lec1 ODE 几何方法

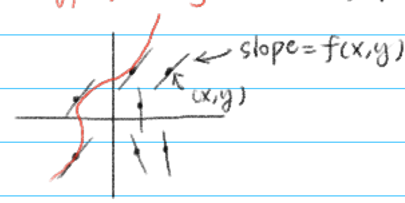
first-order ODE's

$y' = f(x,y)$

$y' = \frac{x}{y}$ $\boxed{y' = x - y^2}$ → bad : unsolvable

$y' = y - x^2$

Geometric view :



Analytic : Geometric :

$y' = f(x,y)$ ⟷ Direction field

$y_1(x)$ sol'n ⟷ Integral curve

$y_1(x)$ Soln to $y' = f(x,y)$ ⟺ graph of $y_1(x)$ is an int. curve

$y_1'(x) = f(x, y_1(x))$ slope of $y_1(x)$ = slope of direction field

at $(x, y_1(x))$ $f(x, y_1(x))$

Draw DIR. FLD

computer : Human :

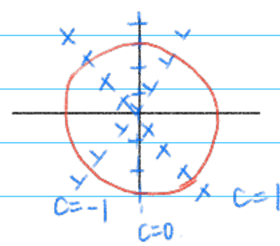
1. Pick (x,y) [equally spaced]
2. $f(x,y)$ – finds
3. Draws ← slope $f(x,y)$

1. Pick slope C
2. $f(x,y) = C$ plot eqn → isocline (equal slope)
3. → slope C

Example 1 : $y' = -\frac{x}{y}$

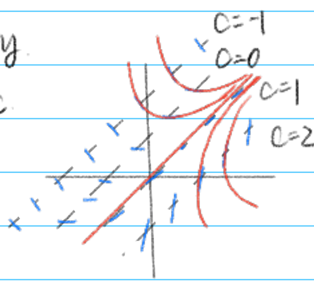
$-\frac{x}{y} = C$ isocline

$\boxed{y = -\frac{1}{C}x}$



Example 2 : $y' = 1 + x - y$

$y = x + 1 - C$



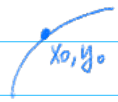
$y(x) \sim x$

$\boxed{\text{Two Integ. curve can't cross}}$ Can't have two slope ⊗

→ Only one direction at one point

$\boxed{\text{Two int. curve cannot be tangent}}$ (tough) ⊗

Exist + Uniqueness:



$y' = f(x,y)$ has one and only one soln. through (x_0, y_0)

HYP: $f(x,y)$ continuous near (x_0, y_0)

$f_y(x,y)$ continuous near (x_0, y_0)

$\frac{dy}{dx} = \frac{1-y}{x}$ → 不是标准形式

not continuous at $x=0$; unique and existence therom can't guarantee at y axis $(x=0)$

$xy' = -y + 1$

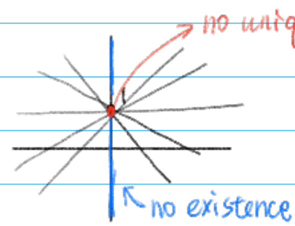
$\frac{dy}{1-y} = \frac{dx}{x}$

$\ln|1-y| = \ln|x| + C_1$

$1-y = cx$

$y = 1 - cx$

Existence:



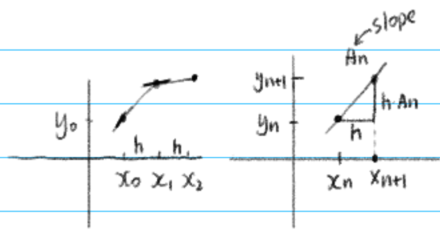
bad points

Lec2 欧拉数值方法及推广

numerical solns

IVP $\begin{cases} y' = f(x,y) & \text{ODE} \\ y(x_0) = y_0 & \text{initial value} \end{cases}$

Euler's method



$y_{n+1} - y_n = h \cdot A_n$

$y_{n+1} = y_n + h \cdot A_n$

h = step size

Euler equation (program)

$\begin{cases} x_{n+1} = x_n + h \\ y_{n+1} = y_n + h \cdot A_n \\ A_n = f(x_n, y_n) \end{cases}$

recursive equation
for computer calculate

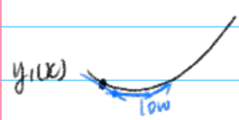
Example.

$\begin{cases} y' = x^2 - y^2 \\ y(0) = 1 \\ h = 0.1 \end{cases}$

n	x_n	y_n	A_n	hA_n
0	0	1	-1	-0.1
1	0.1	0.9	-0.8	-0.08
2	0.2	0.82		

$y(0.2) = 0.82$

Euler too high or too low?



convex 凹 $y'' > 0$ (y' increase all the time)

concave 凸 $y'' < 0$

$y' = x^2 - y^2$

$y'' = 2x - 2yy'$ At (0,1) $y(0) = 1$ $y'(0) = -1$ $y''(0) = -2 \times (-1) = 2$ convex → E. too low at start point

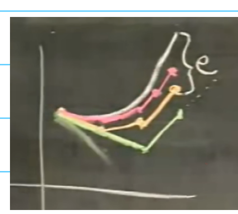
smaller step size h

error e $\nearrow$ depends on

$\boxed{e \sim c_1 h}$

Euler: first-order method

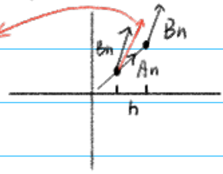
→ halve the h, halve the e



Better method

find a better value than A_n

better slope



$$\begin{cases} x_{n+1} = x_n + h \\ y_{n+1} = y_n + h\left(\frac{A_n+B_n}{2}\right) \end{cases}$$

$$\begin{cases} \tilde{y}_{n+1} = y_n + hA_n \quad \text{(temporary value)} \\ B_n = f(x_{n+1}, \tilde{y}_{n+1}) \end{cases}$$

Heun's –
Improved Euler
Modified Euler

RK2 (2: 2nd-order method)

$e \sim C_2 h^2$

RK4 : standard method : very accurate
Runge-Kutta 4th order-method : $\frac{1}{6}(A_n + 2B_n + 2C_n + D_n)$ ← Super slope

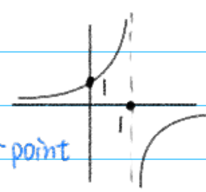
Pitfalls :

#1 you'll find

#2 $y' = y^2$

soln : $y = \frac{1}{c-x}$

↑ singular point at $x = c$



lec3 一阶线性ODE

$\boxed{a(x)y' + b(x)y = c(x)}$

Linear: $ay_1 + by_2 = c$ ($c=0$ homogeneous)

Standard LINEAR form: $\boxed{y' + p(x)y = q(x)}$ *

~~[NOT standard 1st-order form: $y' = p(x)y + q(x)$]~~ ⊗

MODELS

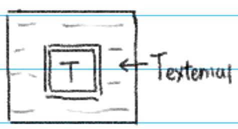
$\boxed{\text{Temp.-conc.}}$ Conduction-diffusion model

$\boxed{\text{Mixing}}$

Decay, bank acc't

some motion prbs

Conduction



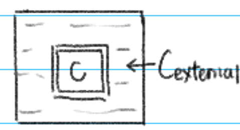
Newton cooling law

$$\boxed{\frac{dT}{dt} = k(T_e - T)}$$

t: time T: Temperature ($k>0$ ← conductivity)

$T(0) = T_0$

Diffusion



(membrane wall)

$$\boxed{\frac{dc}{dt} = k_1(C_e - C)}$$

C: salt conc. inside ($k_1>0$)

Ce: salt conc. outside

General linear eqn: $y' + p(x)y = q(x)$

$$\frac{dT}{dt} + k\cdot T = k\cdot T_e$$

↑ usually const. but not nec.

$y' + p(x)y = q(x)$ Integrating factor: $u(x)$

$uy' + puy = qu$

$\parallel$

$(uy)'$ works if $u' = p\cdot u$

Sep vars: $\ln u = \int p(x)dx$

$\boxed{u = e^{\int p\,dx}}$ Integrating factor

(no arb. constant since only 1 u needed)

Method: $y' + py = q$

* ⓪ standard linear form
① Calculate $e^{\int p dx}$ = int. factor
② Multiply $\boxed{\text{both}}$ sides by $e^{\int p dx}$
③ Integrate

Example1: $xy' - y = x^3$.

⓪ $y' - \frac{1}{x} y = x^2$

① $e^{-\int \frac{1}{x} dx} = \frac{1}{x}$

② $\frac{1}{x} y' - \frac{1}{x^2} y = x$

③ $\int (\frac{1}{x} y)' = \int x$

$\frac{1}{x} y = \frac{x^2}{2} + C$

$y = \frac{x^3}{2} + C \cdot x$

Example2: $(1+\cos x) y' - (\sin x) y = 2x$

⓪ $y' - (\sin x)/(1+\cos x) y = 2x/(1+\cos x)$

① $e^{-\int \sin x/(1+\cos x) dx} = 1 + \cos x$

② $(1+\cos x) y' - (\sin x)/(1+\cos x) y = 2x$

③ $\int [(1+\cos x) \cdot y]' = \int 2x$

$y(1+\cos x) = x^2 + C$

$y = \frac{x^2 + C}{1+\cos x}$

If $y(0) = 1$, $C = 2$

Linear with K constant

Temp: $\frac{dT}{dt} + KT = KT_e$

I.F.: $e^{\int K dt} = e^{Kt}$

$\frac{dT}{dt} \cdot e^{Kt} + K \cdot T e^{Kt} = K \cdot T_e \cdot e^{Kt}$

$\int (e^{Kt} \cdot T)' = \int K \cdot T_e \cdot e^{Kt} + C$

$T = \underbrace{e^{-Kt} \int_0^t K \cdot T_e(t_1) e^{Kt_1} dt_1}_{\text{steady state soln}} + \underbrace{C \cdot e^{-Kt}}_{\substack{\text{disappear when } t \to \infty \\ \to 0 \\ \text{transient}}}$

$T(0) = T_0 \quad C = T_0$

Lec4 一阶方程换元法 Substitution.

Scaling $y' = f(x,y)$

$\boxed{x_1 = \frac{x}{a} \quad y_1 = \frac{y}{b}}$ (a, b constant)

① change units

② make vars. dimensionless (w/o. units)

③ reduce #, or simplify constants

$$\boxed{\frac{dT}{dt} = k(M^4 - T^4)}$$

T: int. temp

M: constant ext. temp

(big temp diff)

$T_1 = \frac{T}{M} \qquad T = MT_1$

$M \cdot \frac{dT_1}{dt} = kM^4(1 - T_1^4)$ less constant

$\boxed{\frac{dT_1}{dt} = k_1(1 - T_1^4)}$ $k_1 = kM^3$ "lumping constants" *

A. Direct: new var = old var. $\quad T_1 = \frac{T}{M} \quad$ e.g. $\int x\sqrt{1-x^2}\,dx \quad u = 1 - x^2$

B. Inverse: old var = new var. $\quad T = M \cdot T_1 \quad$ e.g. $\int \sqrt{1-x^2}\,dx \quad x = \sin u$

DIRECT SUB. Example 1:

Bernoulli equation

$y' = p(x)y + q(x)y^n \quad (n \neq 0)$

$\frac{y'}{y^n} = p(x) \cdot \frac{1}{y^{n-1}} + q(x)$

let $v = \frac{1}{y^{n-1}} \quad v' = (1-n)\frac{1}{y^n} \cdot y'$

$\frac{v'}{1-n} = p(x) \cdot v + q(x)$ linear!

Example 2:

$y' = \frac{y}{x} - y^2$

$\frac{y'}{y^2} = \frac{1}{xy} - 1$

let $\frac{1}{y} = v \quad v' = -\frac{1}{y^2} \cdot y'$

$-v' = \frac{v}{x} - 1$

$v' + \frac{v}{x} = 1$

Int. factor $= e^{\int \frac{1}{x} dx} = e^{\ln x} = x$.

$x \cdot v' + v = x$.

$\int (xv)' = \int x$

$xv = \frac{x^2}{2} + C$

$v = \frac{x}{2} + \frac{C}{x} = \frac{1}{y} \qquad y = \frac{2x}{x^2 + 2C}$

Homogeneous ODE's

$y' = F(y/x)$

Example1: $y' = \frac{x^2 y}{x^3+y^3} = \frac{y/x}{1+(y/x)^3}$ Invariant under zoom : $x \to ax_1$, $y \to ay_1$

$z: xy' = \sqrt{x^2+y^2}$

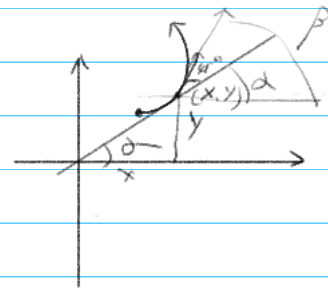
$y' = \sqrt{1+(y/x)^2}$

$y' = F(y/x)$

$z = y/x \quad y = zx \quad y' = z'x + z$

$z'x + z = F(z)$

$x \cdot \frac{dz}{dx} = F(z) - z$

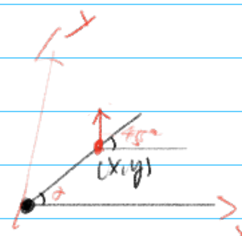


$\tan\alpha = \frac{y}{x}$

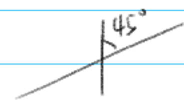
$\alpha + 45 = \beta$

$\tan\beta = \frac{\frac{y}{x}+1}{1-\frac{y}{x}} = \frac{dy}{dx}$

Example :



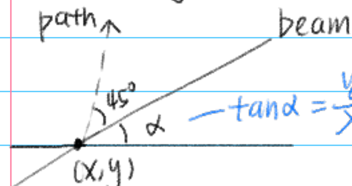
- drug boat goes



beam

What's the boat's path? with known angle

constant 45° to beam



$\tan\alpha = \frac{y}{x}$

$y = y(x)$ unknown graph

$y' = \tan(\alpha + 45°) = \frac{\tan\alpha + \tan\pi/4}{1 - \tan\alpha\tan\pi/4}$

$y' = \frac{y/x + 1}{1 - y/x}$ homogeneous ODE

$z = y/x$

$y = zx; y' = z'x + z$

$z'x + z = \frac{z+1}{1-z}$

$\frac{dz}{dx} \cdot x = \frac{z + 1 - z + z^2}{1-z} = \frac{z^2+1}{1-z}$

$\frac{1-z}{z^2+1} dz = \frac{1}{x} dx$

$\tan^{-1} z - \frac{1}{2}\ln(1+z^2) = \ln x + C$

$\tan^{-1} z = \ln\sqrt{1+z^2} + \ln x + C$

$\tan^{-1}(y/x) = \ln\sqrt{x^2+y^2} + C$

$\theta = \ln r + C$

$r = C_1 e^{\theta}$

Lec5 一阶自治ODE

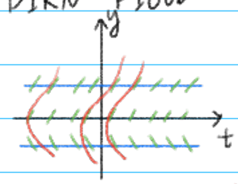
$\frac{dy}{dt} = f(y) \leftarrow$ no t on RHS

autonomous

(sep. var.)

Prob: get quantitative info about solns

DIR'N FIELD



Lines: $y = y_0$

isoclines: slopes: $f(y_0)$

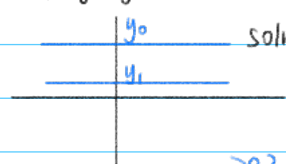
int. curve 平移不改变形状

$\frac{dy}{dt} = f(y)$

Critical point y_0

$f(y_0) = 0$ ① find the critical point

$y = y_0$ is a soln



soln (int. curve)

② Graph $f(y)$ >0? <0? $\frac{dy}{dt}$ increasing

E.g. y = money in acc't

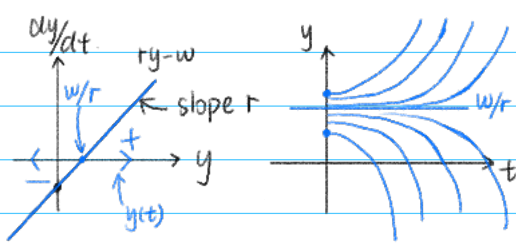
r = (cont) interest rate w = rate of embezzlement

$\frac{dy}{dt} = ry - w$

① crit. pts $ry - w = 0$

$y = \frac{w}{r}$ c.p.

