Computational Economics
Homework 6

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Exercise 1: *Show that $X'X$ is invertible.*

First, I show that $X'X$ is positive definite. A matrix A is positive definite if $z'Az > 0$ $\forall\ z \in \mathbb{R}^n/\{0\}$. Let $z \in \mathbb{R}^n/\{0\}$, then $z'(X'X)z = (Xz)'(Xz) \neq 0$ where the non-zero result follows from the fact that the linear independence of X implies $Xz \neq 0$ $\forall z \in \mathbb{R}^n/\{0\}$. Now, since $(Xz)'(Xz)$ is a square term, it cannot be negative, and assuming X is not a matrix of zeros, then $z'(X'X)z > 0$, and so $X'X$ is positive definite.

Now, the positive definiteness of $X'X$ implies that $\det(X'X) > 0$ by Sylvester's Criterion. A positive determinant implies $X'X$ is non-singular, which implies that $X'X$ is invertible.

Exercise 2: *Show that the projection matrix is the identity when $k = n$*

We have $P = X(X'X)^{-1}X'$. When $k = n$, X is a square, $k \times k$ matrix. Since X has linearly independent columns, they form a set of basis vectors for $\mathbb{R}^k$. Then, by the theorem on page 25 of the notes, $P = X(X'X)^{-1}X'$ is the projection matrix on $\mathbb{R}^k$.

Now, by the Orthogonal Projection Theorem II (OPT II), $Py = y$ iff $y \in S = span(X) = \mathbb{R}^k$. So $Py = y$ $\forall y \in \mathbb{R}^k$. That is, P is the identity matrix for elements in $\mathbb{R}^k$.

Geometric intuition:

The OPT II implies that projecting an element, y, onto the space, S, from which it came leaves y unaltered. We construct projections by taking linear combinations of vectors in S, and $y \in S$ can be constructed from itself.

When $n \neq k$, some elements, x, are outside the space. Projecting x onto the S involves finding the closest point to x inside S. This point, $\hat{x}$, will not be equal to x, and so $\hat{x} = Px \neq x$.

However, when $n = k$, our space is the entire set of k-valued real vectors. Then any vector in $\mathbb{R}^k$ is inside the space. Thus projecting a vector in $\mathbb{R}^k$ onto S is simply projecting onto $\mathbb{R}^k$ itself. Hence, the projection returns the element being projected.

Exercise 3: *Show that projecting $y \in \mathbb{R}^n$ onto $span(\mathbb{1})$ is the mean of the elements of y.*

The projection is $P = X(X'X)^{-1}X'$ where $X = \alpha[1, 1, ..., 1]' = \alpha\mathbb{1}$ for some $\alpha \in \mathbb{R}$. This is the case because $S = span(\mathbb{1})$ is just linear combinations of the one vector, and X must have linearly independent columns. Without loss of generality, suppose $\alpha = 1$.

$$\begin{aligned} P &= \mathbb{1}(\mathbb{1}'\mathbb{1})^{-1} \\ &= \mathbb{1}\left(\sum_{i=1}^{n} 1 \times 1\right)^{-1} \mathbb{1}' \\ &= \mathbb{1}(n)^{-1}\mathbb{1} \\ &= \mathbb{1}\frac{1}{n}\mathbb{1}' \end{aligned}$$

And so

$$\begin{aligned} Py &= \mathbb{1}\frac{1}{n}\mathbb{1}'y \\ &= \mathbb{1}\frac{1}{n}\left(\sum_{i=1}^{n} 1 \times y_i\right) \\ &= \mathbb{1}\bar{y} \end{aligned}$$

where $\bar{y}$ is the mean of the elements of y.

Exercise 4: *Show that if X has a constant column, then elements of $\hat{u}$ sum to 0.*

Let $\tilde{X} = [\mathbb{1},\ X] \in \mathcal{M}(n \times k + 1)$ be the matrix with a ones column. Notice that as long as $k < n$, then $\tilde{X}$ will be linearly independent, like X. Let $S = span(\tilde{X})$. Note that $\mathbb{1} \in S$ since $\mathbb{1} \in col(\tilde{X})$. By OPT II, $P\mathbb{1} = \mathbb{1}$, since $\in S$. Note, also, that since P is symmetric, $P = P'$, so $\mathbb{1}'P' = \mathbb{1}'P = \mathbb{1}'$.

Now, $\hat{u} = My = (I - P)y$. The sum of residuals is then given by

$$\begin{aligned} \mathbb{1}'\hat{u} &= \mathbb{1}'(I - P)y \\ &= (\mathbb{1}'I - \mathbb{1}'P)y \\ &= (\mathbb{1}' - \mathbb{1}')y \\ &= 0'y \\ &= 0 \end{aligned}$$

So with a column of ones in the $\hat{X}$ vector, the sum of residuals always sums to zero.

Exercise 5: Show that if S is nonempty in $\mathbb{R}^n$, then $S \bigcap S^{\perp} = \{0\}$

Note that if $x \in S^{\perp}$, then $< x, y >= 0 \; \forall y \in S$.

Suppose $x \in S \bigcap S^{\perp}$, then $x \in S$ and $x \in S^{\perp}$. Since $x \in S^{\perp}$, then $< x, y >= 0 \; \forall y \in S$. Since $x \in S$, then $< x, x >= 0$. But this is only possible for $x = 0$.

Therefore, $x \in S \bigcap S^{\perp}$ implies that $x = 0$ which implies that $S \bigcap S^{\perp} = \{0\}$.