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HW SET # 6
ANALYTICAL EXC.

Ex.1): Let $A$ be an $(n \times K)$ matrix. We are now going to show that with no further assumptions about $A$, $Null(A) = Null(A^T A)$.

$\Rightarrow$ Suppose $A\vec{x} = \vec{0}$, i.e., $\vec{x} \in Null(A)$

Then $(A^T A)\vec{x} = A^T \vec{0} = \vec{0}$

$$\Rightarrow \vec{x} \text{ is also in } Null(A^T A)$$

$$\Rightarrow Null(A) \subset Null(A^T A)$$

$\Leftarrow$ Suppose $(A^T A)\vec{x} = \vec{0}$, i.e., $\vec{x} \in Null(A^T A)$

Then $\vec{x}^T (A^T A)\vec{x} = x^T \vec{0} = \vec{0}$

$$\Leftrightarrow (\vec{x}^T A^T)(A\vec{x}) = \vec{0}$$

$$\Leftrightarrow (A\vec{x})^T (A\vec{x}) = \vec{0}$$

$$\Leftrightarrow \|A\vec{x}\|^2 = \vec{0}$$

$$\Leftrightarrow A\vec{x} = \vec{0}$$

$$\Rightarrow \vec{x} \text{ is also in } Null(A)$$

$$\Rightarrow Null(A^T A) \subset Null(A)$$

$\Rightarrow$

$$Null(A) = Null(A^T A) \quad \blacksquare$$

Now we use this fact to prove that:
if $A$ is an $(n \times k)$ matrix with linearly independent columns, then $(A^T A)$ is <u>invertible</u>.

Since $A$ has linearly independent columns, then by def. $A\vec{x} = \vec{0}$ has only the trivial solution, i.e., $\vec{x} = \vec{0}$ or $Null(A) = \{\vec{0}\}$.
Therefore, $Null(A^T A) = \{\vec{0}\} = Null(A)$.
$\Rightarrow$ $(A^T A)$ has linearly independent columns.
Since $(A^T A)$ is a $(k \times k)$ matrix (square), then by the INVERTIBLE MATRIX THM $(A^T A)$ must be invertible.

$\blacksquare$

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#: <u>OTHER PROOF METHOD FOR 1):</u>

Let $A$ be an $(n \times k)$ matrix with linearly independent columns.

Then $\vec{x}^T (A^T A) \vec{x} = (\vec{x}^T A^T)(A\vec{x}) = (A\vec{x})^T (A\vec{x}) =$

$$= \|A\vec{x}\|^2 \geq 0,$$

and $\|A\vec{x}\|^2 = 0 \iff A\vec{x} = \vec{0}$

But since $A$ has linearly indep. columns $A\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$

Therefore, $\vec{x}^T (A^T A)\vec{x} = 0 \iff \vec{x} = 0$, and $\vec{x}^T (A^T A)\vec{x} > 0$ otherwise. $\Rightarrow$

$\Rightarrow$ $(A^T A)$ is positive definite

$\Rightarrow$ $(A^T A)$ has all positive eigenvalues.

$\Rightarrow$ $\det(A^T A) \neq 0$ $\Rightarrow$ $(A^T A)$ is invertible.

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Ex. 2) : $P = X(X^T X)^{-1} X^T$ , $y \in \mathbb{R}^n$

If $X$ is $(n \times n)$ with linearly indep. columns,
then $P = X\left(X^{-1}(X^T)^{-1}\right)X^T =$

$$=(X X^{-1})\left[(X^T)^{-1}(X^T)\right] = I \cdot I = I .$$

- By invertible matrix thm if $X$ is square $(n \times n)$ with linearly independent columns, $X$ is invertible and $X^T$ is also invertible.
- Because the $\text{Span}\{col_1(X) \dots col_n(X)\} = \mathbb{R}^n$, then Projection of $y$ onto Span $(X)$ is $y$ itself, because $y$ is already in $\mathbb{R}^n$.

$\Rightarrow$ $X\beta = y$ has always a unique solution.

