Comp Econ Homework 6

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Analytical Exercises

Exercise 1

To show that $\mathbf{X}'\mathbf{X}$ is invertible, I will use the hint and show that the matrix is positive definite. It is a well-known theorem that a positive definite matrix has a positive determinant and therefore it is invertible.

Def: $\mathbf{A} \in \mathcal{M}(k \times k)$ is positive definite if $\mathbf{z}'\mathbf{A}\mathbf{z} > 0\ \forall \mathbf{z} \in \mathbb{R}^k \backslash \mathbf{0}$

Let $\mathbf{z} \in \mathbb{R}^k \backslash \mathbf{0}$. Let $\mathbf{A} = \mathbf{X}'\mathbf{X}$. WTS: $\mathbf{A}$ is positive definite, ie $\mathbf{z}'\mathbf{A}\mathbf{z} > 0$.

In fact,

$$\mathbf{z}'\mathbf{A}\mathbf{z} = \mathbf{z}'\mathbf{X}'\mathbf{X}\mathbf{z} = (\mathbf{X}\mathbf{z})'(\mathbf{X}\mathbf{z}) = ||\mathbf{X}\mathbf{z}||^2 > 0$$

Note that $(\mathbf{X}\mathbf{z})$ is a linear combination of the columns of $\mathbf{X}$ and $\mathbf{X}\mathbf{z} \in \mathbb{R}^k$ Hence, $\mathbf{X}'\mathbf{X}$ is positive definite.

Exercise 2

Let $\mathbf{P} = \mathbf{X}(\mathbf{X}'X)^{-1}\mathbf{X}'$. WTS: if $k = n$, then $\mathbf{P}$ is the identity, $\mathbf{P} = \mathbf{I}$

In other words, WTS: $\forall y \in \mathbb{R}^n$, $\mathbf{P}\mathbf{y} = \mathbf{y}$

Since $k = n$ and X has linearly independent columns, the columns of form a base of $\mathbb{R}^n$, ie span$(\mathbf{X}) = S = \mathbb{R}^n$.

Since $\mathbf{P}$ is the projection matrix, we know that $\mathbf{y} - \mathbf{P}\mathbf{y} \in S^\perp$, and $(\mathbb{R}^n)^\perp = \{\mathbf{0}\}$

Therefore,

$$\begin{aligned} \mathbf{y} - \mathbf{P}\mathbf{y} &= \mathbf{0} \\ \Rightarrow \mathbf{y} &= \mathbf{P}\mathbf{y} \end{aligned}$$

Exercise 3

Let $\mathbf{1} = (1, \ldots, 1)' \in \mathbb{R}^n$ a column vector of ones and $\mathbf{P} = \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'$. Let $y \in \mathbb{R}^n$

Note that $\mathbf{P}\mathbf{y} \in \mathbb{R}^n$.

$$\mathbf{Py} = \underbrace{\mathbf{1}}_{(n\times 1)} \underbrace{(\mathbf{1}'\mathbf{1})^{-1}}_{(1\times n)\times(n\times 1)} \underbrace{(\mathbf{1}'y)}_{(1\times n)\times(n\times 1)}$$

I'll compute each term separately.

The term $\mathbf{1}'y = \sum_{i=1}^{n} y_i$

Note also that $(\mathbf{1}'\mathbf{1}) = n$ and hence $(\mathbf{1}'\mathbf{1})^{-1} = \frac{1}{n}$

Therefore, $(\mathbf{1}'\mathbf{1})^{-1}(\mathbf{1}'y) = \frac{1}{n}\sum_{i=1}^{n} y_i \equiv \bar{y}$

Finally,

$$\mathbf{Py} = \mathbf{1}\bar{y} = \begin{pmatrix} \bar{y} \\ \vdots \\ \bar{y} \end{pmatrix}$$

Exercise 4

$$\begin{aligned}
\hat{\mathbf{u}} &= \mathbf{My} = (\mathbf{I} - \mathbf{P})\mathbf{y} \\
\mathbf{X}'\hat{\mathbf{u}} &= \mathbf{X}'(\mathbf{I} - \mathbf{P})\mathbf{y} \\
&= \mathbf{X}'(\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{y} \\
&= (\mathbf{X}' - \mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')\mathbf{y} \\
&= (\mathbf{X}' - \mathbf{X}')\mathbf{y} \\
&= \mathbf{0}
\end{aligned}$$

Note that the first row of $\mathbf{X}'$ is a row of only ones, and hence $\mathbf{X}'_{1,\cdot}\hat{\mathbf{u}} = 0$ which is the sum of the elements of $\hat{\mathbf{u}}$.

Exercise 5

Let $\mathbf{x} \in S \cap S^{\perp}$. Then, x is perpendicular to itself, hence $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$