Topics in Computational Economics

Background on Linear Algebra

John Stachurski

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Contents

- Vectors and norms
- Spans and linear subspaces
- Linear independence
- Linear systems of equations

Vector Space

An n-vector $\mathbf{x}$ is a tuple of n real numbers:

$$\mathbf{x} = (x_1, \ldots, x_n) \quad \text{where} \quad x_i \in \mathbb{R} \text{ for each } i$$

We can also write $\mathbf{x}$ vertically, like so:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$\mathbb{R}^n :=$ set of all n-vectors

The **sum** of $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$ defined by

$$\mathbf{x} + \mathbf{y} :=: \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} := \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

The **scalar product** of $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$ defined by

$$\alpha \mathbf{x} = \alpha \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} := \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

Coincides with NumPy / Julia notation

```
In [1]: import numpy as np

In [2]: x = np.array((2, 4, 6))

In [3]: y = np.array((10, 10, 10))

In [4]: x + y  # Vector addition
Out[4]: array([12, 14, 16])

In [6]: 2 * x  # Scalar multiplication
Out[6]: array([ 4,  8, 12])
```

Subtraction performed element by element, analogous to addition

$$\mathbf{x} - \mathbf{y} := \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \\ \vdots \\ x_N - y_N \end{pmatrix}$$

Def can be given in terms of addition and scalar multiplication:

$$\mathbf{x} - \mathbf{y} := \mathbf{x} + (-1)\mathbf{y}$$

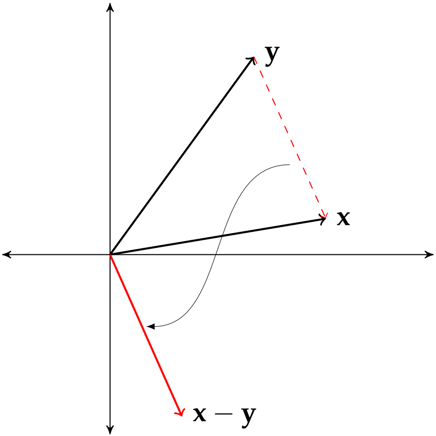


Figure: Difference between vectors

A **linear combination** of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ in $\mathbb{R}^n$ is a vector

$$\mathbf{y} = \alpha_1 \mathbf{x}_1 + \cdots + \alpha_k \mathbf{x}_k$$

where $\alpha_1, \ldots, \alpha_k$ are scalars

- New vectors from old using purely linear operations

Example.

$$0.5 \begin{pmatrix} 6.0 \\ 2.0 \\ 8.0 \end{pmatrix} + 3.0 \begin{pmatrix} 0 \\ 1.0 \\ -1.0 \end{pmatrix} = \begin{pmatrix} 3.0 \\ 4.0 \\ 1.0 \end{pmatrix}$$

Inner Product and Norm

The **inner product** of two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ is

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^{n} x_i y_i$$

Fact. For any $\alpha, \beta \in \mathbb{R}$ and any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, we have

1. $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$
2. $\langle \alpha \mathbf{x}, \beta \mathbf{y} \rangle = \alpha \beta \langle \mathbf{x}' \mathbf{y} \rangle$
3. $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$

The (Euclidean) **norm** of $\mathbf{x} \in \mathbb{R}^n$ is defined as

$$\|\mathbf{x}\| := \sqrt{\mathbf{x}'\mathbf{x}}$$

Fact. For any $\alpha \in \mathbb{R}$ and any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

1. $\|\mathbf{x}\| \geqslant 0$ and $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$
2. $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$
3. $\|\mathbf{x}+\mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|$ (**triangle inequality**)
4. $|\mathbf{x}'\mathbf{y}| \leqslant \|\mathbf{x}\|\|\mathbf{y}\|$ (**Cauchy-Schwarz inequality**)

Linear Independence

Consider a set of vectors $X := \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$

