逆矩阵

$$A=\begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix} \xrightarrow{\text{可逆}} a\neq 0 \quad a\neq b$$

$$[A \mid I] \rightarrow [I \mid A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ a & a & b & 0 & 1 & 0 \\ a & a & a & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} a & b & b & 1 & 0 & 0 \\ 0 & a-b & 0 & -1 & 1 & 0 \\ 0 & 0 & a-b & 0 & -1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & b/a & b/a & 1/a & 0 & 0 \\ 0 & 1 & 0 & -1/(a-b) & 1/(a-b) & 0 \\ 0 & 0 & 1 & 0 & -1/(a-b) & 1/(a-b) \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a-b} & 0 & \frac{-b}{a(a-b)} \\ 0 & 1 & 0 & \frac{-1}{a-b} & \frac{1}{a-b} & 0 \\ 0 & 0 & 1 & 0 & \frac{-1}{a-b} & \frac{1}{a-b} \end{array}\right]$$

$$A^{-1}=\frac{1}{a-b}\begin{bmatrix} 1 & 0 & -\frac{b}{a} \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

LU分解

$$A=\begin{pmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{pmatrix} \xrightarrow[E_{21}=\begin{pmatrix} 1 & & \\ -a & 1 & \\ 0 & 0 & 1 \end{pmatrix}]{} \begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ b & b & a \end{pmatrix} \xrightarrow[E_{31}=\begin{pmatrix} 1 & & \\ 0 & 1 & \\ -b & 0 & 1 \end{pmatrix}]{} \begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & b & a-b \end{pmatrix} \xrightarrow[E_{32}=\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -\frac{b}{a} & 1 \end{pmatrix}]{\text{Assume } a\neq 0} \begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a-b \end{pmatrix} \; U$$

$$E_{32}E_{31}E_{21}A=U$$

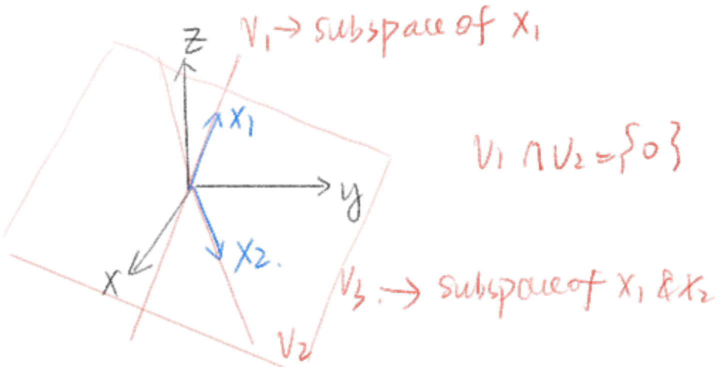
$$A=\underbrace{E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}}\,U$$

$$\Rightarrow \begin{pmatrix} 1 & & \\ a & 1 & \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ 0 & 1 & \\ b & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & \frac{b}{a} & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & \frac{b}{a} & 1 \end{pmatrix} (L) \quad a\neq 0$$

Singular matrix CAN have LU decomposition

三维空间的子空间



解 $Ax=0$ / $Ax=b$.

$\underbrace{x-5y+2z=9}$ a plane in R^3.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} \ \\ 0 \\ 0 \end{bmatrix}}_{P_0} + \underbrace{c_1 \begin{bmatrix} \ \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \ \\ 0 \\ 1 \end{bmatrix}}_{S_0}$$

$c_1=c_2=0$

$\Rightarrow P_0$ is in S

$$\Rightarrow \begin{cases} x-5y+2z=9 \\ y=0 \\ z=0 \end{cases} \quad \text{or} \quad \begin{matrix} y=1 \\ z=0 \end{matrix} \quad \begin{matrix} y=0 \\ z=1 \end{matrix}$$

$$\begin{cases} x-2y-2z=b_1 \\ 2x-5y-4z=b_2 \\ 4x-9y-8z=b_3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 2 & -5 & -4 & b_2 \\ 4 & -9 & -8 & b_3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 0 & -1 & 0 & -2b_1+b_2 \\ 0 & -1 & 0 & -4b_1+b_3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 5b_1-2b_2 \\ 0 & 1 & 0 & 2b_1-b_2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underbrace{\uparrow \ \uparrow}_{\text{pivot}} \ \underbrace{\uparrow}_{\text{free}}$$

