Sorting algorithms

A Ammar, A Scemama, P Reinhardt, Y Damour

November 23, 2024

Sorting for Sorting

- Sorting algorithms are used in everyday applications

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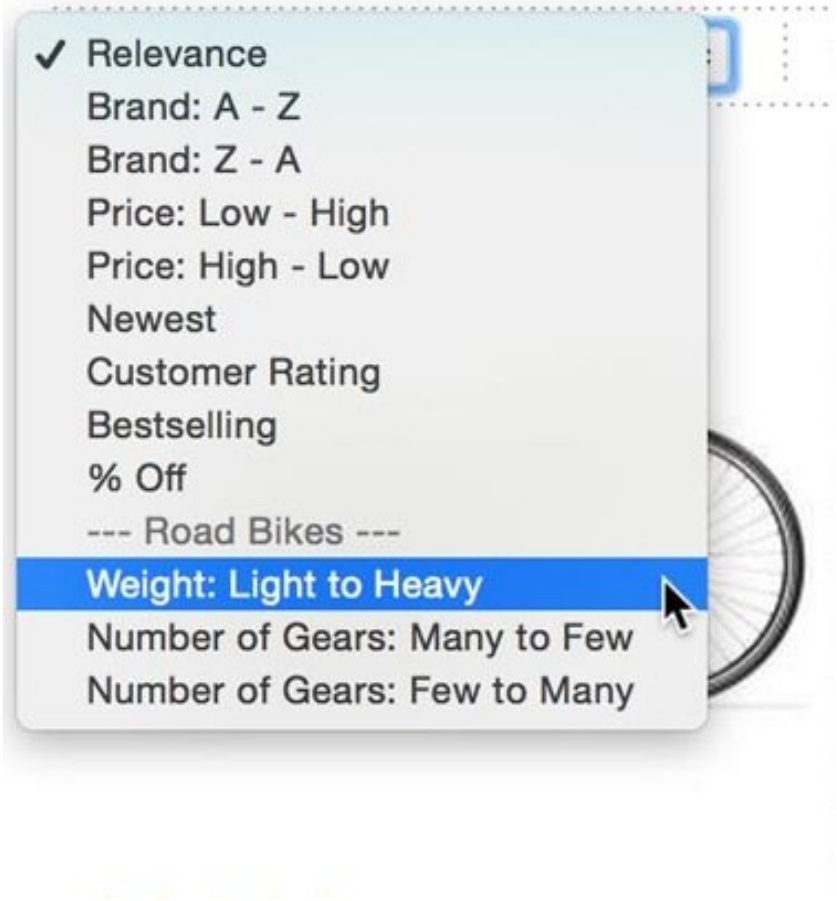

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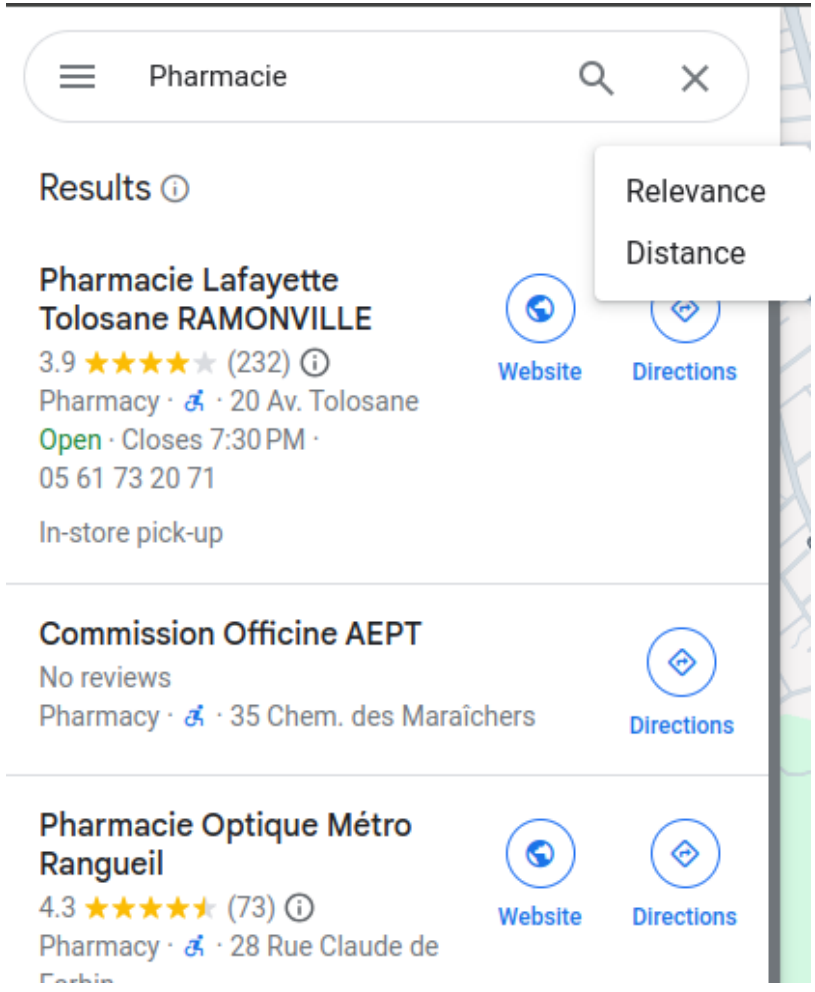


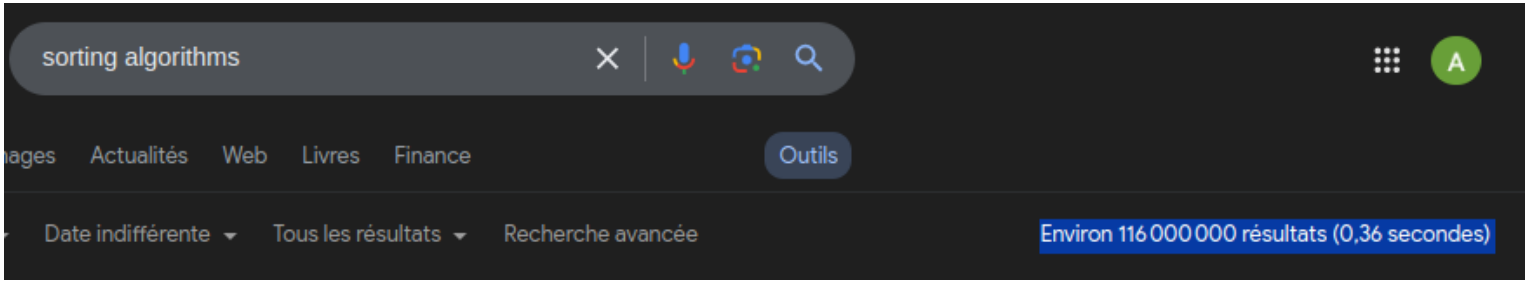
Sorting for Sorting

```
:~/.../slides$ ls -lt

 41231 Nov 13 23:29 main.log
206249 Nov 13 23:29 main.pdf
  1252 Nov 13 23:29 main.aux
   747 Nov 13 23:29 main.nav
     0 Nov 13 23:29 main.snm
     0 Nov 13 23:29 main.toc
     0 Nov 13 23:29 main.out
  4096 Nov 13 23:29 images
  1873 Nov 13 23:27 intro.tex
  1886 Nov 13 22:25 main.tex
 10290 Nov 13 19:55 quick_sort.tex
 10834 Nov 13 19:55 merge_sort.tex
  5904 Nov 13 19:55 bubble_sort.tex
  6337 Nov 13 19:47 radix_sort.tex
   261 Nov 10 23:43 Makefile
:~/.../slides$
```

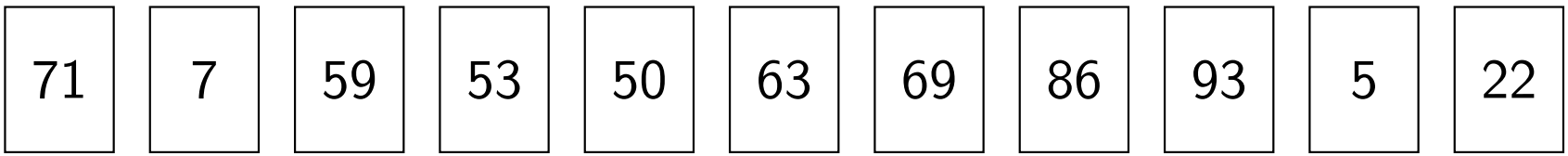
Sorting for Efficiency

- Sorting algorithms are employed for more than just sorting



Linear vs Binary search

- Find 69 ?



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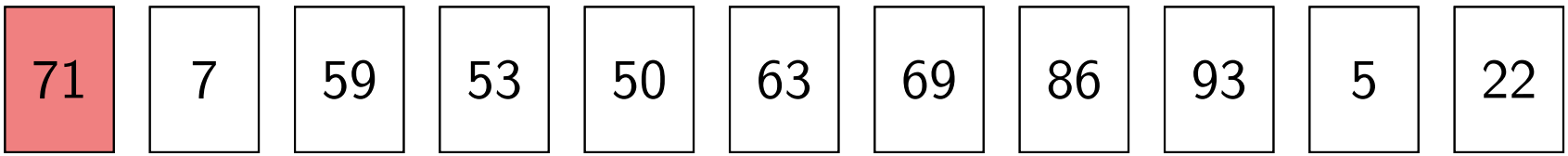

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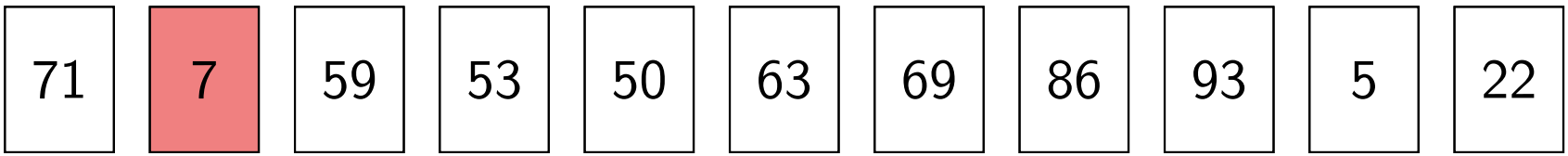

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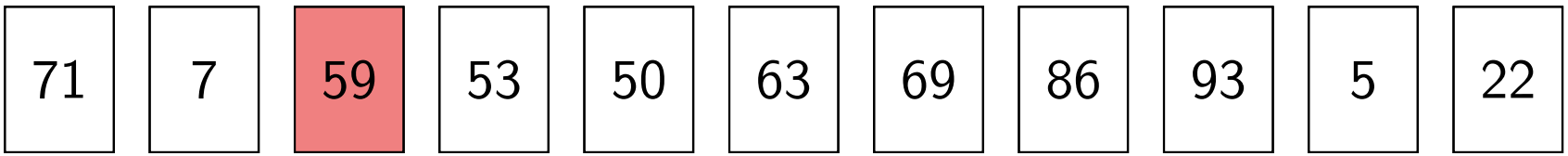

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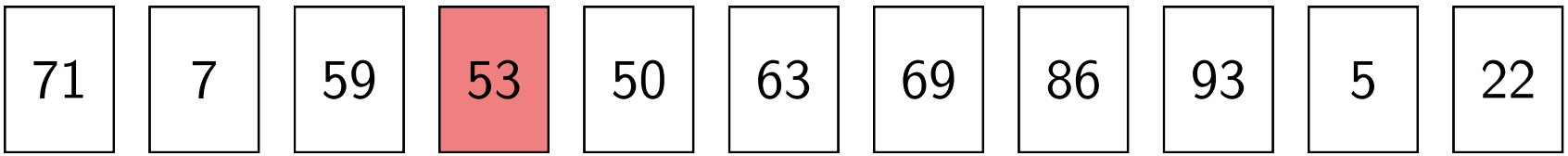

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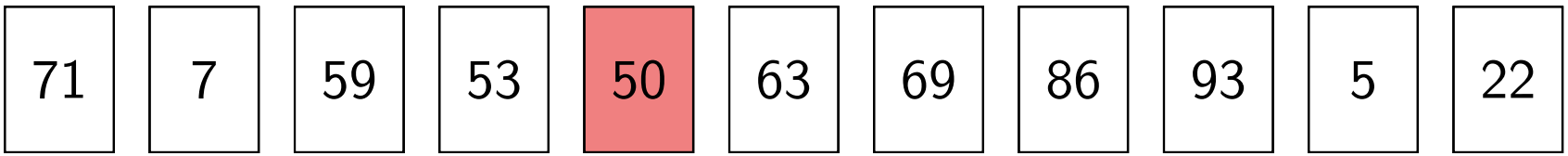

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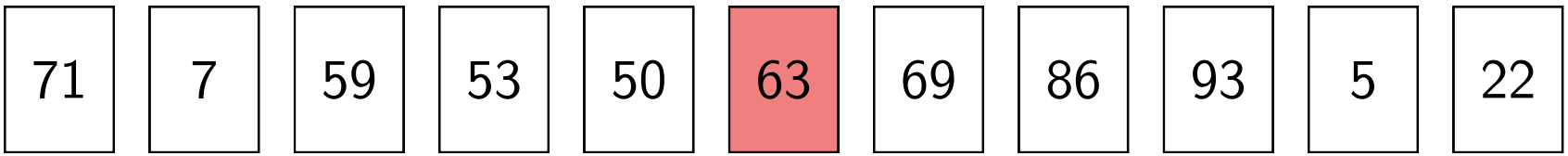

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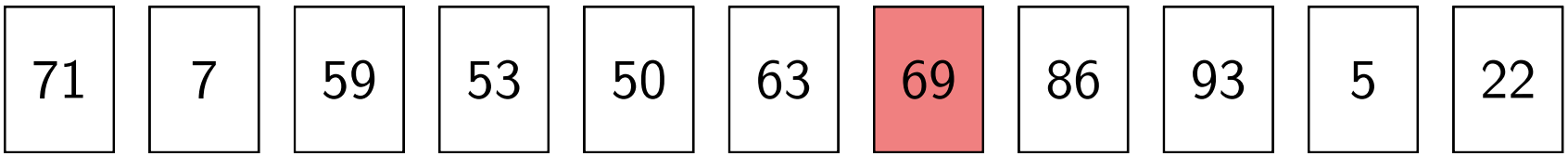

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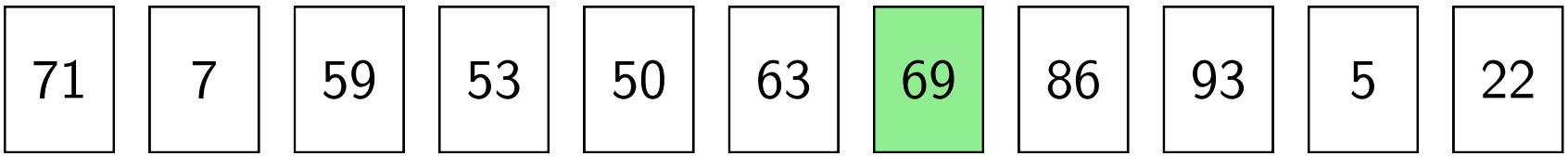

