Topics in Computational Economics

lecture 10

John Stachurski

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Today's Lecture

Some linear / quadratic problems:

- The spectral norm
- Dynamics of moments in VARs
- Convergence of moments
- A linear-quadratic asset pricing model

Comments on HW7

All assignments have equal weighting

Please make sure your notebook can be converted to PDF

Consider using docstrings

My preference: The code should look like the math (see, e.g., [3])

Aim for proofs where every step is clear (see Annex)

Metrizing $\mathcal{M}(n \times k)$

Recall that $\mathcal{M}(n \times k)$ is the vector space of $n \times k$ real matrices

Now we want to add metric/topological properties

- When is matrix $\mathbf{A}$ "close" to matrix $\mathbf{B}$?
- When does $\mathbf{A}_n$ converge to $\mathbf{A}$?
- What does $\sum_{n=1}^{\infty} \mathbf{A}_n$ mean?

To this end, we introduce a norm on $\mathcal{M}(n \times k)$

The Spectral Norm

Given $\mathbf{A} \in \mathcal{M}(n \times k)$, the **spectral norm** of $\mathbf{A}$ is

$$\|\mathbf{A}\| := \sup \left\{ \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|} \, : \, \mathbf{x} \in \mathbb{R}^k, \; \mathbf{x} \neq \mathbf{0} \right\}$$

- LHS is the spectral norm of $\mathbf{A}$
- RHS is ordinary Euclidean vector norms

Below we often just say **norm** of $\mathbf{A}$

Ex. Show that in the supremum we can restrict attention to $\mathbf{x}$ such that $\|\mathbf{x}\| = 1$ without changing the value

Properties of the Spectral Norm

The spectral norm is in fact a norm on $\mathcal{M}(n \times k)$

Fact. For all $\mathbf{A}, \mathbf{B} \in \mathcal{M}(n \times k)$,

1. $\|\mathbf{A}\| \geqslant 0$ and $\|\mathbf{A}\| = 0 \iff \mathbf{A} = \mathbf{0}$
2. $\|\alpha \mathbf{A}\| = |\alpha| \|\mathbf{A}\|$ for any scalar α
3. $\|\mathbf{A} + \mathbf{B}\| \leqslant \|\mathbf{A}\| + \|\mathbf{B}\|$

Another immediate property of the norm is that

$$\|\mathbf{A}\mathbf{x}\| \leqslant \|\mathbf{A}\| \cdot \|\mathbf{x}\| \qquad \forall \, \mathbf{x} \in \mathbb{R}^k$$

Fact. If $\mathbf{AB}$ is well defined, then $\|\mathbf{AB}\| \leqslant \|\mathbf{A}\| \|\mathbf{B}\|$

Proof: Let $\mathbf{A} \in \mathcal{M}(n \times k)$, let $\mathbf{B} \in \mathcal{M}(k \times j)$ and let $\mathbf{x} \in \mathbb{R}^j$

We have

$$\|\mathbf{ABx}\| \leqslant \|\mathbf{A}\| \cdot \|\mathbf{Bx}\| \leqslant \|\mathbf{A}\| \cdot \|\mathbf{B}\| \cdot \|\mathbf{x}\|$$

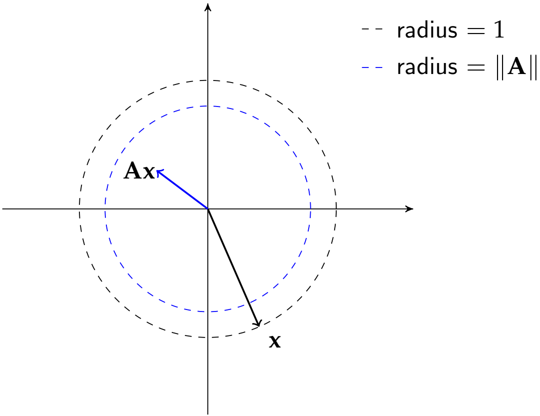
$$\therefore \quad \frac{\|\mathbf{ABx}\|}{\|\mathbf{x}\|} \leqslant \|\mathbf{A}\| \cdot \|\mathbf{B}\|$$