* particular sol:

$Ax=b$

$z=0. \Rightarrow x_p = \begin{bmatrix} 5b_1-2b_2 \\ 2b_1-b_2 \\ 0 \end{bmatrix}$

* All sols:

$\vec{x} = \vec{x}_p + c \cdot \vec{x}_s$

* special sol:

$Ax=0$

$z=1 \quad x_s = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

向量空间的基底与维数

Find the dim of the vector space spanned by the vectors:

(1, 1, -2, 0, -1)

(1, 2, 0, -4, 1)

(0, 1, 3, -3, 2)

(2, 3, 0, -2, 0)

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 1 & 2 & -4 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \text{basis} \quad \text{dim}=3$$

and find the basis for that space

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 3 \\ -2 & 0 & 3 & 0 \\ 0 & -4 & -3 & -2 \\ -1 & 1 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x x x

basis

change the col space

四个基本子空间的计算

$$\text{Suppose } B = \underset{L}{\begin{pmatrix} 1 & & \\ 2 & 1 & \\ -1 & 0 & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 5 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}}$$

Find a basis for and compute the dim of each of the 4 fundamental subspaces

$$B = \begin{pmatrix} 1 & & \\ 2 & 1 & \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$N(B) = c_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -\frac{3}{5} \\ -1 \\ 1 \end{bmatrix}$$

dim = 1

$C(B)$ = linear comb of $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ b \end{bmatrix}$

dim = 2.

$\dim N(B^T) = 1$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\dim C(B^T) = 2$

$$\left\{ \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\begin{pmatrix} 1 & & \\ -2 & 1 & \\ 1 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 5 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

矩阵的空间

Show that the set of 2x3 matrices whose nullspace contains $\begin{bmatrix}2\\1\\1\end{bmatrix}$ is a vector subspace,

and find a basis for it.

What about the set of those whose col space contains $\begin{bmatrix}2\\1\end{bmatrix}$

$$A \cdot \begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \qquad B \cdot \begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \ \checkmark$$

$$(A+B)\begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$cA\begin{bmatrix}2\\1\\1\end{bmatrix} = c\begin{bmatrix}0\\0\end{bmatrix} \ \checkmark$$

$$2a+b+c=0$$

$$[a \ \ b \ \ -2a-b] = [a \ \ 0 \ \ -2a] + [0 \ \ b \ \ -b]$$

正交向量和子空间

S is spanned by $(1\,2\,2\,3)$ and $(1\,3\,3\,2)$

i) Find a basis of $S^\perp$

Can every v in $\mathbb{R}^4$ be writed uniquely in S and $S^\perp$ YES!

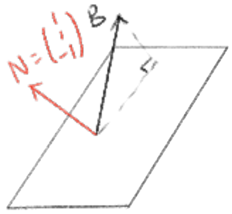
$$A = \begin{bmatrix}1&1\\2&3\\2&3\\3&2\end{bmatrix} \qquad \begin{pmatrix}1&2&2&3\\1&3&3&2\end{pmatrix}x=0 \qquad x = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}-5b\\-a+b\\a\\b\end{pmatrix} = \underbrace{a\begin{pmatrix}0\\-1\\1\\0\end{pmatrix} + b\begin{pmatrix}-5\\1\\0\\1\end{pmatrix}}_{S^\perp}$$

$$\Rightarrow \begin{pmatrix}1&2&2&3\\0&1&1&-1\end{pmatrix}x=0 \qquad x_4=b \quad x_3=a$$

$$\begin{pmatrix}1&1&0&-5\\2&3&-1&1\\2&3&1&0\\3&2&0&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix} = v$$

