

STACHUSKI
PROBLEM SET 6

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EXERCISE 1

Show that $\mathbf{X}'\mathbf{X}$ is invertible.

Answer:

Let $0 \neq \alpha \in \mathbb{R}^k$. Then

$$\begin{aligned}\alpha'(X'X)\alpha &= (X\alpha)'(X\alpha) \\ &= \left(\sum_{j=1}^k \alpha_j x_j \right)' \underbrace{\left(\sum_{j=1}^k \alpha_j x_j \right)}_z \\ &= \sum_{i=1}^n z_i^2 \geq 0\end{aligned}$$

where x_j is the j -th column of matrix X . But since $\{x_1, \dots, x_k\}$ is LI, z as a linear combination of columns of X is nonzero and therefore we have inequality holding strictly. This implies that $X'X$ is *positive definite*.

Hence, all eigenvalues of $(X'X)$ are positive. To see this, let λ_j, z_j be an eigenvalue, eigenvector pair of $(X'X)$. Then

$$\lambda_j z_j = (X'X)z_j \Rightarrow \lambda_j z_j' z_j = z_j'(X'X)z_j > 0$$

but $\lambda_j \|z\|^2 > 0 \Leftrightarrow \lambda_j > 0$. Taking as given the result regarding the relation of determinant of a matrix and its eigenvalues - $\det A = \prod_{j=1}^k \lambda_j$ - we get that determinant of $X'X$ is positive, therefore invertible.

EXERCISE 2

Let $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. Show that if $k = n$, then \mathbf{P} is the identity.

Answer:

If $k = n$, then X is square and, because we assumed that it has LI columns, is invertible. Since X invertible implies X' invertible¹ we have

$$P = X(X'X)^{-1}X' = XX^{-1}(X')^{-1}X' = I$$

The result is also intuitive. From the orthogonal projection theorem we know that Py is the vector in $\text{span}X$ which best approximates y . Here, $\text{span}X = \mathbb{R}^n$, so $Py = y \forall y \in \mathbb{R}^n$, which implies $P = I$

EXERCISE 3

Show that the projection of $\mathbf{y} \in \mathbb{R}^n$ onto $\text{span}\{\mathbf{1}\}$ is the mean of elements of \mathbf{y} .

Answer:

Let $S = \text{span}\{\mathbf{1}\}$. From the lecture notes, we have that $\text{proj } S$ can be represented in a matrix form as $\mathbf{P} = \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'$. Therefore the projection of $\mathbf{y} \in \mathbb{R}^n$ onto S is given by

$$\begin{aligned} \mathbf{P}\mathbf{y} &= \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'\mathbf{y} \\ &= \mathbf{1}(n)^{-1} \sum_{i=1}^n y_i \\ &= \begin{bmatrix} n^{-1} \sum_{i=1}^n y_i \\ \vdots \\ n^{-1} \sum_{i=1}^n y_i \end{bmatrix} \end{aligned}$$

EXERCISE 4

Let $\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{X} \in \mathcal{M}(n \times k)$ have linearly independent columns. Let $S = \text{span}\{x_1, \dots, x_k\}$, $\mathbf{P} = \text{proj } S$ and $\mathbf{M} = \mathbf{I} - \mathbf{P}$. Show that if $x_1 = \mathbf{1}$, then elements of $\hat{\mathbf{u}} = \mathbf{M}\mathbf{y}$ sum to zero.

¹The fact that X is invertible mean that exists X^{-1} such that $XX^{-1} = X^{-1}X = I$ but in that case $(X^{-1})'X' = X'(X^{-1})' = I \Rightarrow (X^{-1})' = (X')^{-1}$

Answer:

By definition of \mathbf{M} , we can rewrite $\hat{u} = y - \mathbf{P}y$. From the **Orthogonal Projection Theorem** we have $y - \mathbf{P}y \perp S$. In particular, \hat{u} must be orthogonal to $\mathbf{1}$, since $\mathbf{1} \in S$. Therefore, the result follows from the realization of $\sum_{i=1}^n \hat{u}_i = \langle \mathbf{1}, \hat{u} \rangle = 0$.

EXERCISE 5

Show that if S is a nonempty subset of \mathbb{R}^n , then $S \cap S^\perp = \{0\}$.

Answer:

The following is true

$$x \in S \cap S^\perp \Rightarrow \langle x, x \rangle = \|x\|^2 = 0 \Leftrightarrow x = 0$$

where the last inequality follows from the properties of the norm induced by $\langle \cdot, \cdot \rangle$.

However, from the assumption on the statement only, we cannot be sure that $S \cap S^\perp \neq \emptyset$ - we would if S was assumed to be a linear subspace. So, we either have $S \cap S^\perp = \emptyset$ if $0 \notin S$ or the desired result if we add the assumption $0 \in S$.