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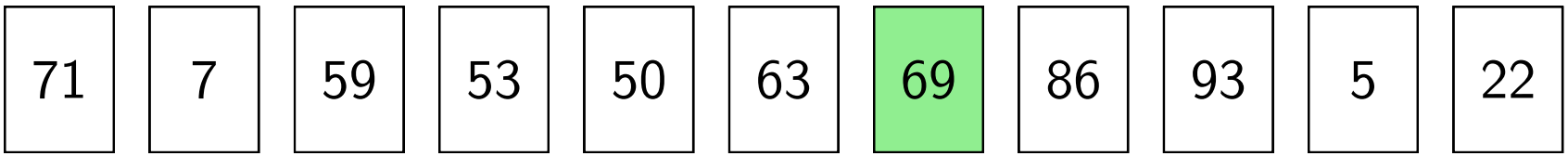

➥ **time scaling** $\mathcal{O}(N)$

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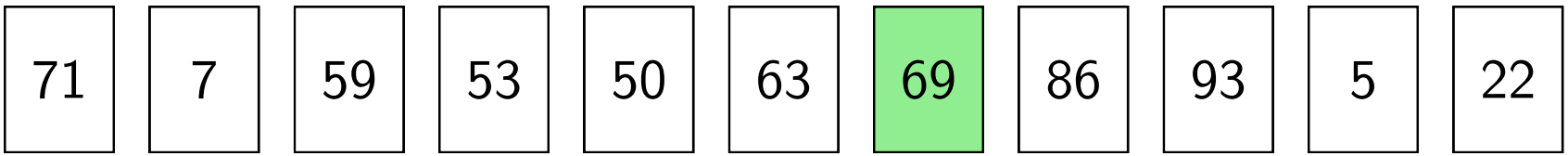

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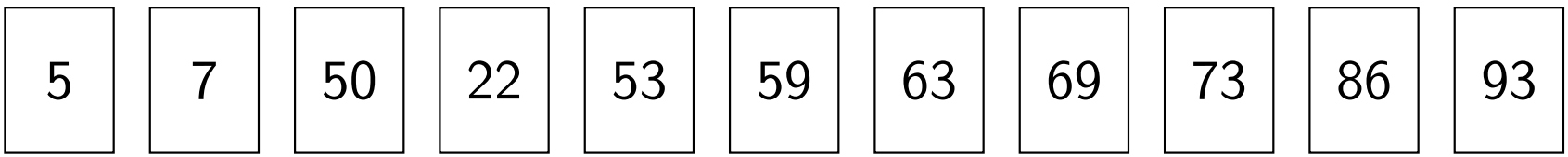

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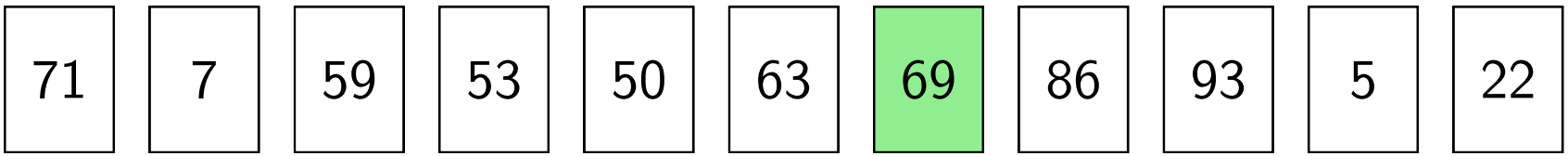

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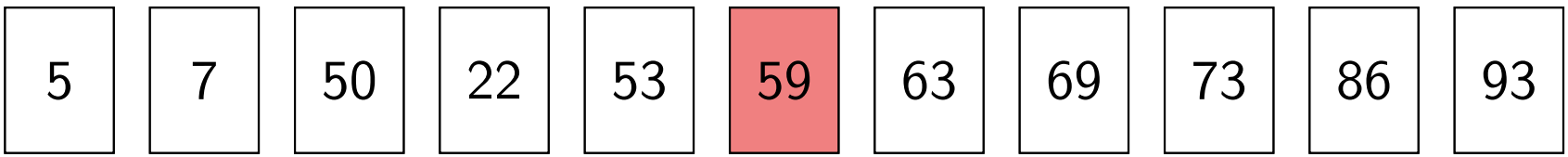

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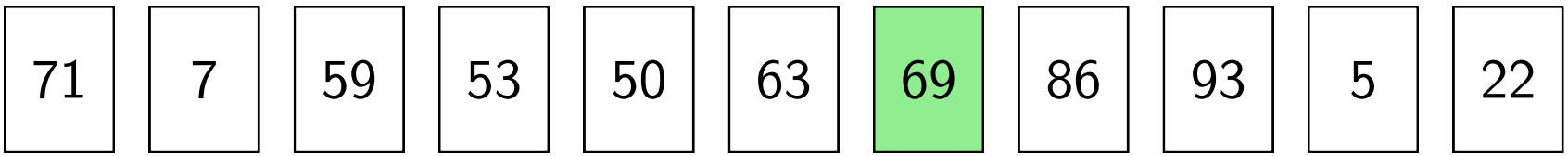

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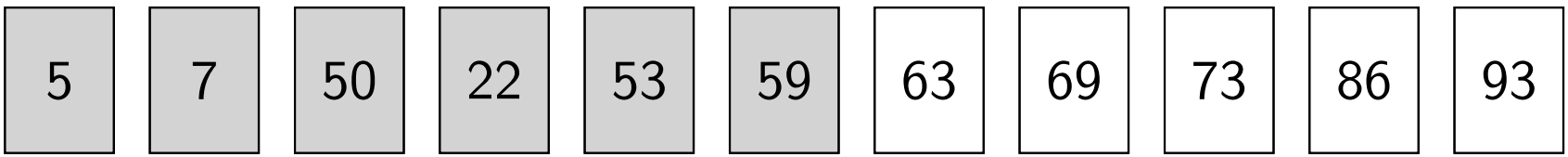

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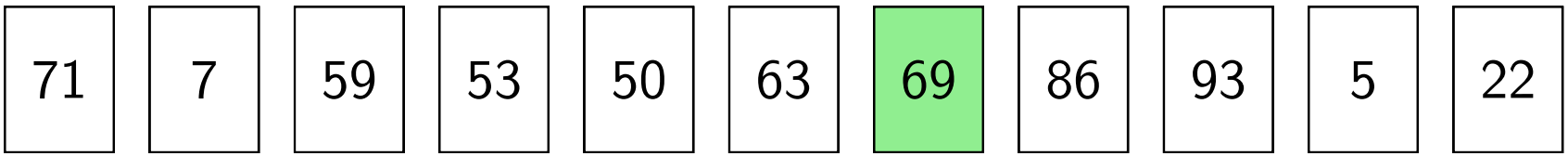

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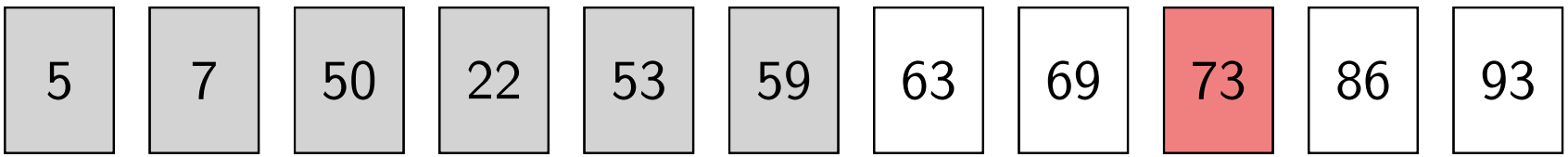

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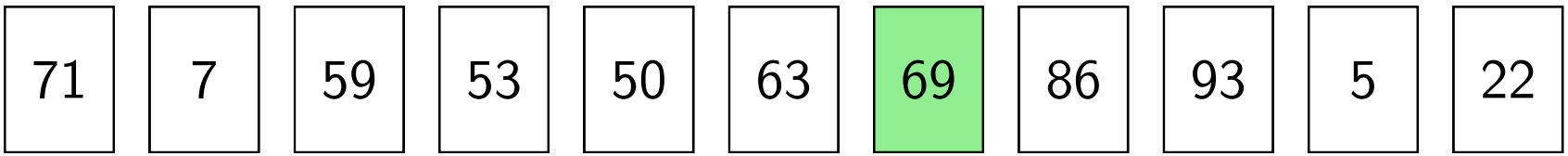

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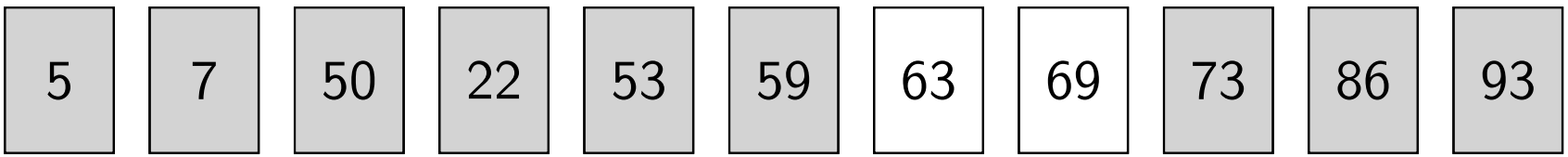

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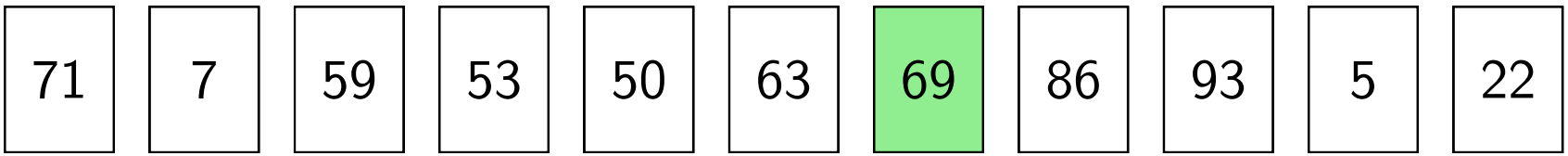

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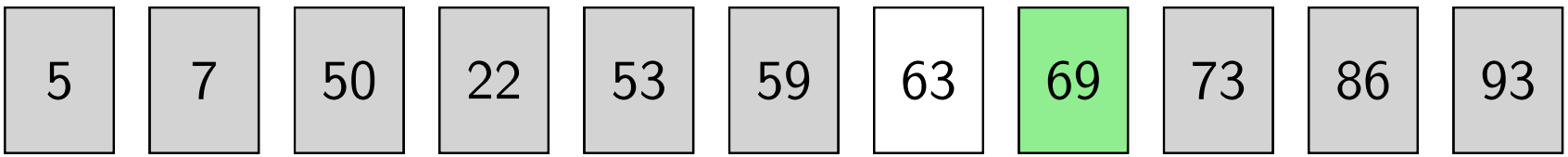

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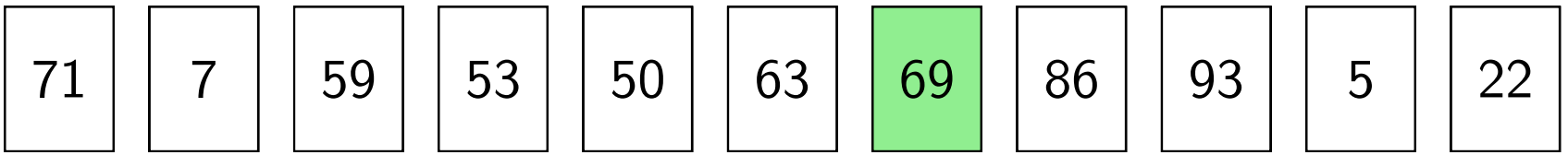

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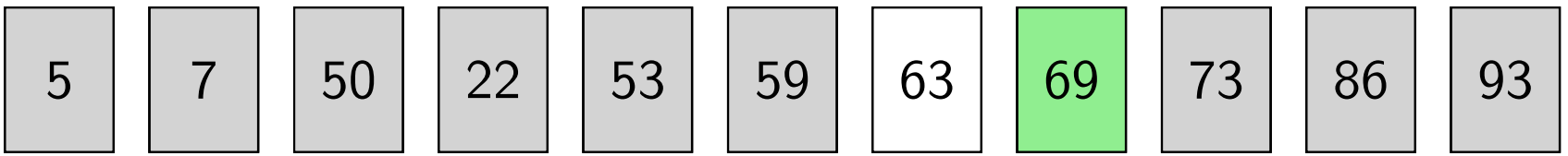
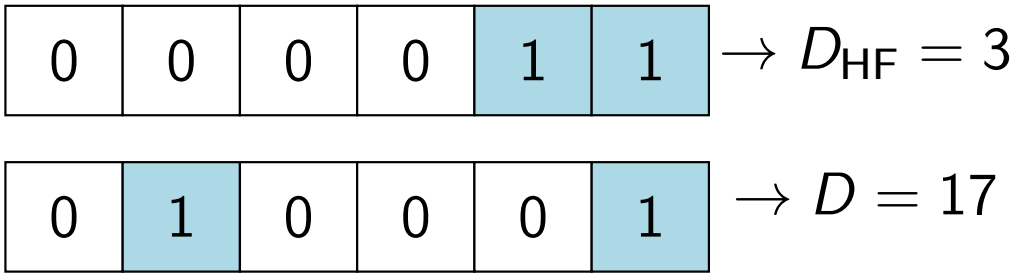


➥ **time scaling** $\mathcal{O}(\log N)$

🧐 $\log(116\,000\,000) \approx 18$ **!!**

Example: CI wavefunction



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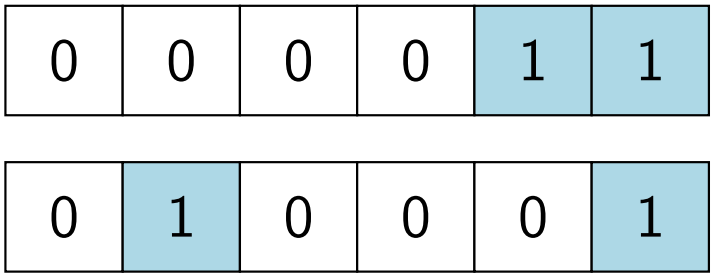

- Overlap between 2 CI wavefunctions

$$\begin{cases} \Psi = \sum_{I=1}^{N} c_I D_I \\ \tilde{\Psi} = \sum_{I=1}^{\tilde{N}} \tilde{c}_I \tilde{D}_I \end{cases} \Rightarrow S = \left\langle \tilde{\Psi} \mid \Psi \right\rangle$$

Example: CI wavefunction

0	0	0	0	1	1

$\rightarrow D_{\text{HF}} = 3$

0	1	0	0	0	1

$\rightarrow D = 17$

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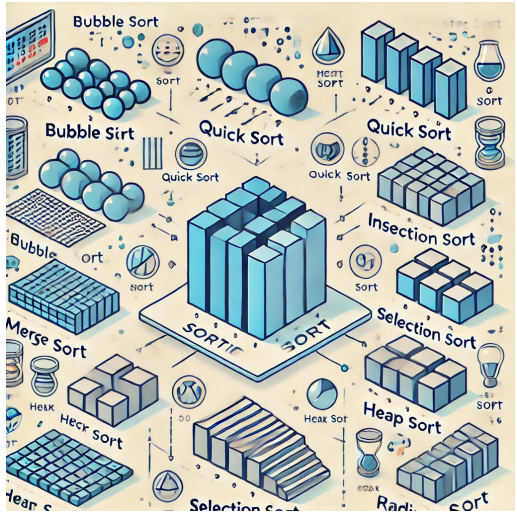
- Naive implementation: $\mathcal{O}(N\tilde{N})$
- Smart implementation: $\mathcal{O}(N \log(N) + \tilde{N} \log(\tilde{N}))$

Best Sorting Algorithm ?