子空间上的投影

Find the orthogonal projection matrix onto the plane: $x+y-z=0$



$$P = A(A^TA)^{-1}A^T$$

$$A = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \quad a_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad a_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad P = \frac{1}{3}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$IB = PB + P_N B$$

$$I = P + P_N. \quad \Rightarrow P = I - P_N = N(N^TN)N^T$$

最小二乘逼近

Find the quadratic equation through the 0 that is a best fit for the points

$(1,1)$ $(2,5)$ $(-1,-2)$ $\qquad ct + dt^2 = y.$

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ -1 & 1 \end{pmatrix} \quad \hat{x} = \begin{pmatrix} c \\ d \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$$

$A \cdot \hat{x} = b$ can't solve.

$$A^TA\hat{x} = A^Tb$$

$$d = -5/2 \quad c = 11/2$$

Gram-Schmidt 正交化

Find q_1, q_2, q_3 (orthonormal) from a, b, c (cols of A).

Then write A as QR $\qquad A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$

$$q_1 = a = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad q_2 = b - a(a^Ta)^{-1}a^Tb = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$q_3 = c - q_2(q_2^Tq_2)^{-1}q_2^Tc - q_1(q_1^Tq_1)^{-1}q_1^Tc$$

$$= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \frac{\begin{bmatrix} 0 & & \\ & 0 & \\ & & 9 \end{bmatrix}}{9}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$A = QR$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

$a\ b\ c \quad q_1\ q_2\ q_3$

$$\begin{cases} a = 1 \cdot q_1 \\ b = 2 \cdot q_1 + 3 \cdot q_2 \\ c = 4\, q_1 + 6\, q_2 + 5 \cdot q_3 \end{cases}$$

特征值和特征向量.

Given the invertible $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{pmatrix}$ find the eigenvalues & eigenvectors of A^2, $A^{-1} - I$.

$Av = \lambda v$

$A(Av) = A \cdot (\lambda v) = \lambda (Av) = \lambda^2 v.$

$A^{-1} A v = A^{-1} \lambda v$

$A^{-1} v = A^{-1} \frac{Av}{\lambda} = A^{-1} A \frac{1}{\lambda} v = \lambda^{-1} v.$

$(A^{-1} - I) v = (\lambda^{-1} - 1) v$

$Av = \lambda v$

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & -2 \\ 0 & 1 & 4-\lambda \end{vmatrix}$

$= (1-\lambda)[(1-\lambda)(4-\lambda)+2] - 2 \cdot 0 + 3 \cdot 0$

$= (1-\lambda)(\lambda^2 - 5\lambda + 6) = 0.$

$= (1-\lambda)(\lambda - 2)(\lambda - 3) = 0.$

$\lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = 3.$

$$\begin{cases} v_1 + 2v_2 + 3v_3 = v_1 \\ v_2 - 2v_3 = v_2 \\ v_2 + 4v_3 = v_3 \end{cases} \Rightarrow \begin{cases} v_1 = 0\ 0\ 1 \\ v_2 = 0. \\ v_3 = 0. \end{cases}$$

矩阵的方幂

Find a formula for C^k where $C=\begin{pmatrix} 2b-a & a-b \\ 2b-2a & 2a-b \end{pmatrix}$; Calculate C^{100} when $a=b=-1$

$\det(C-\lambda I)=(\lambda-a)(\lambda-b)$ $\quad \lambda_1=a \quad \lambda_2=b.$

$C-aI=\begin{pmatrix} 2b-2a & a-b \\ 2b-2a & a-b \end{pmatrix} \quad v_1=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$C-bI=\begin{pmatrix} b-a & a-b \\ 2b-2a & 2a-2b \end{pmatrix} \quad v_2=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$C=S\Lambda S^{-1}=\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$