We can always attain $\mathbf{0}$ as a linear combination of these vectors

Proof: $0\mathbf{x}_1 + \cdots + 0\mathbf{x}_k = \mathbf{0}$

The set X is called linearly independent when this is the only way

That is, $X \subset \mathbb{R}^N$ is called **linearly independent** if

$$\alpha_1 \mathbf{x}_1 + \cdots + \alpha_k \mathbf{x}_k = \mathbf{0} \quad \Longrightarrow \quad \alpha_1 = \cdots = \alpha_k = 0$$

Let $X \subset \mathbb{R}^n$ be any nonempty set

Set of all possible linear combinations of elements of X is called the **span** of X, denoted by $\operatorname{span}(X)$

For finite $X := \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ the span can be expressed as

$$\operatorname{span}(X) := \left\{ \text{ all } \sum_{i=1}^{k} \alpha_k \mathbf{x}_k \text{ such that } (\alpha_1, \ldots, \alpha_K) \in \mathbb{R}^k \right\}$$

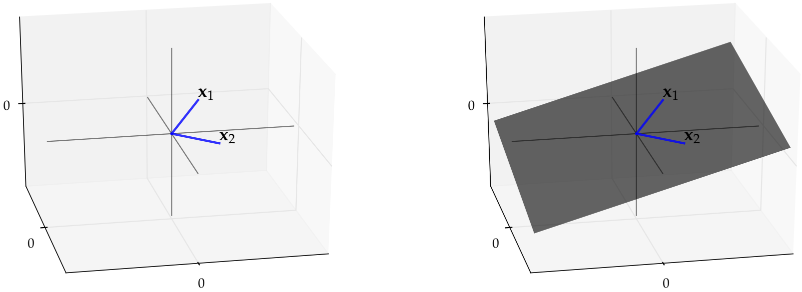


Figure: The span of two vectors in $\mathbb{R}^2$

Example

Consider the vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \subset \mathbb{R}^n$, where

$$\mathbf{e}_1 := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 := \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \mathbf{e}_n := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

That is, $\mathbf{e}_i$ has all zeros except for a 1 as the i-th element

Vectors $\mathbf{e}_1, \ldots, \mathbf{e}_N$ called the **canonical basis vectors** of $\mathbb{R}^N$

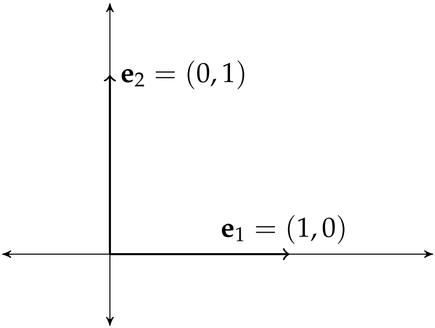


Figure: Canonical basis vectors in $\mathbb{R}^2$

Fact. The span of $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is equal to all of $\mathbb{R}^n$

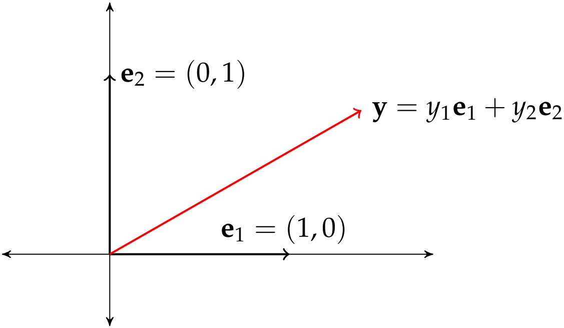
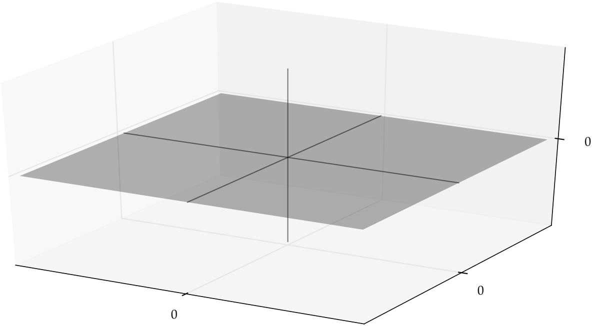