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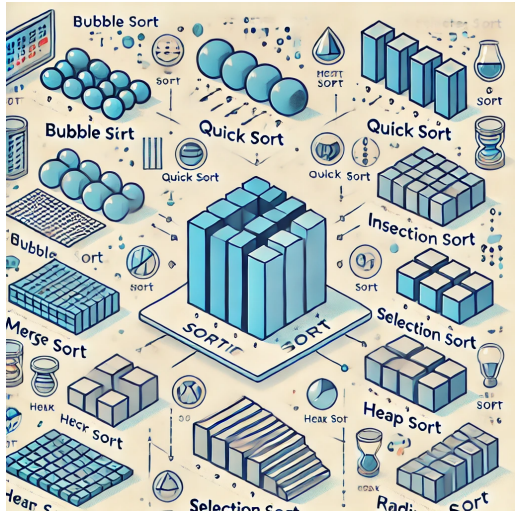

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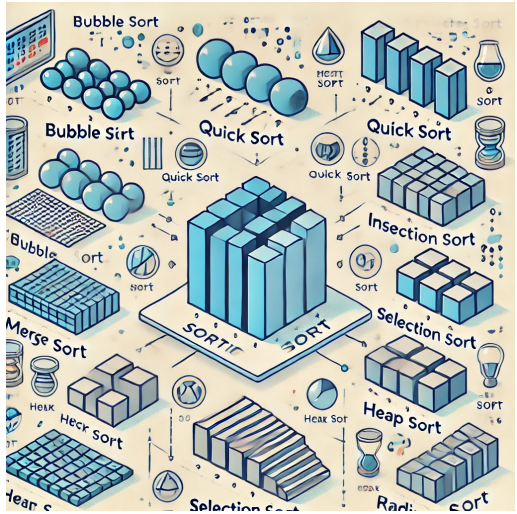

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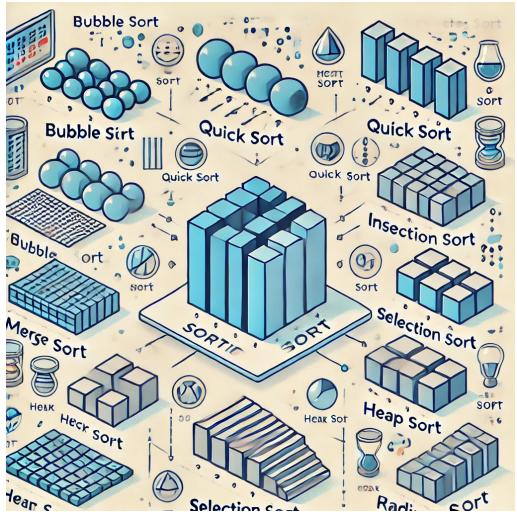

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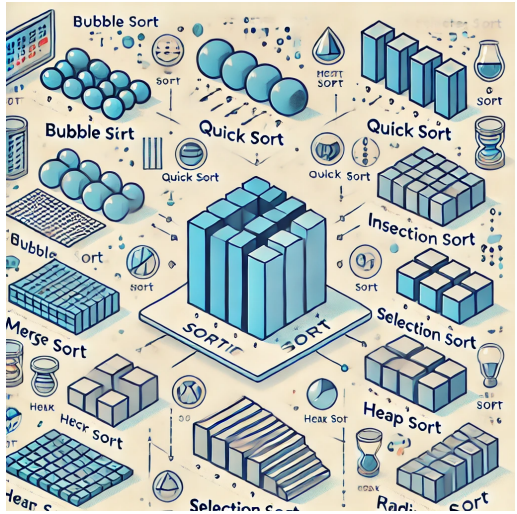

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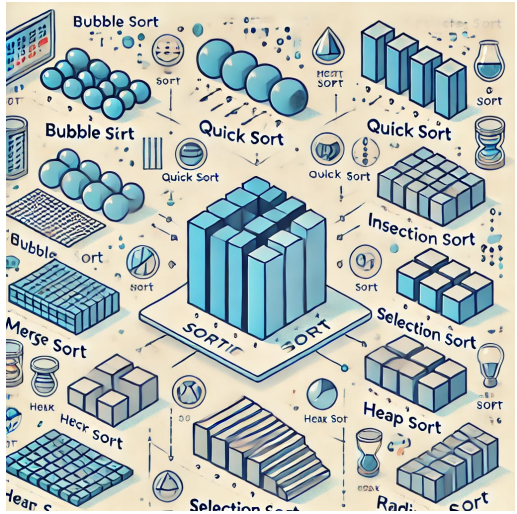

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  - Adaptiveness (best, worst, and average cases)
  - Online vs. Offline sorting

1. Bubble Sort
2. Merge sort
3. Quick sort
4. Radix sort

Bubble Sort

Bubble Sort Algorithm

- Goal: Sort an array of n items
- Algorithm:
  1. Compare first pair of adjacent items
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  4. Reduce n by 1 and go to step 1

Illustration

First pass

25	13	4	7	16

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25	13	4	7	16

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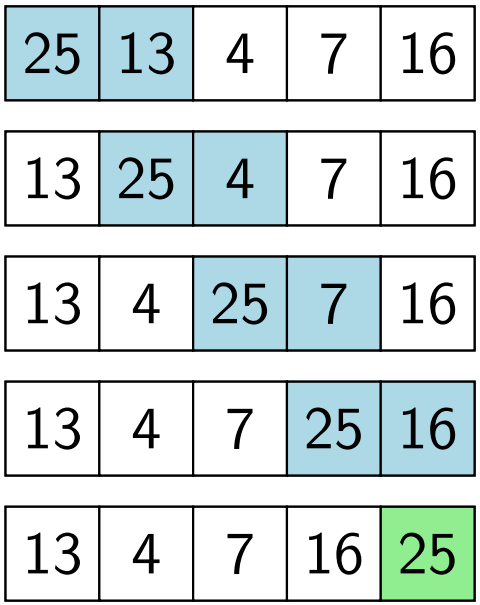

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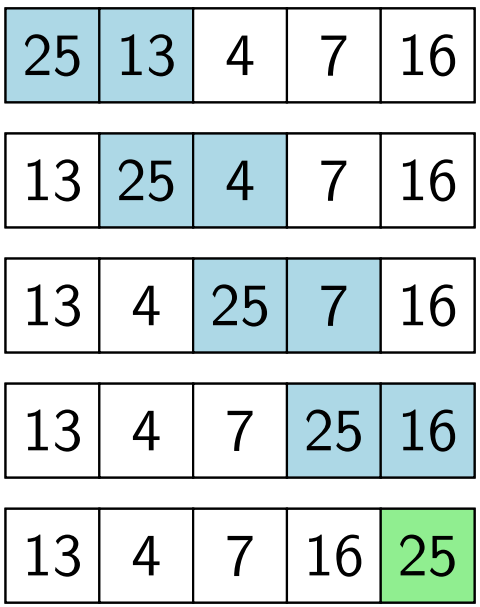
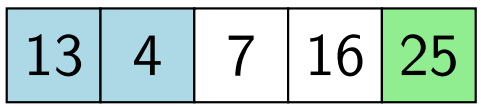


Second pass



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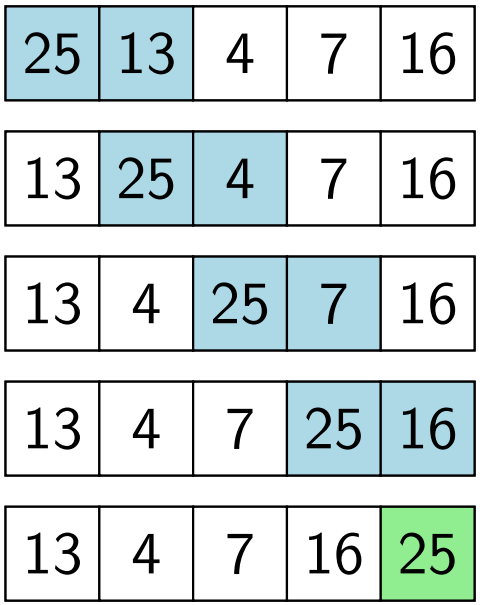

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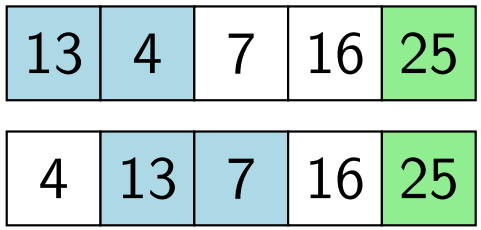

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Second pass

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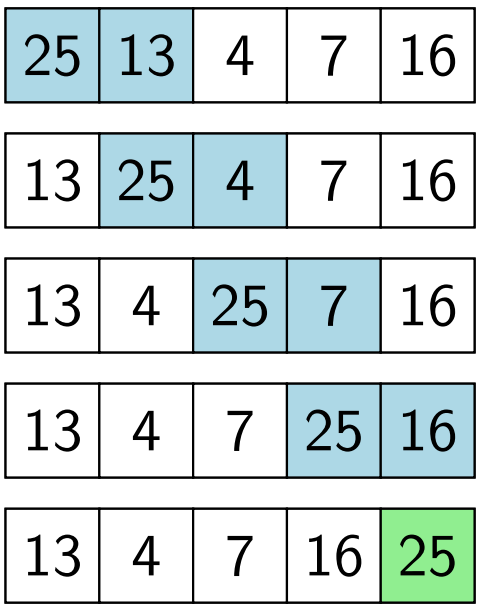

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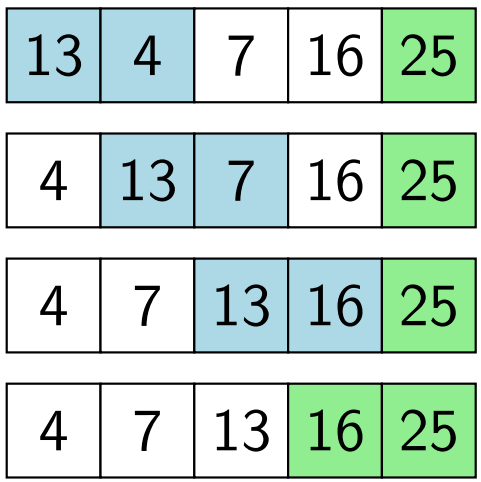

Naive implementation

Data: Array A of n elements
Result: Sorted array A

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# loop over passes
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    for $j = 0$ **to** $n - 2 - i$ **do**

Naive implementation

Data: Array A of n elements
Result: Sorted array A

```
# loop over passes
for i = 0 to n − 2 do
    # compares adjacent elements
    for j = 0 to n − 2 − i do
        # bubble largest element
```

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Data: Array A of n elements
Result: Sorted array A
# loop over passes
for $i = 0$ **to** $n - 2$ **do**
    # compares adjacent elements
    for $j = 0$ **to** $n - 2 - i$ **do**
        # bubble largest element
        if $A[j] > A[j + 1]$ **then**
            Swap $A[j]$ and $A[j + 1]$;

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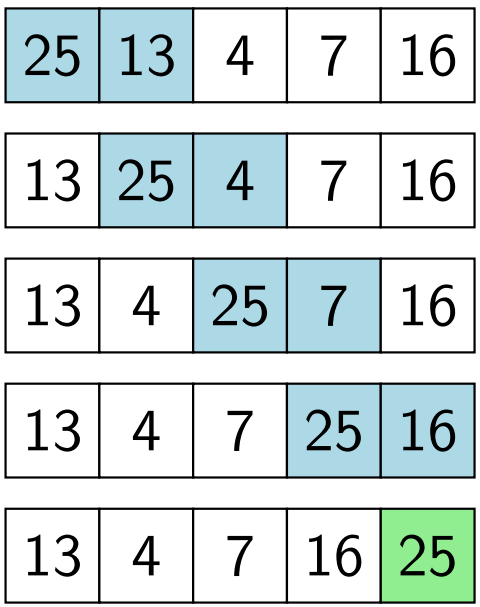
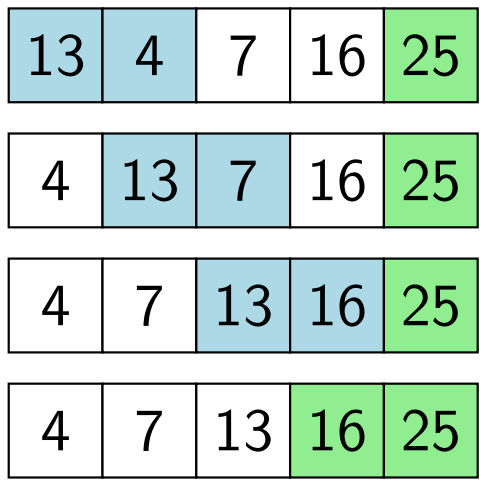
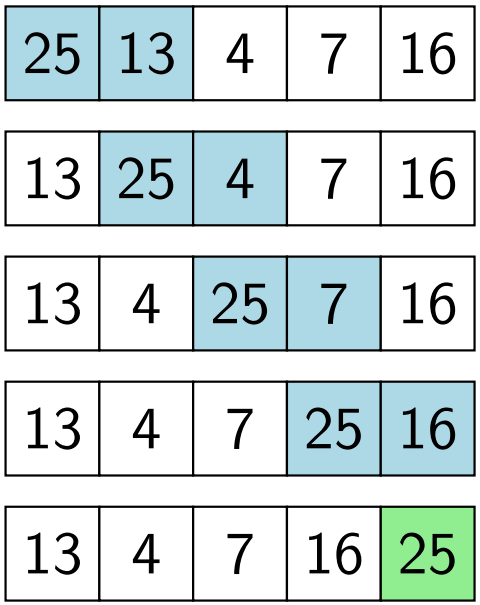


Second pass

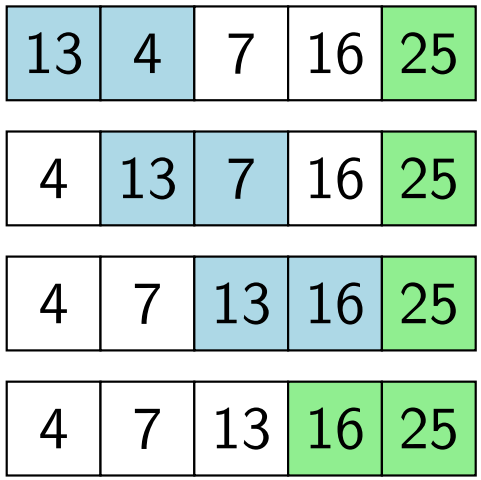


Illustration

First pass



Second pass



Third pass

Illustration

First pass

25	13	4	7	16
13	25	4	7	16
13	4	25	7	16
13	4	7	25	16
13	4	7	16	25

Second pass

13	4	7	16	25
4	13	7	16	25
4	7	13	16	25
4	7	13	16	25

Third pass

4	7	13	16	25
4	7	13	16	25

Illustration

First pass

25	13	4	7	16
13	25	4	7	16
13	4	25	7	16
13	4	7	25	16
13	4	7	16	25

Second pass

13	4	7	16	25
4	13	7	16	25
4	7	13	16	25
4	7	13	16	25

Third pass

4	7	13	16	25
4	7	13	16	25
4	7	13	16	25

Improved implementation

Data: Array A of n elements
Result: Sorted array A
for $i = 0$ **to** $n - 2$ **do**
    is_sorted $=$ true;
    for $j = 0$ **to** $n - 2 - i$ **do**
        if $A[j] > A[j + 1]$ **then**
            Swap $A[j]$ and $A[j + 1]$;
            is_sorted $=$ false;
    if *is_sorted* **then**
        return;

