(s) \to f(s) \text{ in } \mathbb{R}, \forall\, s \in S$$

Show that $\{x_n\} \subset \mathbb{R}$, $x_n \to x$ and $x_n \geqslant 0$ for all n implies $x \geqslant 0$

Conclude that 1 holds

Equivalence

Two metrics ρ and ρ' on M are called **equivalent** if there exist constants K, L such that, for all $x, y \in M$

$$\rho'(x,y) \leqslant K\rho(x,y) \quad \text{and} \quad \rho(x,y) \leqslant L\rho'(x,y)$$

Fact. If (M,ρ) and (M,ρ') are equivalent, then they share the same

- convergent sequences
- Cauchy sequences
- open sets
- closed sets
- compact sets, etc.

Ex. Prove it

Banach Space

Most metric spaces of interest in econ arise as either

- a Banach space
- some subset of a Banach space
- some transformation of a Banach space

Relative to metric spaces, Banach spaces have

- an additional algebraic structure
- some nice additional properties

Let V be a nonempty set with a notion of

- addition (a map $+$ from $V \times V$ to V)
- scalar multiplication (a map $\cdot$ from $\mathbb{R} \times V$ to V)

Called a **vector space** if, $\forall\, u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$,

1. $u + (v + w) = (u + v) + w$
2. $u + v = v + u$
3. exists an element $0 \in V$ s.t. $u + 0 = u$ for all $u \in V$
4. $\forall\, u \in V$, $\exists\, v \in V$ s.t. $u + v = 0$
5. $\alpha(\beta u) = (\alpha\beta)u$
6. $1u = u$
7. $\alpha(u + v) = \alpha u + \alpha v$
8. $(\alpha + \beta)u = \alpha u + \beta u$

Example. $\mathbb{R}^n$ with usual notions of addition and scalar multiplication

- the zero element is the origin

Example. The set of $n \times n$ real matrices with usual notions of addition and scalar multiplication

- the zero element is the matrix of zeros

Example. The set $\mathbb{R}^S$ of all functions $f \colon S \to \mathbb{R}$ with

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) = \alpha f(x)$$

- the zero element is the function identically zero

If V is a vector space and $U \subset V$ satisfies

$$\alpha, \beta \in \mathbb{R} \text{ and } u, v \in U \implies \alpha u + \beta v \in U$$

then U is called a **linear subspace of V**

Example. If S is any set, then $b\mathbb{R}^S$ is a linear subspace of $\mathbb{R}^S$

Example. If S is a metric space, then $c\mathbb{R}^S$ is a linear subspace of $\mathbb{R}^S$

Fact. If V is a vector space and U is a linear subspace of V, then U itself is a vector space

Let V be a vector space

A map $\|\cdot\|: V \to \mathbb{R}$ is called a **norm** on V if, for all $u, v \in V$ and $\alpha \in \mathbb{R}$,

1. $\|u\| \geqslant 0$ and $\|u\| = 0$ if and only if $u = 0$
2. $\|\alpha u\| = |\alpha|\|u\|$
3. $\|u + v\| \leqslant \|u\| + \|v\|$

Example. The Euclidean norm on $\mathbb{R}^n$ is a norm in this sense

Ex. Show: $\rho(u, v) = \|u - v\|$ is a metric on V (induced metric)

The pair $(V, \|\cdot\|)$ is called a **normed linear space**

If complete under the induced metric then called a **Banach space**

Example. $\mathbb{R}^n$ with the Euclidean norm is a Banach space

Example. The set $b\mathbb{R}^S$ is a Banach space under the norm

$$\|f\|_\infty := \sup_{x \in S} |f(x)|$$

Example. The set $cb\mathbb{R}^S$ of continuous functions in $b\mathbb{R}^S$ is a Banach space under the same norm

Example. The space $\ell_p(S)$ is a Banach space under the norm

$$\|f\|_p := \left[\sum_{x \in S} |f(x)|^p \right]^{1/p}$$

Equivalence

Norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on V are called **equivalent** if $\exists\, K, L \geqslant 0$ such that

$$\|v\|_1 \leqslant K\|v\|_2 \quad \text{and} \quad \|v\|_2 \leqslant L\|v\|_1 \quad \text{for all } v \in V$$

Ex. Show that if two norms on V are equivalent then so are the induced metrics

Thm. Any two norms on finite dimensional space are equivalent

Hence, if two metrics on $\mathbb{R}^n$ are both induced by a norm, then they share

- open sets, closed sets,
- compact sets, etc.