②



LOGISTIC Equation

Example: Population behavior $y(t)$

$\frac{dy}{dt} = ky$ k: growth rate

k: constant = simple growth

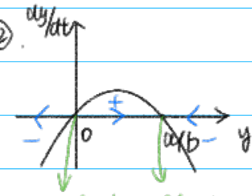
logistic growth: k declines as y increases

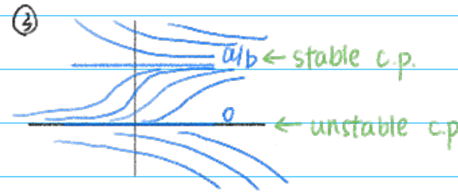
simplest choice: $k = a - by$

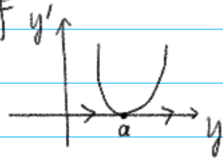
$$\frac{dy}{dt} = ay - by^2$$ sep. vars. partial fraction

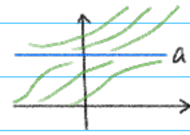
crit. pts. ① $ay - by^2 = 0$

$y = 0 \quad y = a/b$

② 

③ 

If 

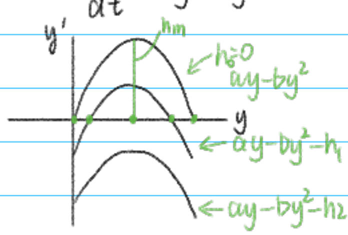


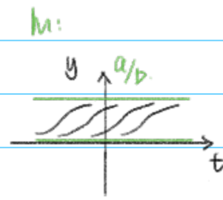
semistable

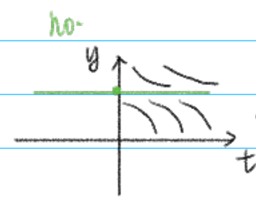
Logistic equation with harvesting

harvest: at constant time rate h

$$\frac{dy}{dt} = ay - by^2 - h$$





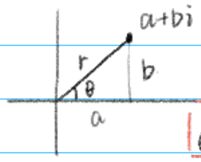


Lec6 复数和复指数

$z = a+bi \quad \bar{z} = a-bi \quad z\cdot\bar{z} = a^2+b^2$

$\frac{2+i}{1-3i}\cdot\frac{1+3i}{1+3i} = \frac{1}{10}(2+i)(1+3i) = \frac{1}{10}(-1+7i)$

Polar rep'n



$a+bi = r\cos\theta + i\cdot r\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ Euler

Exponential

① Exp. law: $a^x \cdot a^y = a^{x+y}$ $\boxed{e^{i\theta_1}\cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)} \ ?}$

② e^{at} $\frac{dy}{dt} = ay$ $\boxed{\frac{d}{d\theta}e^{i\theta} = ie^{i\theta} \ ?}$

Soln $y(0)=1$

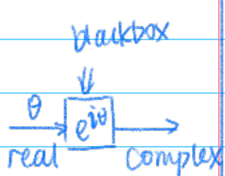
③ infinite series should work out

$e^{i\theta_1}\cdot e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= \cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1 - \sin\theta_1\sin\theta_2$

$= i\sin(\theta_1+\theta_2) + \cos(\theta_1+\theta_2)$

$= e^{i(\theta_1+\theta_2)}$



$e^{i\theta} = \cos\theta + i\sin\theta$

such a complex-valued fn of t:

$u(t) + iv(t) \quad (u(t)=\cos\theta \quad v(t)=\sin\theta)$

$D(u+iv) = Du + iDv$

$\frac{d}{dt}e^{it} = \frac{d}{dt}(\cos t + i\sin t)$

$= -\sin t + i\cos t$

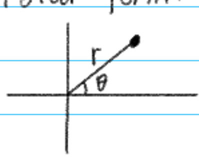
$= ie^{it}$

$e^{i0} = \cos 0 + i\sin 0$

$= 1$

$e^{a+ib} = e^a\cdot e^{ib}$

Polar form: $\alpha = re^{i\theta}$ r: modulus of α



θ: argument of α $\arg(\alpha)$

Advantage of polar form: good for multiplication

$$r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

mult. the moduli, add the arguments

复化: complex domain

Example: Real part of $e^{-x+ix} = \mathrm{Re}\left(e^{(-1+i)x}\right)$

$$\int e^{-x}\cos x \, dx = \mathrm{Re}\int e^{(-1+i)x}\, dx$$

$\underbrace{e^{-x}}_{\text{real}} \cdot \underbrace{\cos x}_{\text{real part of } (e^{ix})}$

$$\int e^{(-1+i)x}\,dx = \frac{e^{(-1+i)x}}{-1+i} = \frac{1}{-1+i}\, e^{-x}(\cos x + i\sin x)$$

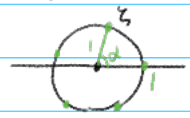
$$= \frac{-1-i}{2e^x}(\cos x + i\sin x)$$

$$= \frac{-1}{2e^x}(\cos x + i\cos x + i\sin x - \sin x)$$

Want Real part only $\quad \frac{1}{2e^x}(-\cos x + \sin x)$

Complex roots:

$\sqrt[n]{1}$: n answers as cx #s



$\zeta = e^{i\cdot\frac{2\pi}{5}} \quad \zeta^5 = e^{i\cdot 2\pi} = 1$ (since 2π and 0 are the same angle)

ζ: zeta

$\alpha = 2\pi/5$

5 5^{th} roots of 1

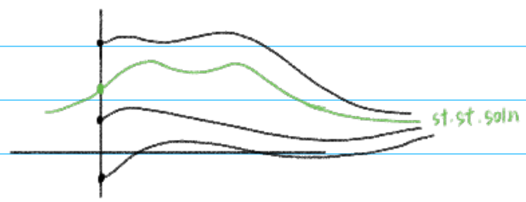
Lec7-8

一阶常系数线性方程

Solve $y' + ky = q(t)$

Sol'n $y = \underbrace{e^{-kt}\int q(t)\cdot e^{kt}\,dt}_{\text{steady state sol'n (sss)}} + \underbrace{Ce^{-kt}}_{0 \text{ as } t\to\infty}$ transient

use $y(0)$

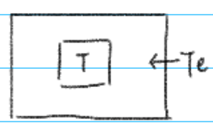


Pick simplest fn

input: $q(t)$

response: $y(t)$ soln to ODE

physical input: $q_e(t)$



$y' + ky = kq_e(t)$

$\boxed{1/k\, y' + y = q_e(t)}$

standard form: input response analysis

Superposition of inputs: $q_1(t) \longrightarrow y_1(t)$

$q_2(t) \longrightarrow y_2(t)$

LINEAR of ODE $\Rightarrow$ $\begin{cases} q_1 + q_2 \longrightarrow y_1 + y_2 \\ cq_1 \longrightarrow cy_1 \end{cases}$

Physical input: trigonometric

$y' + ky = kq_e(t)$, $q_e(t) = \cos(wt)$

w: angular frequency = # complete osc in 2π

Prob: $q_e = \cos(wt)$, find the response

复化 Complexify the problem (some easier to exponentials)

$\tilde{y}' + k\tilde{y} = k\cdot e^{iwt}$ $\quad \tilde{y} = y_1 + iy_2$ (complex sol'n)

Find $\tilde{y}$, then y_1 will solve the original ODE int. f'r. e^{kt}

$$(\tilde{y}'e^{kt})' = k e^{(i\omega+k)t}$$

$$\tilde{y} e^{kt} = \frac{k}{k+\omega i} e^{(i\omega+k)t}.$$

Two methods:

① GO Polar

② GO Cartesian笛标

Cartesian method

$$\boxed{\tilde{y} = \frac{1}{1+\omega i/k} e^{i\omega t}} = \frac{1-\omega i/k}{1+(\omega/k)^2}(\cos\omega t + i\sin\omega t)$$

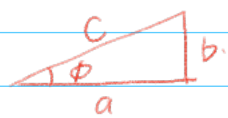
take real part

$$= \frac{k^2-\omega i k}{k^2+\omega^2}(\cos\omega t + i\sin\omega t)$$

$$= 1/(k^2+\omega^2)\,(k^2\cos\omega t + k^2 i\sin t - \omega k i\cos\omega t + \omega k\sin\omega t)$$

$$y_1 = 1/(k^2+\omega^2)\cdot(k^2\cos\omega t + \omega k\sin\omega t) = \frac{1}{1+(\omega/k)^2}(\cos\omega t + \frac{\omega}{k}\sin\omega t)$$

☆ $\boxed{a\cos\theta + b\sin\theta = C\cos(\theta-\phi)}$



$$= \frac{1}{1+(\omega/k)^2}\cdot\sqrt{1+(\omega/k)^2}\cdot\cos(\omega t-\phi)$$

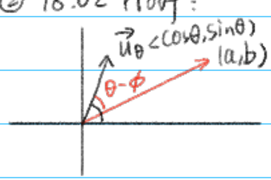
笛卡尔和极坐标系的转化

Proof:

① H.S. Proof: extend the R.H.S = L.H.S

but how about from L.H.S to R.H.S?

② 18.02 Proof:



$$a\cos\theta - b\sin\theta = \langle a,b\rangle\cdot\langle\cos\theta, \sin\theta\rangle$$

$$= |\langle a,b\rangle|\cdot|\langle\cos\theta,\sin\theta\rangle|\cdot\cos(\theta-\phi)$$

$$= |\langle a,b\rangle|\cdot 1\cdot\cos(\theta\ \alpha|$$

$$= C\cos(\theta-\phi)$$

③ 18.03 Proof:

$$(a-bi)(\cos\theta + i\sin\theta)$$

$$= \sqrt{a^2+b^2}\,e^{-i\phi}\cdot e^{i\theta}$$

$$= C\cdot e^{i(\theta-\phi)}$$

take the real part of both sides: $a\cos\theta + b\sin\theta = C\cdot\cos(\theta-\phi)$

Polar method

① Remember α complex

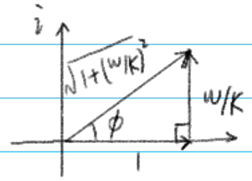
$\frac{1}{\alpha}\cdot\alpha = 1$ $\qquad$ $\arg(1/\alpha) + \arg(\alpha) = 0$

$\left|\frac{1}{\alpha}\right| = \frac{1}{|\alpha|}$ $\qquad$ $\arg(1/\alpha) = -\arg(\alpha)$

$\frac{(ix)^3}{3!} = \frac{i^3 x^3}{3!}$

② Polar form: $\frac{1}{1+i(w/k)} = Ae^{-i\phi}$

$= \frac{1}{\sqrt{1+(w/k)^2}} \cdot e^{-i\phi}$



$e^{i\theta} = \cos\theta + i\sin\theta$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!}$

$\tilde{y} = \frac{1}{1+wi/k} e^{iwt}$

$\phi = \arg(1+i(w/k))$ $\qquad$ $= \frac{1}{\sqrt{1+(w/k)^2}} \cdot e^{-i\phi} \cdot e^{iwt}$

$= \arctan(w/k)$

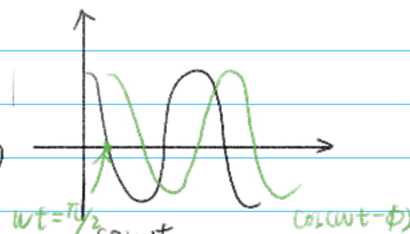
$= \tan^{-1}(w/k)$ $\qquad$ $= \frac{1}{\sqrt{1+(w/k)^2}} \cdot e^{i(wt-\phi)}$

phase lag of $\cos(wt-\phi)$

real answer $y_1 = \underbrace{\frac{1}{\sqrt{1+(w/k)^2}}}_{=A} \cdot \cos(wt-\phi)$

$k\uparrow \quad A\uparrow \quad \phi\downarrow$.

熱傳導速度



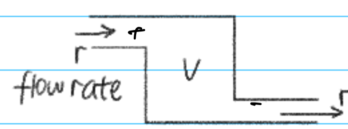
$$\boxed{\begin{array}{l} \text{Basic Linear ODE} \\ ① \ y' + ky = kq_e(t) \\ ② \ y' + ky = q(t) \\ ③ \ y' + p(t)y = q(t) \end{array}}$$

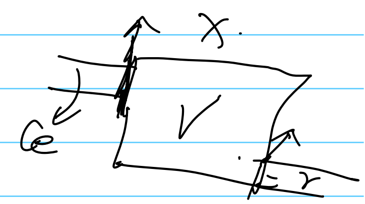
Temp-Conc

cond-Diff

$k>0$

Mixing example





$x(t)$ = amount of salt in tank at time t.

C_e = Conc. of incoming salt

r : flow rate

V : volume

$\frac{dx}{dt} = r \cdot C_e - r$

$\frac{dx}{dt}$ = rate of salt inflow − rate of salt outflow

$= r \cdot C_e - r \cdot \frac{x}{V}$

$\frac{dx}{dt} + r\cdot\frac{x}{V} = r \cdot C_e$ $\qquad$ $C(t) = \frac{x}{V}$ (tank conc.)

$V \cdot \frac{dc}{dt} + r \cdot C(t) = r \cdot C_e$

* $\boxed{\frac{dc}{dt} + \frac{r}{V}C(t) = \frac{r}{V} \cdot C_e}$ $\qquad$ $k = \frac{r}{V}$ basic parameter

Element Decay :

$$A \xrightarrow{\text{decay}} B \xrightarrow{\text{decay}}$$

$$\frac{dB}{dt} = k_1 A - k_2 B$$

$$= k_1 A_0 e^{-k_1 t} - k_2 B$$

$$B' + k_2 B = k_1 A_0 e^{-k_1 t}.$$

人口模型

biology $\frac{dP}{dt} = aP$

economics

If $k < 0$ none of steady-state ; input ; response . applies

$$\frac{dy}{dt} - ay = q(t) \quad a > 0, \; k = -a$$

$$e^{at} \int q(t) e^{-at} dt + Ce^{at} \longrightarrow \pm\infty .$$

not transient .

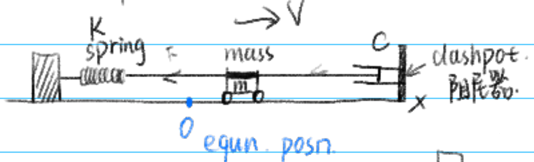
Lec 9,10 二阶常系数齐次线性方程.

Linear 2^{nd}-order ODE with constant coeffs.

$y'' + Ay' + By = 0 \leftarrow$ homog

Assume: gen soln: $y = c_1 y_1 + c_2 y_2$ where y_1 and y_2 are solns

INIT: choose c_1 and c_2

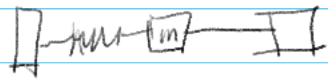


$F = ma = mx''$

x: 位移

$mx'' = -kx - cx$

Force spring dashpot



$mx'' + cx + kx = 0$

$$\boxed{x'' + \frac{c}{m}x + \frac{k}{m}x = 0}$$

阻尼常数 弹簧常数

(typical model)

To solve ODE $\rightarrow$ Find 2 solns (independent)

Basic method:

try $y = e^{rt}$ (t = ind. var.)

Plug in: $r^2 e^{rt} + Are^{rt} + Be^{rt} = 0$

$$\boxed{r^2 + Ar + B = 0}$$

characteristic equation of the system

Case 1: roots $r_1 \neq r_2$ (real)

gen soln: $\boxed{y = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$

Ex: $y'' + 4y' + 3y = 0$ $\quad y(0) = 1 \quad y'(0) = 0$

$r^2 + 4r + 3 = 0$

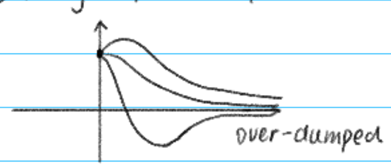
$(r+1)(r+3) = 0$

$y = c_1 e^{-3t} + c_2 e^{-t}$ $\quad y(0) = c_1 + c_2 = 1$

$y' = -3c_1 e^{-3t} - c_2 e^{-t}$ $\quad y'(0) = -3c_1 - c_2 = 0$

$\Rightarrow \begin{cases} c_1 = -1/2 \\ c_2 = 3/2 \end{cases}$

soln: $y = -\frac{1}{2} e^{-3t} + \frac{3}{2} e^{-t}$



Case 2: Complex roots $r = a \pm bi$ under-damped 欠阻尼

We get complex soln. $y = e^{(a+bi)t}$

Thrm: if $u+iv$ cplx soln to $y''+Ay'+By=0$, then u, v are real solns

Proof: $(u+iv)''+A(u+iv)'+B(u+iv)=0$

$$\underbrace{u''+Au'+Bu}_{\text{real part } =0} + i\underbrace{(v''+Av'+Bv)}_{\text{imaginary part } =0} = 0$$

$\rightarrow$ u & v are solns to $x''+Ax'+Bx=0$

soln: $y = e^{at+ibt} = e^{at}(\cos bt)$ real part $\quad e^{at}\cdot e^{ibt}$

$e^{at}(\sin bt)$ imaginary part $\quad = e^{at}(\cos bt + i\sin bt)$

$y = e^{at}(c_1\cos bt + c_2 \sin bt)$

e^{at} ↓ 指数: 控制振荡幅度

$(c_1\cos bt + c_2\sin bt)$ ↓ same frequency ⇒ pure sin oscillation

$y''+4y'+5y=0$

$r^2+4r+5=0$

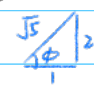
$r = \dfrac{-4\pm\sqrt{-4}}{2} = -2\pm i$

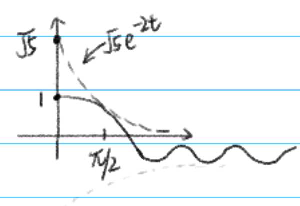
$e^{(-2+i)t} \quad e^{-2t}\cos t \quad e^{-2t}\sin t$

$y = e^{-2t}(c_1\cos t + c_2\sin t) \qquad y(0)=1 \quad y'(0)=0$

trigonometric identity

soln: $y = e^{-2t}(\cos t + 2\sin t) = \sqrt{5}\, e^{-2t}\cos(t-\phi)$





临界阻尼

Case 3: Critical-damped 2 equal roots $r^2+Ar+B=0$

$r=-a$

$(r+a)^2=0$

$r^2+2ar+a^2=0$

ODE: $y''+2ay'+a^2y=0$

soln: $y = e^{-at}$

Know one soln to $y''+py'+qy=0$, there's another $y=y_1 u$

$a^2 | y = e^{-at} \cdot u$

$2a | y' = -ae^{-at} \cdot u + e^{-at} u'$

$1 | y'' = a^2 e^{-at} u - 2a e^{-at} u' + e^{-at} u''$

$0 = 0 + 0 + e^{-at} u''$

$e^{-at} u'' = 0.$

$u'' = 0.$

$u = C_1 t + C_2$ (just t is enough)

2^{nd} soln: $y = e^{(-at) \cdot t}$

Lec 10 Complex roots of $r^2 + br + k = 0$ → oscillation.