Analysis

- Time complexity:
  - Outer loop (i): $0, 1, \ldots, n-2$
  - Inner loop (j):

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  - Total number of iterations: $1 + 2 + \cdots + n - 1 = \frac{n(n-1)}{2}$

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- Total time: $\mathcal{O}(c\, n(n-1)) = \mathcal{O}(n^2 - n) = \mathcal{O}(n^2)$
- We need a constant amount of memory: $\mathcal{O}(1)$

Merge sort

Idea

- Merge Sort is a divide-and-conquer algorithm
  1. Divide: Recursively divide the array into two halves
  2. Conquer: Merge the two halves

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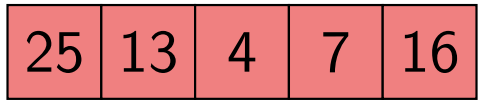
- Analysis (homework 😉):
  - Time Complexity: $\mathcal{O}(n \log n)$
  - Space Complexity: $\mathcal{O}(n)$

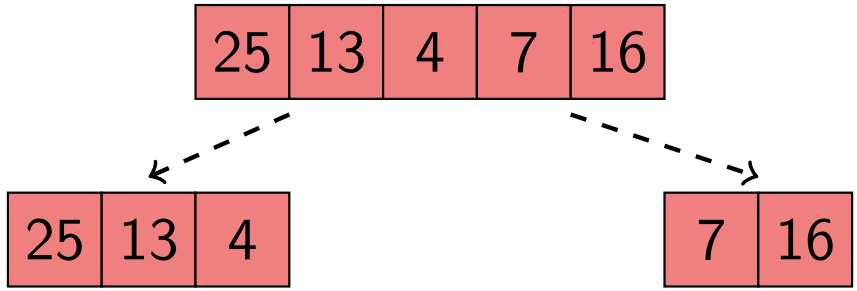
Illustration



Divide

Conquer

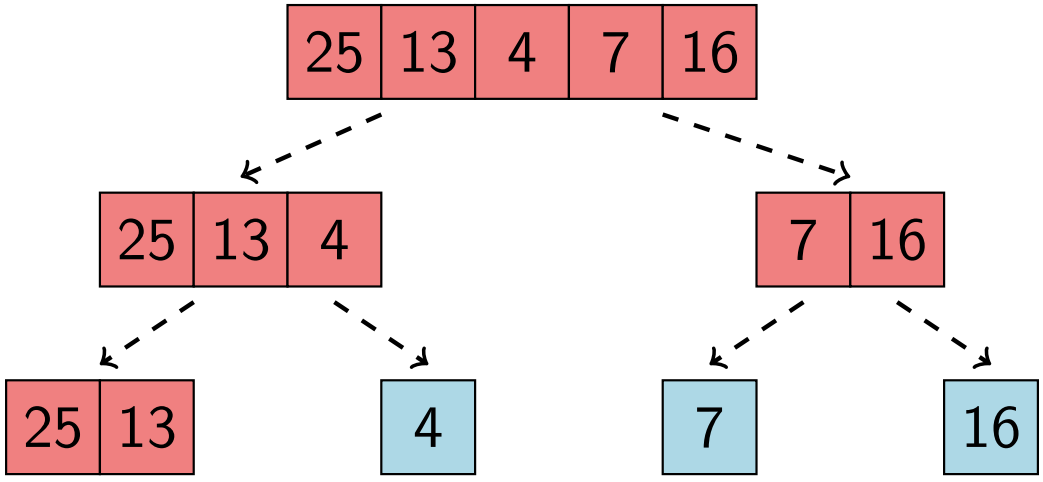
Illustration



Divide

Conquer

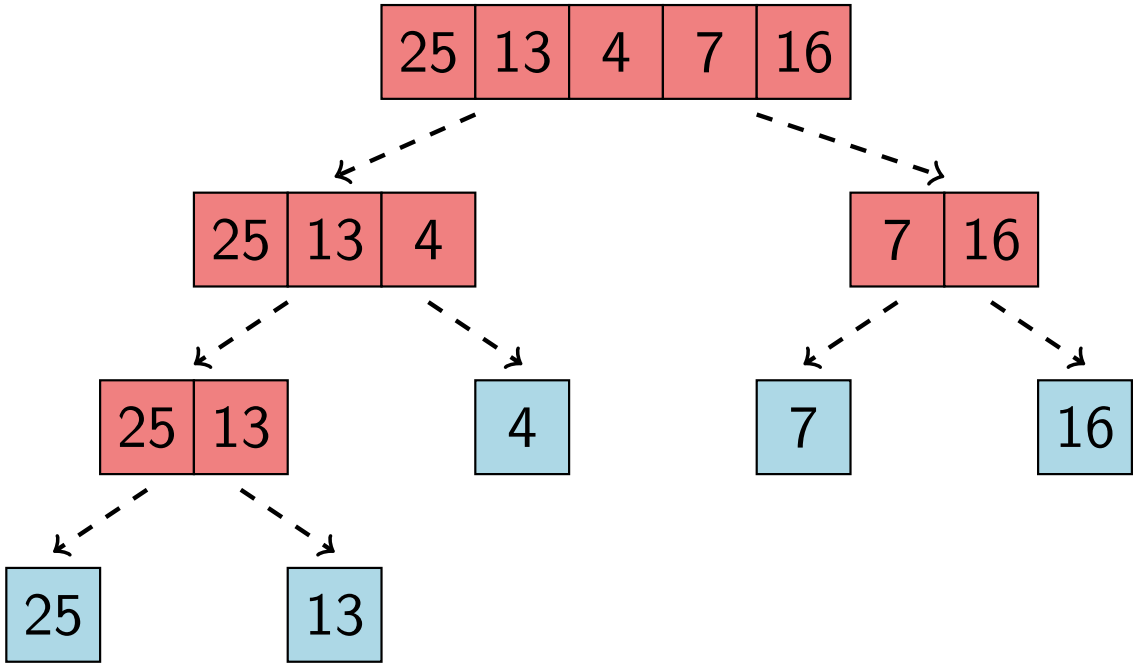
Illustration



Divide

Conquer

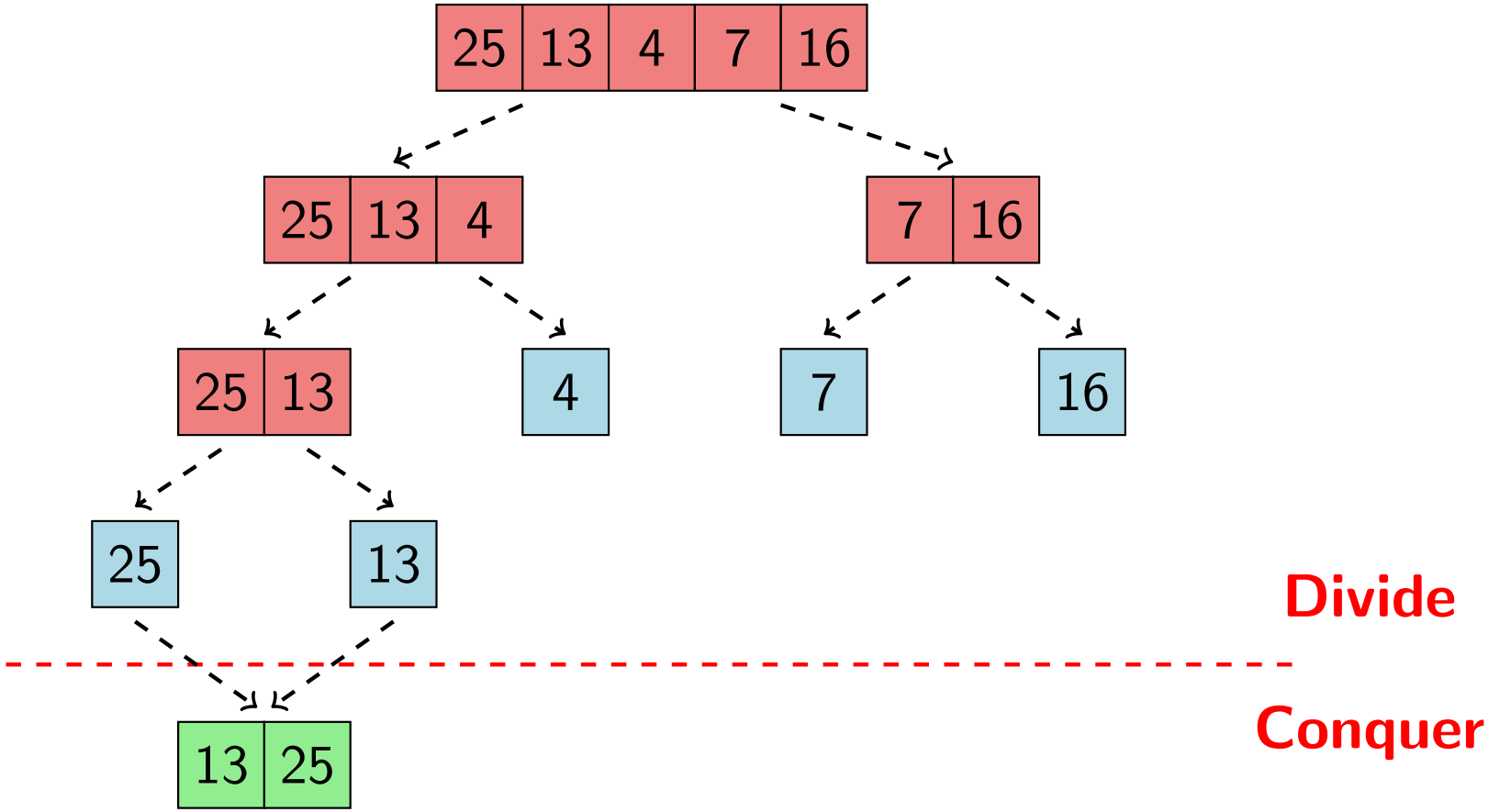
Illustration



Divide

Conquer

Illustration