Figure: Canonical basis vectors in $\mathbb{R}^2$

Example. Consider the set

$$P := \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1, x_2 \in \mathbb{R}\}$$

Graphically, $P =$ flat plane in $\mathbb{R}^3$, where height coordinate $= 0$



Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the canonical basis vectors in $\mathbb{R}^3$

Claim: $\text{span}\{\mathbf{e}_1, \mathbf{e}_2\} = P$

Proof:

Let $\mathbf{x} = (x_1, x_2, 0)$ be any element of P

We can write $\mathbf{x}$ as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$$

In other words, $P \subset \text{span}\{\mathbf{e}_1, \mathbf{e}_2\}$

Conversely (check it) we have $\text{span}\{\mathbf{e}_1, \mathbf{e}_2\} \subset P$

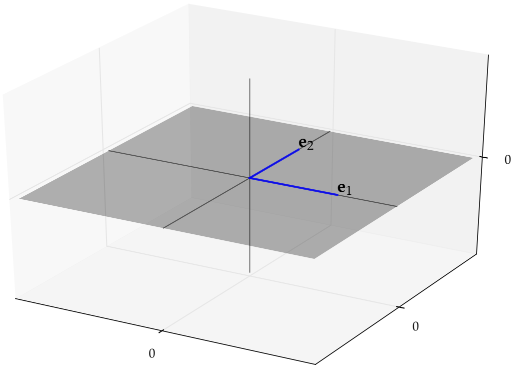


Figure: $\text{span}\{\mathbf{e}_1, \mathbf{e}_2\} = P$

Linear Subspaces

A nonempty $S \subset \mathbb{R}^N$ called a **linear subspace** of $\mathbb{R}^N$ if

$$\mathbf{x}, \mathbf{y} \in S \;\text{ and }\; \alpha, \beta \in \mathbb{R} \quad \implies \quad \alpha\mathbf{x} + \beta\mathbf{y} \in S$$

In other words, $S \subset \mathbb{R}^N$ is "closed" under vector addition and scalar multiplication

Note: Sometimes we just say **subspace**...

Example. Fix $\mathbf{a} \in \mathbb{R}^N$ and let $A := \{\mathbf{x} \in \mathbb{R}^N : \langle \mathbf{a}, \mathbf{x} \rangle = 0\}$

The set A is a linear subspace of $\mathbb{R}^N$

Proof: Let $\mathbf{x}, \mathbf{y} \in A$ and let $\alpha, \beta \in \mathbb{R}$

We must show that $\mathbf{z} := \alpha \mathbf{x} + \beta \mathbf{y} \in A$

Equivalently, that $\langle \mathbf{a}, \mathbf{z} \rangle = 0$

True because

$$\langle \mathbf{a}, \mathbf{z} \rangle = \langle \mathbf{a}, \alpha \mathbf{x} + \beta \mathbf{y} \rangle = \alpha \langle \mathbf{a}, \mathbf{x} \rangle + \beta \langle \mathbf{a}, \mathbf{y} \rangle = 0 + 0 = 0$$

Ex. Show: $\operatorname{span}(Z)$ is a linear subspace, $\forall$ nonempty $Z \subset \mathbb{R}^N$

Fact. If S and S' are two linear subspaces of $\mathbb{R}^N$, then $S \cap S'$ is also a linear subspace of $\mathbb{R}^N$.

