Quantitative Economics: Problem Set 6

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March 24, 2016

Problem 1

Suppose we have a $k \times 1$ vector $\mathbf{v}$ such that $\mathbf{v} \neq \mathbf{0}$. Then:

$$\mathbf{v}'\mathbf{X}'\mathbf{X}\mathbf{v} = (\mathbf{X}\mathbf{v})'\mathbf{X}\mathbf{v} > 0$$

because this is equal to the sum of the squares of the elements in the vector $\mathbf{X}\mathbf{v}$. Therefore, by definition, $\mathbf{X}'\mathbf{X}$ is positive definite. Positive definite matrices have positive determinants, which mean they are invertible.

Problem 2

If $k = n$, then $\text{span}\{\mathbf{X}\} = \mathbb{R}^n$. This means that $\text{span}\{\mathbf{X}\}^\perp = \{0\}$, which means that $\mathbf{M} = 0$. Then $\mathbf{P} = \mathbf{I} - \mathbf{M}$ implies that $\mathbf{P}$ must be the identity.

Problem 3

Suppose that $\mathbf{X}$ is an $n \times 1$ vector of ones. Then $\mathbf{X}'\mathbf{X} = n$ and

$$\begin{aligned}
\mathbf{P} &= \frac{\mathbf{X}\mathbf{X}'}{n} \\
&= \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}
\end{aligned}$$

So the projection is going to be:

$$\mathbf{P}\mathbf{y} = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & & \vdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} y_i \\ \vdots \\ \frac{1}{n}\sum_{i=1}^{n} y_i \end{pmatrix}$$

Problem 4

The sum of residuals is:

$$\mathbf{1}'(\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{X}')\mathbf{y}$$

If $\mathbf{X}$ includes an intercept, then we know that projecting a vector of ones onto the column space of $\mathbf{X}$ will produce a vector of ones: $\mathbf{P1} = \mathbf{1}$. We can re-write this as:

$$\begin{aligned}\mathbf{1}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-\mathbf{1}} &= \mathbf{1}' \\ &= \mathbf{1}'\mathbf{I}\end{aligned}$$

But this means that $\mathbf{1}'\mathbf{I} - \mathbf{1}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{X}' = 0$, which means that the sum of the residuals is zero.

Problem 5

Take a vector $\mathbf{v}$ such that $\mathbf{v} \in S$ and $\mathbf{v} \in S^{\perp}$. Then $\mathbf{x} \cdot \mathbf{x} = 0$, and since this is the sum of squares of the elements of $\mathbf{x}$, the only way that can happen is if $\mathbf{x} = 0$.