$r = a \pm bi$

$r = a + bi$ soln: $e^{(a+ib)t}$ Re $e^{at} \cos bt = y_1$; Imaginary $e^{at} \sin bt = y_2$.

if $r = a - bi$ soln: $e^{(a-ib)t}$ $e^{at} \cos(-bt) = y_1'$; $e^{at} \sin(-bt) = y_2'$

gen soln: $y = C_1 y_1 + C_2 y_2$

$$\boxed{y = \underset{cx\ (cx)}{C_1} e^{(a+bi)t} + \underset{cx\ (cx)}{C_2} e^{(a-bi)t}}$$

Which are real solns?

Ans: by hack - multip all at, make Imaginary part = 0

$u + iv$ — $v = 0$; change i to $-i$; see if it stays the same

chge $i \to -1$, give cond'ns on the $C_1 + C_2$

$C_1 e^{(a+bi)t} + C_2 e^{(a-bi)t}$ $i \to -i$

$\overline{C_1} e^{(a-bi)t} + \overline{C_2} e^{(a+bi)t}$. want the same: $C_2 = \overline{C_1}$ ($\overline{C_2} = C_1$)

(cx conj.)

$(c + id) \cdot e^{(a+bi)t} + (c - id) e^{(a-bi)t}$. change this to old form (cos & sin)

[many sci + eng write soln this way]

HACK. write it all out

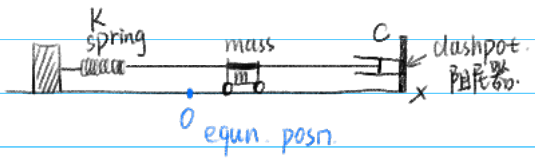
反过来的Euler formula

$\cos a = \frac{e^{ia} + e^{-ia}}{2}$

$\sin a = \frac{e^{ia} - e^{-ia}}{2i}$

NICER: $e^{at} [c(e^{ibt} + e^{-ibt}) + id(e^{ibt} - e^{-ibt})]$

$= e^{at} (2c \cdot \cos bt + 2id \cdot \sin bt)$



$mx'' = -kx - cx$

$x'' + \frac{c}{m}x' + \frac{k}{m}x = 0$

$y'' + 2py' + w_0^2 y = 0$

oscillations: $r^2 + 2pr + w_0^2 = 0$

$r = -p \pm \sqrt{p^2 - w_0^2}$

$p=0$ undamped 无阻尼. $y'' + w_0^2 y = 0$ ← circular frequency

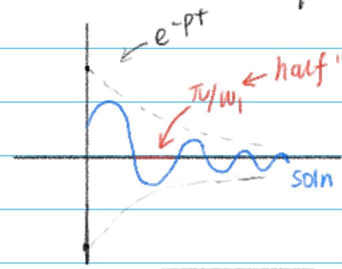
soln: $r = \pm i w_0$

$y = C_1 \cos w_0 t + C_2 \sin w_0 t$

$= A\cos(w_0 t - \phi)$

damped case.

get oscillation $p^2 - w_0^2 < 0 \iff p < w_0$



w_1: pseudofrequency

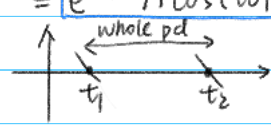
if $c\uparrow$(damping); $w_1 \downarrow$ (?)

$r = -p \pm \sqrt{-(w_0^2 - p^2)} = -p \pm \sqrt{-w_1^2}$

soln: $e^{-pt}(C_1 \cos w_1 t + C_2 \sin w_1 t)$

$= \boxed{e^{-pt} A\cos(w_1 t - \phi)}$ W_1: pseudo-frequency

crosses



$\cos(w_1 t_1 - \phi) = 0$

$w_1 t_1 - \phi = \pi/2$.

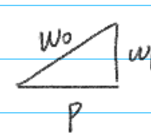
$w_1(t_1 + 2\pi/w_1) - \phi = \pi/2 + 2\pi$

p : only depend on ODE ($c/2m$)

ϕ } dep. on
A } initial conditions.

W_1 = only dep. on ODE $W_0^2 - p^2 = w_1^2$

damping/spring



Lec11 二阶齐次线性方程相关理论

$y''+p(x)y'+q(x)y=0$ linear in y'', y', y

sol'n method:

find 2 ind't solns $y_1\ y_2$ $\quad y_2 \neq c y_1$ $\quad (y_1=0, y_2$ nonzero$)$

$y_1 \neq c' y_2$

All sol'ns are: $y = c_1 y_1 + c_2 y_2$

Q1: Why are $c_1 y_1 + c_2 y_2$ sol'n?
Q2: Why all the sol'n?

叠加定理. Q1: superposition principle: $y_1\ y_2$ solns to lin. homog. ODE

Proof. $y''+py'+qy=0$.

D: differentiation $\quad D^2 y + pDy + qy = 0$

$(D^2+pD+q)y=0$

微分算子 Linear operator $L = D^2+pD+q$ $\quad u(x) \rightarrow \boxed{L} \rightarrow v(x)=0$. $\quad$ ✱ the operator is linear

$Ly=0$

Linear $\quad L(u_1+u_2) = L(u_1)+L(u_2)$

$L(cu) = cL(u)$

[Ex: D linear $\quad (u_1+u_2)' = u_1' + u_2'$

linear operator $\quad (cu)' = cu'$

Proof of superp'n ODE: $\boxed{Ly=0}$

$L(c_1y_1+c_2y_2) = L(c_1y_1)+L(c_2y_2)$

$= c_1\underbrace{L(y_1)}_{0} + c_2\underbrace{L(y_2)}_{0}$

Q2:

Solving the IVP (fit init. values)

Thrm: $\{c_1y_1+c_2y_2\}$ is enough to satisfy any init values

Proof: $\left.\begin{array}{l} y(x_0)=a \\ y'(x_0)=b. \end{array}\right\}$ init

$y = c_1y_1 + c_2y_2$

$y' = c_1y_1' + c_2y_2'$

Plug in $x=x_0$

$$\begin{cases} C_1 y_1(x_0) + C_2 y_2(x_0) = a \\ C_1 y_1'(x_0) + C_2 y_2'(x_0) = b \end{cases}$$ 证明这个方程组一定有解

C_1, C_2 variables.

Solvable for $C_1\ C_2$ if $\det \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}_{x_0} \neq 0$. 满秩.

Wronskian

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ → u_1 and u_2 are indep. vectors.

$\{c_1 y_1 + c_2 y_2\} = \{c_1' u_1 + c_2' u_2\}$. all sol'ns

---

看不懂;!

?

Find normalised sol'ns

(at 0 → could be x_0)

Y_1: $Y_1(0)=1$, $Y_1'(0)=0$ Y_2: $Y_2(0)=0$, $Y_2'(0)=1$

Ex: $y'' + y = 0$. $y'' - y = 0$.

$y_1 = \cos x = Y_1$ $y_1 = e^x$

$y_2 = \sin x = Y_2$ $y_2 = e^{-x}$

gen soln: $c_1 e^x + c_2 e^{-x} = y$

$c_1 e^x - c_2 e^{-x} = y'$

$C_1 = C_2 = 1/2$

If $Y_1 . Y_2$ normalized at 0

soln to IVP ODE + $\begin{cases} y(0) = a = y_0 \\ y(0) = b = y_0' \end{cases}$

$y = y_0 Y_1 + y_0' Y_2$

实分析 Existence + Unique Thrm

$y'' + py' + qy = 0$ p, q con't for all x

There's one and only one soln s.t. $\begin{cases} y(0) = A \\ y'(0) = B \end{cases}$

want All sol'ns to ODE

claim. $\{c_1 Y_1 + c_2 Y_2\}$ are all the solns

Proof: given soln $u(x)$ $u(0) = u_0$, $u'(0) = u_0'$ $\Rightarrow u = u_0 Y_1 + u_0' Y_2$ by uni-ex thrm

then $u_0 Y_1 + u_0' Y_2$ satisfies init values

Lec 12 二阶非齐次线性方程及解的结构 Inhomog. eqn.

$y''+p(x)y'+q(x)y=f(x)$ (x = time often)

input, signal, driving term, forcing term

soln $y(x)$ = response, output

passive system → homo: $y''+p(x)y'+q(x)y=0$ assoc. homog. equn / reduced equn.

soln: $y_c=c_1y_1+c_2y_2$ complementary soln.

y_h. soln to homog. equn.

forced system Examples: $mx''+bx'+kx=f(x)$ spring-mass-dashpot

external force

Thm: $\boxed{Ly=f(x)}$ L: linear operator

Soln: y_p+y_c

$y=y_p+\underline{c_1y_1+c_2y_2}$.

particular: anyone will do. particular soln of col space

soln of null space.

To prove: ① All the $y_p+c_1y_1+c_2y_2$ are soln's

Proof: $L(y_p+c_1y_1+c_2y_2)=L(y_p)+\underbrace{L(c_1y_1+c_2y_2)}_{=0}$

$=Lf(x)$

Satisfy $L(y)=f(x)$

To prove: no other solns

$u(x)$: soln. $L(u)=f(x)$

$L(y_p)=f(x)$

$L(u-y_p)=0$ $u-y_p=c_1\tilde{y}_1+c_2\tilde{y}_2 \Leftrightarrow u=y_p+c_1\tilde{y}_1+c_2\tilde{y}_2$

∴ u is not a new soln, it's just one of y's

$y'+ky=q(t)$

soln: $y=\underbrace{e^{-kt}\int q(t)e^{kt}dt}_{y_p}+\underbrace{c\cdot e^{-kt}}_{y_c}$ $y_c'+ky_c=0$

$k>0$. $y=$ steady state + transient

y_p $y_c\to 0$ as $t\to\infty$.

$k<0$ the above is meaningless

$$\boxed{y'' + Ay' + By = f(t)}$$

$$y = y_p + \underbrace{c_1 y_1 + c_2 y_2}_{\text{uses init conditions}}$$

Q: when does $c_1 y_1 + c_2 y_2 \to 0$ as $t \to \infty$ for all c_1, c_2

If this is so, ODE called stable $\quad y = \underbrace{y_p}_{\text{S.S.S.}} + \underbrace{c_1 y_1 + c_2 y_2}_{\text{transient}}$

char. roots	soln	stability condition
$r_1 \neq r_2$	$c_1 e^{r_1 t} + c_2 e^{r_2 t}$	$r_1 < 0$ $r_2 < 0$.
$r_1 = r_2$	$(c_1 + c_2 t) \cdot e^{r_1 t}$	$r_1 < 0$
$r = a \pm bi$	$e^{at} \cdot (c_1 \cos bt + c_2 \sin bt)$	$a < 0$

* ODE $y'' + Ay' + By = f(t)$
is stable if all the characteristic roots have $\boxed{\text{negative real part}}$

Lec13 非齐次ODE: $p(D)y = e^{ax}$

$$\boxed{y'' + Ay' + By = f(x)}$$

Find a part. soln y_p; gen soln: $y = y_p + c_1 y_1 + c_2 y_2$

Important $f(x)$

① e^{ax} (often $a<0$) ② $\begin{cases} \sin wx \\ \cos wx \end{cases}$ ③ $\begin{cases} e^{ax}\sin wx \\ e^{ax}\cos wx \end{cases}$

all special cases of $e^{(a+iw)x} = \boxed{e^{\alpha x}}$ input

$p(D) = D^2 + AD + B$

$$\underbrace{(D^2 + AD + B)}_{p(D)} y = f(x)$$

$$\boxed{p(D)\, e^{\alpha x} = p(\alpha) e^{\alpha x}}$$ substitution law

Proof: $(D^2 + AD + B) \cdot e^{\alpha x} = D^2 e^{\alpha x} + AD \cdot e^{\alpha x} + B e^{\alpha x}$

$= \alpha^2 e^{\alpha x} + A\alpha \cdot e^{\alpha x} + B \cdot e^{\alpha x}$

$= p(\alpha) \cdot e^{\alpha x}$

Exponential input therom: $f(x) \to \exp(e^{\alpha x})$

$$\boxed{y_p = \frac{e^{\alpha x}}{p(\alpha)}}$$

Proof: $p(D)y = e^{\alpha x}$?

$p(D) \cdot \frac{e^{\alpha x}}{p(\alpha)} = \frac{p(\alpha) e^{\alpha x}}{\underbrace{p(\alpha)}_{\text{constant}}} = e^{\alpha x}$

Assume $p(\alpha) \neq 0$

What if $p(\alpha) = 0$?

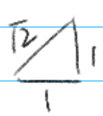
Ex. $y'' - y' + 2y = 10 e^{-x} \sin x$, find par. soln

imaginary part

$(D^2 - D + 2)\tilde{y} = 10 e^{(-1+i)x}$

$\tilde{y}_p = \frac{10 \cdot e^{(-1+i) \cdot x}}{(-1+i)^2 - (-1+i) + 2} = \frac{10 \cdot e^{(-1+i)x}}{3(1-i)} = \frac{10}{3} \cdot \frac{1+i}{2} \cdot e^{-x} \cdot (\cos x + i \sin x)$

$y_p = \mathrm{Im}(\tilde{y}_p) = \frac{5e^{-x}}{3}(\cos x + \sin x)$



$= \frac{5}{3} e^{-x} \cdot \sqrt{2} \cos(x - \pi/4)$

$\lambda^2 + 2\lambda - 3 = 0$

$y'' + 2y' - 3y = e^{2x}$

$\frac{1}{D^2 + 2D - 3} e^{2x} = \frac{1}{5} e^{2x}$

$y'' + 2y' - 3y = x e^{2x}$

Exponential shift formula:

$$\boxed{p(D)\cdot e^{ax}u(x) = e^{ax}p(D+a)u(x)}$$

数学归纳法

mathematical induction

Proof: let $p(D)=D$.

$$D\cdot e^{ax}u = e^{ax}\cdot u' + a e^{ax}\cdot u$$
$$= e^{ax}(Du+au)$$
$$= e^{ax}(D+a)u$$
$$= e^{ax}p(D+a)u$$

let $p(D)=D^2$

$$D^2 e^{ax}u = D(D(e^{ax}u)) = D(e^{ax}(D+a)\cdot u)$$
$$= e^{ax}(D+a)[(D+a)\cdot u]$$
$$= e^{ax}(D+a)^2 u.$$

$(D^2+AD+B)y = e^{ax}$ (a: complex)

$p(a)=0 \leftarrow a$ simple root of $p(D)$

$$\boxed{y_p = \frac{x e^{ax}}{p'(a)}}$$

if a double root

$$\boxed{y_p = \frac{x^2\cdot e^{ax}}{p''(a)}}$$

Proof: (simple root)

$p(D) = (D-b)(D-a)$ $b\neq a$.

$p'(D) = (D-a)+(D-b)$

$p'(a) = (a-a)+a-b = a-b$

$$p(D)\cdot\frac{x\cdot e^{ax}}{p'(a)} \overset{?}{=} e^{ax}$$

$$L.H.S = e^{ax}p(D+a)\cdot\frac{x}{p'(a)}$$
$$= e^{ax}(D+a-b)D\,\frac{x}{p'(a)}$$
$$= e^{ax}[(D+a-b)1]\cdot\frac{Dx}{p'(a)}$$
$$= e^{ax}\cdot\frac{(a-b)\cdot 1}{a-b}$$
$$= e^{ax}$$

Exercise: $y''-3y'+2y = e^x$. 1 is simple root. $D^2-3D+2=0$.

$p'(D) = 2D-3$

$p'(1) = -1$

$$y_p = \frac{xe^x}{-1} = -xe^x$$

Lec14 共振 resonance

$p(D)$ { formal polyn in D.
linear operator on $y(t)$

$f(t) = e^{\alpha t}$ (α: complex)

$$y_p = \frac{e^{\alpha t}}{p(\alpha)}, \quad p(\alpha) \neq 0$$

$$= \frac{t e^{\alpha t}}{p'(\alpha)}, \quad p(\alpha) = 0, \ p'(\alpha) \neq 0; \ \alpha: \text{single root}$$

Resonance

$y'' + w_0^2 y = \cos w_1 t$ — input $w_1 \neq w_0$ frequency different

w_0: 单摆/弹簧的固有频率/任何振动系统

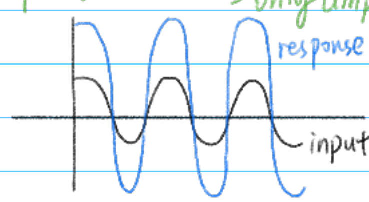
$$(D^2 + w_0^2) y = \cos w_1 t \xrightarrow{\text{complexify}} (D^2 + w_0^2)\tilde{y} = e^{i w_1 t} \quad \text{(Real part)}$$

$$\tilde{y}_1 = \frac{e^{i w_1 t}}{(i w_1)^2 + w_0^2} = \frac{e^{i w_1 t}}{w_0^2 - w_1^2}$$

$$\text{Real part: } y_1 = \frac{\cos w_1 t}{w_0^2 - w_1^2}$$

Response — only amplitude is changed

$w_1 \approx w_0$ ampli. large

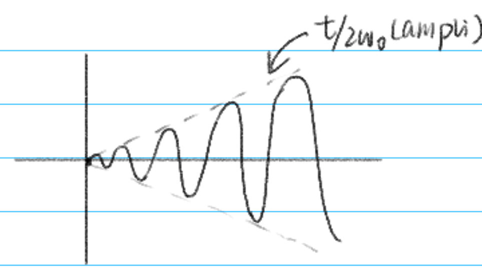


$w_1 = w_0$ $\quad (D^2 + w_0^2) y = \cos w_0 t$

cx $\quad (D^2 + w_0^2)\tilde{y} = e^{i w_0 t}$

$$\tilde{y}_p = \frac{t e^{i w_0 t}}{2 i w_0}$$

$$\mathrm{Re}(\tilde{y}_p) = \frac{t \sin(w_0 t)}{2 w_0} = y_p$$



$$(D^2 + w_0^2) y = \cos w_1 t$$

$$\text{Real part: } y_1 = \frac{\cos w_1 t}{w_0^2 - w_1^2} \left| - \frac{\cos w_0 t}{w_0^2 - w_1^2} \right|_{y_c} = \frac{\cos w_1 t - \cos w_0 t}{w_0^2 - w_1^2}$$

Response — only amplitude is changed

洛必达: var $\to w_1$

$$\lim_{w_1 \to w_0} \frac{\cos w_1 t - \cos w_0 t}{w_0^2 - w_1^2} = \lim_{w_1 \to w_0} \frac{-t \sin(w_1 t)}{-2 w_1} = \frac{t \sin(w_0 t)}{2 w_0}$$

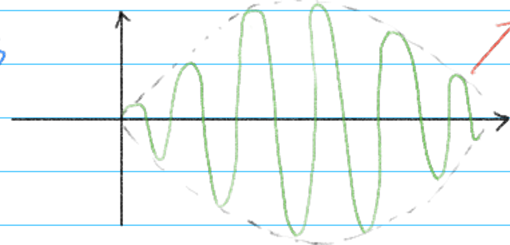
Geometric meaning:

$\cos B - \cos A = 2\sin\frac{A-B}{2}\cdot\sin\frac{A+B}{2}$

$$\frac{\cos w_1 t - \cos w_0 t}{w_0^2 - w_1^2} = \underbrace{\frac{2}{w_0^2-w_1^2}\cdot\sin\frac{(w_0-w_1)t}{2}}_{\text{varying amplitude}}\cdot\sin\frac{(w_0+w_1)t}{2}$$

→ frequency is small

BEATS



→ 多多叫

Damped resonance.