Proof: Let S and S' be two linear subspaces of $\mathbb{R}^N$

Fix $\mathbf{x}, \mathbf{y} \in S \cap S'$ and $\alpha, \beta \in \mathbb{R}$

We claim that $\mathbf{z} := \alpha \mathbf{x} + \beta \mathbf{y} \in S \cap S'$

- Since $\mathbf{x}, \mathbf{y} \in S$ and S is a linear subspace we have $\mathbf{z} \in S$
- Since $\mathbf{x}, \mathbf{y} \in S'$ and S' is a linear subspace we have $\mathbf{z} \in S'$

Therefore $\mathbf{z} \in S \cap S'$

Other examples of linear subspaces

- Lines through the origin in $\mathbb{R}^2$ and $\mathbb{R}^3$
- Planes through the origin in $\mathbb{R}^3$

Ex. Let S be a linear subspace of $\mathbb{R}^N$. Show that

1. $\mathbf{0} \in S$
2. If $X \subset S$, then $\operatorname{span}(X) \subset S$
3. $\operatorname{span}(S) = S$

Theorem. If linear subspace S is spanned by k vectors, then S every linearly independent subset of S contains no more than k vectors

Linear Functions

A function $T \colon \mathbb{R}^n \to \mathbb{R}^k$ is called **linear** if

$$T(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y})$$

for all $\alpha, \beta \in \mathbb{R}$ and all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$

Let $\mathcal{M}(n \times k) :=$ all $n \times k$ real matrices

Fact. For every linear $T \colon \mathbb{R}^n \to \mathbb{R}^k$, there exists a unique $\mathbf{A} \in \mathcal{M}(n \times k)$ such that

$$T\mathbf{x} = \mathbf{A}\mathbf{x} \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^k$$

Basis of a Subspace

Let S be a linear subspace of $\mathbb{R}^n$

Recall that $\mathbf{b}_1, \ldots, \mathbf{b}_k \in S$ form a **basis** of S if

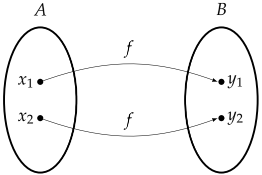
$$\forall\, \mathbf{x} \in S,\ \exists \text{ unique scalars } \alpha_1, \ldots, \alpha_k \text{ s.t. } \mathbf{x} = \sum_{i=1}^{k} \alpha_i \mathbf{b}_i$$

Theorem. $\mathbf{b}_1, \ldots, \mathbf{b}_k \in S$ form a basis of S if and only if they are linearly independent and their span equals S

Solving Equations

We want to find the x that solves $f(x) = y$

Ideal case: f is a bijection



Equivalent:

- f is a bijection
- each $y \in B$ has a unique preimage
- $f(x) = y$ has a unique solution x for each y

Linear Equations

Now consider linear system $\mathbf{Ax} = \mathbf{b}$, where

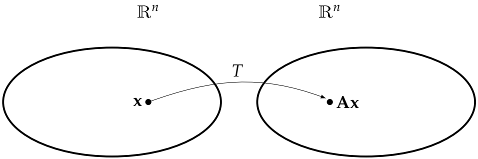
- $\mathbf{A} \in \mathcal{M}(n \times n)$
- $\mathbf{b} \in \mathbb{R}^n$
- We seek a solution $\mathbf{x} \in \mathbb{R}^n$

Note: number of equations $=$ number of unknowns

Does such a solution exist?

If so is it unique?

Let T be defined by $T\mathbf{x} = \mathbf{A}\mathbf{x}$



Equivalent:

1. $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for any given $\mathbf{b}$
2. $T\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for any given $\mathbf{b}$
3. T is a bijection

So when is T a bijection?

Note that T is a linear map

Can that help us determine when T is a bijection?

Is there anything special about linear bijections?

