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Part 1: Analytic Exercises

Exercise 1. Prove that **X'X** is invertible

To do so I will first prove that if a square matrix is positive definite, then it is invertible.
Let $\mathbf{A}$ be a square matrix.
Fact (1): $\mathbf{A}$ is non-singular iff it is invertible.
Fact (2): $\mathbf{A}$ is non-singular iff its determinant is non-zero.
Fact (3): If $\mathbf{A}$ is positive definite, then it has a unique decomposition $\mathbf{A} = \mathbf{VDV}^T$ where $\mathbf{D}$ is a diagonal matrix with the eigenvalues and $\mathbf{V}$ is an orthogonal matrix ($\mathbf{V}^T\mathbf{V} = \mathbf{I}$).
Fact (4): If $\mathbf{A}$ is positive definite, then it has positive eigenvalues.
Proof: $\mathbf{Az} = \lambda\mathbf{z} \implies \mathbf{z'Az} = \lambda\mathbf{z'z}$ and since LHS is positve and $\mathbf{z'z} > 0$, then $\lambda > 0$
Fact (5): If $\mathbf{A}$ is positive definite, then it has a non-zero determinant.
Proof: By the single value decomposition, $|\mathbf{A}| = |\mathbf{VDV}^T| = |\mathbf{V}|^2|\mathbf{D}| = |\mathbf{D}| > 0$ [by Fact (4)].
Fact (6): By all the previous, if a matrix $\mathbf{A}$ is positive definite, then it is invertible.

So I have to show that the matrix **X'X** is positive definite.
We know that a matrix $\mathbf{M}$ is positive definite if and only if $\mathbf{z'Mz} > 0$ for all $\mathbf{z} \neq \mathbf{0}$.

Let $\mathbf{X}$ be of dimension nxk. Then we have that: $\mathbf{z'(X'X)z} = (\sum_{i=1}^{k} \mathbf{x}_i z_i)^T(\sum_{i=1}^{k} \mathbf{x}_i z_i) = \mathbf{y}^T\mathbf{y}$. Note that by the L.I. property, since $\mathbf{z} \neq \mathbf{0}$, then $\mathbf{y} \neq \mathbf{0}$, so $\mathbf{y}^T\mathbf{y} > 0$. Then, **X'X** is positive definite and then invertible.

Exercise 2. Prove that P is the identity

Since X is square and L.I., by fact in slide 33 it is non-singular, and by fact it is invertible. Note that this makes sense since the columns of X conform a basis for the whole of $\mathbb{R}^n$, which means that for any equation of the type $Xy = b$ there exists a unique solution and is given by $y = X^{-1}b$.
Then, $P = X(X'X)^{-1}X' = XX^{-1}X'^{-1}X' = I$. This makes sense by the definition of P. Recall

$$\hat{y} \equiv \arg\min_{z \in \mathbb{R}^n} \|y - z\| = Py \tag{1}$$

and in this particular case, we can take that norm all the way to 0, which, by definition of metric spaces, can only happen if $\hat{y} = y \iff P = I$

Exercise 3. .

Since the basis we are working with is a singleton (vector of ones), we get Span $\{\mathbf{1}\} \equiv \{\text{all } \alpha\mathbf{1} \text{ s.t. } \alpha \in \mathbb{R}\}$. Note that in this case we can rewrite the definition of the projection as the minimization of the sum of the squared residuals [due to definition of Euclidean Distance]:

$$\arg\min_{\mathbf{z}\in \text{span}\{\mathbf{1}\}} \sum_{i=1}^{n} (y_i - z)^2 \tag{2}$$

And from FOC it is direct that $z = \frac{1}{n}\sum_i^n y_i$

Exercise 4. .

If the matrix of regressors contains the intercept, then the sum of the residuals equals 0. The proof is direct when solving for the vector of regression coefficients. Recall

$$\hat{\beta} \equiv \arg\min_{\mathbf{b}\in\mathbb{R}^k} \sum_{n=1}^{N} [y_n - \mathbf{b'x}_n]^2 \tag{3}$$

Letting $\mathbf{x} = [\mathbf{1} \quad \mathbf{z}]$, the FOC yields: $\sum_{n=1}^{N}[y_n - c - \hat{\mathbf{b}}'\mathbf{z}_n] = \sum_{n=1}^{N} \hat{\mathbf{u}}_n = 0 \implies c = \bar{y} - \hat{\mathbf{b}}'\bar{\mathbf{z}}$

Note that if $c \neq 0$ then by omitting the constant we would have a positive sum.

Note that by theorem $\hat{\mathbf{u}}$ is orthogonal to S. In particular, it is orthogonal to the vector of constants. This means that the inner product between the residuals and the vector of ones is 0, and since is nothing else than the N times the mean of the residuals, we arrive to the same conclusion.

Exercise 5. .

By definition, $S \cap S^{\perp} = \{\mathbf{x} \in S \text{ s.t. } \mathbf{x} \perp S\}$. In particular, for an element $\mathbf{x} \in S \cap S^{\perp}$ we have that $\langle x, x\rangle = 0$. Since this is the square of a norm, $\langle x, x\rangle = \|\mathbf{x}\|^2 = 0 \implies \mathbf{x} = \mathbf{0}$ by definition of a norm.