$mx'' + cx' + kx = f(t)$

*(Divide by m) $x'' + 2px' + w_0^2 x = f(t)$

valid: $x'' + bx' + kx = f(t)$

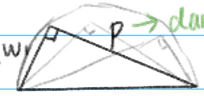
阶 $x'' + bx + kx = f(x)$

阻尼系数

w_0: nat'l undamped freq.

w_1: nat'l damped freq (pseudofreq.)

?



p → damping

damping frequency

w_0 → 弹簧自身的固有频率

from char. roots of damp eqn.

$y'' + 2py' + w_0^2 y = \cos wt$

Prob: which w gives maximum amplitude for the response?

$w_1 = \sqrt{w_0^2 - p^2}$

ans: $w_r = \sqrt{w_0^2 - 2p^2}$

Lec 15 傅里叶级数引入 Fourier Series

$y''+ay'+by=f(t)$ → periodic

soln: $y(t)$ → response Input

$f(t)$: exp'l. sin. cos

$f(t)$ → periodic, period 2π

$$\boxed{f(t)=c_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)}$$

$$f(t)=\frac{a_0}{2}+a_1\cos(wt)+b_1\sin wt+a_2\cos(2wt)+b_2\sin(2wt)+\cdots$$

$$\begin{cases} a_n=\frac{2}{T}\int_{t_0}^{t_0+T}f(t)\cos(nwt)\,dt \\ b_n=\frac{2}{T}\int_{t_0}^{t_0+T}f(t)\sin(nwt)\,dt \end{cases}$$

Input Response.

superposition principle: ODE is $\boxed{\text{linear}}$!

$b_n\sin nt \rightsquigarrow b_n y_n^{(s)}(t)$

$a_n\cos nt \longrightarrow a_n y_n^{(c)}(t)$

$f(t) \rightsquigarrow \sum a_n y_n^{(c)}(t)+b_n y_n^{(s)}(t)+C_1$

Program today:

given $f(t)$, 2π as period. Find its Fourier Series

两个 fn 正交.
$\int_0^{2\pi}u(t)v(t)\,dt=0$

Def. $u(t)$, $v(t)$ fns on $[-\pi,\pi]$ (say 2π is a period)

Orthogonal on $[-\pi,\pi]$ if $\boxed{\int_{-\pi}^{\pi}u(t)\cdot v(t)=0}$

Thm: $\begin{cases}\sin nt & n=1,\cdots\infty \\ \cos mt & m=0,\cdots\infty\end{cases}$

any 2 distinct ones are orthogonal on $[-\pi,\pi]$

Include:

$$\left.\begin{matrix}\int_{-\pi}^{\pi}\sin^2 nt\,dt \\ \int_{-\pi}^{\pi}\cos^2 nt\,dt\end{matrix}\right\}\pi.$$

① Trig. Identities

② complex exp'ls

③ Use ODE

Proof: $m\neq n$ $\sin nt$ $\cos nt$.

satisfy: $u_n''+n^2u_n=0$

u_n v_m be any 2 of these fn's

$u_n''=-n^2u_n$

partial derivative

$$\int_{-\pi}^{\pi}u_n''v_m\,dt=\underbrace{u_n'v_m\Big]_{-\pi}^{\pi}}_{=0}-\int_{-\pi}^{\pi}u_n'v_m'\,dt$$

$$\int_{-\pi}^{\pi}u_n''v_m\,dt=-n^2\int_{-\pi}^{\pi}u_nv_m\,dt \qquad \int_{-\pi}^{\pi}v_m''u_n\,dt=-m^2\int_{-\pi}^{\pi}v_mu_n\,dt$$

$$\therefore \int_{-\pi}^{\pi}u_nv_m\,dt=0 \quad (m\neq n)$$

period 2π $\boxed{f(t) = c_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt}$

find a_n. b_n

$$f(t) = \cdots a_k \cos kt + \cdots + a_n \cos nt$$

$$\int_{-\pi}^{\pi} f(t) \cos nt \, dt = \underbrace{\cdots}_{=0} \underbrace{\int_{-\pi}^{\pi} a_k \cos kt \cos nt \, dt}_{=0} + \cdots + \int_{-\pi}^{\pi} a_n \cos^2 nt \, dt \rightarrow a_n \pi$$

only survive

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt.$$

$$f(t) = c_0 + \cdots + a_n \cos nt$$

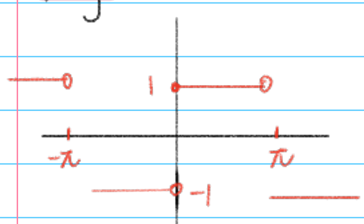
$$\int_{-\pi}^{\pi} f(t) dt = 2\pi c_0 + \cdots \underbrace{\int_{-\pi}^{\pi} a_n \cos nt \, dt}_{=0} + \cdots$$

$\sin nt$

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, dt = \frac{a_0}{2}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt.$$

Keys



$$a_n = 0$$

$$b_n = -\int_{-\pi}^{0} \sin nt \, dt + \int_{0}^{\pi} \sin nt \, dt$$

$$= \frac{1 - \cos n\pi}{n} + \frac{1 - \cos n\pi}{n}$$

$$= \frac{2}{n} \cdot (1 - \cos n\pi)$$

$\cos n\pi$: -1 n, odd; 1 n, even $\} (-1)^n$.

$$= \begin{cases} \frac{2}{n} \cdot 2 & (\text{odd}) \\ 0 & (\text{even}) \end{cases}$$

Example. Let $f(t)$ be the period 2 function, which is defined on the window $[-1,1)$ by $f(t) = |t|$. Compute the Fourier series of $f(t)$.

The graph of $f(t)$ below shows why this function is called either a triangle wave or a continuous sawtooth function.

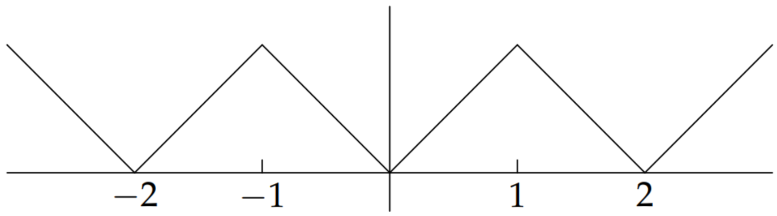


Figure 1: The period 2 triangle wave.

$P=2 \quad L=1 \quad \frac{n\pi}{L} = n\pi.$

$a_n = \frac{2}{2}\int_{-1}^{1} |t|(\cos(n\pi t))\,dt$

$= 2\int_0^1 t\cos n\pi t\,dt.$

$= \frac{2}{n^2\pi^2}((-1)^n - 1) = \begin{cases} -\frac{4}{n^2\pi^2} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even.} \end{cases}$

$a_0 = 1$

Lec16 傅里叶级数拓展

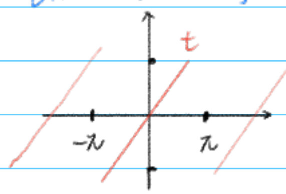
$f(t)=\frac{a_0}{2}+\sum_1^{\infty} a_n\cos nt+b_n\sin nt$. periodic (2π)

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$.

$f(t)=g(t) \Rightarrow$ F.S for f = F.S for g

Since we have formulas for a_n and b_n



① Shorten the calculation

② Extend F.S

$\sin nt$ odd, $\cos nt$ even

$\begin{cases} f(t) \text{ odd} \to \text{All } a_n\text{'s ARE } 0 \\ f(t) \text{ even} \to f(t)=\frac{a_0}{2}+\sum_1^{\infty} a_n\cos nt \quad (\text{All } b_n\text{'s ARE } 0) \end{cases}$

$f(-t)=f(t)$

$\uparrow$

F.S.

$\frac{a_0}{2}+\sum a_n\cos nt-b_n\sin nt=f(-t)$

$\frac{a_0}{2}+\sum a_n\cos nt+b_n\sin nt=f(t)$

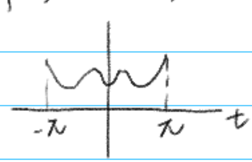
Use Uniqueness of F.

$\therefore b_n=-b_n \quad b_n=0$

$f(t)$ even: simplify calc. of a_n.

even x even = even

$f(t)\cos nt$ is also even



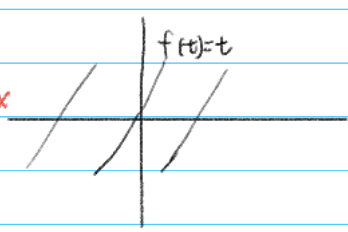
$\therefore a_n=\frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,dt. \quad (b_n=0)$

odd x odd = even

If $f(t)$ odd, $b_n=\frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt \quad (a_n=0)$

odd x odd = even

$\int uv'dx=uv-\int u'v\,dx$



f(t)

odd

$b_n=\frac{2}{\pi}\int_0^{\pi} t\sin nt\,dt$

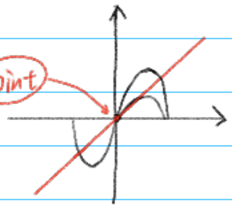
$=\frac{2}{\pi}\left[-t\frac{\cos nt}{n}\Big|_0^{\pi}-\int_0^{\pi}\frac{-\cos nt}{n}dt\right]$

$=\frac{2}{\pi}\left[\frac{-\pi}{n}(-1)^n+\frac{\sin nt}{n^2}\Big|_0^{\pi}\right]=\frac{2}{n}(-1)^{n+1}$

F.S. for $f(t) = 2\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt = 2(\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \cdots)$

Taylor series: fn close to the basepoint

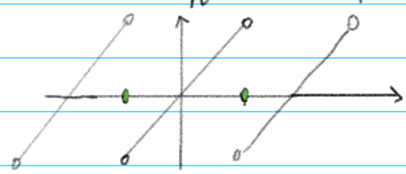
F.S: what a fn look like over a whole interval i.e. $[-\pi,\pi]$



Thm: a) If f is continuous at t_0, then $f(t)$ = some of F.S at t_0 (F.S. converge)

b) If f has a jump discon't at t_0, then $f(t) = \cdots$ F.S converges to the midpoint of the jump

$2\cdot\sum \frac{(-1)^{n+1}}{n}\sin nt = F(t)$ ← what's graph?



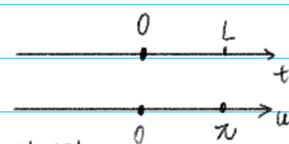
Extension #1 period is $2L$

use nat'l fns $\cos n\frac{\pi}{L}t$

$\sin n\frac{\pi}{L}t$.

↑ period $0, \pm L, \pm 2L$.

Usually $L=1$



$t = \frac{L}{\pi}u$

$u = \frac{\pi}{L}t$.

$f(t) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi}{L}t + b_n \sin\frac{n\pi}{L}t$.

$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi}{L}t\,dt$

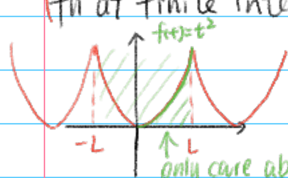
$f(t)$ even, period $2L$.

$a_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi}{L}t\,dt$

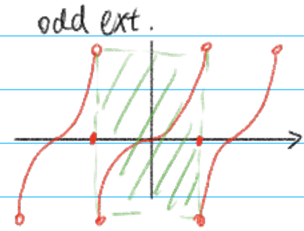
etc. for $f(t)$ odd.

Extension #2 $f(t)$ defined on $[0,L]$

(fn at finite interval) odd ext.

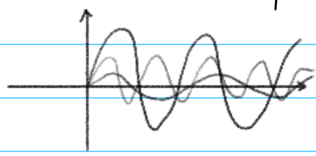


cosine series

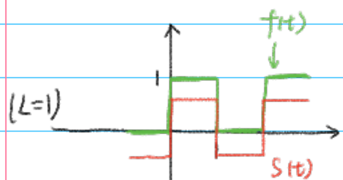


sin series

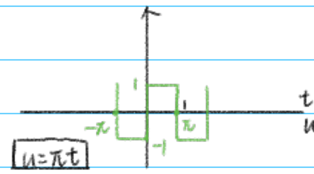
Lec17 **利用傅里叶级数求特解**



C+E+G $\qquad f(t) = b_1 \sin at + b_2 \sin bt + \cdots$



$\begin{cases} \sin n\pi t \\ \cos n\pi t \end{cases}$



$g(u) = \frac{4}{\pi} \sum \frac{\sin nu}{n}$

not even or odd

$$\left.\begin{aligned} f(t) &= S(t) + \tfrac{1}{2} \\ S(t) &= \tfrac{1}{2} g(u) \end{aligned}\right\} \Rightarrow f(t) = \frac{1}{2} + \frac{1}{2} \cdot \frac{4}{\pi} \sum \frac{\sin n\pi t}{n}$$

$x'' + \omega_0^2 x = f(t)$ solve this - find a p.s x_p

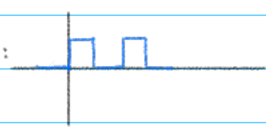
can find x_p, if $f(t)$ is $\cos\omega t, \sin\omega t \to$ use $cx \cdot e^{i\omega t}$ $\quad \boxed{\dfrac{\begin{cases}\cos\omega t \\ \sin\omega t\end{cases}}{\underbrace{\omega_0^2}_{系统的固有频率} - \underbrace{\omega}_{\text{driving frequency}}}}$ $\Rightarrow$ if $\omega_0 \to \omega \Rightarrow$ Response is very large

$\Rightarrow$ Resonance

If $f(t) = \dfrac{a_0}{2} + \sum_1^\infty a_n \cos\omega_n t + b_n \sin\omega_n t \quad \left(\omega_n = \dfrac{n\pi}{L}\right)$

then $x_p = \dfrac{a_0}{2\omega_0^2} + \sum_1^\infty \dfrac{a_n \cos\omega_n t}{\omega_0^2 - \omega_n^2} + \dfrac{b_n \sin\omega_n t}{\omega_0^2 - \omega_n^2}$

Input: f(t)

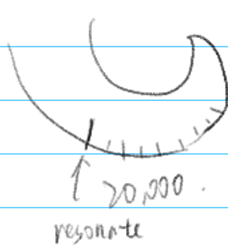


$x'' + \omega_0^2 x = f(t)$

Response: $\dfrac{1}{2\omega_0^2} + \dfrac{2}{\pi} \sum_{n\,odd} \dfrac{\sin n\pi t}{(\omega_0^2 - (n\pi)^2)n}$ $\quad$ frequency $\to 3 \Rightarrow$ most important

$x_p(t) \approx .005 + .6\left(\dfrac{\sin\pi t}{91} + \dfrac{\sin 3\pi t}{45} + \dfrac{\sin 5\pi t}{635} + \cdots\right)$

system picks out the frequency closest to its natural frequency



耳蜗

Book's method (notes) Use different F.S. term by term

Assume soln of form: $x_p = C_0 + \sum^{\infty} C_n \sin n\pi t$ what are the C_n's ?

Subst. into the ODE: $\underline{x_p'' = \sum^{\infty} -C_n (n\pi)^2 \sin n\pi t}$

sum of C_n's with $n\pi t$ $= F.S\ f(t)$

By equating coeff. get C_n.

$$C_0 + \sum_1^{\infty} C_n (w_0^2 - (n\pi)^2) \sin n\pi t.$$

$$= \frac{1}{2} + \frac{2}{\pi} \sum_{n\,odd} \frac{\sin n\pi t}{n}$$

$$C_0 = \frac{1}{2w_0^2} \quad C_n = \frac{2}{\pi} \cdot \frac{1}{n} \cdot \frac{1}{w_0^2 - (n\pi)^2}$$

If $g(x)$ is a piecewise continuous periodic function and $2L$ is a period, then

$$g(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{L}\right) + a_2 \cos\left(\frac{2\pi x}{L}\right) + \cdots + b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + \cdots$$

The Fourier coefficients are defined as the numbers fitting into this expression. They can be calculated using the integral formulas

$$a_n = \frac{1}{L}\int_{-L}^{L} g(x) \cos\left(\frac{n\pi x}{L}\right) dx\,, \qquad b_n = \frac{1}{L}\int_{-L}^{L} g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$\ddot{x} + \omega_n^2 x = A\cos(\omega t)$ has solution $A\dfrac{\cos(\omega t)}{\omega_n^2 - \omega^2}$ and

$\ddot{x} + \omega_n^2 x = A\sin(\omega t)$ has solution $A\dfrac{\sin(\omega t)}{\omega_n^2 - \omega^2}$ as long as $\omega \neq \omega_n$.

