Topics in Computational Economics

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Exercise 1

Firstly, note that as the collection of vectors in X are linearly independent, X has column rank equal to k. Therefore $ker(X) = \{0\}$, that is $\forall\ v \in \mathbb{R}^n$ the condition $Xv = 0 \implies v = 0$. Secondly, take the product norm (inherited from the dot product) and note that $\|v\| = 0 \implies v = 0$ by definition. Thirdly by the properties of the transpose $\langle Xv, w\rangle = \langle v, X^T w\rangle$. Now these three facts imply that the following is true $\forall v \neq 0$:

$$0 < \|Xv\| = \langle Xv, Xv\rangle = \langle v, X^T X v\rangle = \langle X^T X v, v\rangle \tag{1}$$

implying that $X^T X$ is positive definite. As a positive definite matrix only has positive eigenvalues it is invertible (the determinant is nonzero).

Exercise 2

The collection of vectors in X are assumed to be linearly independent, therefore they must form a basis of $\mathbb{R}^n$. This immediatly implies that X spans $\mathbb{R}^n$. Combined with the fact that $\forall\ v \in S$, $Pv = v$ (where S is the subspace spanned by X, P is the projection to S) we have that $Pv = v$, $\forall\ v \in \mathbb{R}^n$, that is, P is the identity. A more direct proof would be to invert P, and show that it is equal to P, but the previous proof also conveys intuition.

Exercise 3

Directly calculate:

$$P = 1_{n\times1}(1_{n\times1}^T 1_{n\times1})^{-1} 1_{n\times1}^T = \frac{1_{n\times1} 1_{n\times1}^T}{n} = \frac{1_{n\times n}}{n} \tag{2}$$

Therefore for any $y \in \mathbb{R}^n$

$$Py = \frac{1_{n \times n}}{n} y = \begin{bmatrix} \frac{\sum_{i=1}^{n} y_i}{n} \\ \frac{\sum_{i=1}^{n} y_i}{n} \\ \vdots \\ \frac{\sum_{i=1}^{n} y_i}{n} \end{bmatrix} = \begin{bmatrix} \bar{y} \\ \bar{y} \\ \vdots \\ \bar{y} \end{bmatrix} \tag{3}$$

Exercise 4

Firstly, note that multiplying the residual maker M from the right with a matrix formed by any collections of vectors from X would result in 0. To see this take X itself and to select the relevant columns, multiply it from the right with a "selector" matrix S, whose only nonzero elements (namely ones) are on the diagonal:

$$MXS = (I - X(X^T X)^{-1} X^T)XS = XS - XS = 0 \tag{4}$$

Implying that $M1 = 0$. As M is symmetric this implies $1^T M = 0$ and so $1^T My = 0$, proving the claim.

Exercise 5

Show by double inclusion.

$S \cap S^T \supset \{0\}$

For any subspace S, 0 must be contained in it as taking a (degenerate) linear combination with weights $(0, 0)$ of any two vectors in S must be contained. As S^T is a subspace too, this direction is proved.

$S \cap S^T \subset \{0\}$

Take any $v \in S \cap S^T$. By definition it must hold that $v \in S$ and $v \perp S$. The latter implies that $\langle v, x \rangle = 0$ for any $x \in S$, including v itself. That is $\|v\| = 0$ and by the properties of the norm we have $v = 0$