Called the **submultiplicative property**

Implication: $\|\mathbf{A}^j\| \leqslant \|\mathbf{A}\|^j$ for any $j \in \mathbb{N}$ and $\mathbf{A} \in \mathcal{M}(n \times n)$

If $\|\mathbf{A}\| \leqslant 1$ then $\mathbf{A}$ is called **nonexpansive**

If $\|\mathbf{A}\| < 1$ then $\mathbf{A}$ is called **contractive**



Fact. $(\mathcal{M}(n \times k), \| \cdot \|)$ is a Banach space

Proof: Vector space properties already stated

Norm properties shown on last slide

Remains to show that the space is complete

But every finite dimensional normed linear space is complete

Distance, Convergence, etc.

Once we have a norm, we have an induced metric

$$d(\mathbf{A}, \mathbf{B}) = \|\mathbf{A} - \mathbf{B}\|$$

All metric properties defined in terms of this distance

For example, let $\{\mathbf{A}_j\}$ and $\mathbf{A}$ be in $\mathcal{M}(n \times k)$

If $\|\mathbf{A}_j - \mathbf{A}\| \to 0$ then we say that $\mathbf{A}_j$ **converges** to $\mathbf{A}$

Similarly,

$$\sum_{j=1}^{\infty} \mathbf{A}_j = \mathbf{B} \quad \iff \quad \lim_{J \to \infty} \left\| \sum_{j=1}^{J} \mathbf{A}_j - \mathbf{B} \right\| = 0$$

For $\mathbf{A} \in \mathcal{M}(n \times n)$, the **spectral radius** is

$$\rho(\mathbf{A}) := \max\{|\lambda| : \lambda \text{ an eigenvalue of } \mathbf{A}\}$$

Ex. Show that, for all $\mathbf{A} \in \mathcal{M}(n \times n)$, we have

1. $\|\mathbf{A}\| = \sqrt{\rho(\mathbf{A}'\mathbf{A})}$
2. $\|\mathbf{A}'\| = \|\mathbf{A}\|$ and $\rho(\mathbf{A}') = \rho(\mathbf{A})$

Gelfand's formula states that, for all $\mathbf{A} \in \mathcal{M}(n \times n)$,

$$\|\mathbf{A}^k\|^{1/k} \to \rho(\mathbf{A}) \quad \text{as} \quad k \to \infty$$

Ex. Use Gelfand's formula to show that

$$\rho(\mathbf{A}) < 1 \implies \|\mathbf{A}^k\| \to 0$$

Neumann Series Lemma

Let $\mathbf{A} \in \mathcal{M}(n \times n)$

Fact. (**Neumann series lemma.**) If $\rho(\mathbf{A}) < 1$, then $\mathbf{I} - \mathbf{A}$ is nonsingular and

$$(\mathbf{I} - \mathbf{A})^{-1} = \sum_{j=0}^{\infty} \mathbf{A}^j$$

Example. If $\rho(\mathbf{A}) < 1$, then $\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}$ has the unique solution

$$\mathbf{x}^* = \sum_{j=0}^{\infty} \mathbf{A}^j \mathbf{b}$$

Proof of the NSL

Ex. Show that $B_J := \sum_{j=0}^{J} \mathbf{A}^j$ is Cauchy and hence $\sum_{j=0}^{\infty} \mathbf{A}^j$ exists

Now observe that $(\mathbf{I} - \mathbf{A}) \sum_{j=0}^{\infty} \mathbf{A}^j = \mathbf{I}$, since

$$
\begin{aligned}
\left\| (\mathbf{I} - \mathbf{A}) \sum_{j=0}^{\infty} \mathbf{A}^j - \mathbf{I} \right\| &= \left\| (\mathbf{I} - \mathbf{A}) \lim_{J \to \infty} \sum_{j=0}^{J} \mathbf{A}^j - \mathbf{I} \right\| \\
&= \lim_{J \to \infty} \left\| (\mathbf{I} - \mathbf{A}) \sum_{j=0}^{J} \mathbf{A}^j - \mathbf{I} \right\| \\
&= \lim_{J \to \infty} \left\| \mathbf{A}^{J+1} \right\| = 0
\end{aligned}
$$

Global Stability of Vector-Valued Systems

Let $\mathbf{A} \in \mathcal{M}(n \times n)$ and consider the dynamic model

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{b}, \quad \mathbf{x}_0 \text{ given}$$

When is $U\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}$ globally stable in $(\mathbb{R}^n, \| \cdot \|)$?