$C^k=S\Lambda^k S^{-1}=\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}=\begin{pmatrix} a^k & b^k \\ 2a^k & b^k \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}=\begin{pmatrix} 2b^k-a^k & a^k-b^k \\ 2b^k-2a^k & 2a^k-b^k \end{pmatrix}$

$$C^{100}=I.$$

对称矩阵与正定矩阵

a). A invertible. $\Leftrightarrow \det A=\lambda_1\cdots\lambda_n \neq 0$

A is pos definite $\Leftrightarrow \lambda_1,\lambda_2,\dots,\lambda_n>0.$

$\det(A)=\lambda_1.\lambda_2\cdots\lambda_n>0 \neq 0$

b). The only pos definite projection matrix is $P=I$.

P is projection eig. values of $P=0$ or 1

c). $D=\mathrm{diag}(d_1,d_2\cdots d_n)$

For any vector $x\neq 0$, $x^TDx>0$

$x^T=(x_1x_2\cdots x_n)$ $\quad x^TDx=d_1x_1^2+d_2x_2^2+\cdots+d_nx_n^2>0$

For which value of c is $B=\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2+c\end{bmatrix}$

- pos definite? $c>0$
- pos semidefinite? $c\geq 0$

* det test

2, $\det\begin{bmatrix}2&-1\\-1&2\end{bmatrix}=3$, $\det B = 2[2(2+c)-1]+[(-2-c)-1]-(1+2)$

$= 2(3+2c)+(-c-3)-3 = 3c$

* pivots test

$$\begin{bmatrix}2&-1&-1\\0&3/2&-3/2\\0&0&c\end{bmatrix}$$

* energy test

$$[x\ y\ z]\, B \begin{bmatrix}x\\y\\z\end{bmatrix} \geq 0$$

奇异值分解

SVD of $C=\begin{pmatrix}5&5\\-1&7\end{pmatrix}$

$C=U\Sigma V^T$ $\quad C^TC=\begin{pmatrix}26&18\\18&74\end{pmatrix}$ $\quad \lambda_1=20$ $\quad v_1=\begin{pmatrix}-3/\sqrt{10}\\1/\sqrt{10}\end{pmatrix}$

$\left\{\begin{array}{l} C^TC = V\Sigma^T\Sigma V^T \\ CV = U\Sigma. \end{array}\right.$ 对称矩阵的特征值分解 $\quad \lambda_2=80$ $\quad v_2=\begin{pmatrix}1/\sqrt{10}\\3/\sqrt{10}\end{pmatrix}$

$\Sigma=\begin{pmatrix}2\sqrt{5}&0\\0&4\sqrt{5}\end{pmatrix}$ $\quad V=\begin{pmatrix}-3/\sqrt{10}&1/\sqrt{10}\\1/\sqrt{10}&3/\sqrt{10}\end{pmatrix}$

$$C\cdot V=\begin{pmatrix}5&5\\-1&7\end{pmatrix}\begin{pmatrix}-3/\sqrt{10}&1/\sqrt{10}\\1/\sqrt{10}&3/\sqrt{10}\end{pmatrix}=\frac{1}{\sqrt{10}}\begin{pmatrix}5&5\\-1&7\end{pmatrix}\begin{pmatrix}-3&1\\1&3\end{pmatrix}$$

$$=\frac{1}{\sqrt{10}}\begin{pmatrix}-10&20\\10&20\end{pmatrix}=\begin{pmatrix}-\sqrt{10}&2\sqrt{10}\\\sqrt{10}&2\sqrt{10}\end{pmatrix}$$

$$C\cdot V\cdot\Sigma^{-1}=\begin{pmatrix}-\sqrt{10}&2\sqrt{10}\\\sqrt{10}&2\sqrt{10}\end{pmatrix}\begin{pmatrix}\frac{1}{2\sqrt{5}}&0\\0&\frac{1}{4\sqrt{5}}\end{pmatrix}=\begin{pmatrix}-\sqrt{2}/2&\sqrt{2}/2\\\sqrt{2}/2&\sqrt{2}/2\end{pmatrix}$$