Lec 19 拉普拉斯变换引入

幂级数

power series: $\sum_0^\infty a_n x^n = A(x)$

$\sum_0^\infty a(n) x^n = A(x)$

$a(n) \rightsquigarrow A(x)$

$1 \rightsquigarrow \frac{1}{1-x}$, $|x|<1$ (converge)

$\frac{1}{n!} \rightsquigarrow e^x$

continuous analog $\int_0^\infty a(t) x^t dt = A(x) = A(e^{-s})$

$n = 0, 1, 2 \cdots$ $\boxed{\int_0^\infty f(t) e^{-st} dt = F(s)}$

$\downarrow$ Laplace transform

$t \quad 0 \le t < \infty$

$x = e^{\ln x}$; $x^t = (e^{\ln x})^t$

$0 < x < 1 \quad \ln x < 0$

$s = -\ln x \quad x = e^{-s}$

$f(t) \rightarrow \boxed{\text{transf}} \rightarrow g(s)$

$f(t) \rightarrow \boxed{\text{operator}} \rightarrow g(t)$

Linear transform $\mathcal{L}$ $\begin{cases} \mathcal{L}(f(t)) = F(s) \\ f(t) \rightsquigarrow F(s) \end{cases}$

$\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$

$\mathcal{L}(cf) = c\mathcal{L}(f)$

$\boxed{f(t) = 1 \rightsquigarrow ? \quad \frac{1}{s}}$ $\int_0^\infty 1 \cdot e^{-st} dt = \lim_{R\to\infty} \int_0^R e^{-st} dt$

$\int_0^R e^{-st} dt = \left. \frac{e^{-st}}{-s} \right]_0^R = -\frac{1}{s} \cdot (e^{-Rs} - 1)$

$= \lim_{R\to\infty} \frac{1}{s} \cdot (1 - e^{-Rs}) = \frac{1}{s}$, if s is positive

$\boxed{\begin{array}{l} e^{at} \rightsquigarrow \frac{1}{s-a} \\ e^{at} f(t) \rightsquigarrow F(s-a) \end{array}}$ $\int_0^\infty e^{at} f(t) e^{-st} \cdot dt = \int_0^\infty f(t) e^{-(s-a)t} dt = F(s-a)$

shift by a

Exp'l shift law

if $s-a>0$ (Assuming $F(s), s>0$)

can use also for $e^{(a+bi)t} \rightsquigarrow \frac{1}{s-(a+bi)}$

backwards

Euler formula

$\boxed{\cos at = \frac{1}{2}(e^{iat} + e^{-iat})}$

$\mathcal{L}(\cos at) = \frac{1}{2}\left(\frac{1}{s-ai} + \frac{1}{s+ai}\right)$

$= \frac{1}{2} \cdot \frac{2s}{s^2+a^2}$

$= \frac{s}{s^2+a^2}$

$\begin{cases} \cos at \rightsquigarrow \frac{s}{s^2+a^2} & s>0 \\ \sin at \rightsquigarrow \frac{a}{s^2+a^2} & s>0 \end{cases}$

$\frac{1}{s(s+3)} = \frac{1/3}{s} - \frac{1/3}{s+3}$

$\rightsquigarrow \mathcal{L}^{-1} \qquad \rightsquigarrow \mathcal{L}^{-1}$

$? \qquad \frac{1}{3} - \frac{1}{3} e^{-3t}$

$\int uv'dx = uv - \int u'v\,dx$

$$t^n \rightsquigarrow \frac{n!}{s^{n+1}}$$

$$\int_0^\infty t^n e^{-st}\,dt = \left[t^n \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty n t^{n-1} \frac{e^{-st}}{-s}\,dt$$

↑ ↑
DIFF INT

$$= \lim_{t\to\infty} t^n \frac{e^{-st}}{-s} - 0 \quad (s>0)$$

$$\frac{n}{s}\int_0^n t^{n-1}\cdot e^{-st}\,dt = \frac{n}{s}\mathcal{L}(t^{n-1})$$

$$= \frac{-1}{s}\lim_{t\to\infty} \frac{t^n}{e^{st}}$$

$$= -\frac{1}{s}\lim_{t\to\infty} \frac{nt^{n-1}}{se^{st}} \xrightarrow[\text{many times}]{\text{洛必达}} -\frac{1}{s}\lim_{t\to\infty} \frac{n(n-1)\cdots 1\cdot t^0}{s^n e^{st}} = 0$$

$$\mathcal{L}(t^n) = \frac{n}{s}\mathcal{L}(t^{n-1}) = \cdots = \frac{n(n-1)\cdots 1}{s^n}\mathcal{L}(t^0) = \frac{n!}{s^{n+1}}$$

Lec20 **拉氏变换求解线性ODE**

$f(t)$ of 'exponential type' has Laplace transformation

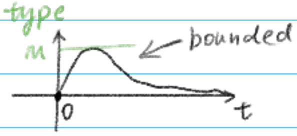
$|f(t)| \le Ce^{kt}$ $c, k > 0$ some pos. constant

$|\sin t| \le 1 \cdot e^{0t}$

$t^n \le M \cdot e^t$ for some M, all $t > 0$

is of exp'l type

$\frac{t^n}{e^t}$



$\frac{t^n}{e^t} \to 0$ as $t \to \infty$ by L'Hop

going too rapidly

$1/t$ $\int_0^\infty e^{-st} \frac{1}{t} dt$ not converge not exp'l type

e^{t^2} not exp'l type $e^{t^2} > e^{kt}$, no matter how big k is

$$\boxed{y'' + Ay' + By = h(t) \quad y(0) = y_0,\ y'(0) = y_0'} \quad \text{IVP}$$

$$\downarrow \mathcal{L}$$

$y(t)$ soln $\rightsquigarrow Y(s)$ $\boxed{\text{Alg. eqn. in } Y(s)} \xrightarrow{\text{solve for } Y} \boxed{Y = \frac{p(s)}{q(s)}} \xrightarrow{\mathcal{L}^{-1}} \boxed{y = y(t)}$

Integration by part

$$\mathcal{L}(f'(t)) = \int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty -se^{-st} \cdot f(t)\,dt$$

DIFF

$$= 0 - \underbrace{f(0)}_{\text{init value}} + s F(s)$$

$\int uv'dx = uv - \int u'v dx$

$\lim_{t \to \infty} \frac{f(t)}{e^{st}} = 0$ since $f(t)$ is of "exp'l type"

$s > k$

$f'(t) \rightsquigarrow sF(s) - f(0)$

exp'l type

$f''(t) = [f'(t)]' = s\mathcal{L}(f'(t)) - f'(0)$

$= s(s \cdot F(s) - f(0)) - f'(0)$

$f''(t) \rightsquigarrow s^2 F(s) - sf(0) - f'(0)$

Ex $y''-y=e^{-t}$, $y(0)=1$, $y'(0)=0$

$\Big\} \mathcal{L}$

$$s^2Y - s\cdot 1 - 0 - Y = \frac{1}{s+1}$$

$$(s^2-1)Y = \frac{1}{s+1}+s = \frac{1+s^2+s}{s+1}$$

$$Y = \frac{s^2+s+1}{(s+1)^2(s-1)}$$

$$\frac{s^2+s+1}{(s+1)^2(s-1)} = \frac{(s+1)^2-s}{(s+1)^2(s-1)}$$

$$= \frac{(s+1)^2-\frac{1}{2}(s+1)-\frac{1}{2}(s-1)}{(s+1)^2(s-1)}$$

shift rule.

$$= -\frac{1/2}{(s+1)^2}+\frac{0}{s+1}+\frac{1}{s-1}-\frac{1/2}{(s+1)(s-1)}$$

$t \rightsquigarrow \frac{1}{s^2}$

$te^{-t} \rightsquigarrow \frac{1}{(s+1)^2}$

$$= -\frac{1}{2(s+1)^2}+\frac{3/4}{s-1}+\frac{1/4}{s+1}$$

$$\overset{\mathcal{L}^{-1}}{\rightsquigarrow} = \underbrace{-\frac{1}{2}t\cdot e^{-t}}_{y_p} + \underbrace{\frac{3}{4}e^{t}+\frac{1}{4}e^{-t}}_{y_c}$$

Rules for the Laplace transform

Definition:	$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$ for $\operatorname{Re}(s) \gg 0$.
Linearity:	$\mathcal{L}[af(t)+bg(t)] = aF(s)+bG(s)$.
$\mathcal{L}^{-1}$:	$F(s)$ essentially determines $f(t)$ for $t>0$.
s-shift rule:	$\mathcal{L}[e^{rt}f(t)] = F(s-r)$.
s-derivative rule:	$\mathcal{L}[tf(t)] = -F'(s)$.
t-derivative rule:	$\mathcal{L}[f'(t)] = sF(s)-f(0^-)$.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s}, \quad \mathcal{L}[\delta(t-a)] = e^{-as}$$

$$\mathcal{L}[e^{rt}] = \frac{1}{s-r}, \quad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}, \quad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

Dec 21 卷积 Convolution

$f(t) * g(t)$

$$\left.\begin{array}{l} F(s) = \int_0^\infty e^{-st} f(t)\,dt \\ G(s) = \int_0^\infty e^{-st} g(t)\,dt \end{array}\right\} F(x)\cdot G(x) = \int_0^\infty e^{-st} (f*g)\,dt$$

$f*g = g*f$

since $FG = GF$

$$F(x) = \sum_0^\infty a_n x^n$$

$$G(x) = \sum_0^\infty b_n x^n$$

$$F(x)\cdot G(x) = \sum_0^\infty c_n x^n \qquad c_n = \{a_i b_j\}$$

$$n \to t \qquad x \to e^{-s}$$

$$\boxed{f(t) * g(t) = \int_0^t f(u) g(t-u)\,du}$$

$$f*g \xrightarrow[\mathcal{L}]{} F(s)G(s)$$

Ex: $t^2 * t = \int_0^t u^2 \cdot (t-u)\,du$

$$\downarrow \mathcal{L} \quad = \frac{u^3}{3}t - \frac{u^4}{4}\Big]_0^t = \frac{t^4}{12}$$

$$\frac{2}{s^3}\cdot\frac{1}{s^2} = \frac{2}{s^5} \qquad \mathcal{L}^{-1}\left(\frac{2}{s^5}\right) = \frac{1}{12}\mathcal{L}^{-1}\left(\frac{4!}{s^5}\right) = \frac{t^4}{12}$$

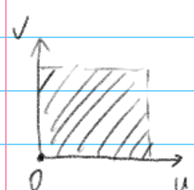
Ex: $f(t) * 1 = \int_0^t f(u)\cdot 1\,du$

Double Integral

二重积分

$$F(s)\cdot G(s) = \int_0^\infty e^{-su} f(u)\,du \cdot \int_0^\infty e^{-sv} g(v)\,dv$$

$$\boxed{= \int_0^\infty\int_0^\infty e^{-s(u+v)} f(u) g(v)\,du\,dv}$$



let $u+v=t$. $v = t-u$.

Jacobian = 1

雅可比

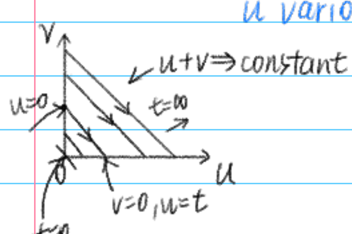
$$du\,dv = \frac{\partial(u,v)}{\partial(u,t)}\,du\,dt$$

$u=u$, $v=t-u$ $\quad J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$

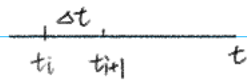
$$= \int_0^\infty\int_0^t e^{-st} f(u) g(t-u)\,du\,dt$$

$$= \int_0^\infty e^{-st} \int_0^t f(u) g(t-u)\,du\,dt = \int_0^\infty e^{-st} (f*g)\,dt$$

u various, t held fixed $\quad t=u+v$; $u+v$ is fixed



Radio waste $f(t)$ dump rate t = years



Amount dumped $[t_i, t_{i+1}] \approx f(t)\Delta t$

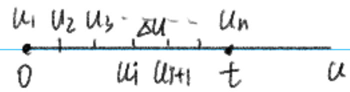
Problem: Start $t=0$

At time t, how much radioactive waste is in the pile? (decay)

decay for the time of t ; k fixed (assume)

$A_0 \cdot e^{-kt}$ = amount left at time t

init amt.



Amt dumped in $[u_i, u_{i+1}] \approx f(u_i)\Delta u$ → length of time it had on pile

By time t, it has decayed $f(u_i)\Delta u \cdot e^{-k(t-u_i)}$

Total amt left at time t $\approx \sum_{\substack{i=1 \\ u_i=0}}^{u_n=t} f(u_i)\, e^{-k(t-u_i)}\Delta u$

let $\Delta u \to 0$ $\to \int_0^t f(u)\, e^{-k(t-u)}\, du = f(t) * e^{-kt}$

#1 Dump garbage (undecay) $= f(t) * 1$ ← constant (undecay)

$= \int_0^t f(u)\, du$

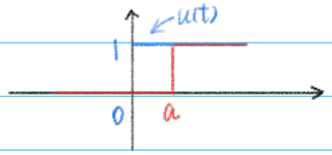
#2 $f(t) * t$ what grows like t? # kg of chicken at time t

↑ production rate for new chicken (kg)

↑ linear growth of baby chicks

Lec 22 **拉氏变换处理不连续函数**

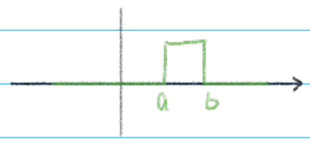
Jump discontinuous



$u(t)$ " unit step "

$u(0)$ undefined

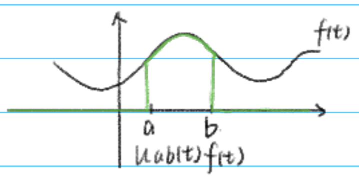
$u_a(t) = u(t-a)$



$u_{ab}(t)$: " unit box "

$u_{ab}(t) = u_a(t) - u_b(t)$

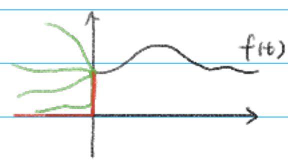
$= u(t-a) - u(t-b)$



$\mathcal{L}(u(t)) = \int_0^\infty e^{-st} \overset{=1}{u(t)} dt = \frac{1}{s} = \mathcal{L}(1) \quad s>0$

? $\mathcal{L}^{-1}(\frac{1}{s}) =$

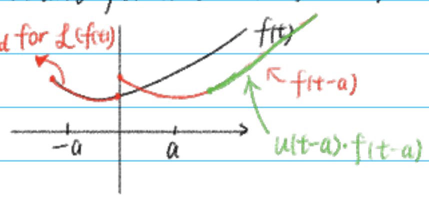
$\begin{cases} f(t) \xrightarrow{\mathcal{L}} F(s) \\ F(s) \xrightarrow{\mathcal{L}^{-1}} u(t) \cdot f(t) \end{cases}$



make $\mathcal{L}^{-1}$ unique ($<0 \to 0$) $\quad \int_0^\infty e^{-st} f(t) dt$.

want formula: $\mathcal{L}(f(t-a))$ in terms of $\mathcal{L}(f(t))$ doesn't exist

1. 去掉t<0的值 not used for $\mathcal{L}(f(t))$



replace t by $t_1 = t-a$

$$\boxed{\begin{array}{ll} \text{Right formula: } & u(t-a)f(t-a) \leadsto e^{-as}\mathcal{L}(f(t)) = F(s) \\ \text{t-axis trans: } & u(t-a)f(t) \leadsto e^{-as}\mathcal{L}(f(t+a)) \end{array}}$$

$> u(t-a)f(t-a+a) \leadsto u(t_1)f(t_1+a) = e^{-as}\mathcal{L}(f(t+a))$ $\quad t_1 = t+a$

$u(t-a)f(t-a)$

$\leadsto \int_0^\infty e^{-st} u(t-a) f(t-a) dt \qquad t_1 = t-a \quad t = t_1 + a$

$= \int_{-a}^\infty e^{-s(t_1+a)} u(t_1) f(t_1) dt_1$

$= e^{-as} \int_{-a}^\infty e^{-st_1} u(t_1) f(t_1) dt_1$

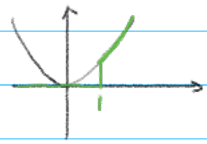
$= e^{-as} \int_0^\infty e^{-st_1} u(t_1) f(t_1) dt_1$ (since $u(t_1) = 0$ for $t_1 < 0$)

$= e^{-as} F(s)$

$u_{ab}(t) = u(t-a) - u(t-b)$

$\xrightarrow[\mathcal{L}]{} \dfrac{e^{-as}}{s} - \dfrac{e^{-bs}}{s}$

Eg. $u(t-1)\cdot t^2 = e^{-s}\mathcal{L}[(t+1)^2] = e^{-s}\mathcal{L}(t^2+2t+1) = e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)$

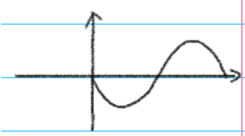


Ex: $\mathcal{L}^{-1}\left(\dfrac{1+e^{-\pi s}}{s^2+1}\right) = \dfrac{1}{s^2+1} + \dfrac{e^{-\pi s}}{s^2+1}$

$\dfrac{1}{s^2+1} \longrightarrow u(t)\sin t.$

$\dfrac{e^{-\pi s}}{s^2+1} \longrightarrow u(t-\pi)\sin(t-\pi)$

Ans: $\underset{t\geq 0}{u(t)\sin t} + \underset{t\geq\pi}{u(t-\pi)\sin(t-\pi)}$



$f(t) = \begin{cases} \sin t & 0\leq t<\pi. \\ 0 & \pi\leq t \end{cases}$

$\sin(t-\pi) = -\sin t$

Lec23 **Delta 函数**

Input: unit impulse

$f(t)$ force

impulse of $f(t)$ on $[a,b] = \int_a^b f(t)\,dt$.