Linear Equations

Fact. If T is a linear function from $\mathbb{R}^n$ to $\mathbb{R}^n$, then the following are all equivalent:

1. T is a bijection
2. T is onto
3. T is one-to-one
4. $V := \{T\mathbf{e}_1, \ldots, T\mathbf{e}_n\}$ is linearly independent

If hold we say that T is **nonsingular** ($\iff$ linear bijection)

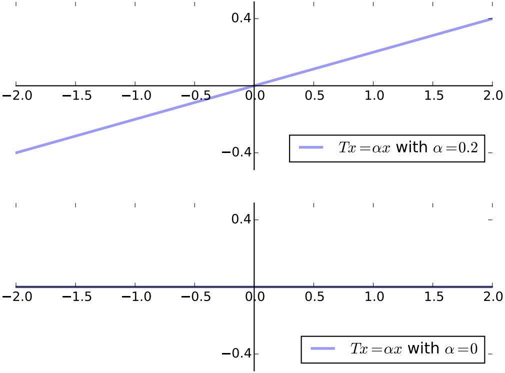


Figure: The case of $n = 1$, nonsingular and singular

Let's use this information to study $\mathbf{A}\mathbf{x} = \mathbf{b}$

Unique solution always exists if $T\mathbf{x} = \mathbf{A}\mathbf{x}$ is a bijection

We have conditions for when T is a bijection

But we want conditions stated in terms of $\mathbf{A}$, not T

Fact. For $\mathbf{A} \in \mathcal{M}(n \times n)$, all of the following are equivalent:

1. For each $\mathbf{b} \in \mathbb{R}^n$, the equation $\mathbf{Ax} = \mathbf{b}$ has a unique solution
2. For each $\mathbf{b} \in \mathbb{R}^n$, the equation $\mathbf{Ax} = \mathbf{b}$ has a solution
3. If $\mathbf{Ax} = \mathbf{Ay}$, then $\mathbf{x} = \mathbf{y}$
4. The columns of $\mathbf{A}$ are linearly independent

If hold we say that $\mathbf{A}$ is **nonsingular**

Also equivalent:

- The linear span of the columns of $\mathbf{A}$ is all of $\mathbb{R}^n$
- $\operatorname{rank}(\mathbf{A}) = n$

All equivalent ways of saying that $T\mathbf{x} = \mathbf{A}\mathbf{x}$ is a bijection

Example. For condition 2 the equivalence is

for each $\mathbf{b} \in \mathbb{R}^n$, the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution

$$\iff \forall\, \mathbf{b} \in \mathbb{R}^n,\ \exists\, \mathbf{x} \in \mathbb{R}^n \text{ s.t. } \mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\iff \forall\, \mathbf{b} \in \mathbb{R}^n,\ \exists\, \mathbf{x} \in \mathbb{R}^n \text{ s.t. } T\mathbf{x} = \mathbf{b}$$

$$\iff T \text{ is onto}$$

Since T is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$,

$$\iff T \text{ is a bijection}$$

Example. For condition 3 the equivalence is

$$\text{if } \mathbf{Ax} = \mathbf{Ay}, \text{ then } \mathbf{x} = \mathbf{y}$$

$$\iff \text{if } T\mathbf{x} = T\mathbf{y}, \text{ then } \mathbf{x} = \mathbf{y}$$

$$\iff T \text{ is one-to-one}$$

Since T is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$,

$$\iff T \text{ is a bijection}$$

Now consider condition 4:

$$\text{The columns of } \mathbf{A} \text{ are linearly independent}$$

Let $\mathbf{e}_i$ be the i-th canonical basis vector in $\mathbb{R}^n$

Observe that $\mathbf{A}\mathbf{e}_i = \mathrm{col}_i(\mathbf{A})$

$$\therefore \quad T\mathbf{e}_i = \mathrm{col}_i(\mathbf{A})$$

$$\therefore \quad V := \{T\mathbf{e}_1, \ldots, T\mathbf{e}_n\} = \text{ columns of } \mathbf{A}$$

And V is linearly independent if and only if T is a bijection

Inverse Matrices

Let $\mathbf{A} \in \mathcal{M}(n \times n)$. If

$$\mathbf{B} \in \mathcal{M}(n \times n) \quad \text{and} \quad \mathbf{AB} = \mathbf{BA} = \mathbf{I}$$