Fixed Point Theorems

Let (M, ρ) be a metric space and let $T \colon M \to M$

A **fixed point** of T is a point $x^* \in M$ such that $Tx^* = x^*$

Example. If $f(x) = x^2$ on $\mathbb{R}$, then 0 and 1 are fixed points

Example. If $f(x) = x + 1$ on $\mathbb{R}$, then f has no fixed points on $\mathbb{R}$

The map T is called a **uniform contraction** on (M,ρ) if

$$\exists\, \alpha < 1 \quad \text{such that} \quad \rho(Tx,Ty) \leqslant \alpha\rho(x,y), \quad \forall x,y \in M$$

Example. $f(x) = \alpha x + b$ on metric space $(\mathbb{R}, |\cdot|)$ with $|\alpha| < 1$, since

$$|f(x) - f(y)| = |\alpha x - \alpha y| = |\alpha||x - y|$$

Fact. Every uniform contraction T is continuous on M

Proof: If $x_n \to x$ in (M,ρ), then

$$\rho(Tx_n, Tx) \leqslant \alpha\rho(x_n, x) \to 0$$

Fact. If T is a uniform contraction on (M, ρ) and $x \in M$, then the trajectory $\{T^k x\}$ is Cauchy

Sketch of proof: Along the trajectory $\{T^k x\}$ from x, we have

$$\begin{aligned}
\rho(T^{k+1}x, T^k x) &\leqslant \alpha \rho(T^k x, T^{k-1}x) \\
&\leqslant \alpha^2 \rho(T^{k-1}x, T^{k-2}x) \\
&\vdots \\
&\leqslant \alpha^k \rho(Tx, x)
\end{aligned}$$

Banach's Fixed Point Theorem

Theorem. If (M,ρ) is complete and T is a uniform contraction, then T has a unique fixed point x^* in M and

$$\rho(T^k x, x^*) \leqslant \alpha^k \rho(x, x^*), \quad \forall k \in \mathbb{N}, \forall x \in M$$

Proof sketch: Pick any $x \in M$

The sequence $\{T^k x\}$ is Cauchy and hence converges to some x^*

The point x^* is a fixed point, since

$$Tx^* = T(\lim_k T^k x) = \lim_k T(T^k x) = \lim_k T^{k+1} x = x^*$$

The fixed point is unique (proof in weaker setting below)

Higher Order Contractions

Theorem. If (M, ρ) is complete and T^n is a uniform contraction for some $n \in \mathbb{N}$, then

1. T has a unique fixed point x^* in M
2. $T^k x \to x^*$ as $k \to \infty$ for all $x \in M$ (i.e., **globally stable** f.p.)

Partial proof: By Banach fixed point theorem, T^n has a fixed point x^* in M

In fact x^* is also a fixed point of T, since

$$\rho(Tx^*, x^*) = \rho(TT^n x^*, T^n x^*) = \rho(T^n T x^*, T^n x^*) \leqslant \alpha \rho(Tx^*, x^*)$$

$$\therefore \quad \rho(Tx^*, x^*) = 0$$

Dropping Uniformity

T is called **contracting** on (M, ρ) if

$$\rho(Tx, Ty) < \rho(x, y), \quad \forall x \neq y$$

Clearly weaker than

$$\exists\, \alpha < 1 \text{ s.t. } \rho(Tx, Ty) \leqslant \alpha \rho(x, y), \quad \forall x, y \in M \tag{1}$$

Note: Contracting does not imply existence of a fixed point

Example.

- $M = [0, \infty)$ with $\rho(x, y) = |x - y|$
- $Tx = x + e^{-x}$

Let T be contracting on (M, ρ)

Theorem. If M is compact, then T has a unique, globally stable fixed point $x^* \in M$

Proof of uniqueness: Suppose that

- x, y are both fixed points
- x and y are distinct

Then

$$\rho(Tx, Ty) = \rho(x, y) \quad \text{and} \quad \rho(Tx, Ty) < \rho(x, y)$$

Contradiction

Proof of existence:

Define $r\colon M \to \mathbb{R}$ by $r(x) = \rho(Tx, x)$

Ex. Show that r is continuous

- Hint: First show $|\rho(x,y) - \rho(x',y')| \leqslant \rho(x,x') + \rho(y,y')$

Since M is compact, r has a minimizer x^* in M

We claim that $Tx^* = x^*$

Must be true, because if not then

$$r(Tx^*) = \rho(T^2x^*, Tx^*) < \rho(Tx^*, x^*) = r(x^*)$$

Contradiction

Lagrange Stability and Contractions

What if M is not compact?

A map $T\colon M \to M$ is called **Lagrange stable** if the trajectory of x is precompact for every $x \in M$.

- $B \subset M$ called **precompact** if it is contained in a compact set

Example.

- Lagrange stability of $Tx = ax$ depends on $|a| \leqslant 1$ or > 1

Theorem. If T is Lagrange stable and contracting on (M, ρ), then T has a unique and globally stable fixed point in M

Sketch of proof:

Pick $x \in M$

Let $\Gamma(x)$ be the closure of the trajectory of x

By Lagrange stability, $\Gamma(x)$ is compact

Ex: Show that T maps $\Gamma(x)$ into itself

Note that T is contracting on $\Gamma(x)$

Hence a fixed point exists in $\Gamma(x)$

The fixed point is unique because T is contracting on all of M