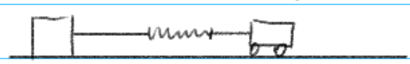
$f(t) = $ constant F

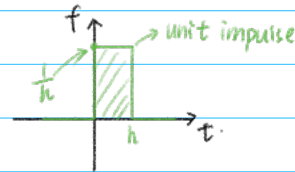
impulse $= F\cdot(b-a)$

$\delta(t)$: unit impulse

$u(t)$ = unit step fn

$\vec{F}$ $t=[0,2)$



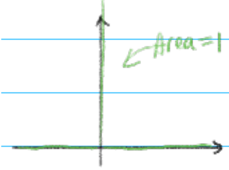


$u(t-a)g(t-a) \rightsquigarrow e^{-as}G(s)$

$y''+y = \boxed{\frac{1}{h}u_{0h}(t)} = \frac{1}{h}\cdot[u(t)-u(t-h)] \xrightarrow{\mathcal{L}} \frac{1}{h}\left[\frac{1}{s} - \frac{e^{-hs}}{s}\right]$

$\lim\limits_{h\to 0} \frac{1-e^{-hs}}{hs} = \lim\limits_{u\to 0} \frac{1-e^{-u}}{u} \to 1$

non-fn $\delta(t)$



$\frac{1}{h}u_{0h}(t) \xrightarrow{\mathcal{L}} \frac{1-e^{-hs}}{hs}$

$\downarrow \qquad\qquad \downarrow h\to 0$

$\delta(t) \xrightarrow{\mathcal{L}} 1$

$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

$u(t)f(t) * \delta(t) \xrightarrow{\mathcal{L}} F(s)\cdot \underset{\substack{\parallel \\ 1}}{G(t)}$

$\overset{\parallel}{u(t)\cdot f(t)} \xrightarrow{\mathcal{L}} F(s)$.

In convolution operation: $\delta(t)$'s just like I (identity)

$\boxed{u'(t) = \delta(t)}$

$y''+y = A\delta(t-\frac{\pi}{2}) \qquad y(0)=1 \quad y'(0)=0$

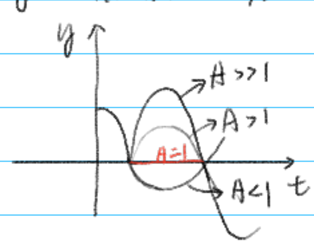
Kicked with impulse A. at $t=\pi/2$

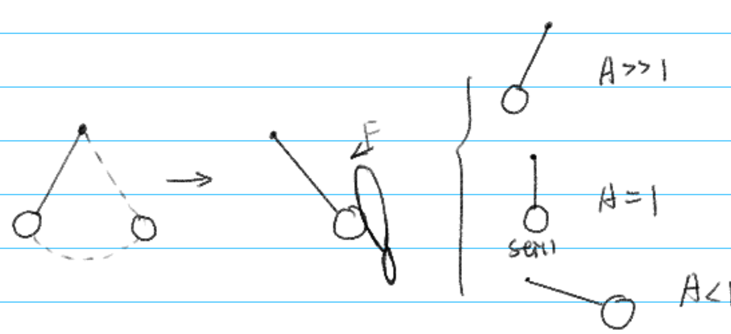
$s^2Y - s + Y = A\cdot e^{-\pi/2\cdot s}\cdot 1$

$Y = \frac{s}{s^2+1} + \frac{Ae^{-\pi/2\cdot s}}{s^2+1}$

$y = \mathcal{L}^{-1}(Y) = \cos t + u(t-\pi/2)\cdot A\sin(t-\pi/2)$

$y = \begin{cases} \cos t & 0\le t\le \pi/2 \\ (1-A)\cos t & \pi/2 \le t. \end{cases}$





$$y'' + ay' + by = f(t) \qquad y(0)=0 \quad y'(0)=0$$

Input

$$s^2 y + asy + by = F(s)$$

$$y = F(s) \cdot \frac{1}{s^2+as+b} \; W(s)$$

→ transfer fn.

$$\Big\downarrow \mathcal{L}^{-1}$$

$w(t)$ → weigh fn of the system

$$y(t) = f(t) * w(t) = \int_0^t f(u) w(t-u)\, du$$

What $w(t)$ really?

$$y'' + ay' + by = \delta(t) \qquad y(0)=0=y'(0)$$

Kick mass at $t=0$, unit impulse

$$s^2 y + asy + by = 1$$

$$y = \frac{1}{s^2+as+b} = W(s) \xrightarrow{\mathcal{L}^{-1}} w(t)$$

Transfer function, **Green's Formula, Laplace Transform of Convolution**

1. Green's Formula in Time and Frequency

When we studied convolution we learned Green's formula. This says, the IVP

$$p(D)x = f(t), \quad \text{with rest IC} \tag{1}$$

has solution

$$x(t) = (w * f)(t), \text{ where } w(t) \text{ is the weight function.} \tag{2}$$

$p(D)w(t) = \delta(t)$

(Remember, the weight function is the same as the unit impulse response.) $w(t)$

The Laplace transform changes these equations to ones in the frequency variable s.

$$p(s)X(s) = F(s) \tag{3}$$

$\frac{1}{p(s)} = W(s)$

$$X(s) = \frac{1}{p(s)}F(s) = W(s)F(s), \tag{4}$$

where $W(s)$ is the transfer function. $\mathcal{L}(w(t)) = W(s)$

Equation (2) is *Green's formula in time* and (4) is *Green's formula in frequency*. In words, viewed from the t side, the solution to (1) is the convolution of the weight function and the input. Viewed from the s side, the solution is the product of the transfer function and the input.

2. Convolution

Comparing equations (2) and (4) we see that

$$\mathcal{L}(w * f) = W(s) \cdot F(s). \tag{5}$$

It appears that Laplace transforms convolution into multiplication. Technically, equation (5) only applies when one of the functions is the weight function, but the formula holds in general.

Theorem: For any two functions $f(t)$ and $g(t)$ with Laplace transforms $F(s)$ and $G(s)$ we have

$$\mathcal{L}(f * g) = F(s) \cdot G(s). \tag{6}$$

Remarks:
1. This theorem gives us another way to prove convolution is commutative. It is just the commutivity of regular multiplication on the s-side.

$$\mathcal{L}(f * g) = F \cdot G = G \cdot F = \mathcal{L}(g * f).$$

2. In fact, the theorem helps solidify our claim that convolution is a type of multiplication, because viewed from the frequency side it *is* multiplication.

Proof: The proof is a nice exercise in switching the order of integration. We won't use 0^- and t^+ in the integrals, since they would just clutter the exposition. It is an amusing exercise to put them in and see that they transform correctly as we manipulate the integrals.

We start by writing $\mathcal{L}(f * g)$ as the convolution integral followed by the Laplace integral.

$$\begin{aligned}
\mathcal{L}(f * g) &= \int_0^\infty (f * g)(t) e^{-st}\, dt \\
&= \int_0^\infty \int_0^t f(t-u) g(u) e^{-st}\, du\, dt.
\end{aligned}$$

Next, we change the order of integration (see the figure below).

$$= \int_0^\infty \int_u^\infty f(t-u) g(u) e^{-st}\, dt\, du.$$

Finally, change variables in the inner integral: substitute $v = t - u$, $dv = dt$, (u a constant)

$$\begin{aligned}
&= \int_0^\infty \int_0^\infty f(v) g(u) e^{-s(v+u)}\, dv\, du \\
&= \int_0^\infty f(v) e^{-sv}\, dv \int_0^\infty g(u) e^{-su}\, du \\
&= F(s) G(s).
\end{aligned}$$

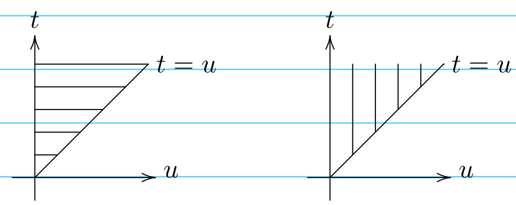


Fig. 1. Changing the order from $du\, dt$ to $dt\, du$.

3. Integration Rule

If differentiation on the time side leads to multiplication by s on the frequency side then we should expect integration in time to lead to division by s. If $f(t)$ is a function with Laplace transform $F(s)$ then the integration rule states:

$$\mathcal{L}\left(\int_{0^-}^{t^+} f(\tau)\,d\tau\right) = \frac{F(s)}{s}.$$

Proof: One way to prove this is using the t-derivative rule. Let's be clever and use convolution instead. The integral is exactly $f(t) * 1$. Thus,

$$\mathcal{L}\left(\int_{0^-}^{t^+} f(\tau)\,d\tau\right) = \mathcal{L}(f * 1) = F(s) * \mathcal{L}(1) = \frac{F(s)}{s}.$$

This is what we needed to show.

Definition of Poles

1. Rational Functions

A **rational function** is a ratio of polynomials $q(s)/p(s)$.

Examples. The following are all rational functions. $(s^2+1)/(s^3+3s+1)$, $1/(ms^2+bs+k)$, $s^2+1+(s^2+1)/1$.

If the numerator $q(s)$ and the denominator $p(s)$ have no roots in common, then the rational function $q(s)/p(s)$ is in **reduced form**

Example. The three functions in the example above are all in reduced form.

Example. $(s-2)/(s^2-4)$ is not in reduced form, because $s=2$ is a root of both numerator and denominator. We can rewrite this in reduced form as

$$\frac{s-2}{s^2-4} = \frac{s-2}{(s-2)(s+2)} = \frac{1}{s+2}.$$

2. Poles

For a rational function in reduced form the **poles** are the values of s where the denominator is equal to zero; or, in other words, the points where the rational function is not defined. We allow the poles to be complex numbers here.

Examples. a) The function $1/(s^2+8s+7)$ has poles at $s=-1$ and $s=-7$.

b) The function $(s-2)/(s^2-4) = 1/(s+2)$ has only one pole, $s=-2$.

c) The function $1/(s^2+4)$ has poles at $s=\pm 2i$.

d) The function s^2+1 has no poles.

e) The function $1/(s^2+8s+7)(s^2+4)$ has poles at -1, -7, $\pm 2i$. (Notice that this function is the product of the functions in (a) and (c) and that its poles are the union of poles from (a) and (c).)

Remark. For ODE's with system function of the form $1/p(s)$, the poles are just the roots of $p(s)$. These are the familiar characteristic roots, which are important as we have seen.

3. Graphs Near Poles

We start by considering the function $F_1(s) = \frac{1}{s}$. This is well defined for every complex s except $s=0$. To visualize $F_1(s)$ we might try to graph it. However it will be simpler, and yet still show everything we need, if we graph $|F_1(s)|$ instead.

To start really simply, let's just graph $|F_1(s)| = \frac{1}{|s|}$ for s real (rather than complex).

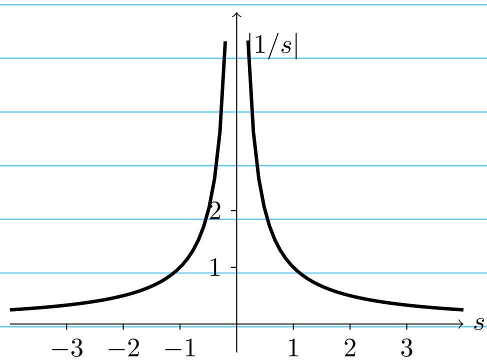


Figure 1: Graph of $\frac{1}{|s|}$ for s real.

Now let's do the same thing for $F_2(s) = 1/(s^2 - 4)$. The roots of the denominator are $s = \pm 2$, so the graph of $|F_2(s)| = \frac{1}{|s^2-4|}$ has vertical asymptotes at $s = \pm 2$.

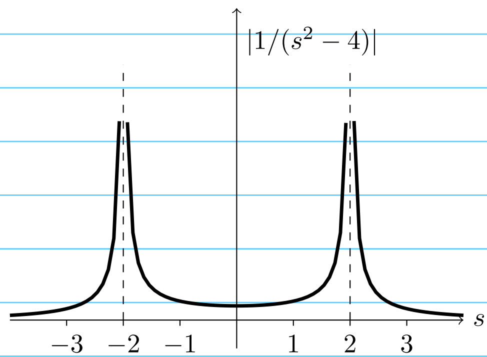


Figure 2: Graph of $\frac{1}{|s^2-4|}$ for s real.

As noted, the vertical asymptotes occur at values of s where the denominator of our function is 0. These are what we defined as the poles.

- $F_1(s) = \frac{1}{s}$ has a single pole at $s = 0$.
- $F_2(s) = \frac{1}{s^2-4}$ has two poles, one each at $s = \pm 2$.

Looking at Figures 1 and 2 you might be reminded of a tent. The poles of the tent are exactly the vertical asympotes which sit at the poles of the function.

Let's now try to graph $|F_1(s)|$ and $|F_2(s)|$ when we allow s to be complex. If $s = a + ib$ then $F_1(s)$ depends on two variables a and b, so the graph requires three dimensions: two for a and b, and one more (the vertical axis) for the value of $|F_1(s)|$. The graphs are shown in Figure 3 below. They are 3D versions of the graphs above in Figures 1 and 2. At each pole there is a conical shape rising to infinity, and far from the poles the function fall off to 0.

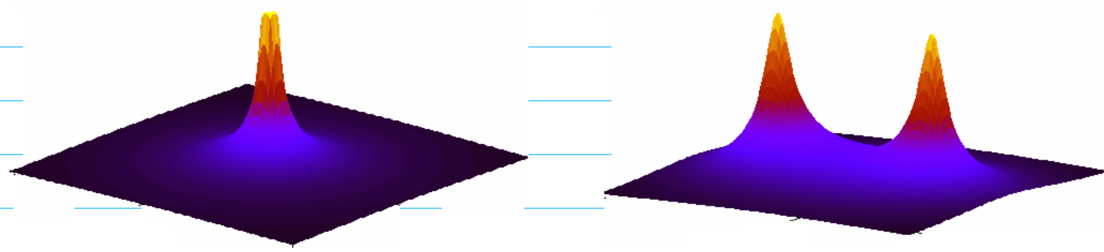


Figure 3: The graphs of $|1/s|$ and $1/|s^2 - 4|$.

Roughly speaking, the poles tell you the shape of the graph of a function $|F(s)|$: it is *large near the poles*. In the typical pole diagams seen in practice, the $|F(s)|$ is also small far away from the poles.

4. Poles and Exponential Growth Rate

If $a > 0$, the exponential function $f_1(t) = e^{at}$ grows rapidly to infinity as $t \to \infty$. Likewise the function $f_2(t) = e^{at} \sin bt$ is oscillatory with the amplitude of the oscillations growing exponentially to infinity as $t \to \infty$. In both cases we call a the *exponential growth rate* of the function.

The formal definition is the following

Definition: The **exponential growth rate** of a function $f(t)$ is the smallest value a such that

$$\lim_{t\to\infty} \frac{f(t)}{e^{bt}} = 0 \quad \text{for all } b > a. \tag{1}$$

In words, this says $f(t)$ grows slower than any exponential with growth rate larger than a.

Examples.

1. e^{2t} has exponential growth rate 2.

2. e^{-2t} has exponential growth rate -2. A negative growth rate means that the function is *decaying* exponentially to zero as $t \to \infty$.

3. $f(t) = 1$ has exponential growth rate 0.

4. $\cos t$ has exponential growth rate 0. This follows because $\lim_{t\to\infty} \frac{\cos t}{e^{bt}} = 0$ for all positive b.

5. $f(t) = t$ has exponential growth rate 0. This may be surprising because $f(t)$ grows to infinity. But it grows linearly, which is slower than any positive exponential growth rate.

6. $f(t) = e^{t^2}$ does not have an exponential growth rate since it grows faster than any exponential.

Poles and Exponential Growth Rate
We have the following theorem connecting poles and exponential growth rate.

Theorem: The exponential growth rate of the function $f(t)$ is the largest real part of all the poles of its Laplace transform $F(s)$.

Examples. We'll check the theorem in a few cases.

1. $f(t) = e^{3t}$ clearly has exponential growth rate equal to 3. Its Laplace transform is $1/(s-3)$ which has a single pole at $s = 3$,and this agrees with the exponential growth rate of $f(t)$.
2. Let $f(t) = t$, then $F(s) = 1/s^2$. $F(s)$ has one pole at $s = 0$. This matches the exponential growth rate zero found in (5) from the previous set of examples.
3. Consider the function $f(t) = 3e^{2t} + 5e^t + 7e^{-8t}$. The Laplace transform is $F(s) = 3/(s-2) + 5/(s-1) + 7/(s+8)$, which has poles at $s = 2, 1, -8$. The largest of these is 2. (Don't be fooled by the absolute value of -8, since $2 > -8$, the largest pole is 2.) Thus, the exponential growth rate is 2. We can also see this directly from the formula for the function. It is clear that the $3e^{2t}$ term determines the growth rate since it is the dominant term as $t \to \infty$.
4. Consider the function $f(t) = e^{-t}\cos 2t + 3e^{-2t}$ The Laplace transform is $F(s) = \frac{s}{(s+1)^2+4} + \frac{3}{s+2}$. This has poles $s = -1 \pm 2i$, -2. The largest real part among these is -1, so the exponential growth rate is -1.