We know that if $\rho(\mathbf{A}) < 1$, then U has a unique fixed point

$$\mathbf{x}^* = \sum_{j=0}^{\infty} \mathbf{A}^j \mathbf{b}$$

How about global stability?

Simple algebra gives

$$U^k \mathbf{x} = \mathbf{A}^k \mathbf{x} + \mathbf{A}^{k-1} \mathbf{b} + \cdots + \mathbf{b}$$

Hence, for any $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$, we have

$$\begin{aligned}
\|U^k \mathbf{x} - U^k \mathbf{y}\| &= \left\| \mathbf{A}^k \mathbf{x} - \mathbf{A}^k \mathbf{y} \right\| \\
&= \left\| \mathbf{A}^k (\mathbf{x} - \mathbf{y}) \right\| \\
&\leqslant \|\mathbf{A}^k\| \cdot \|\mathbf{x} - \mathbf{y}\|
\end{aligned}$$

Thus $\rho(\mathbf{A}) < 1$ implies U^k is a uniform contraction for some $k \in \mathbb{N}$

Therefore U is globally stable on $(\mathbb{R}^n, \| \cdot \|)$

Stochastic Models: Dynamics of Moments

Consider the system

- $\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{b} + \mathbf{C}\mathbf{w}_{t+1}$ with $\mathbf{x}_0$ given
- $\mathbf{w}_t$ is a martingale difference seq (MDS) with

$$\mathbb{E}_t[\mathbf{w}_{t+1}] = \mathbb{E}[\mathbf{w}_{t+1}] = \mathbf{0} \quad \text{and} \quad \mathbb{E}_t[\mathbf{w}_{t+1}\mathbf{w}'_{t+1}] = \mathbf{I}$$

What is the time path of the first two moments

- $\boldsymbol{\mu}_t := \mathbb{E}[\mathbf{x}_t]$
- $\boldsymbol{\Sigma}_t := \mathrm{var}[\mathbf{x}_t] := \mathbb{E}[(\mathbf{x}_t - \boldsymbol{\mu}_t)(\mathbf{x}_t - \boldsymbol{\mu}_t)']$

Dynamics of the Mean

First, regarding $\boldsymbol{\mu}_t$, take expectations over

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{b} + \mathbf{C}\mathbf{w}_{t+1}$$

to get

$$\boldsymbol{\mu}_{t+1} = \mathbf{A}\boldsymbol{\mu}_t + \mathbf{b}$$

Fact. If $\rho(\mathbf{A}) < 1$, then $\{\boldsymbol{\mu}_t\}$ converges to the unique fixed point

$$\boldsymbol{\mu}^* = \sum_{i=0}^{\infty} \mathbf{A}^i \mathbf{b}$$

regardless of $\boldsymbol{\mu}_0$

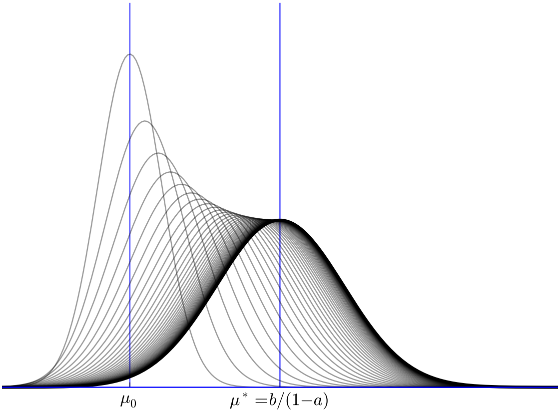


Figure: Convergence of μ_t to μ^* in the scalar model

Dynamics of the Variance

Consider again

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{b} + \mathbf{C}\mathbf{w}_{t+1}$$

We want a similar law of motion for $\boldsymbol{\Sigma}_t := \mathrm{var}[\mathbf{x}_t]$

Comment: Under the MDS assumption, $\mathbb{E}\left[\mathbf{x}_t \mathbf{w}'_{t+1}\right] = \mathbf{0}$ for all t