Then

- $\mathbf{B}$ is called the **inverse** of $\mathbf{A}$, and written $\mathbf{A}^{-1}$
- $\mathbf{A}$ is called **invertible**

Fact. $\mathbf{A} \in \mathcal{M}(n \times n)$ is nonsingular if and only if it is invertible

Remark

- $\mathbf{A}^{-1}$ is just the matrix corresponding to the linear map T^{-1}

Fact. If $\mathbf{A} \in \mathcal{M}(n \times n)$ is nonsingular and $\mathbf{b} \in \mathbb{R}^n$, then

$$\mathbf{x}_b := \mathbf{A}^{-1}\mathbf{b}$$

is the unique solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$

Proof: Since $\mathbf{A}$ is nonsingular we already know that

- a solution exists
- the solution is unique

To show that $\mathbf{x}_b$ is the solution we must show that $\mathbf{A}\mathbf{x}_b = \mathbf{b}$

To see this, observe that

$$\mathbf{A}\mathbf{x}_b = \mathbf{A}\mathbf{A}^{-1}\mathbf{b} = \mathbf{I}\mathbf{b} = \mathbf{b}$$

Fact. If $\mathbf{A}, \mathbf{B} \in \mathcal{M}(n \times n)$ are both nonsingular, then

- $\mathbf{AB}$ is also nonsingular and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Long but interesting proof: Let T, U be the linear maps corresponding to $\mathbf{A}$ and $\mathbf{B}$

True:

- $T \circ U$ is the linear map corresponding to $\mathbf{AB}$
- linear $\circ$ linear $=$ linear
- bijection $\circ$ bijection $=$ bijection, $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

$$\therefore \quad (T \circ U)^{-1} = U^{-1} \circ T^{-1}$$

$$\therefore \quad (\mathbf{AB})^{-1} \text{ exists, equals } \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Example. Consider a one good linear market system

$$q = a - bp \qquad \text{(demand)}$$

$$q = c + dp \qquad \text{(supply)}$$

Treating q and p as the unknowns, let's write in matrix form as

$$\begin{pmatrix} 1 & b \\ 1 & -d \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

A unique solution exists whenever the columns are linearly independent

- means that $(b, -d)$ is not a scalar multiple of $\mathbf{1}$
- means that $b \neq -d$

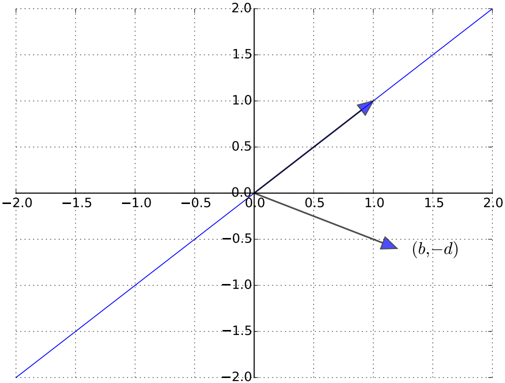


Figure: $(b, -d)$ is not a scalar multiple of $\mathbf{1}$

Let's set $a = 5$, $b = 2$, $c = 1$, $d = 1.5$

- nonsingular, unique solution exists

The matrix system is

$$\mathbf{A} := \begin{pmatrix} 1 & 2 \\ 1 & -1.5 \end{pmatrix}, \quad \mathbf{x} := \begin{pmatrix} q \\ p \end{pmatrix}, \quad \mathbf{b} := \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Solutions should be

- $p = 8/7 \approx 1.14285$
- $q = 19/7 \approx 2.71428$

```
In [1]: from scipy.linalg import inv

In [2]: A = [[1, 2],
             [1, -1.5]]
In [3]: b = [5, 1]

In [4]: q, p = inv(A) @ b

In [5]: p
Out[5]: 1.1428571428571428

In [6]: q
Out[6]: 2.7142857142857144
```

Typically it's better to use solve

```
In [7]: from scipy.linalg import solve

In [8]: solve(A, b)
Out[8]: array([ 2.71428571,  1.14285714])

In [9]: A @ solve(A, b)
Out[9]: array([ 5.,  1.])

In [10]: b
Out[10]: [5, 1]
```

Why is `x = solve(A, b)` better than `x = inv(A) @ b` ?

