Homework Set 6 Analysis

Chase Coleman

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In all of the following, let X be an element of $\mathcal{M}(n \times k)$ with linearly independent columns.

Exercise 1

Question:

Show that $X'X$ is invertible

Solution:

We want to show that $X'X$ is invertible. We will first show that $X'X$ is positive definite.

A matrix $\Gamma \in \mathcal{M}(n \times n)$ is positive definite if and only if $z'\Gamma z \geq 0$ for all non-zero vectors $z \in \mathcal{R}^n$.

Note for any non-zero $z \in \mathcal{R}^n$ we could write[1]

$$\begin{aligned} z'(X'X)z &= (Xz)'(Xz) \\ &= \sum_{m=1}^{n} \left(\sum_{j=1}^{n} X_{m,j} z_j \right)^2 \\ &> 0 \end{aligned}$$

Thus $X'X$ is strictly positive definite. A strictly positive definite matrix has strictly positive eigenvalues which implies the determinant is strictly positive.

All matrices with non-zero determinants are invertible thus $X'X$ is invertible

$\square$

Exercise 2

Question:

Let $P = X(X'X)^{-1}X'$. Show that if $k = n$, then P is the identity. (Justify your steps.) Explain the geometric intuition in terms of the orthogonal projection theorem.

Solution:

The intuition behind the result we will show is that when there are n linearly independent columns in X and each column is a vector in $\mathcal{R}^n$ then the set of vectors spans $\mathcal{R}^n$. This implies any n-dimensional vector, $y \in \mathcal{R}^n$, is in the column space – i.e. the projection of the n-dimensional vector will be the vector itself.

We now can verify our intuition. Let $P = X(X'X)^{-1}X'$. When $X \in \mathcal{M}(n \times n)$ with linearly independent columns[2] we can rewrite this equation using the fact that for invertible matrices $A, B \in \mathcal{M}(n \times n)$ that

[1]We appeal to the linear independence of the columns of X to get the strict inequality

[2]Having n linearly independent columns in n-dimensional space means that X is of full rank and thus is invertible

$(AB)^{-1} = B^{-1}A^{-1}$.

$$
\begin{aligned}
P &= X(X'X)^{-1}X' \\
&= X(X)^{-1}(X')^{-1}X' \\
&= I
\end{aligned}
$$

□

Exercise 3

Question:

Show that the projection of $y \in \mathcal{R}^n$ onto span$\{1\}$ is the mean of the elements of y.

Solution:

The only column in X is a vector of ones, $X = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$. We can then get the projection of $y \in \mathcal{R}^n$ onto X by:

$$
\begin{aligned}
\tilde{y} = Py &= X(X'X)^{-1}X'y \\
&= \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \left(\begin{bmatrix} 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \dots \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & \dots & 1 \end{bmatrix} y \\
&= \frac{1}{N} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \dots & 1 \end{bmatrix} y \\
&= \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \frac{1}{N} \sum_{i=1}^{n} y_i \\
&= \begin{bmatrix} \bar{y} \\ \vdots \\ \bar{y} \end{bmatrix}
\end{aligned}
$$

□

Exercise 4

Question:

It's well-known that for regressions with a constant term, the vector of residuals always sums to zero. To prove this, let $y \in \mathcal{R}^n$ and let $X \in \mathcal{M}(n \times k)$ have linearly independent columns. Let $S = \text{span}(X)$, $P = \text{proj}S$, and $M = I - P$. Let $\hat{u} = My$. Show that if X has a constant column, then the elements of $\hat{u}$ sum to 0. If possible give an argument based around orthogonal projection.

Solution:

Define $\hat{y} = Py$. We know that $\forall x \in \text{span}(X)$ that the inner-product $< y - \hat{y}, x >= 0$ because $y - \hat{y}$ is orthogonal to $\text{span}(X)$.

Note, since X has a constant column (a vector of ones), then $1 \equiv \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \text{span}(X)$ Our previous observation implies

$$
\begin{aligned}
< y - \tilde{y}, 1 > &= 0 \\
\Rightarrow < y - Py, 1 > &= 0 \\
\Rightarrow \sum_{i=1}^{n} (y_i - (Py)_i) = \sum_{i=1}^{n} ((I - P)y)_i &= 0 \\
\Rightarrow \sum_{i=1}^{n} My = \sum_{i=1}^{n} \hat{u} = 0
\end{aligned}
$$

$\square$

Exercise 5

Question:

Show that if S is a nonempty linear subspace of $\mathcal{R}^n$, then $S \cap S^{\perp} = \{0\}$.

Solution:

Assume by $\Rightarrow\Leftarrow$ that $\exists x \in S \cap S^{\perp}$ such that $x \neq 0$.

Notice that $S^{\perp} \equiv \{y \in \mathcal{R}^n | < y, S >= 0\}$

Since $x \in S$ and $x \in S^{\perp}$ then $< x, x >= 0$. However this is only true when $x = 0$ thus we have a contradiction. This means that if S is a nonempty linear subspace of $\mathcal{R}^n$ then $S \cap S^{\perp} = \{0\}$

$\square$