Proof:

$$\mathbb{E}\left[\mathbf{x}_t \mathbf{w}'_{t+1}\right] = \mathbb{E}\left[\mathbb{E}_t[\mathbf{x}_t \mathbf{w}'_{t+1}]\right] = \mathbb{E}\left[\mathbf{x}_t \mathbb{E}_t[\mathbf{w}'_{t+1}]\right] = \mathbb{E}\left[\mathbf{0}\right] = \mathbf{0}$$

By definition,

$$\begin{aligned}\operatorname{var}[\mathbf{x}_{t+1}] &= \mathbb{E}\left[(\mathbf{x}_{t+1} - \boldsymbol{\mu}_{t+1})(\mathbf{x}_{t+1} - \boldsymbol{\mu}_{t+1})'\right] \\ &= \mathbb{E}\left[(\mathbf{A}(\mathbf{x}_t - \boldsymbol{\mu}_t) + \mathbf{C}\mathbf{w}_{t+1})(\mathbf{A}(\mathbf{x}_t - \boldsymbol{\mu}_t) + \mathbf{C}\mathbf{w}_{t+1})'\right]\end{aligned}$$

The right hand side is equal to

$$\begin{aligned}&\mathbb{E}\left[\mathbf{A}(\mathbf{x}_t - \boldsymbol{\mu}_t)(\mathbf{x}_t - \boldsymbol{\mu}_t)'\mathbf{A}'\right] + \mathbb{E}\left[\mathbf{A}(\mathbf{x}_t - \boldsymbol{\mu}_t)\mathbf{w}_{t+1}'\mathbf{C}'\right] \\ &\qquad + \mathbb{E}\left[\mathbf{C}\mathbf{w}_{t+1}(\mathbf{x}_t - \boldsymbol{\mu}_t)'\mathbf{A}'\right] + \mathbb{E}\left[\mathbf{C}\mathbf{w}_{t+1}\mathbf{w}_{t+1}'\mathbf{C}'\right]\end{aligned}$$

Some further manipulations (check) lead to

$$\boldsymbol{\Sigma}_{t+1} = \mathbf{A}\boldsymbol{\Sigma}_t\mathbf{A}' + \mathbf{C}\mathbf{C}'$$

The variance covariance matrices follow

$$\mathbf{\Sigma}_{t+1} = \mathbf{A}\mathbf{\Sigma}_t\mathbf{A}' + \mathbf{C}\mathbf{C}' \tag{1}$$

A steady state of this system is a $\mathbf{\Sigma}$ satisfying

$$\mathbf{\Sigma} = \mathbf{A}\mathbf{\Sigma}\mathbf{A}' + \mathbf{C}\mathbf{C}' \tag{2}$$

Fact. If $\rho(\mathbf{A}) < 1$, then

1. (2) has exactly one solution $\mathbf{\Sigma}^* \in \mathcal{M}(n \times n)$
2. The sequence in (1) satisfies

$$\mathbf{\Sigma}_t \to \mathbf{\Sigma}^* \qquad (t \to \infty)$$

for all initial $\mathbf{\Sigma}_0$ in $\mathcal{M}(n \times n)$

Will give the proof as a generic result for the **discrete Lyapunov equation**

$$\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{A}' + \mathbf{M} \tag{3}$$

Here all matrices are in $\mathcal{M}(n \times n)$ and $\mathbf{X}$ is the unknown

Given $\mathbf{A}$ and $\mathbf{M}$, define the corresponding **Lyapunov operator**

$$L\mathbf{X} = \mathbf{A}\mathbf{X}\mathbf{A}' + \mathbf{M} \tag{4}$$

Fact. If $\rho(\mathbf{A}) < 1$, then (3) has a unique solution $\mathbf{X}^*$ in $\mathcal{M}(n \times n)$, and

$$L^k\mathbf{X} \to \mathbf{X}^* \text{ as } k \to \infty, \quad \forall\, \mathbf{X} \in \mathcal{M}(n \times n)$$

Proof: Suffices to show that L^k is a uniform contraction on $(\mathcal{M}(n \times n), \|\cdot\|)$ for some $k \in \mathbb{N}$