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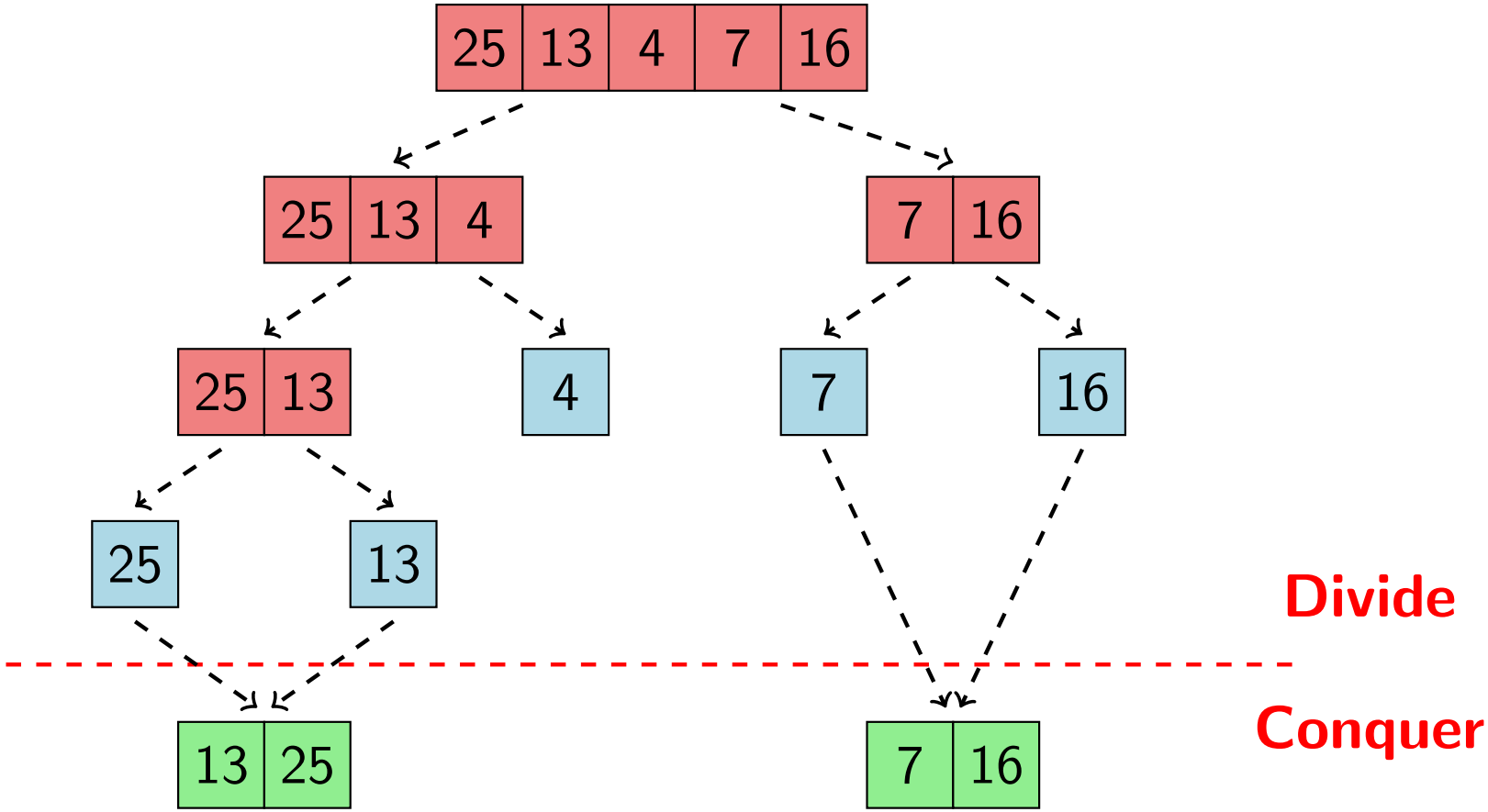
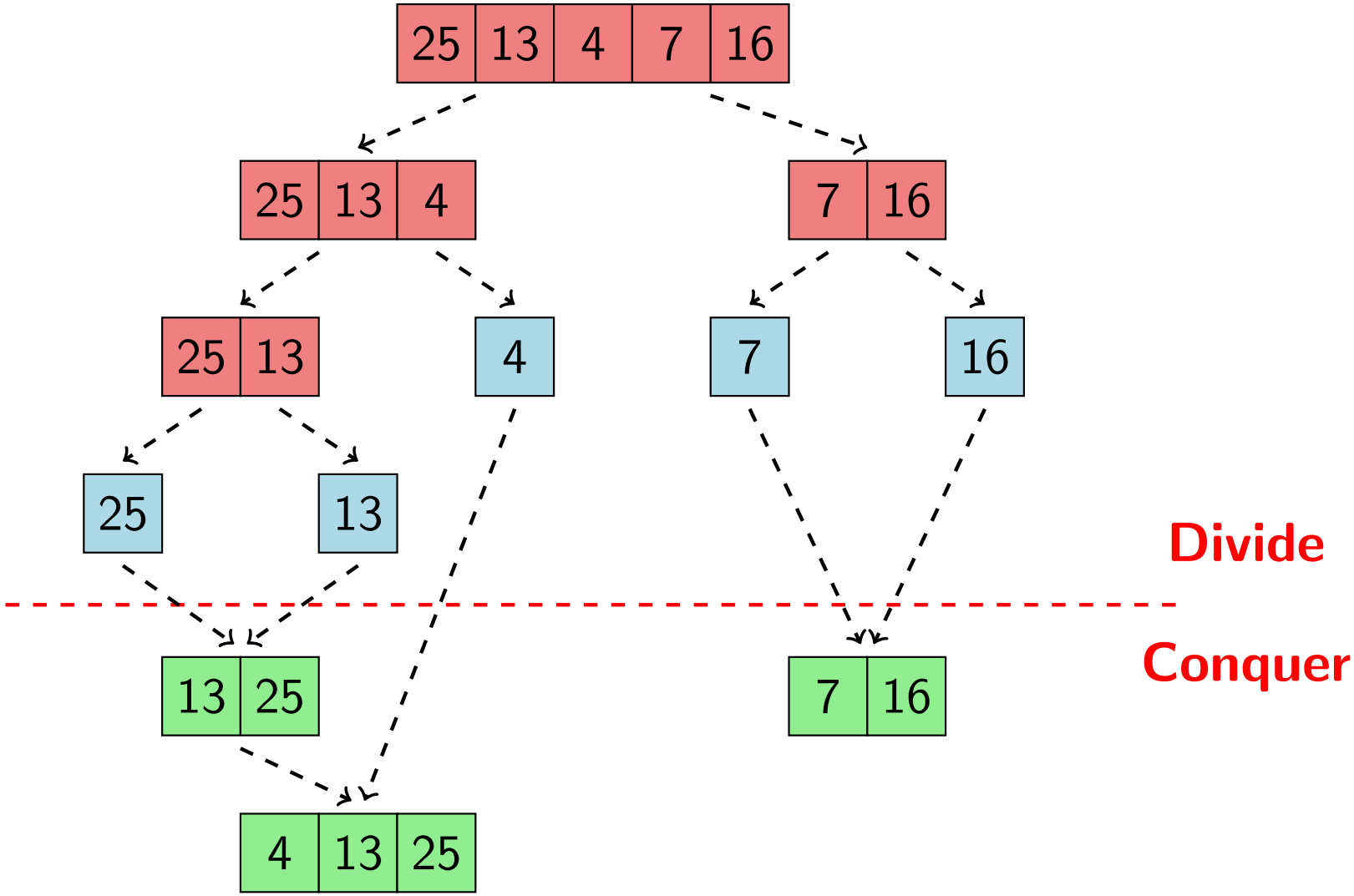
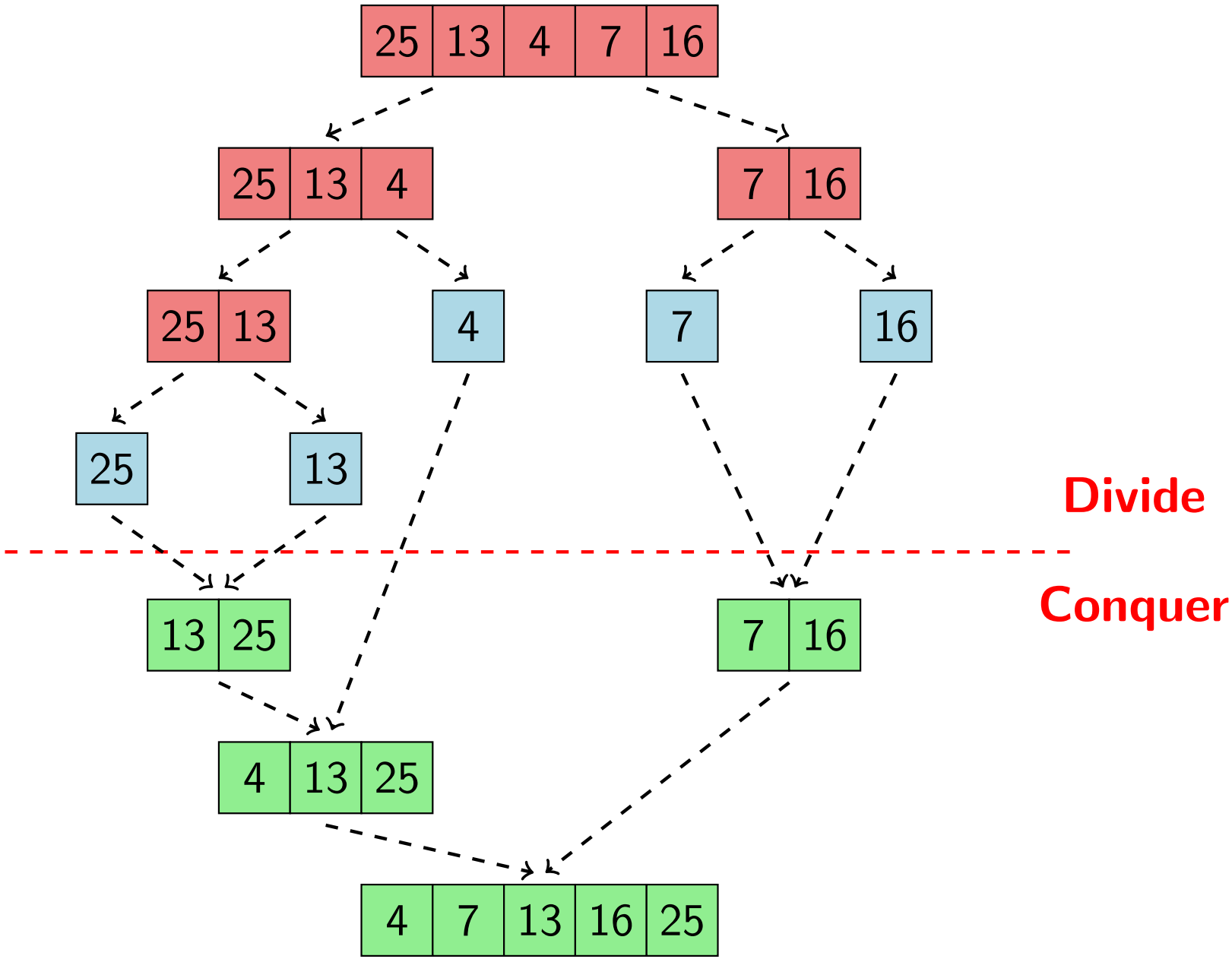


Illustration



Illustration



MergeSort Algorithm

Data: Array A, left and right indices
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    $\text{mid} = \text{left} + \left\lfloor \frac{\text{right} - \text{left}}{2} \right\rfloor$;

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# to stop recursion
if left $<$ right **then**
    # middle index
    mid $=$ left $+ \left\lfloor \frac{\text{right}-\text{left}}{2} \right\rfloor$;
    # recursively sort the two halves
    MergeSort(A, left, mid);
    MergeSort(A, mid+1, right);

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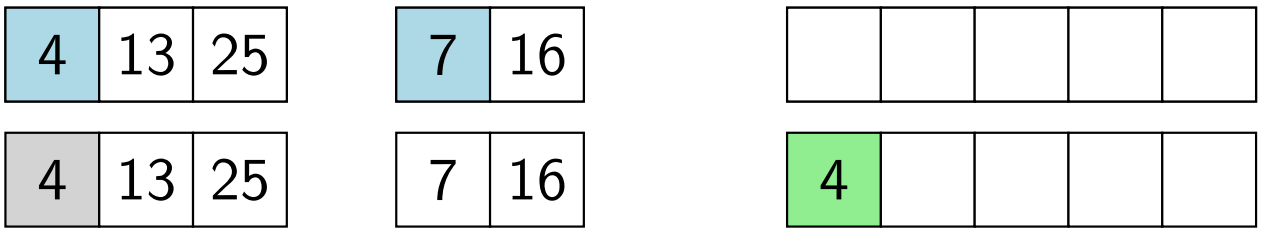
# to stop recursion
if left $<$ right **then**
    # middle index
    mid $=$ left $+ \left\lfloor \frac{\text{right}-\text{left}}{2} \right\rfloor$;
    # recursively sort the two halves
    MergeSort(A, left, mid);
    MergeSort(A, mid+1, right);
    # merge the two halves
    merge(A, left, mid, right);