Note that in item (4) in this set of examples the growth rate is negative because $f(t)$ actually *decays* to 0 as $t \to \infty$. We have the following
Rule:
1. If $f(t)$ has a negative exponential growth rate then $f(t) \to 0$ as $t \to \infty$.

2. If $f(t)$ has a positive exponential growth rate then $f(t) \to \infty$ as $t \to \infty$.

5. An Example of What the Poles Don't Tell Us

Consider an arbitrary function $f(t)$ with Laplace transform $F(s)$ and $a > 0$. Shift $f(t)$ to produce $g(t) = u(t-a)f(t-a)$, which has Laplace transform $G(s) = e^{-as}F(s)$. Since e^{-as} does not have any poles, $G(s)$ and $F(s)$ have exactly the same poles. That is, the poles can't detect this type of shift in time.

Lec24. 一阶常微分方程组及几何解释 First-order

$x' = f(x, y; t)$ x, y dependent vars.

$y' = g(x, y; t)$ t indep.

Linear system

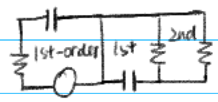
$x' = ax + by + r_1(t)$ a) a, b, c, d constant

$y' = cx + dy + r_2(t)$ (const-coeff system)

b) a, b, c, d can also be fn of t

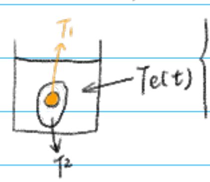
Init conditions: $\boxed{x(t_0) = x_0,\ y(t_0) = y_0}$

When to use multiple eqns?

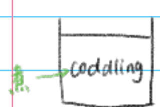


3 equations. → 4th order.

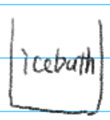
Example:



$$\begin{cases} \dfrac{dT_1}{dt} = a(T_2 - T_1) \\ \dfrac{dT_2}{dt} = a(T_1 - T_2) + b(T_e - T_2) \end{cases} \qquad \begin{cases} T_1' = -aT_1 + aT_2 \\ T_2' = aT_1 - (a+b)T_2 + bT_e(t) \end{cases}$$



$T_e = 100 \cdot e^{-kt}$



$T_e = 0$

$a = 2,\ b = 3.$

$$\begin{cases} T_1' = -2T_1 + 2T_2 \\ T_2' = 2T_1 - 5T_2 \end{cases}$$

$T_1(0) = 40$

$T_2(0) = 45$

eliminate T_2: $T_2 = \dfrac{T_1' + 2T_1}{2}$

$\rightarrow \left(\dfrac{T_1' + 2T_1}{\cancel{2}}\right)' = 4T_1 - 5\left(\dfrac{T_1' + 2T_1}{\cancel{2}}\right)$

2 eqns ⇒ 2-nd order $\quad T_1'' + 2T_1' = 4T_1 - 5T_1' - 10T_1$

$T_1'' + 7T_1' + 6T_1 = 0$

All co-eff. must be positive ⇒ All var. come to 0 (physical problem)

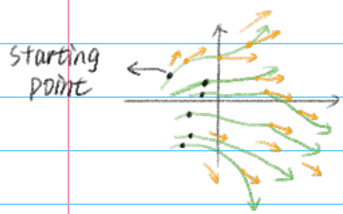
$$\begin{cases} T_1(t) = c_1 e^{-6t} + c_2 e^{-t} \\ T_2 = \frac{1}{2} c_1 e^{-t} - 2c_2 e^{-6t} \end{cases} \overset{\text{init}}{\Longrightarrow} \begin{cases} c_1 = 50 \\ c_2 = -10 \end{cases}$$

$\boxed{\text{autonomous system}}$ (no 't' ← indep var. on RHS)

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

Soln: $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ parametrised curve $\begin{cases} x'(t) \\ y'(t) \end{cases}$ velocity of soln at time t

方向场



system of 2 1-st order eqns (autonomous)

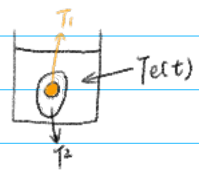
⇒ velocity field

soln

⇒ a parametrised curve with the right velocity everywhere

Lec25-26 常系数齐次线性方程组

D eqn 的历史是从1950年开始的，用 matrix 解ODE



$$\begin{cases} x=T_1 \\ y=T_2 \end{cases} \quad \boxed{\begin{cases} x' = -2x+2y \\ y' = 2x-5y \end{cases}} \quad \boxed{\begin{aligned} x &= c_1 e^{-t} + c_2 e^{-6t} \\ y &= \tfrac{c_1}{2} e^{-t} - 2c_2 e^{-6t} \end{aligned}}$$

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}} \quad \boxed{\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1/2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t}}$$

tried soln.

$$\cancel{\begin{cases} x = a_1 e^{\lambda_1 t} \\ y = a_2 e^{\lambda_2 t} \end{cases}} \text{ (same } \lambda\text{'s)} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} \Rightarrow \text{plug in to ODE}$$

$$\lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cancel{e^{\lambda t}} = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cancel{e^{\lambda t}}$$

$$\begin{cases} \lambda a_1 = -2a_1 + 2a_2 \\ \lambda a_2 = 2a_1 - 5a_2 \end{cases} \Rightarrow \begin{cases} (-2-\lambda)a_1 + 2a_2 = 0 \\ 2a_1 - (\lambda+5)a_2 = 0 \end{cases}$$

$a_1, a_2 \neq 0$
non-zero soln

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 \\ 2 & -\lambda-5 \end{vmatrix} = 0$$

$$(\lambda+2)(\lambda+5) - 4 = 0$$

$$\boxed{\lambda^2 + 7\lambda + 6 = 0} \quad \text{charatter eqn.}$$

$$\lambda_1 = -1 \quad \lambda_2 = -6$$

$$\lambda = -1 \quad \begin{cases} -a_1 + 2a_2 = 0 \\ 2a_1 - 4a_2 = 0 \end{cases} \checkmark \Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} \qquad \lambda = -6 \quad \begin{cases} 4a_1 + 2a_2 = 0 \\ 2a_1 + a_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t}$$

(singular)

$$\boxed{\tilde{c}_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + \tilde{c}_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t}}$$

$$c_1/2 = \tilde{c}_1 \qquad c_2 = \tilde{c}_2$$

In general:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Trial: $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t}$

subst: $\lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ A

Homos system: $\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$

$$(a-\lambda)(d-\lambda) - cb = 0$$

$$\star \quad \boxed{\lambda^2 - \underbrace{(a+d)}_{\text{trace}A}\lambda + \underbrace{ad-bc}_{\det A} = 0} \quad \text{Char. eqn. of the matrix } A$$

roots: $\lambda_1, \lambda_2 \Rightarrow$ real & distinct

$\Rightarrow$ eigenvalues of A (characteristic values, proper value)

property

Invented by German

For each λ_i, find associated $\vec{\alpha}_i = \begin{pmatrix} a_{1i} \\ a_{2i} \end{pmatrix}$ by solving system.

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

eigenvectors belonging to λ_i

gen soln: $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} e^{\lambda_2 t}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \vec{x}$ $\boxed{\vec{x}' = A\vec{x}}$

trial soln $\vec{x} = \vec{\alpha} e^{\lambda t}$

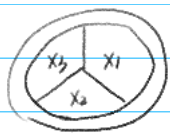
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$ $\lambda \vec{\alpha} \cancel{e^{\lambda t}} = A \vec{\alpha} \cancel{e^{\lambda t}}$

$\boxed{A\vec{\alpha} = \lambda \vec{\alpha}}$

$(A - \lambda I)\vec{\alpha} = 0$

$\boxed{\begin{matrix} \mathrm{Det}(A-\lambda I) = 0 \\ \text{roots: eigenvalues} \end{matrix}}$ ← char. eqn.

特征值有重根的情况.



x_i: temp in tank i

$x_i(t) \to$ find these fn's.

$x_1' = a(x_3 - x_1) + a(x_2 - x_1)$

$x_1' = -2ax_1 + ax_2 + ax_3$ (take $a=1$) $\boxed{\begin{cases} x_1' = -2x_1 + x_2 + x_3 \\ x_2' = x_1 - 2x_2 + x_3 \\ x_3' = x_1 + x_2 - 2x_3 \end{cases}}$

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} = -(\lambda+2)^3 + 2 + 3(\lambda+2) = 0$$

$$\lambda^3 + 6\lambda^2 + 9\lambda = 0$$

$$\lambda(\lambda+3)^2 = 0$$

$\lambda = 0$ $\vec{\alpha}$? $(a_1, a_2, a_3)^T$ $\begin{cases} -2a_1 + a_2 + a_3 = 0 \\ a_1 - 2a_2 + a_3 = 0 \\ a_1 + a_2 - 2a_3 = 0 \end{cases}$

soln: $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} e^{0t} \to$ constant soln $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = -3$ $a_1 + a_2 + a_3 = 0$ ×3 $\underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-3t}}_{\vec{\alpha}_1 \text{ soln}}$ $\underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-3t}}_{\vec{\alpha}_2 \text{ soln.}}$

$$x(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-3t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{0t}$$

complete eigenvalue;

λ repeated eigenvalue

you can find enough ind. eigenvectors to make up needed # of indep. solns.

defective, otherwise

Thm: A real $n\times n$ matrix which is symmetric ($A^T=A$)

All its eigenvalues are complete

特征值为复数

- calculate cx. eigenvalues
- form solns: $\vec{\alpha}e^{(a+bi)t}$
- Take the Re. Img parts $\Rightarrow$ to get 2 real solns

Example:

geog是 nerotic

$$\begin{cases} x'=x+2y \\ y'=-x-y \end{cases}$$

x: Susan to Geog's love

y: Geoge to Susan's love

char. eqn: $\lambda^2+1=0$

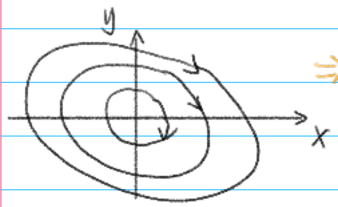
e-value: $\lambda=\pm i$

system for e-vector: $\begin{cases} (1-i)a_1+2a_2=0 \\ -a_1+(-1-i)a_2=0 \end{cases} \Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}=\begin{pmatrix} 1 \\ \frac{-1-i}{2} \end{pmatrix}$

soln: $\begin{pmatrix} 1 \\ \frac{-1-i}{2} \end{pmatrix}e^{it}=\left[\begin{pmatrix} 1 \\ -1/2 \end{pmatrix}+i\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}\right](\cos t+i\sin t)$

Re: $\begin{pmatrix} 1 \\ -1/2 \end{pmatrix}\cos t-\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}\sin t=\begin{pmatrix} x \\ y \end{pmatrix}\rightarrow\begin{cases} x=\cos t \\ y=-\frac{1}{2}(\cos t+\sin t) \end{cases}$

Img: $\begin{pmatrix} 1 \\ -1/2 \end{pmatrix}\sin t+\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}\cos t=\begin{pmatrix} x \\ y \end{pmatrix}\Rightarrow\begin{cases} x=\sin t \\ y=-\frac{1}{2}(\sin t-\cos t) \end{cases}$



$\Rightarrow$ their love goes round and round stretching on for 26 episodes

Lec27 2x2常系数齐次线性方程组作图 Phase Portraits

博弈论, 军备竞赛

$$\begin{cases} x' = -x + by \\ y' = cx - 3y \end{cases}$$ Mass N.H

x, y } departures from the normal amount of tourism advertising budget

$$\boxed{\begin{cases} x' = -x + 2y \\ y' = -3y \end{cases}}$$

$A = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix}$ $\lambda^2 + 4\lambda + 3 = 0$

$\lambda_1 = -3$ $\lambda_2 = -1$ e-values

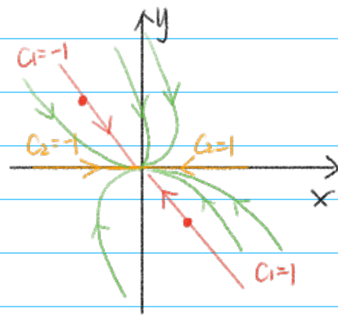
$\vec{\alpha}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$; $\vec{\alpha}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}$$

parametric eqn.

$$\begin{cases} x = c_1 e^{-3t} + c_2 e^{-t} \\ y = -c_1 e^{-3t} \end{cases}$$

1. 4 easy solns : Easy : $\begin{cases} c_1 = \pm 1, c_2 = 0 \\ c_1 = 0, c_2 = \pm 1 \end{cases}$
2. Fill in the rest



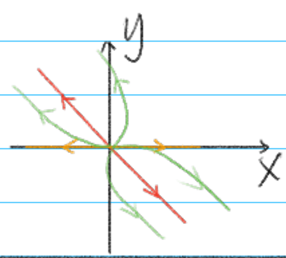
Nodal Sink

asymptotically stable

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}$$

dominant as $t \to -\infty$ dominant as $t \to \infty$.

when λ's are positive $\vec{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t}$



unstable

---

end up spending ∞ dollars, but no state exceed the other

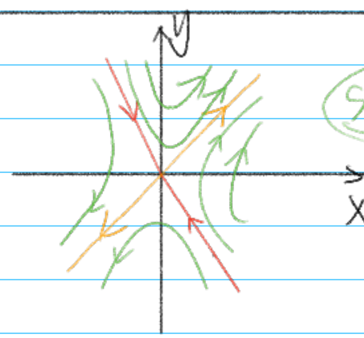
$$\boxed{\begin{cases} x' = -x + 3y \\ y' = 5x - 3y \end{cases}}$$ $A = \begin{pmatrix} -1 & 3 \\ 5 & -3 \end{pmatrix}$

$\lambda^2 + 4\lambda - 12 = 0$

$\lambda_1 = -6$ $\lambda_2 = 2$

$\vec{\alpha}_1 = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ $\vec{\alpha}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$c_1 \begin{pmatrix} 3 \\ -5 \end{pmatrix} e^{-6t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}.$$



Saddle

unstable

$$\begin{cases} x' = -x - y \\ y' = 2x - 3y \end{cases}$$

$A = \begin{pmatrix} -1 & -1 \\ 2 & -3 \end{pmatrix}$ $\lambda^2 + 4\lambda + 5 = 0$

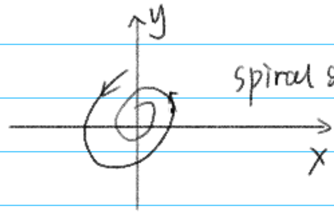
$\lambda = -2 \pm i$

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \left[\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cos t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sin t \right] e^{-2t} + C_2 (\text{similar})$$

As a curve — bounded, periodic 2π

satisfy $Ax^2 + By^2 + Cxy = D \Rightarrow$ hyperbola, parabolas, conic section (ellipses) → bounded

finally the budgets go to the same for 2 states



spiral sink

READ clockwise / counter clockwise

Lec28. 常系数非齐次线性方程组.

theory $\vec{x}' = A\vec{x}$ A: constant

$\boxed{\text{Thrm A}}$ Gen soln to $\vec{x}' = A\vec{x}$ is $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$

where $\vec{x}_1, \vec{x}_2$ are indep. soln's

Easy: All these are soln's (linearity + superposn)

Hard: no other soln's

$\boxed{\text{Thrm B}}$ Wronskian of 2 soln's

$W(\vec{x}_1, \vec{x}_2) := |\vec{x}_1 \ \vec{x}_2|$ (det)

Either $W(t)$ ↗ $\equiv 0$ (if $\vec{x}_1, \vec{x}_2$ dep.)

or ↘ never 0 (for any t value) ($\vec{x}_1, \vec{x}_2$ indep.)

Fundamental Matrix $\mathbb{X}$ for $\vec{x}' = A\vec{x}$

$\mathbb{X} := [\vec{x}_1 \ \vec{x}_2]$

$\vec{x}_1, \vec{x}_2$ ind. cols

Properties: ① $|\mathbb{X}| \neq 0$ for any t

② $\mathbb{X}' = A\mathbb{X} \iff [\vec{x}_1', \vec{x}_2'] = A[\vec{x}_1, \vec{x}_2] = [A\vec{x}_1, A\vec{x}_2]$

$\iff \vec{x}_1' = A\vec{x}_1, \ \vec{x}_2' = A\vec{x}_2$

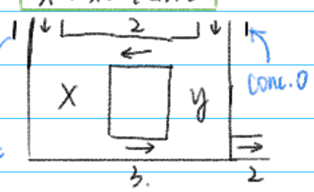
Inhomog. systems

$$\begin{cases} x' = ax + by + r_1(t) \\ y' = cx + dy + r_2(t) \end{cases} \qquad \boxed{\vec{x}' = A\vec{x} + \vec{r}(t)}$$

$\boxed{\text{Thrm C}}$ $\boxed{\vec{x}_{gen} = \vec{x}_c + \vec{x}_p}$

$\boxed{x'' + x = \tan t}$

Mixing problem



flow rate: L/hour

$x' = -3x + 2y + 5e^{-t}$

$y' = 3x - 4y$

$$\vec{x}' = \begin{bmatrix} -3 & 2 \\ 3 & -4 \end{bmatrix} \vec{x} + \begin{pmatrix} 5e^{-t} \\ 0 \end{pmatrix}$$

Method to solve $\vec{x}' = A\vec{x} + \vec{r}$

Variation of parameters

find $\vec{x}_p = v_1(t)\vec{x}_1 + v_2(t)\vec{x}_2$

$\vec{x}_p = \mathbb{X}\vec{v}$ $\qquad \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\qquad \vec{v}$: write on the right.