From the definition,

$$L^k \mathbf{X} = \mathbf{A}^k \mathbf{X} (\mathbf{A}^k)' + \mathbf{A}^{k-1} \mathbf{M} (\mathbf{A}^{k-1})' + \cdots + \mathbf{M}$$

Hence, for any $\mathbf{X}, \mathbf{Y}$ in $\mathcal{M}(n \times n)$, we have

$$\begin{aligned} \|L^k \mathbf{X} - L^k \mathbf{Y}\| &= \left\| \mathbf{A}^k \mathbf{X} (\mathbf{A}^k)' - \mathbf{A}^k \mathbf{Y} (\mathbf{A}^k)' \right\| \\ &= \left\| \mathbf{A}^k (\mathbf{X} - \mathbf{Y}) (\mathbf{A}^k)' \right\| \\ &\leqslant \|\mathbf{A}^k\| \cdot \|\mathbf{X} - \mathbf{Y}\| \cdot \|(\mathbf{A}^k)'\| \end{aligned}$$

Transposes don't change norms, so $\|(\mathbf{A}^k)'\| = \|\mathbf{A}^k\|$ and hence

$$\|L^k\mathbf{X} - L^k\mathbf{Y}\| \leqslant \|\mathbf{A}^k\|^2\|\mathbf{X} - \mathbf{Y}\|$$

Since $\rho(\mathbf{A}) < 1$, we can find $k \in \mathbb{N}$, $\lambda < 1$ such that

$$\|L^k\mathbf{X} - L^k\mathbf{Y}\| \leqslant \lambda\|\mathbf{X} - \mathbf{Y}\| \quad \text{for all } \mathbf{X}, \mathbf{Y} \in \mathcal{M}(n \times m)$$

Note: Gives an algorithm for computing $\mathbf{X}^*$

(Not always the best one)

Application: LQ Risk Neutral Asset Pricing

Recall the risk neutral asset pricing formula

$$p_t = \beta \mathbb{E}_t[d_{t+1} + p_{t+1}]$$

Here

- p_k is price at time k
- $\beta \in (0, 1)$ discounts next period values to current
- $\mathbb{E}_t$ is time t conditional expectation

The price p_t is the endogenous process we want to solve for

This time we'll take an LQ model for dividends

In particular, assume that

$$d_t = \mathbf{x}_t' \mathbf{D} \mathbf{x}_t \text{ for some positive definite } \mathbf{D}$$

Here $\{\mathbf{x}_t\}$ is the **state process** with dynamics

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{C}\mathbf{w}_{t+1}$$

As before,

- $\{\mathbf{w}_t\}$ is an MDS
- $\mathbb{E}_t[\mathbf{w}_{t+1}\mathbf{w}_{t+1}'] = \mathbf{I}$ for all t

Preliminary: Predicting Quadratics

Fact. If $\mathbf{H} \in \mathcal{M}(n \times n)$ and $\{\mathbf{x}_t\}$ is as above, then

$$\mathbb{E}_t[\mathbf{x}_{t+1}' \mathbf{H} \mathbf{x}_{t+1}] = \mathbf{x}_t' \mathbf{A}' \mathbf{H} \mathbf{A} \mathbf{x}_t + \operatorname{trace}(\mathbf{C}' \mathbf{H} \mathbf{C})$$

Ex. Prove it

Prices as Functions of the State

As before, we conjecture that

$$p_t = p(\mathbf{x}_t) \quad \text{for some function } p$$

Another leap: guess that prices are a **quadratic** in $\mathbf{x}_t$

In particular, we guess that

$$p(\mathbf{x}) = \mathbf{x}' \mathbf{P} \mathbf{x} + \delta$$

for some positive definite $\mathbf{P}$ and scalar δ

Substituting

$$p_t = \mathbf{x}_t' \mathbf{P} \mathbf{x}_t + \delta \quad \text{and} \quad d_t = \mathbf{x}_t' \mathbf{D} \mathbf{x}_t$$

into

$$p_t = \beta \mathbb{E}_t [d_{t+1} + p_{t+1}]$$

gives

$$\begin{aligned}
\mathbf{x}_t' \mathbf{P} \mathbf{x}_t + \delta &= \beta \mathbb{E}_t [\mathbf{x}_{t+1}' \mathbf{D} \mathbf{x}_{t+1} + \mathbf{x}_{t+1}' \mathbf{P} \mathbf{x}_{t+1} + \delta] \\
&= \beta \mathbb{E}_t [\mathbf{x}_{t+1}' (\mathbf{D} + \mathbf{P}) \mathbf{x}_{t+1}] + \beta \delta \\
&= \beta \mathbf{x}_t' \mathbf{A}' (\mathbf{D} + \mathbf{P}) \mathbf{A} \mathbf{x}_t + \beta \operatorname{trace}(\mathbf{C}' (\mathbf{D} + \mathbf{P}) \mathbf{C}) + \beta \delta
\end{aligned}$$