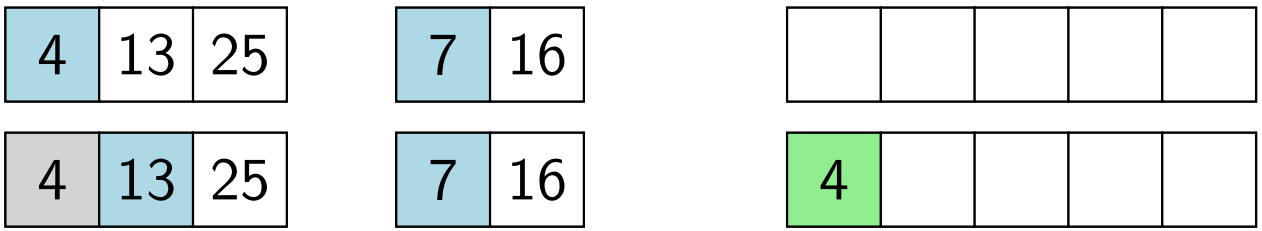
Merge step

Merge step

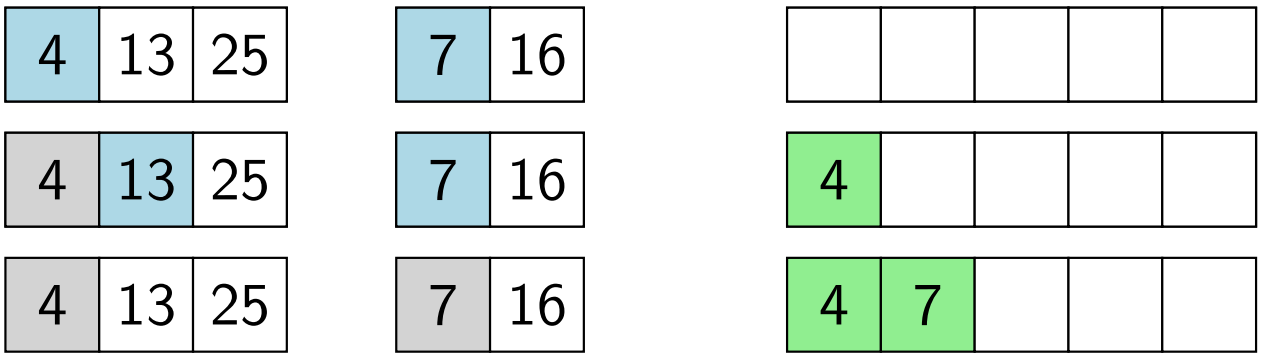
Merge step



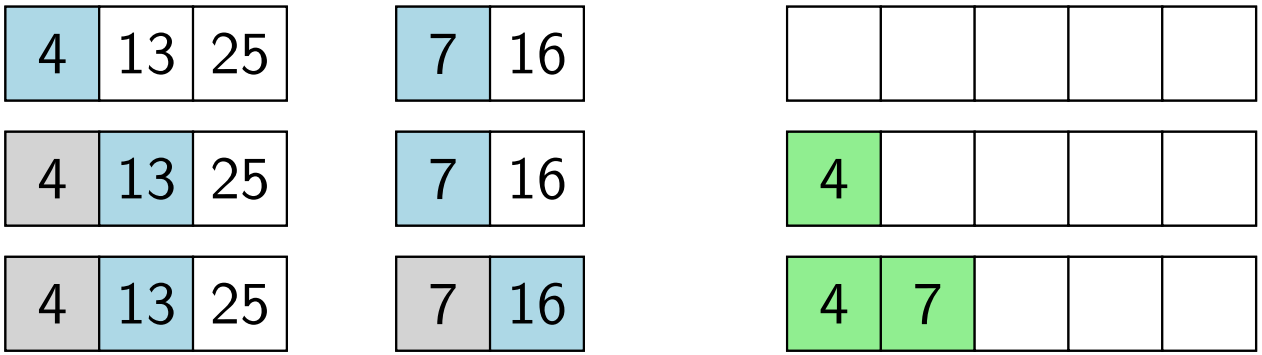
Merge step



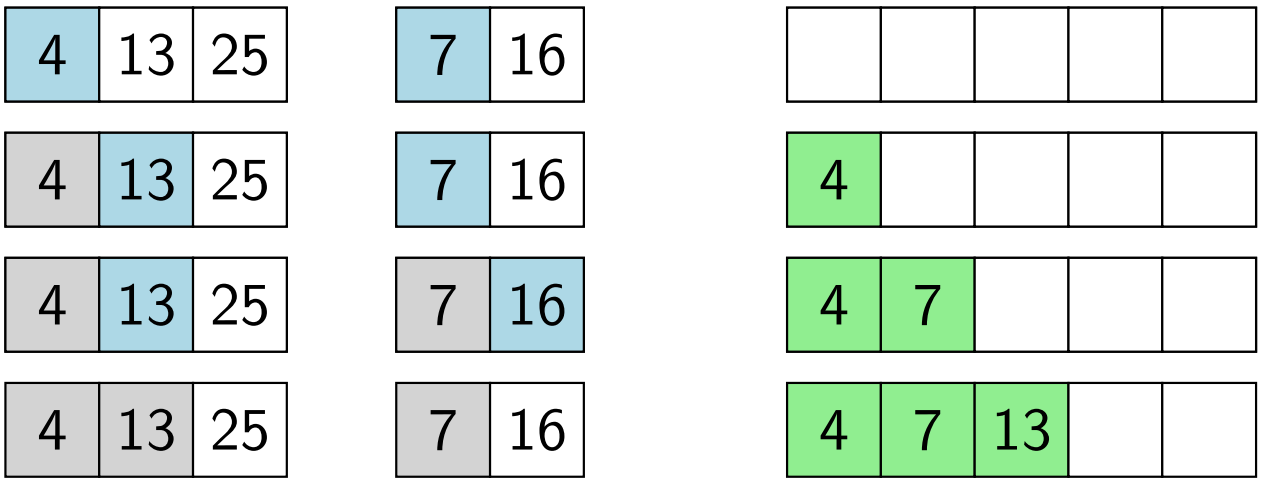
Merge step



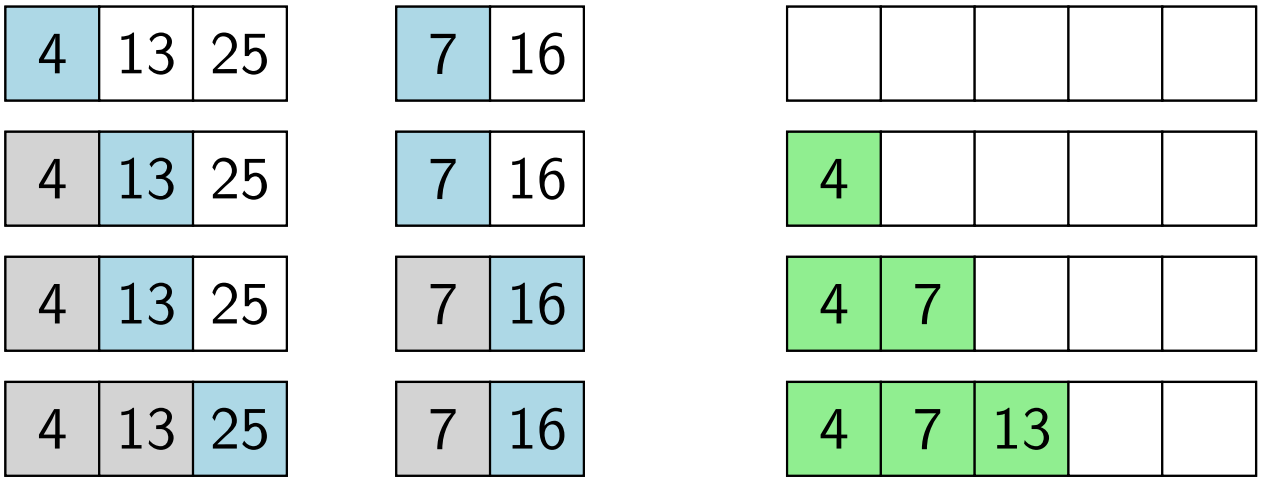
Merge step



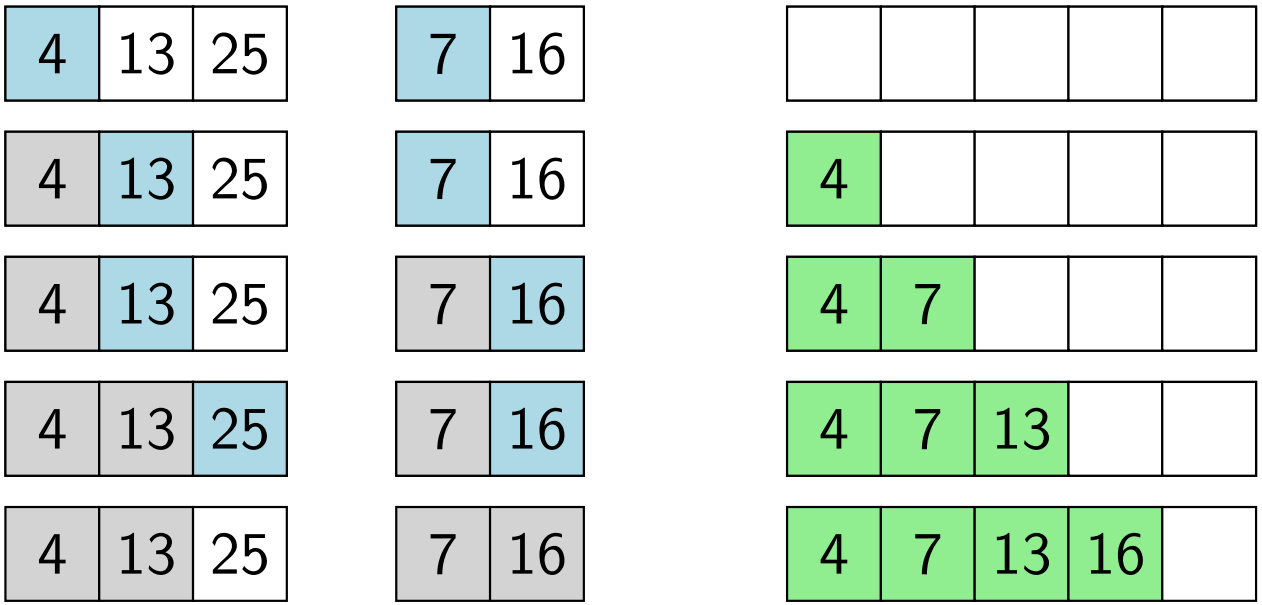
Merge step



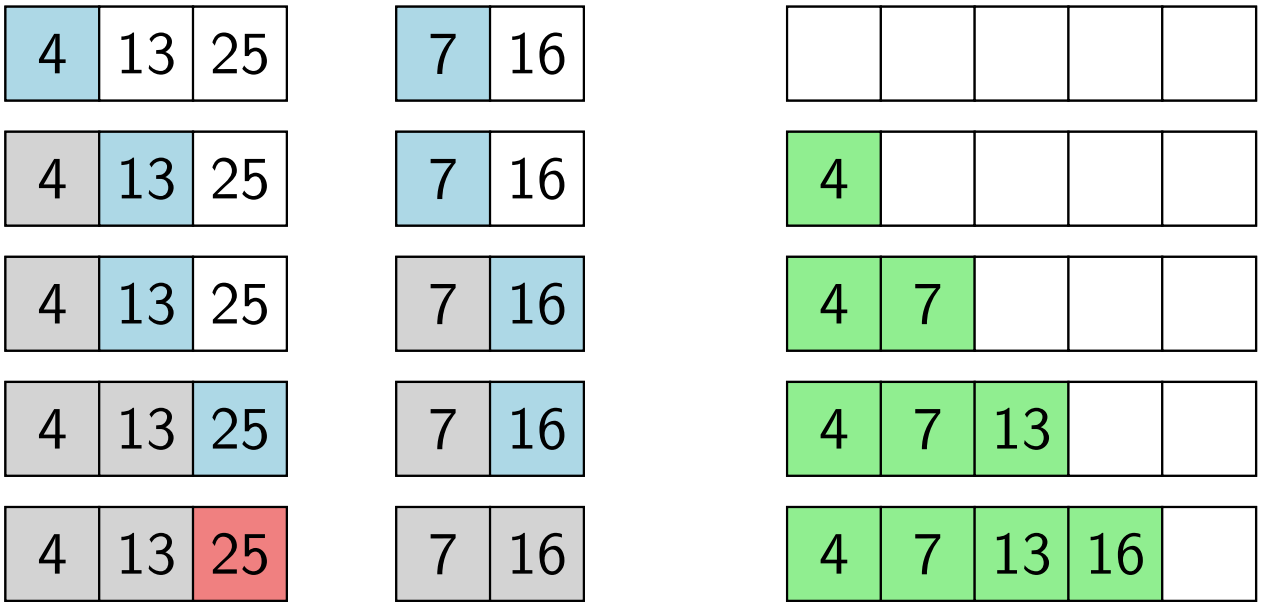
Merge step



Merge step


4 13 25
7 16
4 13 25
7 16
4 13 25
7 16
4 13 25
7 16
4 13 25
7 16
4
4 7
4 7 13
4 7 13 16

Merge step


4 13 25
7 16
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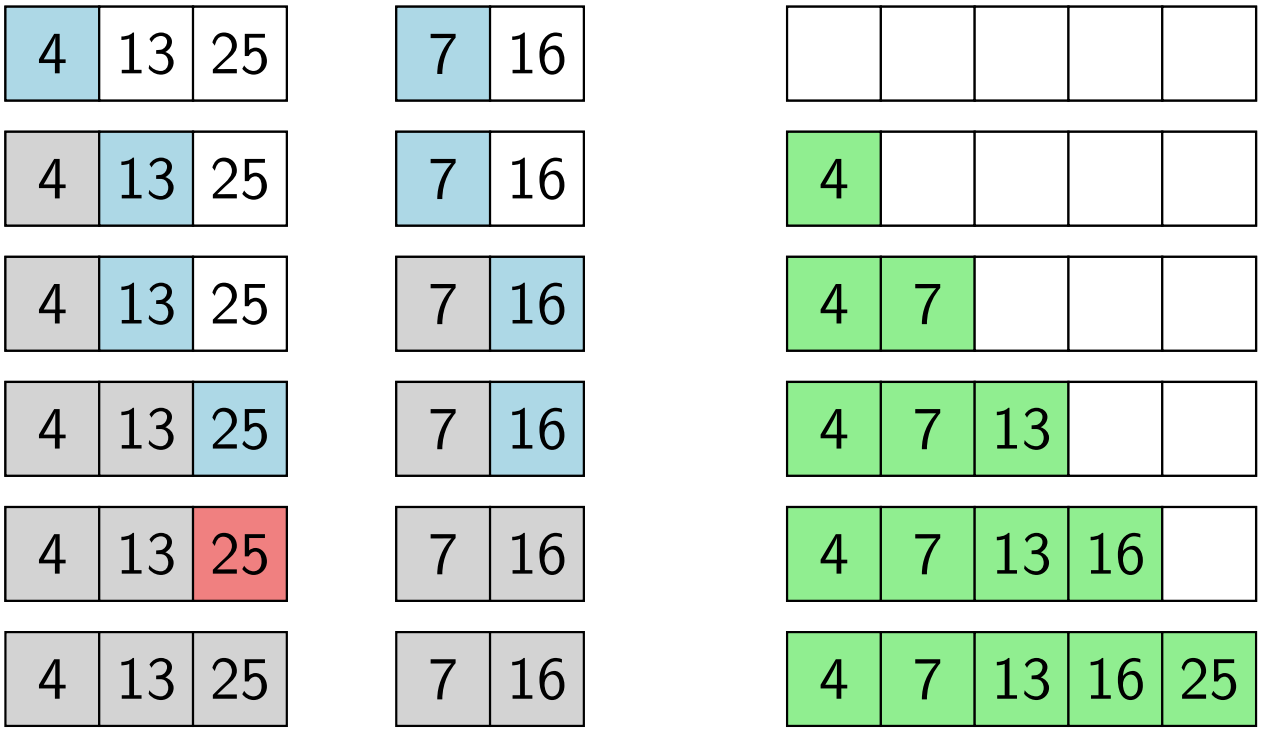
7 16
4 13 25
7 16
4 13 25
7 16
4 13 25
7 16
4
4 7
4 7 13
4 7 13 16

Merge step


4 13 25
7 16
4 13 25
7 16
4 13 25
7 16
4 13 25
7 16
4 13 25
7 16
4 13 25
7 16
4
4 7
4 7 13
4 7 13 16
4 7 13 16 25

Quick sort

Idea

- Quick Sort is a divide-and-conquer algorithm
  1. Divide Step: choose a "pivot"
     - the pivot divides the array into elements $< p$ and those $\geq p$
  2. Conquer Step: Do nothing!

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- *Merge Sort* spends more time in the *conquer step*, while *Quick Sort* spends more time in the *divide step*

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- *Merge Sort* spends more time in the *conquer step*, while *Quick Sort* spends more time in the *divide step*
- Analysis (homework 😉):
  - Time Complexity: $\mathcal{O}(n \log n)$
  - Space Complexity: $\mathcal{O}(\log n)$

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if left $<$ right **then**
    # partition the array
    piv_index $=$ partition(A, left, right);

QuickSort Algorithm

Data: Array A, left and right indices
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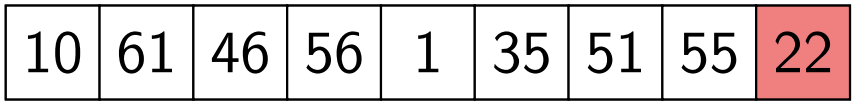
```
# to stop recursion
if left < right then
    # partition the array
    piv_index = partition(A, left, right);
    # recursively sort the two halves
    QuickSort(A, left, piv_index - 1);
    QuickSort(A, piv_index + 1, right);
```

Illustration

10	61	46	56	1	35	51	55	22

Illustration



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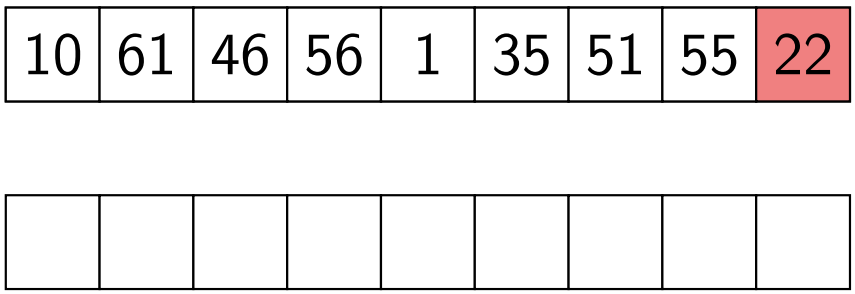

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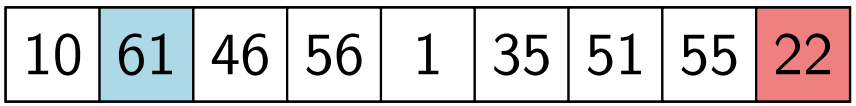

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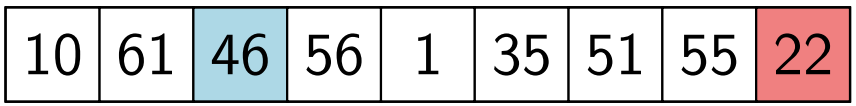

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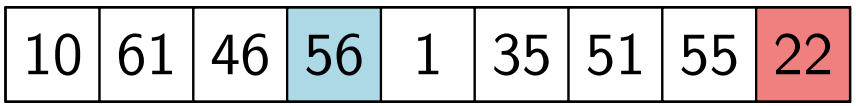
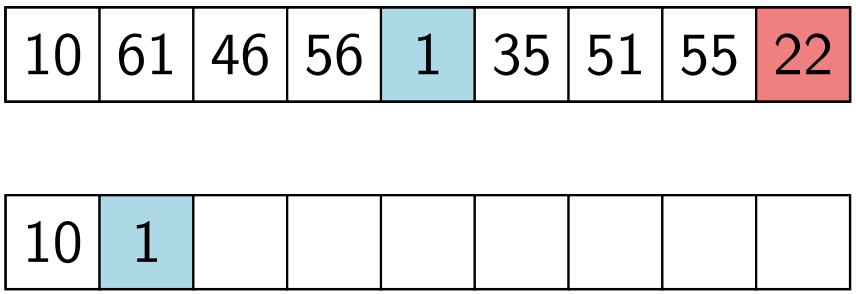
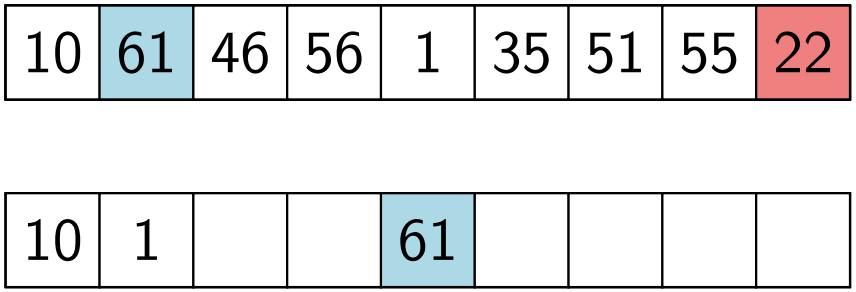


