Computational Economics
HW6 – Solutions

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March 24, 2016

Exercise 1

CLAIM: If $\mathbf{X} \in \mathcal{M}(n \times k)$ with linearly independent columns, then $\mathbf{X}'\mathbf{X}$ is invertible.

PROOF:
First show that the $k \times k$ symmetric matrix $\mathbf{X}'\mathbf{X}$ is positive definite. To this end, let $\mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and write the quadratic form

$$\mathbf{y}'(\mathbf{X}'\mathbf{X})\mathbf{y} = (\mathbf{X}\mathbf{y})'(\mathbf{X}\mathbf{y}) = \|\mathbf{X}\mathbf{y}\|^2 > 0$$

The strict inequality follows from the fact that $\mathbf{X}\mathbf{y} \neq \mathbf{0}$, because the columns of $\mathbf{X}$ are linearly independent and $\mathbf{y} \neq \mathbf{0}$. The spectrum of a symmetric, positive definite matrix contains only strictly positive eigenvalues, so the determinant (given by the product of the eigenvalues) is nonzero and the matrix $\mathbf{X}'\mathbf{X}$ is invertible.

Exercise 2

CLAIM: Let $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$, where $\mathbf{X} \in \mathcal{M}(n \times k)$ with linearly independent columns. If $k = n$, then $\mathbf{P} = \mathbf{I}_n$.

PROOF:
If $\mathbf{X}$ is square, the fact that its columns are linearly independent implies that $\text{rank}(\mathbf{X}) = n$ and s both $\mathbf{X}$ and $\mathbf{X}'$ are invertible. Then,

$$\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \mathbf{X}\mathbf{X}^{-1}\left(\mathbf{X}'\right)^{-1}\mathbf{X}' = \mathbf{I}_n$$

Intuition: The columns of $\mathbf{X}$ constitute a basis for $\mathbb{R}^n$, because they are linearly independent and there is n of them (i.e. the dimension of the space). Moreover, since X is square, $\mathbf{P}$ is an $n \times n$ matrix, hence it projects vectors $\mathbf{y} \in \mathbb{R}^n$ onto $\mathbb{R}^n$, hence $\mathbf{P}\mathbf{y} = \mathbf{y}$.

Exercise 3

CLAIM: The projection of $\mathbf{y} \in \mathbb{R}^n$ onto span$\{\mathbf{1}\}$ is the mean of the elements of $\mathbf{y}$.

PROOF:
Using the formula for the projection matrix associated with the $\mathbf{1}$ vector, the projection of $\mathbf{y} \in \mathbb{R}^n$ onto span$\{\mathbf{1}\}$ is

$$\mathbf{P}\mathbf{y} = \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'\mathbf{y} = \frac{1}{n}\mathbf{1}\mathbf{1}'\mathbf{y} = \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{y}_i\right)\mathbf{1}$$

Exercise 4

CLAIM: Let $\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{X} \in \mathcal{M}(n \times k)$ with linearly independent columns. Let $S = \text{span}(\mathbf{X})$ and $\hat{\mathbf{u}} = \mathbf{y} - \mathbf{P}\mathbf{y}$. If $\mathbf{X}$ has a constant column, then elements of $\hat{\mathbf{u}}$ sum to 0.

PROOF:
Since $\hat{\mathbf{u}} = \mathbf{y} - \mathbf{P}\mathbf{y}$, where $\mathbf{P}$ is the projection operator onto S, we have $\hat{\mathbf{u}} \perp S$. col$(\mathbf{X})$ is a basis for S, so it follows that $\hat{\mathbf{u}} \perp \text{col}_i\mathbf{X}$ for every $i = 1, \ldots, k$. If one of the columns is a constant $\alpha\mathbf{1}$ with $\alpha \in \mathbb{R} \setminus \{0\}$, then

$$\hat{\mathbf{u}} \perp \alpha\mathbf{1} \quad \Leftrightarrow \quad \langle \hat{\mathbf{u}}, \alpha\mathbf{1} \rangle = 0 \quad \Leftrightarrow \quad \sum_{i=1}^{n}\hat{\mathbf{u}}_i = 0$$

Exercise 5

CLAIM: If S is a nonempty linear subspace of $\mathbb{R}^n$, then $S \cap S^{\perp} = \{\mathbf{0}\}$.

PROOF:
First, S and $S^{\perp}$ are both linear subspaces of $\mathbb{R}^n$, hence $\mathbf{0} \in S \cap S^{\perp}$. Suppose now that there is another vector $\mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ in the intersection of S and $S^{\perp}$. $\mathbf{y} \in S^{\perp}$ implies $\mathbf{y} \perp S$, so if we add this to $\mathbf{y} \in S$, we arrive at

$$\mathbf{y} \perp \mathbf{y} \quad \Leftrightarrow \quad \langle \mathbf{y}, \mathbf{y} \rangle = 0 \quad \Leftrightarrow \quad \mathbf{y} = \mathbf{0}$$

Contradiction.