So, we seek a pair $\mathbf{P} \in \mathcal{M}(n \times n)$, $\delta \in \mathbb{R}$ such that

$$\mathbf{x}'\mathbf{P}\mathbf{x} + \delta = \beta \mathbf{x}'\mathbf{A}'(\mathbf{D} + \mathbf{P})\mathbf{A}\mathbf{x} + \beta \operatorname{trace}(\mathbf{C}'(\mathbf{D} + \mathbf{P})\mathbf{C}) + \beta\delta$$

for any $\mathbf{x} \in \mathbb{R}^n$

Suppose exists $\mathbf{P}^* \in \mathcal{M}(n \times n)$ such that

$$\mathbf{P}^* = \beta \mathbf{A}'(\mathbf{D} + \mathbf{P}^*)\mathbf{A}$$

Claim: If this is true and

$$\delta^* := \frac{\beta}{1 - \beta} \operatorname{trace}(\mathbf{C}'(\mathbf{D} + \mathbf{P}^*)\mathbf{C})$$

then the pair $\mathbf{P}^*, \delta^*$ solves the above equation for any $\mathbf{x}$

Proof: By hypothesis, $\mathbf{P}^* = \beta \mathbf{A}'(\mathbf{D} + \mathbf{P}^*)\mathbf{A}$

$$\therefore \quad \mathbf{x}'\mathbf{P}^*\mathbf{x} = \beta \mathbf{x}'\mathbf{A}'(\mathbf{D} + \mathbf{P}^*)\mathbf{A}\mathbf{x}$$

$$\therefore \quad \mathbf{x}'\mathbf{P}^*\mathbf{x} + \delta^* = \beta \mathbf{x}'\mathbf{A}'(\mathbf{D} + \mathbf{P}^*)\mathbf{A}\mathbf{x} + \delta^*$$

To complete the proof, sufficies to show that

$$\delta^* = \beta \operatorname{trace}(\mathbf{C}'(\mathbf{D} + \mathbf{P}^*)\mathbf{C}) + \beta\delta^*$$

This is true from definition of δ^*

Last step: Find $\mathbf{P} \in \mathcal{M}(n \times n)$ that solves

$$\mathbf{P} = \beta \mathbf{A}'(\mathbf{D} + \mathbf{P})\mathbf{A} \tag{5}$$

Claim: A unique solution to (5) exists whenever $\rho(\mathbf{A}) < 1/\sqrt{\beta}$

Proof: Letting $\mathbf{M} := \beta \mathbf{A}'\mathbf{D}\mathbf{A}$ and $\mathbf{\Lambda} := \sqrt{\beta}\mathbf{A}'$, we can express (5) as

$$\mathbf{P} = \mathbf{\Lambda}\mathbf{P}\mathbf{\Lambda}' + \mathbf{M}$$

- A discrete Lyapunov equation in $\mathbf{P}$

Since $\rho(\mathbf{\Lambda}) < 1$, a unique solution $\mathbf{P}^*$ exists

Asset Pricing Summary

We have shown that

$$\rho(\mathbf{A}) < \frac{1}{\sqrt{\beta}} \implies \mathbf{P} = \beta \mathbf{A}'(\mathbf{D} + \mathbf{P})\mathbf{A} \text{ has a unique solution}$$

The solution $\mathbf{P}^*$ and associated δ^* gives the pricing function

$$p^*(\mathbf{x}) := \mathbf{x}'\mathbf{P}^*\mathbf{x} + \delta^*$$

This pricing function satisfies the risk neutral asset pricing equation

Ex. Show that $\mathbf{P}$ is positive definite whenever $\mathbf{A}$ is nonsingular