Quantitative Economics - NYU
Homework 6 - Part 1

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Exercise 1

Let $z \in \mathbb{R}^k, z \neq \mathbf{0}$. Linear independence of the columns of X implies $y = Xz \neq \mathbf{0}$. Then $y'y > 0$, or $(Xz)'Xz = z'X'Xz > 0$, that is to say, $X'X$ is positive definite. Finally positive definitiveness implies non-zero determinant, hence invertibility.

Exercise 2

If $k = n$, then $S = span(X) = \mathbb{R}^n$ and $S^{\perp} = \{\mathbf{0}\}$. Let $y \in \mathbb{R}^n$. Since $My \in S^{\perp}$, then $My = \mathbf{0}$, which in turn implies $(I - P)y = \mathbf{0}$. As y has been chosen arbitrarily, we conclude $P = I$. As the columns of X are linearly independent and $n = k$, the columns of X form a basis for $\mathbb{R}^n$. We then project a vector in $\mathbb{R}^n$ onto the entire $\mathbb{R}^n$, obtaining the same vector.

Exercise 3

Let $\mathbf{1}$ be a column vector of ones of length n. Then $P = \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'$. Let $y \in \mathbb{R}^n$. Now we observe that $\mathbf{1}'\mathbf{1} = n$ and $\mathbf{1}'y = \sum_{i=1}^n y_i$. Hence $P = \frac{1}{n}\mathbf{1}\mathbf{1}'$ and $Py = \frac{1}{n}\mathbf{1}\mathbf{1}'y = \mathbf{1}\frac{\sum_{i=1}^n y_i}{n} = \mathbf{1}\bar{y}$

Exercise 4

Since $\hat{u} \in S^{\perp}, X'\hat{u} = \mathbf{0}$. Let x_j^* be the constant column in X. Then the previous equality implies $\langle x_j^*, \hat{u} \rangle = 0$, i.e. $k\mathbf{1}'\hat{u} = k\sum_{i=1}^n \hat{u}_i = 0$ for $k > 0$.

Exercise 5

By contradiction let us suppose $\exists x \in S \cap S^{\perp}$ such that $x \neq \mathbf{0}$. Then $\langle x, x \rangle = 0$ by property of the orthogonal complement. Hence $\sum_{i=1}^n x_i^2 = 0$, which in turn implies $x_i = 0$ for all i, a contradiction.