1: Exercise 1

Proof. Notice that to show $X'X$ is invertible is equivalent to show that $X'X$ is positive definite.

Given any vector $z \in \mathbb{R}\backslash\{0\}$, $z'(X'X)z = (Xz)'(Xz) = ||X'z||^2 \geq 0$.

Recall that X has linearly independent columns, so $Xz \neq 0$. Therefore we have

$$z'(X'X)z = (Xz)'(Xz) = ||X'z||^2 > 0$$

□

2: Exercise 2

Proof. Let S be a space spanned by X. Since $n = k$ and X has linearly independent columns, $rank(X) = n = k$. Then for any vector $y \in \mathbb{R}^n$, y lies in the space S. Therefore the projection of y on the space X is itself. Now we have $Py = y$.

$$(P - I)y = 0$$

Since this equation holds for any nonzero vector y, we can conclude that $P = I$. □

3: Exercise 3

Proof. Let $\mathbf{1} = (1, \ldots, 1)'$ with n elements. Then $\mathbf{P} = \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'$. Given any $y = (y_1, \ldots, y_n)'$, we have

$$\mathbf{P}y = \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'y$$

Notice that $(\mathbf{1}'\mathbf{1})^{-1} = \frac{1}{n}$ and $\mathbf{1}'y = \sum_{i=1}^{n} y_i$. Therefore we have

$$\mathbf{P}y = \begin{pmatrix} \bar{y} \\ \vdots \\ \bar{y} \end{pmatrix}_{n\times 1}$$

□

4: Exercise 4

Proof. $\hat{u} = My = (I - P)y$

$$X'\hat{u} = X'(I - P)y = X'y - X'y = 0$$

If X has a constant column, then the first row of X' should be a vector of constant 1. Let

$$X' = \begin{bmatrix} x_{const} \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

We can easily find that $X_{const}\hat{u} = 0$. So elements of vector $\hat{u}$ sum to zero. $\square$

5: Exercise 5

Proof. Pick any $x \in S \bigcap S^{\perp} \subseteq \mathbb{R}^n$, x is a vector. Notice that $x \in S$ and $x \perp S \Longrightarrow x \perp x$. Then we have

$$x \cdot x = 0 \Longrightarrow x = \{0\}$$

$\square$