Comp Econ Homework 6
Analytical Exercises

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Assume that $\mathbf{X} \in \mathcal{M}(n \times k)$ with linearly independent columns.

Exercise 1.

Show that $\mathbf{X}'\mathbf{X}$ is invertible.

Following the hint, first show that $\mathbf{X}'\mathbf{X} \in \mathcal{M}(k \times k)$ is positive definite. For this, take any $\mathbf{b} \in \mathbb{R}^k \setminus \mathbf{0}$. Then,

$$\mathbf{b}'\mathbf{X}'\mathbf{X}\mathbf{b} = (\mathbf{X}\mathbf{b})'\mathbf{X}\mathbf{b} = (\sum_{i=1}^{k} \text{col}_i(\mathbf{X})\mathbf{b}_i)'(\sum_{i=1}^{k} \text{col}_i(\mathbf{X})\mathbf{b}_i) = \left\| \sum_{i=1}^{k} \text{col}_i(\mathbf{X})\mathbf{b}_i \right\|^2 > 0.$$

The last strict inequality follows from the fact that the columns of $\mathbf{X}$ are linearly independent and $\mathbf{b} \neq \mathbf{0}$. That means that the linear combination of the columns can not be the zero vector. Hence, $\mathbf{X}'\mathbf{X}$ is positive definite.

This implies that all the eigenvalues of the matrix $\mathbf{X}'\mathbf{X}$ are positive. To see, take any eigenvector $\mathbf{v}$ and eigenvalue λ. Then using the definition of eigenvectors and pre-multiplying by $\mathbf{v}$ yields

$$\mathbf{v}'\mathbf{X}'\mathbf{X}\mathbf{v} = \mathbf{v}'\lambda\mathbf{v} = \lambda\|\mathbf{v}\|^2 > 0 \quad \Longrightarrow \quad \lambda > 0.$$

As $\det(\mathbf{X}'\mathbf{X}) = \Pi_{i=1}^{k}\lambda_i$, we have that the determinant is strictly positive and $\mathbf{X}'\mathbf{X}$ is invertible.

Exercise 2.

Let $\mathbf{P} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ and suppose that $n = k$. Note, that in this case we can factorize the inverse of the product $(\mathbf{X}'\mathbf{X})^{-1} = \mathbf{X}^{-1}(\mathbf{X}^{-1})'$ while also using the fact that the inverse of the transpose is equal to the transpose of the inverse. Then,

$$\mathbf{P} = \mathbf{X}\mathbf{X}^{-1}(\mathbf{X}^{-1})'\mathbf{X}' = \mathbf{I}_n\mathbf{I}_n = \mathbf{I}_n.$$

The geometric interpretation is more intuitive. Since the columns of $\mathbf{X}$ are linearly independent and there are n of them they span the whole space of $\mathbb{R}^n$. That is, the operator $\mathbf{P}$ is going to project vectors in $\mathbb{R}^n$ to the (non-strict) linear subspace $\mathbb{R}^n$. As a result, we can only start off with vectors in the subspace where we are projecting—it is also the whole space—which the linear operator $\mathbf{P}$ is going to leave unaltered. It acts like the identity matrix.

Exercise 3.

We are projecting onto the span of $\mathbf{1} \in \mathbb{R}^n$. The corresponding projection matrix is given by

$$\mathbf{P_1} = \mathbf{1}(\mathbf{1}'\mathbf{1})^{-1}\mathbf{1}' = \mathbf{1}\frac{1}{n}\mathbf{1}' = \frac{1}{n}\mathbf{1}\mathbf{1}'.$$

The matrix $\mathbf{1}\mathbf{1}'$ is in $\mathcal{M}(n \times n)$ and consists only of entries of 1. Hence, for $\mathbf{y} \in \mathbb{R}^n$ we have

$$\mathbf{P_1}\mathbf{y} = \bar{\mathbf{y}}$$

where $\bar{y} = \frac{1}{n}\sum_{i=1}^n y_i$.

Exercise 4.

Denote again the column vector of ones by $\mathbf{1}$. The annihilator $\mathbf{M}$ by definition projects every $\mathbf{z} \in S$ to $\mathbf{0}$. Furthermore, we know that $\mathbf{1} \in S$ as it is one of the column vectors of $\mathbf{X}$—which column space is S. Hence, $\mathbf{M}\mathbf{1} = \mathbf{0}$. Now we can piece together the solution.

$$\sum_{i=1}^n \hat{u}_i = \mathbf{1}'\mathbf{u} = \mathbf{1}'\mathbf{M}\mathbf{y} = \mathbf{y}'\mathbf{M}\mathbf{1} = \mathbf{y}'\mathbf{0} = 0$$

Where the third equality follows from taking the transpose of a scalar.

Exercise 5.

Show that if S is a nonempty linear subspace of $\mathbb{R}^n$ then $S \cap S^\perp = \{\mathbf{0}\}$. Let proj$S = \mathbf{P}$ and proj$S^\perp = \mathbf{M}$.

Suppose otherwise and take $\mathbf{0} \neq \mathbf{z} \in S \cap S^\perp$.

Since $\mathbf{z} \in S^\perp$ we have that $\mathbf{Mz} = \mathbf{z}$. But $\mathbf{z}$ is also in S and as such $\mathbf{Pz} = \mathbf{z}$. Using the connection between $\mathbf{P}$ and $\mathbf{M}$ we get

$$\mathbf{z} = \mathbf{Mz} = (\mathbf{I} - \mathbf{P})\mathbf{z} = \mathbf{z} - \mathbf{Pz} = \mathbf{z} - \mathbf{z} = \mathbf{0}.$$

This contradicts with $\mathbf{0} \neq \mathbf{z}$.