Comp Econ Homework 7

Date 25th March

Due Date: 31st March

We begin with some preamble on risk-neutral asset pricing. As in the lecture, let dividends be given by $D_t = d(X_t)$ for some state process $\{X_t\}$. The state process takes values in a countable set S, called the *state space*. We let the transition probabilities for the state be given by

$$q(x, y) := \mathbb{P}\{X_{t+1} = y \mid X_t = x\} \qquad (x, y \in S)$$

As discussed in the lecture, an *equilibrium price function* is a function p on S that satisfies

$$p(x) = \beta \sum_{y \in S} [d(y) + p(y)] q(x, y) \tag{1}$$

for all $x \in S$. Assume in all of what follows that $\beta \in (0, 1)$ and that d is any function from S to $(0, \infty)$.

Below I state a theorem on existence and uniqueness that is similar to the classic work of Lucas (1978) on asset pricing, although I'll stick to the risk neutral case to simplify the discussion.

Incidentally, this theorem is not the best we can do, but it's proof is relatively easy. (We'll look at more sophisticated results later.) Please read carefully, so you can apply similar ideas in the exercise.

Theorem 0.1. *If $d \in b\mathbb{R}^S$, then there exists a unique equilibrium price function $p^* \in b\mathbb{R}^S$. Moreover, if p is any function in $b\mathbb{R}^S$, then $d_\infty(T^k p, p^*) \to 0$ as $k \to \infty$, where*

$$Tp(x) = \beta \sum_{y \in S} [d(y) + p(y)] q(x, y) \qquad (x \in S) \tag{2}$$

Proof. It suffices to show that T is a uniform contraction on $(b\mathbb{R}^S, d_\infty)$ whenever $d \in b\mathbb{R}^S$. Assuming that $d \in b\mathbb{R}^S$, to show that T has the stated properties, we must first show that T does in fact map $b\mathbb{R}^S$ into itself. To see this, observe that, for any given $p \in b\mathbb{R}^S$ and any $x \in S$,

$$
\begin{aligned}
|Tp(x)| &= \beta \left| \sum_{y \in S} [d(y) + p(y)] q(x,y) \right| \\
&\leq \beta \sum_{y \in S} |d(y) + p(y)| q(x,y) \\
&\leq \beta \sum_{y \in S} (\|d\|_\infty + \|p\|_\infty) q(x,y) \\
&= \beta(\|d\|_\infty + \|p\|_\infty)
\end{aligned}
$$

This bound is uniform over x, so taking the supremum on the left hand side gives $\|Tp\|_\infty < \infty$. In other words, $Tp \in b\mathbb{R}^S$.

To see that T is a uniform contraction, pick any $p, p' \in b\mathbb{R}^S$ and let x be any element of S. We have

$$
\begin{aligned}
|Tp(x) - Tp'(x)| &= \beta \left| \sum_{y \in S} [p(x) - p'(x)] q(x,y) \right| \\
&\leq \beta \sum_{y \in S} \left| p(x) - p'(x) \right| q(x,y) \\
&\leq \beta \sum_{y \in S} \|p - p'\|_\infty q(x,y) = \beta \|p - p'\|_\infty
\end{aligned}
$$

Since $x \in S$ was arbitrary, we have

$$
\|Tp - Tp'\|_\infty \leq \beta \|p - p'\|_\infty
$$

Hence T is a uniform contraction on $(b\mathbb{R}^S, d_\infty)$, as claimed. $\square$

Exercise 1. Provide a Python or Julia function that takes a specification of the primitives and returns an approximation to the equilibrium price function. You can assume that S is finite. The computation method should be iterative, based around the Banach contraction mapping theorem.[1] That

[1] The iterative method is not always best, especially for small state spaces, but it's the one we'll use in this particular exercise. This is to prepare us for the next exercise, where the iterative method is the only obvious method.

is, choose some initial $p \in b\mathbb{R}^S$ and repeatedly apply T to obtain $T^k p$ for $k = 1, 2, \ldots$. As a stopping rule for the iteration, stop when successive iterates are close in the sense of d_∞. (Experiment with different tolerances and choose a number that gives you accuracy up to three or four decimal places.)

To test your code, suppose that the state process for the economy has three states: normal growth, mild recession and severe recession. The transition probabilities $q_1(x, y)$ are given by

	NG at $t+1$	MR at $t+1$	SR at $t+1$
NG at t	0.971	0.029	0
MR at t	0.145	0.778	0.077
SR at t	0	0.508	0.492

(This set of transition probabilities was estimated by James Hamilton from business cycle data.) Set dividends to 1 in state NG and zero otherwise. Set $\beta = 0.98$. Report the corresponding equilibrium price function.

Now repeat, with new transition probabilities $q_2(x, y)$ given by

	NG at $t+1$	MR at $t+1$	SR at $t+1$
NG at t	0.871	0.029	0.1
MR at t	0.145	0.778	0.077
SR at t	0	0.508	0.492

How do your answers change? Can you give any interpretation?

(If you're new to these kinds of problems and need clarification on any points, you can stop by my office and ask.)

Now consider the asset pricing problem with heterogeneous beliefs described in Ljungqvist and Sargent (RMT4, p. 537). The ideas are originally due to Harrison and Kreps (1978). Here we treat only the mathematical and computational aspects of the problem. See the discussion in either of these references for motivation.

The equilibrium condition is modified to

$$p(x) = \beta \max_{i \in \{1,2\}} \left\{ \sum_{y \in S} [d(y) + p(y)] q_i(x, y) \right\} \qquad (x \in S) \tag{3}$$

where q_1 and q_2 represent different beliefs about state transition probabilities.

Exercise 2. Give conditions under which (3) has a unique solution in $b\mathbb{R}^S$ by extending our earlier analysis (i.e., the case of homogeneous beliefs). Do not assume that S is finite. Give a proof that your conditions are sufficient.

Remark: One possibly helpful hint is the following inequality, which holds for any sequences $\{a_i\}$ and $\{b_i\}$.

$$|\max_i a_i - \max_i b_i| \leq \max_i |a_i - b_i|$$

You can check this if you like (the proof is not that hard), but you can also just take it as given and use it to help you solve the exercise.

Exercise 3. Write a second function that does the same job as the function you wrote for exercise 1, but now for the heterogeneous beliefs case.[2] For the computational problem you can assume that S is finite, similar to exercise 1. To test your code, let the state space be as in the test problem in exercise 1, let q_1 and q_2 be the transition probabilities given in that exercise, and let d and β be as specified in that exercise. Compute the equilibrium price function.

Please submit your work as a single Jupyter notebook with the usual naming convention.

References

LUCAS, R. E. (1978): "Asset prices in an exchange economy," *Econometrica*, 1429–1445.

[2] It would be natural to try to write just one function that can handle either case, but for this assignment please just write two separate functions.