- The former uses LU decomposition
- More numerically stable

Intuition:

- In the case of `x = inv(A) @ b` we need to compute
  - the $n \times n$ object `inv(A)`
  - the $n \times 1$ object `x`
- In `x = solve(A, b)` the inverse is not computed

Less floating point operations, more accuracy

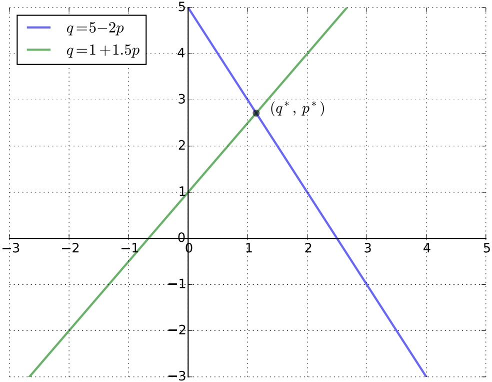


Figure: Equilibrium (p^*, q^*) in the one good case

Of course if $\mathbf{A}$ is singular we hit problems

```
In [1]: import numpy as np
In [2]: from scipy.linalg import solve

In [3]: A = [[0, 2, 4],
   ...:       [1, 4, 8],
   ...:       [0, 3, 6]]

In [4]: b = (1, 2, 0)

In [5]: A, b = np.asarray(A), np.asarray(b)

In [6]: solve(A, b)
```

Output:

```
LinAlgError        Traceback (most recent call last)
<ipython-input-8-4fb5f41eaf7c> in <module>()
----> 1 solve(A, b)
/home/john/anaconda/lib/python2.7/site-packages/scipy/linal
     97           return x
     98       if info > 0:
---> 99           raise LinAlgError("singular matrix")
    100       raise ValueError('illegal value in %d-th argume
LinAlgError: singular matrix
```

The problem is that $\mathbf{A}$ is singular (not nonsingular)

- In particular, $\mathrm{col}_3(\mathbf{A}) = 2\,\mathrm{col}_2(\mathbf{A})$

Determinants

Let $S(N)$ be set of all bijections from $\{1, \ldots, N\}$ to itself

For $\pi \in S(N)$ we define the **signature** of π as

$$\operatorname{sgn}(\pi) := \prod_{m<n} \frac{\pi(m) - \pi(n)}{m - n}$$

The **determinant** of $N \times N$ matrix $\mathbf{A}$ is then given as

$$\det(\mathbf{A}) := \sum_{\pi \in S(N)} \operatorname{sgn}(\pi) \prod_{n=1}^{N} a_{\pi(n)n}$$

- You **don't** need to understand or remember this...

Important facts concerning the determinant

Fact. If $\mathbf{I}$ is the $N \times N$ identity, $\mathbf{A}$ and $\mathbf{B}$ are $N \times N$ matrices and $\alpha \in \mathbb{R}$, then

1. $\det(\mathbf{I}) = 1$
2. $\mathbf{A}$ is nonsingular if and only if $\det(\mathbf{A}) \neq 0$
3. $\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$
4. $\det(\alpha \mathbf{A}) = \alpha^N \det(\mathbf{A})$
5. $\det(\mathbf{A}^{-1}) = (\det(\mathbf{A}))^{-1}$

```
In [1]: import numpy as np

In [2]: A = np.random.randn(2, 2)  # Random matrix

In [3]: A
Out[3]:
array([[-0.70120551,  0.57088203],
       [ 0.40757074, -0.72769741]])

In [4]: np.linalg.det(A)
Out[4]: 0.27759063032043652

In [5]: 1.0 / np.linalg.det(np.linalg.inv(A))
Out[5]: 0.27759063032043652
```