Homework 6 Comp Econ

Analytical Exercises

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March 24, 2014

Analytical Exercises

Exercise 1 $X \in \mathcal{M}(n \times k)$. Proving that $X'X \in \mathcal{M}(k \times k)$ is invertible is equivalent to proving that $X'X$ is non singular (linearly independent columns), which is equivalent to proving that $\det(X'X) \neq 0$. Let's prove the latter statement.

First, I prove that $X'X$ is positive definite.
Take any $z \in \mathbb{R}^k \setminus \{0\}$. Because the columns of $X \in \mathcal{M}(n \times k)$ are linearly independent and $z \in \mathbb{R}^k \setminus \{0\}$, by taking the contrapositive in the definition of linear independence, we have that $Xz \neq 0_{n\times 1}$. This implies that the Euclidian norm $||Xz|| > 0$ (since $||Xz|| \geq 0$ and $Xz \neq 0_{n\times 1}$). Thus $||Xz||^2 > 0$, which is equivalent to $(Xz)'Xz > 0$, thus $z'X'Xz > 0$. That is, $X'X$ is positive definite.

Second, I prove that all eigenvalues of the positive definite matrix $X'X$ are positive. Consider the eigenvector $z \in \mathbb{R}^k \setminus \{0\}$ associated with the eigenvalue $\lambda \in \mathbb{R}$ in the eigendecomposition of $A \equiv X'X$. By definition, $Az = \lambda z$. Hence $z'Az = z'\lambda z = \lambda z'z$, and thus $\lambda = \frac{z'z}{z'Az} > 0$, because the quadratic form in a nonzero vector $z'z$ (or norm squared) is positive and A is positive definite. This holds for every λ and z.

Third and last, I prove $\det(X'X) > 0$. Consider the eigendecomposition of $X'X = Q\Lambda Q^{-1}$, where Q is a matrix whose columns are the eigenvectors of $X'X$ and Λ is a diagonal matrix containing the eigenvalues associated with these eigenvectors. Using successively the commutativity property of the determinant and its definition, we have $\det(X'X) = \det(Q\Lambda Q^{-1}) = \det(Q^{-1}Q\Lambda) = \det(\Lambda) = \prod_{i=1}^{n} \Lambda_{ii} > 0$. QED.

Exercise 2 The projection matrix $P = X(X'X)^{-1}X'$ is $(n \times n)$ because $X \in \mathcal{M}(n \times n)$. Let $z \in \mathbb{R}^n$ (column vector). We have that $Pz = X(X'X)^{-1}X'z = XX^{-1}(X')^{-1}X'z = I_{n\times 1}I_{n\times 1}z = z$, i.e. P is the identity. Here, X is invertible because its columns are linearly independent (exercise 1).

Geometric intuition. The matrix $X \in \mathcal{M}(n \times n)$ has n linearly independent columns. So $\{\text{col}_1(X), ...\text{col}_n(X)\}$ is a basis of $\mathbb{R}^n$. Thus Py is the orthogonal projection of a vector $y \in \mathbb{R}^n$ on span(X) = span $\{\text{col}_1(X), ...\text{col}_n(X)\} = \mathbb{R}^n$. Obviously this projection is the vector y is itself.

Exercise 3 By definition, span$(\mathbf{1}) = \{x \in \mathbb{R}^n : x = \sum_{i=1}^{n} \alpha_i \text{ for } \alpha_1, ...\alpha_n \in \mathbb{R}\}$, where $\mathbf{1}$ is an $(n \times 1)$ column vector of 1's.

The projection matrix on span$(\mathbf{1})$ is $P = \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}'$. The projection of $y \in \mathbb{R}^n$ on span$(\mathbf{1})$ is an $(n \times 1)$ vector of the mean of the elements of y because of the following:

$$Py = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \left(\begin{pmatrix} 1 \cdots 1 \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 \cdots 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \frac{1}{n} \sum_{i=1}^{n} y_i = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} y_i \\ \vdots \\ \frac{1}{n}\sum_{i=1}^{n} y_i \end{pmatrix}$$

Exercise 4 Let $\hat{u} \in \mathbb{R}^n$ be the residual from the orthogonal projection of $y \in \mathbb{R}^n$ on $S =$ span $\{\text{col}_1(X), ...\text{col}_k(X)\}$. Suppose that $\text{col}_1(X) = \mathbf{1} \in \mathbb{R}^n$. We want to show that $\sum_{i=1}^{n} \hat{u}_i =$
$\begin{pmatrix} 1 \cdots 1 \end{pmatrix} \begin{pmatrix} \hat{u}_1 \\ \vdots \\ \hat{u}_n \end{pmatrix} = \langle \hat{u}, \mathbf{1} \rangle = 0.$

From the orthogonal projection theorem, $\hat{u} = (I-P)y = My \perp S$, where $S =$ span $\{\text{col}_1(X), ...\text{col}_n(X)\}$. This implies that $\hat{u} \perp z$, $\forall z \in S$. In particular, $\hat{u} \perp \text{col}_1(X) = \mathbf{1}$. That is, $\langle \hat{u}, \mathbf{1} \rangle = 0$. QED.

Exercise 5 Take a linear subspace $S \neq \emptyset \subset \mathbb{R}^n$. Note that if S is a non empty linear subspace, then it contains at least the element $\{0\}$. Indeed, suppose $x \in S \neq \emptyset$. Then $x + (-1).x = 0 \in S$. Now, we want to show $S \cap S^{\perp} = \{0\}$.

I proved that $\{0\} \in S$. Because $S^{\perp}$ is a linear subspace, by the same token $\{0\} \in S^{\perp}$. So $\{0\} \subseteq S \cap S^{\perp}$.
Now let $x \in S \cap S^{\perp}$. $x \in S$, so by the definition of the orthogonal complement of S, $\langle x, z \rangle = 0$ $\forall z \in S^{\perp}$. In particular for $z = x$ (recall $x \in S^{\perp}$), we have $\langle x, x \rangle = 0$. That is, $||x||^2 = 0$, which is true if and only if $x = 0$. Thus $S \cap S^{\perp} \subseteq \{0\}$. QED.