Substitude into the system, see what $\vec{v}_1, \vec{v}_2$

$\vec{x}_p' = A\vec{x}_p + \vec{r}(t)$

$\vec{X}' = A\vec{X}$

$(\mathbb{X}\cdot\vec{v})' = A(\mathbb{X}\cdot\vec{v}) + r(t)$

$\cancel{\mathbb{X}'\vec{v}} + \mathbb{X}\cdot\vec{v}' = \cancel{A(\mathbb{X}\cdot\vec{v})} + r(t)$

$\mathbb{X}\vec{v}' = \vec{r}$

$\vec{v}' = \mathbb{X}^{-1}\vec{r}$

$\vec{v} = \int \mathbb{X}^{-1}\vec{r}\,dt$ (integral each entry)

$\vec{x}_p = \mathbb{X}\int \mathbb{X}^{-1}\vec{r}\,dt$ (one xp good enough)

Lec29 矩阵指数及应用

$\boxed{\vec{x}' = A\vec{x}}$

fundamental matrix for A:

$X = [\vec{x}_1\ \vec{x}_2]$ (2 ind. solns)

Basic propertis of X

① $|X| \neq 0$ for any t

② $X' = AX$ (cols solve the system)

Write gen solns to system

$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$

gen soln $\boxed{\vec{x} = Xc} = [\vec{x}_1\ \vec{x}_2]\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

What do all FM's look like?

$[X\vec{c}_1\ \ X\vec{c}_2] = X[\vec{c}_1\ \vec{c}_2] \to C$ 2x2 matrix.

most gen soln $= XC \quad |C| \neq 0$.

$\vec{x}' = A\vec{x}$ The soln is given by $\boxed{\text{Formula}}$

1x1 case: $x' = ax$: soln is $\boxed{x = ce^{at}}$

Check: $e^{at} = 1 + at + a^2 t^2/2! + a^3 \cdot t^3/3!$

$\frac{de^{at}}{dt} = a + a^2 t + a^3 t^2/2!$

A fundamental Matrix for $\vec{x}' = A\vec{x}$

soln is

$e^{At} = I_2 + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} \cdots$ Sum of 2x2 matrices

② satisfies $X' = AX$: similar to 1x1 case

① $\overset{\text{Det}}{}\ |X(0)| = |I_2| = 1 \neq 0$

$\begin{cases} x' = y \\ y' = x \end{cases}$

$x' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\vec{x} \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^n$.

$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}t + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{t^2}{2!} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{t^3}{3!} + \cdots$

$= \begin{bmatrix} 1 + \frac{t^2}{2!} + \cdots & t + \frac{t^3}{3!} \\ t + \frac{t^3}{3!} + \cdots & 1 + \frac{t^2}{2!} \cdots \end{bmatrix}$

$= \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} = \begin{bmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{bmatrix}\frac{1}{2}$

IVP $\vec{x}' = A\vec{x} \quad \vec{x}(0) = \vec{x}_0$. Find $\vec{x}(t)$

gen soln. $\vec{x} = e^{At}\vec{c}$.

$\vec{x}(0) = e^{A \cdot 0}\vec{c}$

$\vec{x}_0 = I_2\vec{c} = \vec{c} \qquad \boxed{\vec{x} = e^{At} \cdot \vec{x}_0}$ ☆

$e^{A+B} \neq e^A \cdot e^B$

True in special cases: $AB=BA$ $\begin{cases} ① \ A = cI = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \\ ② \ B = -A \longleftarrow e^{A-A} = e^A \cdot e^{-A} \\ ③ \ B = A^{-1} \end{cases}$ show $(e^A)^{-1} = e^{-A}$

Calc. e^{At}: ① series (too hard)

② $\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$ use exp'l law

③ $\underset{\text{FM}}{X} \cdot X(0)^{-1} = e^{At}$ $\boxed{X(0) \cdot X(0)^{-1} = I_2}$ e^{At} have the same 2 properties

The equation $\dot{\mathbf{u}} = A\mathbf{u}$ (or the matrix A) is

"stable" if all solutions tend to $\mathbf{0}$ as $t \to \infty$.

"unstable" if most solutions grow without bound as $t \to \infty$.

"neutrally stable" otherwise.

A **fundamental matrix** for a square matrix A is a matrix of functions, $\Phi(t)$, whose columns are linearly independent solutions to $\dot{\mathbf{u}} = A\mathbf{u}$.

The fundamental matrix whose value at $t = 0$ is the identity matrix is the **matrix exponential** e^{At}. It can be computed from any fundamental matrix $\Phi(t)$:

$$e^{At} = \Phi(t)\Phi(0)^{-1}.$$

$\vec{X} = e^{At} \cdot C = e^{At} \cdot \vec{X}(0)$

The solution to $\dot{\mathbf{u}} = A\mathbf{u}$ with initial condition $\mathbf{u}(0)$ is given by $e^{At}\mathbf{u}(0)$.

If $\mathbf{q}$ is constant, and A is invertible, then $\mathbf{u_p}(t) = -A^{-1}\mathbf{q}$ is a solution to the inhomogeneous equation $\dot{\mathbf{u}} = A\mathbf{u} + \mathbf{q}$. The general solution is $\mathbf{u_p} + \mathbf{u_h}$, where $\mathbf{u_h}$ is the general solution of the associated homogeneous equation $\dot{\mathbf{u}} = A\mathbf{u}$.

$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$

Lecho 方程组的解耦 decoupling

$\boxed{\vec{x}' = A\vec{x}'}$

2 methods of solving ODE.

① eigenvalues, e-vectors

② $e^{At}\vec{x}_0$ $\quad e^{At} = \Phi(t)\,\Phi(0)^{-1}$

} The language's been changed

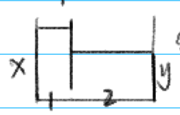
③ Decoupling. $\begin{cases} u = ax + by \\ v = cx + dy \end{cases}$ find u, v

$\Downarrow$ in uv-coords. (decoupled)

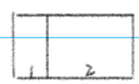
$\begin{cases} u' = k_1 u \\ v' = k_2 v \end{cases}$

couple: x, y occur at both RHS, LHS $\Rightarrow$ cannot solve seperately for x, y.

Example:



side view.



top view

flow rate through the hole: [ℓ/s] $\propto$ area of the hole $\cdot$ height difference.

$\begin{cases} x' = c(y - x) \\ 2y' = c(x - y) \end{cases}$ take $c = 2 \Rightarrow \begin{cases} x' = -2x + 2y \\ 2y' = x - y \end{cases}$ (ht diff $\propto$ pressure of hole)

heigh diff. (var.) $= x - y \Rightarrow$ more natural var.

$u = x + 2y$ (total amt water)

$v = x - y$ ($\propto$ pressure)

New system: (in u, v) $\begin{cases} u' = x' + 2y' = 0 \\ v' = x' - y' = -3x + 3y \end{cases} \Rightarrow \begin{cases} u' = 0 \\ v' = -3v \end{cases}$ decouple!

soln: $\begin{cases} u = c_1 \\ v = c_2 e^{-3t} \end{cases}$

In terms of x, y:

$\begin{cases} x = \frac{1}{3}(u + 2v) = \frac{1}{3}(c_1 + 2c_2 e^{-3t}) \\ y = \frac{1}{3}(u - v) = \frac{1}{3}(c_1 - c_2 e^{-3t}) \end{cases}$

$$\vec{x} = \frac{1}{3}c_1\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{3}c_2\begin{pmatrix}2\\-1\end{pmatrix}e^{-3t}$$

Decouple: e-values real, complete (assume need)

$\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}a_1' & a_2' \\ b_1' & b_2'\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$ NEED $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}a_1 & a_2 \\ b_1 & b_2\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}$

?

D - decoupling mx

$D^{-1} = E \quad E = (\vec{\alpha}_1, \vec{\alpha}_2)$: cols are 2 e-vectors.

$\vec{\alpha}_1 \longleftarrow \begin{pmatrix}1\\0\end{pmatrix}$

$\vec{\alpha}_2 \longleftarrow \begin{pmatrix}0\\1\end{pmatrix}$

$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}a_1 & a_2 \\ b_1 & b_2\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}$, with columns $\vec{\alpha}_1$, $\vec{\alpha}_2$; $\begin{pmatrix}1\\0\end{pmatrix} \longrightarrow \begin{pmatrix}a_1\\b_1\end{pmatrix}$

Substitude into $\vec{x}' = A\vec{x}$ to see if it's decoupled in u, v coord.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

$$(A - \lambda_1 I)\vec{\alpha}_1 = 0 \qquad \lambda_1, \vec{\alpha}_1$$

$$\boxed{A\vec{\alpha}_1 = \lambda_1 \vec{\alpha}_1}$$

$$AE = A[\vec{\alpha}_1 \vec{\alpha}_2] = [A\vec{\alpha}_1, A\vec{\alpha}_2]$$
$$= [\lambda_1\vec{\alpha}_1, \lambda_2\vec{\alpha}_2]$$
$$= [\vec{\alpha}_1 \vec{\alpha}_2]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \longrightarrow E\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\vec{X}' = A\vec{X}$$

make substitution: $\vec{X} = E\vec{u} \qquad \vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$

$$\cancel{E}\vec{u}' = AE\vec{u} = \cancel{E}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\vec{u}$$
$$\vec{u}' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\vec{u}$$

$$\begin{cases} u' = \lambda_1 u \\ v' = \lambda_2 v. \end{cases} \Rightarrow \begin{cases} u = C_1 e^{\lambda_1 t} \\ v = C_2 e^{\lambda_2 t}. \end{cases}$$

Decouple: $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Ev+Ev: $\lambda^2 + 3\lambda = 0$.

$$\lambda_1 = 0 \quad \lambda_2 = -3$$
$$\vec{\alpha}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{\alpha}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \qquad D = E^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \cdot \frac{1}{3}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} u = \frac{1}{3}(x + 2y) \\ v = \frac{1}{3}(-x + y) \end{cases} \Rightarrow \begin{cases} u' = 0 \\ v' = -3 \cdot v \end{cases}$$

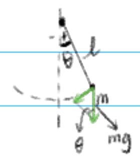
Lec31 非线性自治方程组及作图 automous non-linear

$x' = f(x,y)$

$y' = g(x,y)$

Problem: sketch its trajectories

Do it with example: non-linear problem (lightly damped)



$m\vec{a} = \vec{F}$

$m \underbrace{l\theta''}_{a} = -mg\sin\theta - c_1 \underbrace{l\theta'}_{v}$

$\theta'' + \frac{c_1}{m}\theta' + \frac{g}{l}\sin\theta = 0$

$\boxed{\theta'' + c\theta' + k\sin\theta = 0}$

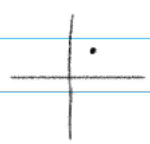
$\boxed{\theta' = w \quad w' = -k\sin\theta - c\theta'}$ $c=1 \quad k=2$

$\boxed{\begin{aligned}\theta' &= w \\ w' &= -2\sin\theta - w\end{aligned}}$ crit. pts: $\begin{cases}\theta = 0, \pm\pi, \pm 2\pi \\ w = 0\end{cases}$

1. Look for crit. points of the system

(x_0, y_0) $\quad f(x_0, y_0) = 0$ $\quad \begin{cases} x = x_0 \\ y = y_0 \end{cases}$

$g(x_0, y_0) = 0$ for all time



Solve $\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$

② For each cr. point (x_0, y_0)

lin'ze system near (x_0, y_0) crit. pts.

Lin'ze at $(0,0)$

$\begin{cases} \theta' = w \\ w' = -2\theta - w \end{cases}$ since $\sin\theta \approx \theta$, if $\theta \approx 0$

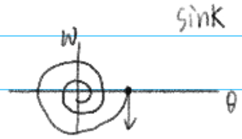
$\begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$

$\lambda^2 + \lambda + 2 = 0$

$\lambda = \frac{-1 \pm \sqrt{-7}}{2}$ spiral (cx roots)

sink (since $\lambda = -\frac{1}{2} + bi$)

钟摆运动



$\ast$ Jacobian matrix $J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_0$ Calculate at (x_0, y_0)

This is the matrix of the lin'ed system

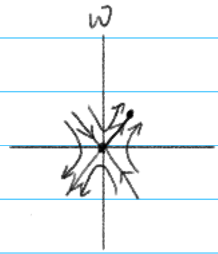
Via J: $J = \begin{pmatrix} 0 & 1 \\ -2\cos\theta & -1 \end{pmatrix}$ at $(\pi, 0)$ $J_0 = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$

at $(0,0)$: $J_0 = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}$

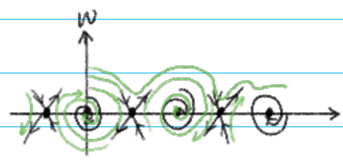
$\lambda^2 + \lambda - 2 = 0$

$\lambda_1 = -2 \quad \lambda_2 = 1$

$x(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}$.



(3) Big picture: plog trajs around each cr. pt and then add some



The Exponential Matrix

The work in the preceding note with fundamental matrices was valid for any linear homogeneous square system of ODE's,

$$\mathbf{x}' = A(t)\,\mathbf{x}\,.$$

However, if the system has *constant coefficients*, i.e., the matrix A is a constant matrix, the results are usually expressed by using the exponential matrix, which we now define.

Recall that if x is any real number, then

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots\,. \tag{1}$$

Definition 3 Given an $n \times n$ constant matrix A, the **exponential matrix** e^A is the $n \times n$ matrix defined by

$$e^A = I + A + \frac{A^2}{2!} + \ldots + \frac{A^n}{n!} + \ldots\,. \tag{2}$$

Each term on the right side of (2) is an $n \times n$ matrix adding up the ij-th entry of each of these matrices gives you an infinite series whose sum is the ij-th entry of e^A. (The series always converges.)

In the applications, an independent variable t is usually included:

$$e^{At} = I + A\,t + A^2\,\frac{t^2}{2!} + \ldots + A^n\,\frac{t^n}{n!} + \ldots\,. \tag{3}$$

This is not a new definition, it's just (2) above applied to the matrix $A\,t$ in which every element of A has been multiplied by t, since for example

$$(At)^2 = At \cdot At = A \cdot A \cdot t^2 = A^2 t^2.$$

Try out (2) and (3) on these two examples (the second is very easy, since it is not an infinite series).

Example 3A. Let $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Show: $e^A = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}$; and $e^{At} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}$

Example 3B. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, show: $e^A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

What's the point of the exponential matrix? The answer is given by the theorem below, which says that the exponential matrix provides a royal road to the solution of a square system with constant coefficients: no eigenvectors, no eigenvalues, you just write down the answer!

Theorem 3 *Let A be a square constant matrix. Then*

(1) *(a) $e^{At} = \widetilde{\Phi}_0(t)$, the normalized fundamental matrix at 0;*

(2) *(b) the unique solution to the IVP $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ is $\mathbf{x} = e^{At}\mathbf{x}_0$.*

Proof. Recall that in the previous note we saw that if $\widetilde{\Phi}_0(t)$ is the normalized fundamental matrix then

$$\text{The solution to the IVP}: \quad \mathbf{x}' = A(t)\,\mathbf{x},\ \ \mathbf{x}(0) = \mathbf{x}_0 \quad \text{is} \quad \mathbf{x}(t) = \widetilde{\Phi}_0(t)\mathbf{x}_0. \tag{4}$$

Statement (2) follows immediately from (1), in view of (4).

We prove (1) is true by using the fact that if $t_0 = 0$ then the normalized fundamental matrix has $\Phi(0) = I$. Letting $\Phi = e^{At}$, we must show $\Phi' = A\Phi$ and $\Phi(0) = I$.

The second of these follows from substituting $t = 0$ into the infinite series definition (3) for e^{At}.

To show $\Phi' = A\Phi$, we assume that we can differentiate the series (3) term-by-term; then we have for the individual terms

$$\frac{d}{dt} A^n \frac{t^n}{n!} = A^n \cdot \frac{t^{n-1}}{(n-1)!},$$

since A^n is a constant matrix. Differentiating (3) term-by-term then gives

$$\begin{aligned} \frac{d\Phi}{dt} = \frac{d}{dt} e^{At} &= A + A^2 t + \ldots + A^n \frac{t^{n-1}}{(n-1)!} + \ldots \\ &= A\,e^{At} = A\,\Phi\,. \end{aligned} \tag{5}$$

Calculation of e^{At}.

The main use of the exponential matrix is in Theorem 3 — writing down explicitly the solution to an IVP. If e^{At} has to be calculated for a specific system, several techniques are available.

a) In simple cases, it can be calculated directly as an infinite series of matrices.

b) It can always be calculated, according to Theorem 3, as the normalized fundamental matrix $\widetilde{\Phi}_0(t)$, using (11): $\widetilde{\Phi}_0(t) = \Phi(t)\Phi(0)^{-1}$.

c) A third technique uses the exponential law

$$e^{(B+C)t} = e^{Bt}e^{Ct}, \qquad \textit{valid if } BC = CB. \tag{6}$$

To use it, one looks for constant matrices B and C such that

$$A = B + C, \qquad BC = CB, \qquad e^{Bt} \text{ and } e^{Ct} \text{ are computable;} \tag{7}$$

then

$$e^{At} = e^{Bt}e^{Ct}. \tag{8}$$

Example 3C. Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Solve $\mathbf{x}' = A\,\mathbf{x}$, $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, using e^{At}.

Solution. We set $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; then (7) is satisfied, and

$$e^{At} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

by (8) and Examples 3A and 3B. Therefore, by Theorem 3 (2), we get

$$\mathbf{x} = e^{At}\mathbf{x}_0 = e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = e^{2t} \begin{pmatrix} 1 + 2t \\ 2 \end{pmatrix}.$$