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$$y = \underset{z \in \mathbb{R}^n}{\operatorname{argmin}} \| y - z \|$$

Ex.3) : $y \in \mathbb{R}^n$ , $X$ is an $(n \times k)$ matrix.

$$\vec{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$$

$$\Rightarrow \quad P = \vec{1}\left(\vec{1}^T \vec{1}\right)^{-1} \vec{1}^T$$

$$\Rightarrow \quad Py = \vec{1}\left(\vec{1}^T \vec{1}\right)^{-1} \vec{1}^T y =$$

$$= \vec{1}\left(\sum_{i=1}^{n} 1\right)^{-1} \vec{1}^T y =$$

$$= \vec{1} \ (n)^{-1} \vec{1}^T y = \frac{1}{n} \vec{1} \vec{1}^T y =$$

$$= \frac{1}{n} \vec{1} \sum_{i=1}^{n} 1 \cdot y_i = \frac{1}{n} \vec{1} \cdot \sum_{i=1}^{n} y_i =$$

$$= \frac{1}{n} \sum_{i=1}^{n} y_i \cdot \vec{1} = \bar{y} \cdot \vec{1} = \begin{bmatrix} \bar{y} \\ \bar{y} \\ \vdots \\ \bar{y} \end{bmatrix} \in \mathbb{R}^n$$

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Ex. 4) : $y \in \mathbb{R}^n$ , $X \in M(n \times k)$ with linearly independent columns.

$S = span(X)$ , $P = proj(S)$ , $M = I - P = proj(S^\perp)$

$\hat{u} = My$ , $\vec{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$

• We want to show that $\sum_{i=1}^{n} u_i = 0$

Then $\sum_{i=1}^{n} u_i = \vec{1}^T \hat{u} = \vec{1}^T (I - P) y = ⊛$

$\Rightarrow\!\!>$

- We will now prove 3 facts:

1) $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$, with $A$ invertible matrix:

PROOF: $\bullet\ A^T\left(A^{-1}\right)^T = \left(A^{-1}A\right)^T = I^T = I$

$\bullet\ \left(A^{-1}\right)^T A^T = \left(A A^{-1}\right)^T = I^T = I$ $\blacksquare$

2) $X$ $n \times K$ with linearly indep. cols:

$\Rightarrow \left(X^T X\right)$ is symmetric

PROOF: $\left(X^T X\right)^T = X^T \left(X^T\right)^T = X^T X$ $\blacksquare$

3) $P = X\left(X^T X\right)^{-1} X^T$ is symmetric:

PROOF: $P^T = \left[X\left(X^T X\right)^{-1} X^T\right]^T = \left[\left[X\left(X^T X\right)^{-1}\right]X^T\right]^T =$

$= \left[X\left[X\left(X^T X\right)^{-1}\right]^T\right] = \left[X\left[\left(X^T X\right)^{-1}\right]^T X^T\right] =$

(by 1) + 2)) $= \left[X\left[\left(X^T X\right)^T\right]^{-1} X^T\right] = \left[X\left[X^T X\right]^{-1} X^T\right] =$

$= P$ $\blacksquare$

Therefore: $\circledast$ $\sum_{i=1}^{n} u_i = \vec{1}^T \hat{u} = \vec{1}^T\left(I - P\right)y =$

$= \left(1^T I - 1^T P\right)y = \left(1^T - 1^T P^T\right)y =$

[Because $\vec{1}$ is already in Span($X$)] $= \left(1^T - \left(P\vec{1}\right)^T\right)y = \left(\vec{1}^T - \vec{1}^T\right)y = 0$ $\blacksquare$

Ex. 5): $S$ linear subspace:

1) $\vec{0} \in S$
2) $\alpha \vec{x} + \beta \vec{y} \in S \quad \forall \vec{x}, \vec{y} \in S$ and $\alpha, \beta \in \mathbb{R}$.

$S^{\perp} = \{ \vec{t} \mid \vec{t} \cdot \vec{s} = 0,\ \forall \vec{s} \in S \}$ is also a subspace.

- We want to show that $S \cap S^{\perp} = \{\vec{0}\}$
- **PROOF:** Suppose $\exists\ \vec{x} \neq \vec{0}$ that is in both $S$ and $S^{\perp}$.

Then, it must be that

$$\vec{x} \cdot \vec{x} = 0$$

But $\vec{x} \cdot \vec{x} = \|\vec{x}\|^2 = 0 \iff \vec{x} = \vec{0}$

$$\Rightarrow\Leftarrow$$

Contradiction.

Therefore, it must be that $S \cap S^{\perp} = \{\vec{0}\}$