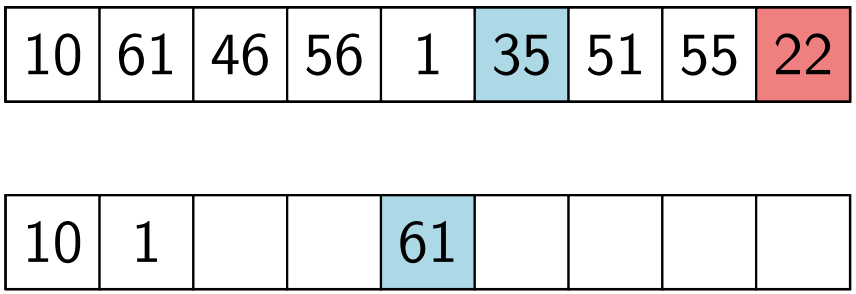
Illustration



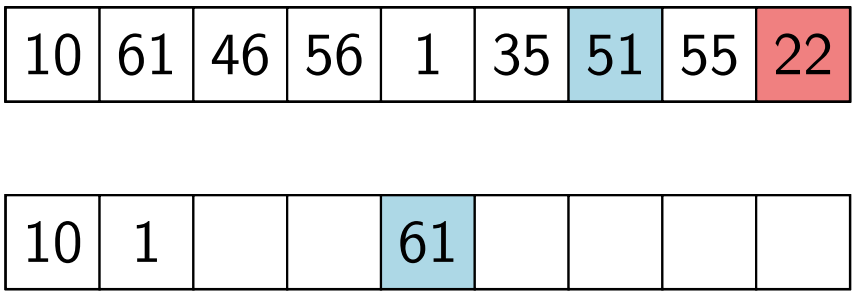
Illustration



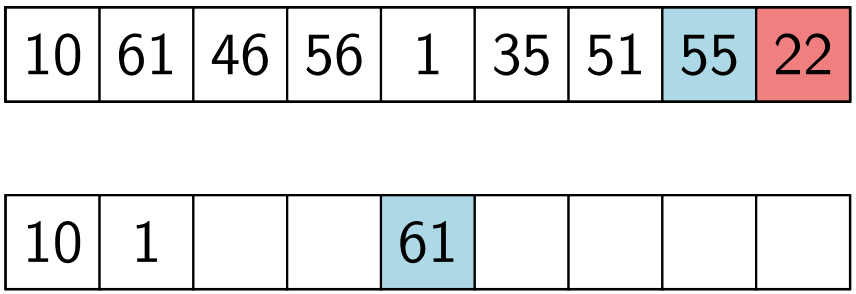
Illustration



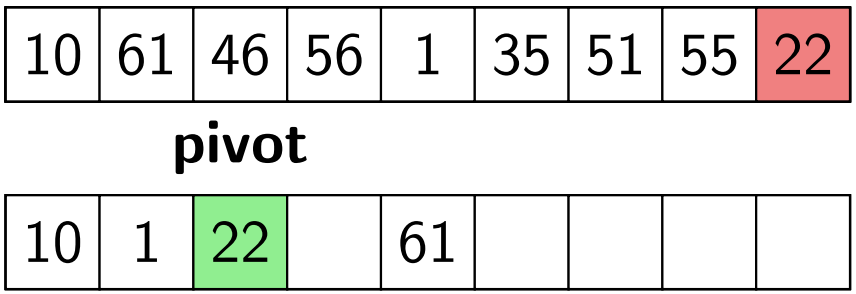
Illustration



Illustration



Illustration



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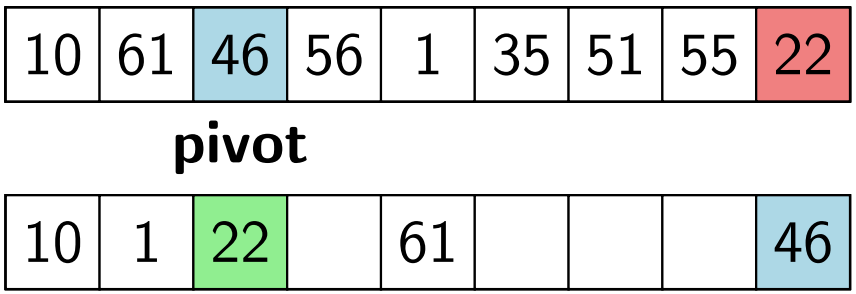

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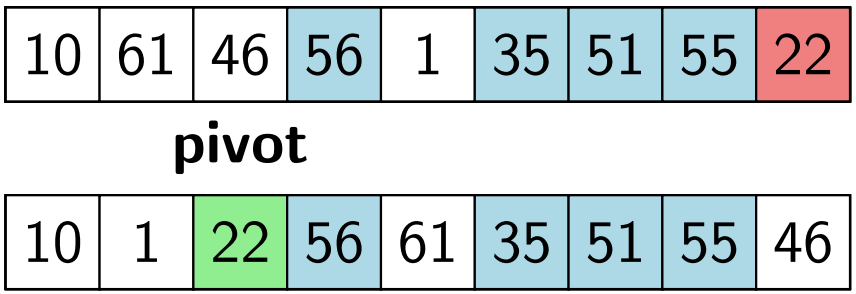

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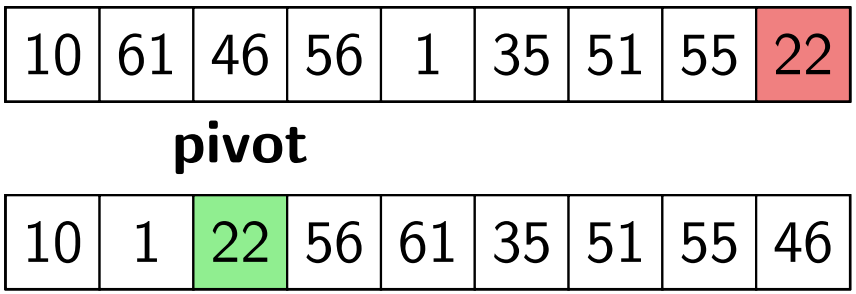

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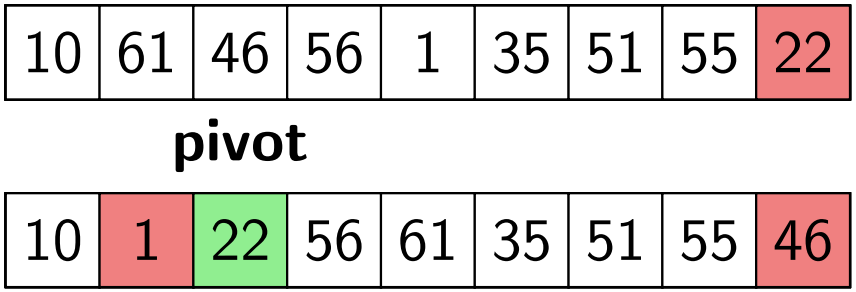

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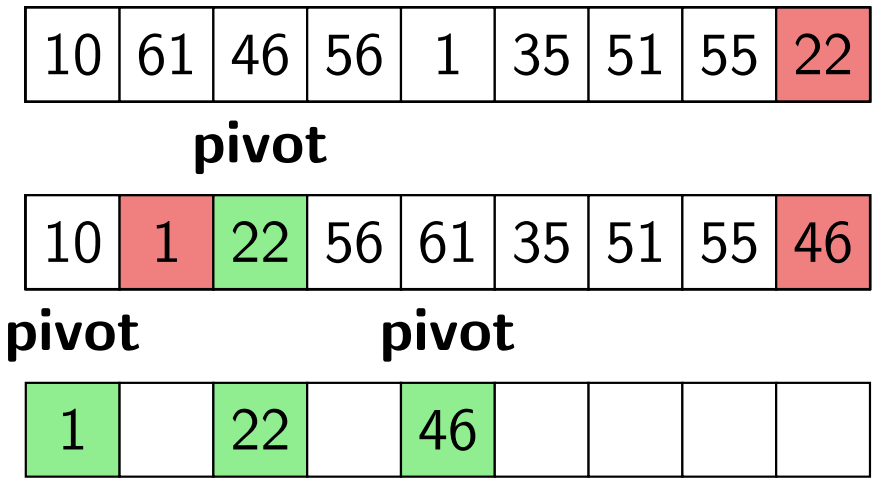

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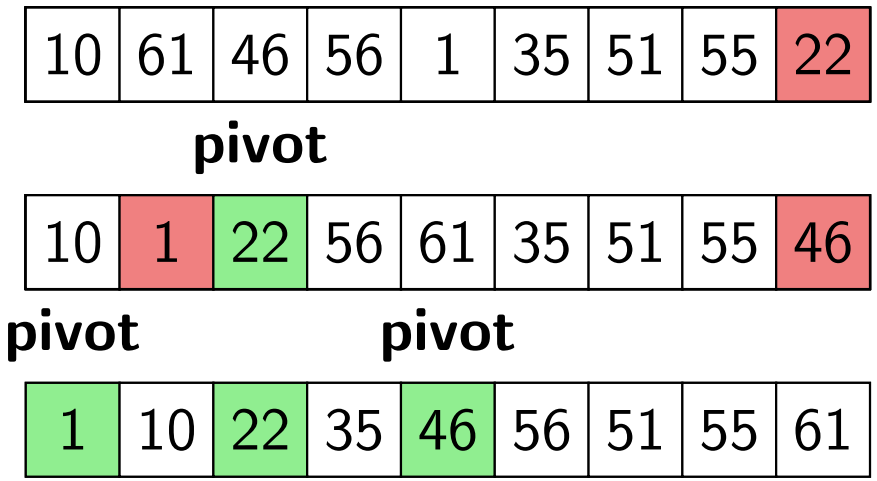

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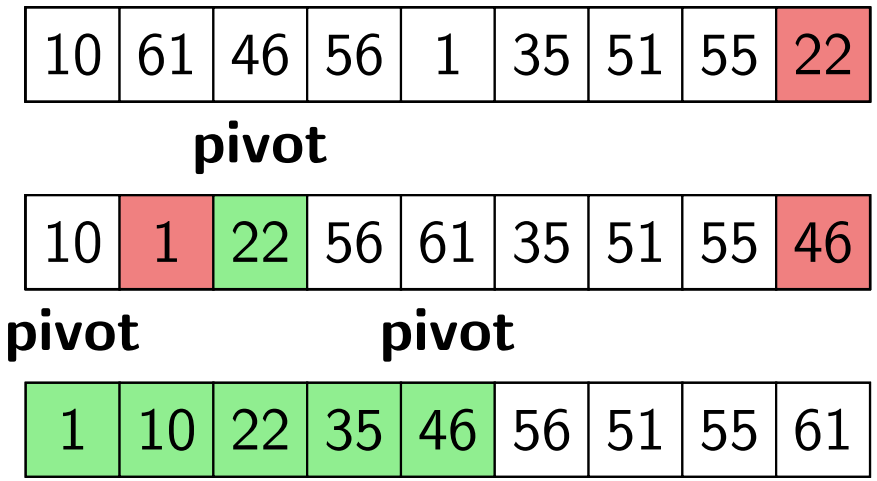
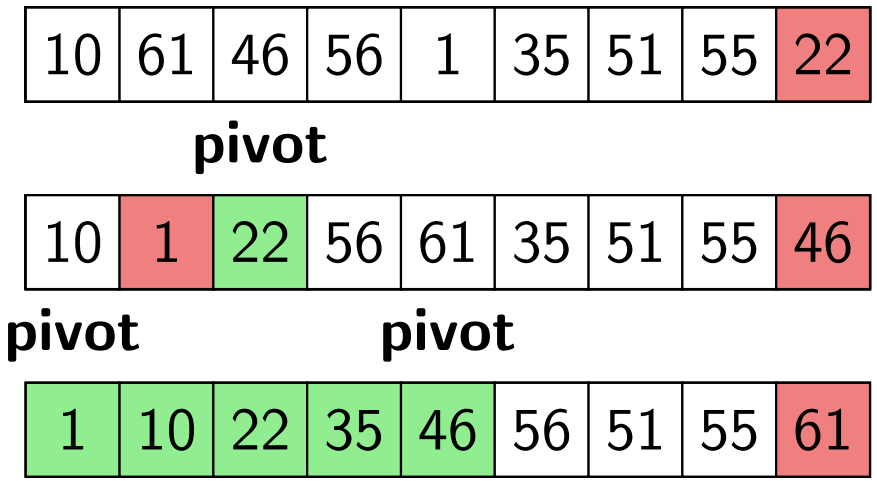
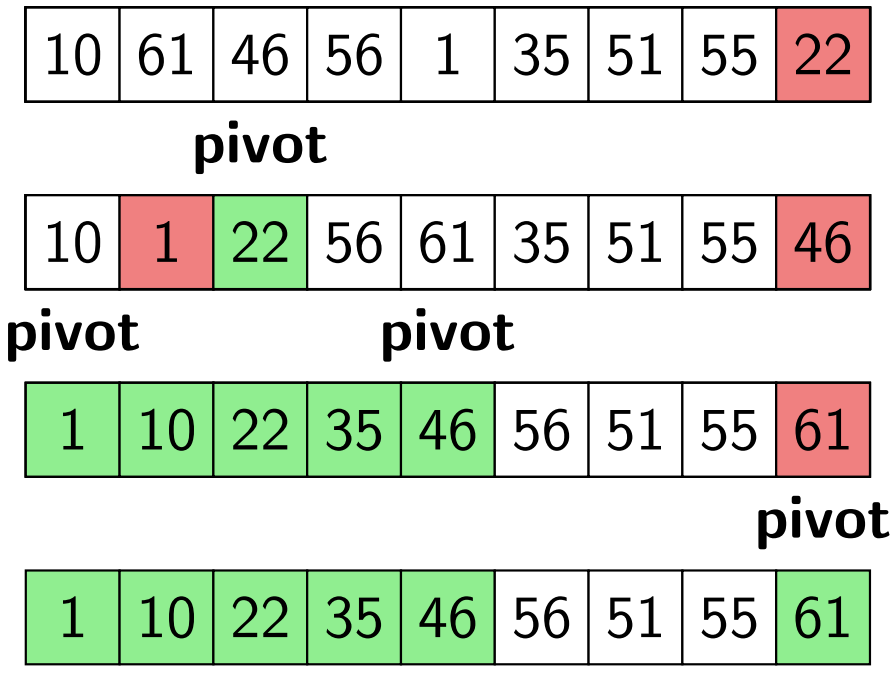


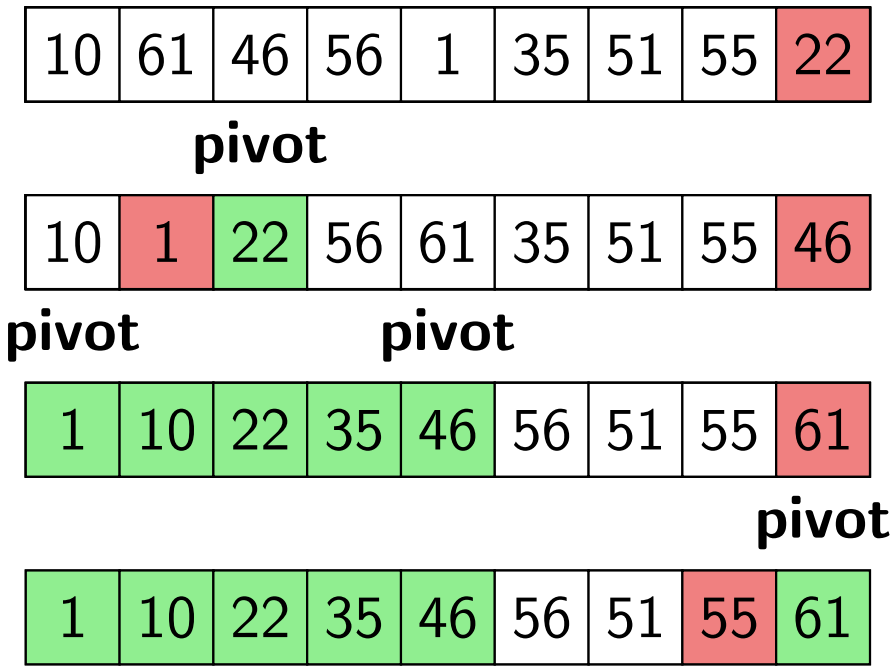
Illustration



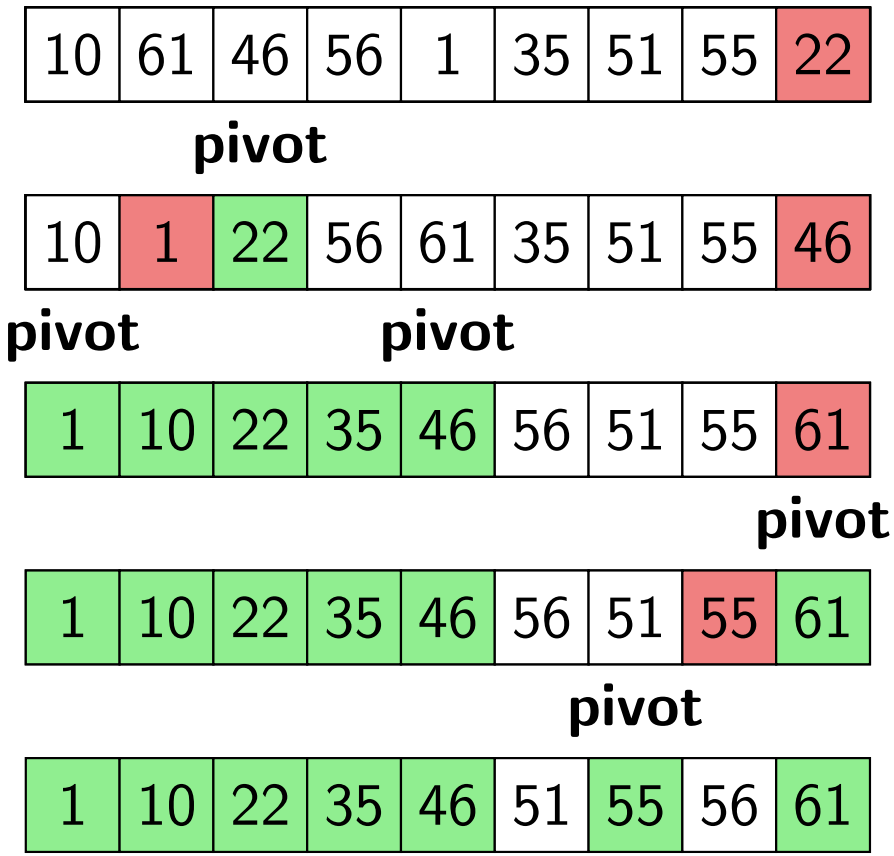
Illustration



Illustration



Illustration



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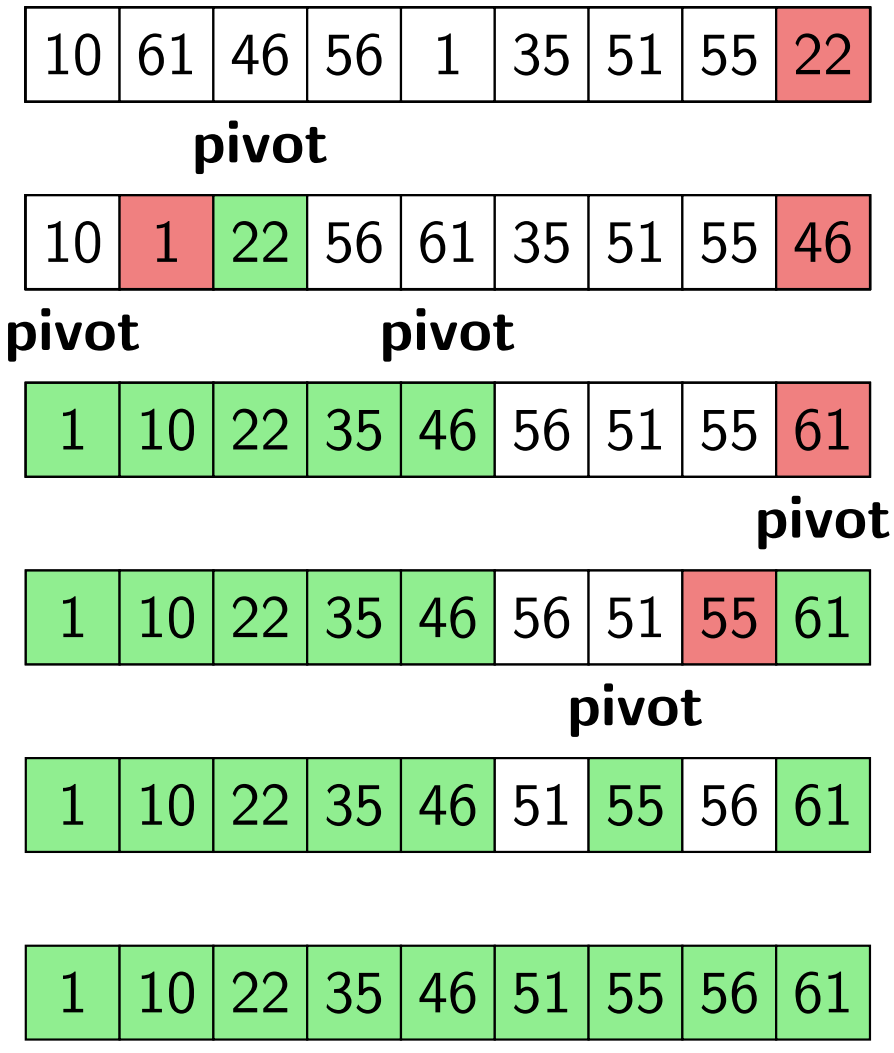

Radix sort

Idea

- Radix (root): base in which we express an integer
  - Radix 10, Radix 2, ...
  - from right(LSD $\rightarrow$ MSD), from left (MSD $\rightarrow$ LSD)

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- *Counting Sort*:
  - Non-comparative sorting (no **direct** use of $<, >$)
  - treats data as a "character" string (digit, bit, ...)
- Analysis (homework 😉):
  - Time Complexity: $\mathcal{O}(nk)$, (k: # of bits of largest number)
  - Space Complexity: $\mathcal{O}(n+k)$, $\mathcal{O}(nk)$, ...

Illustration

3	6	7	8	2

Illustration

3	6	7	8	2

0011	0110	0111	1000	0010

Illustration

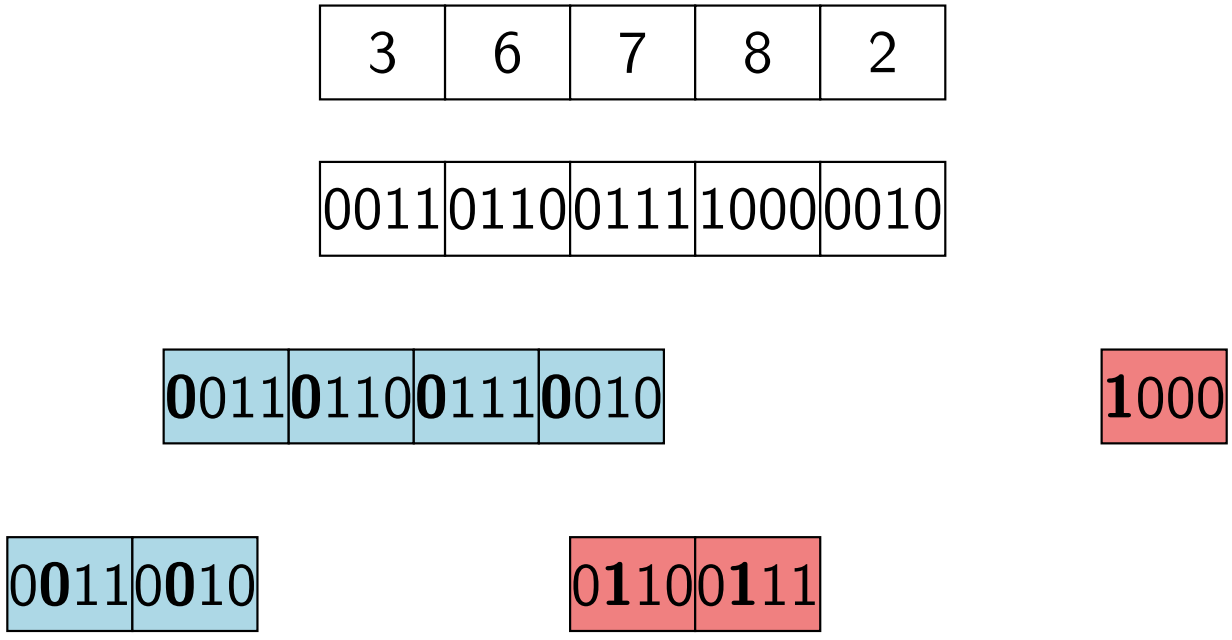
3	6	7	8	2

0011	0110	0111	1000	0010

0011	**0**110	**0**111	**0**010

1000

Illustration



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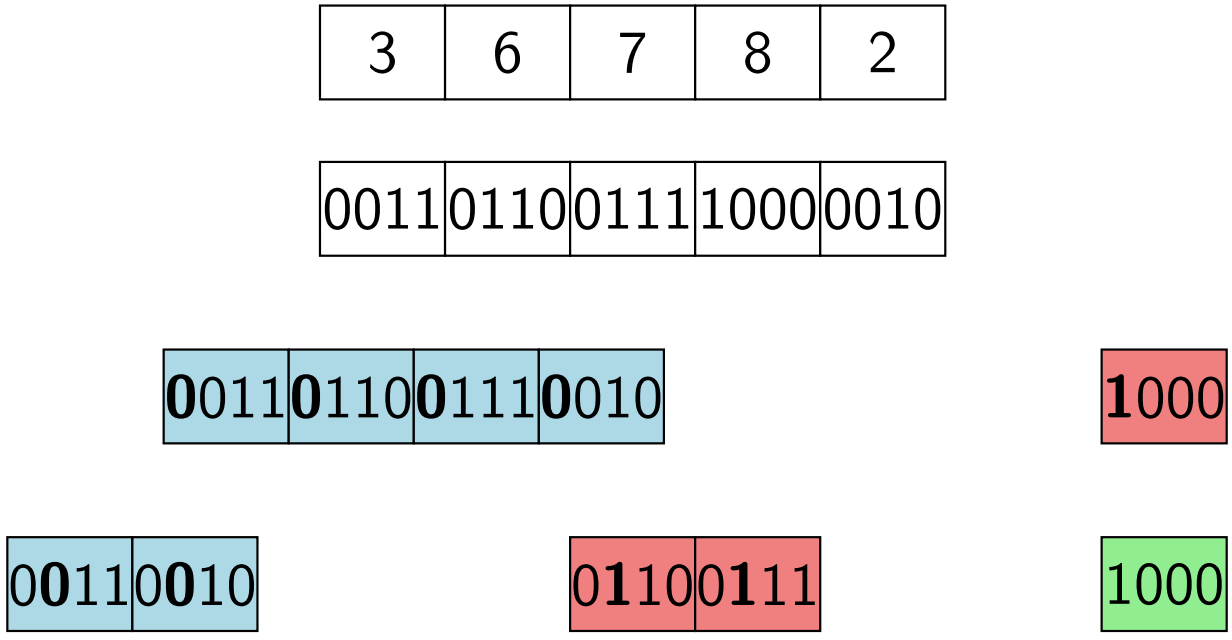

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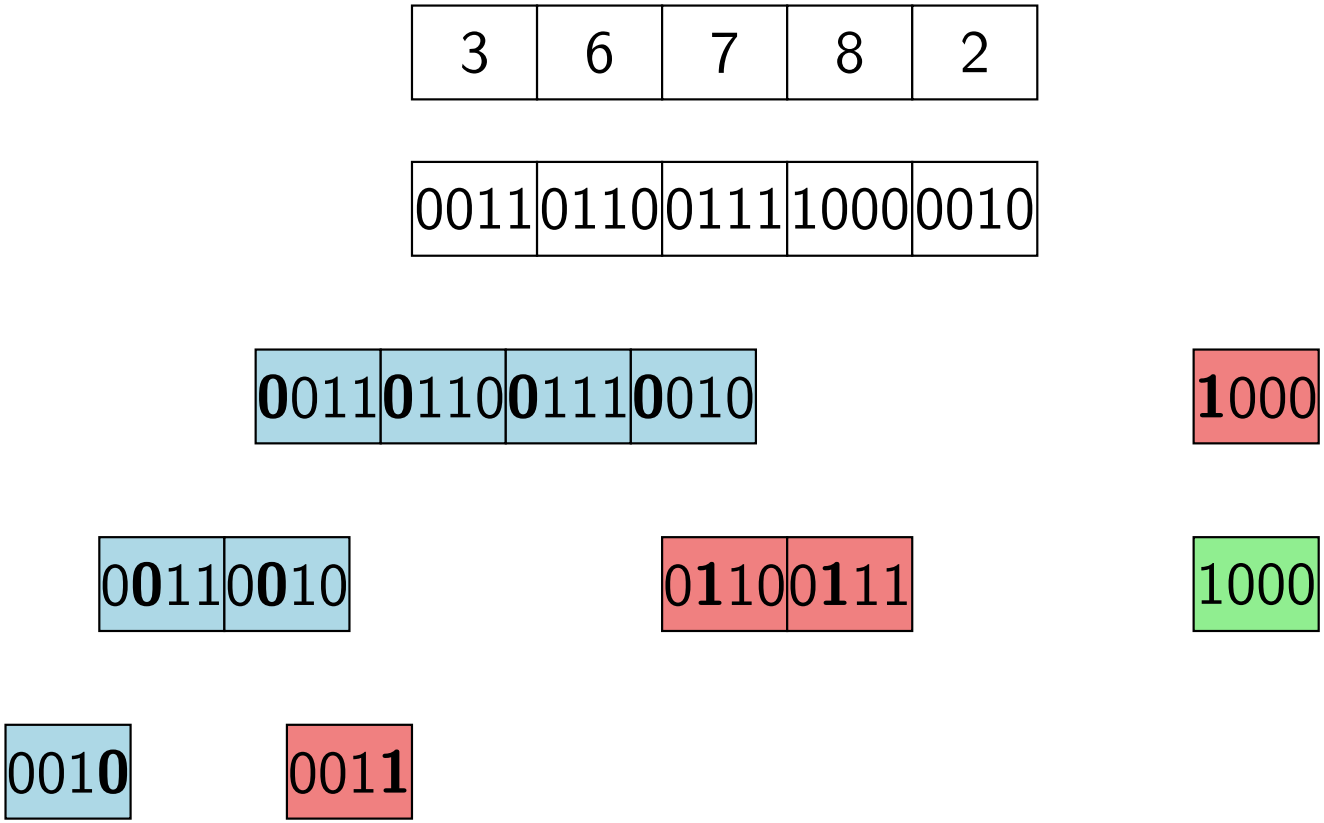

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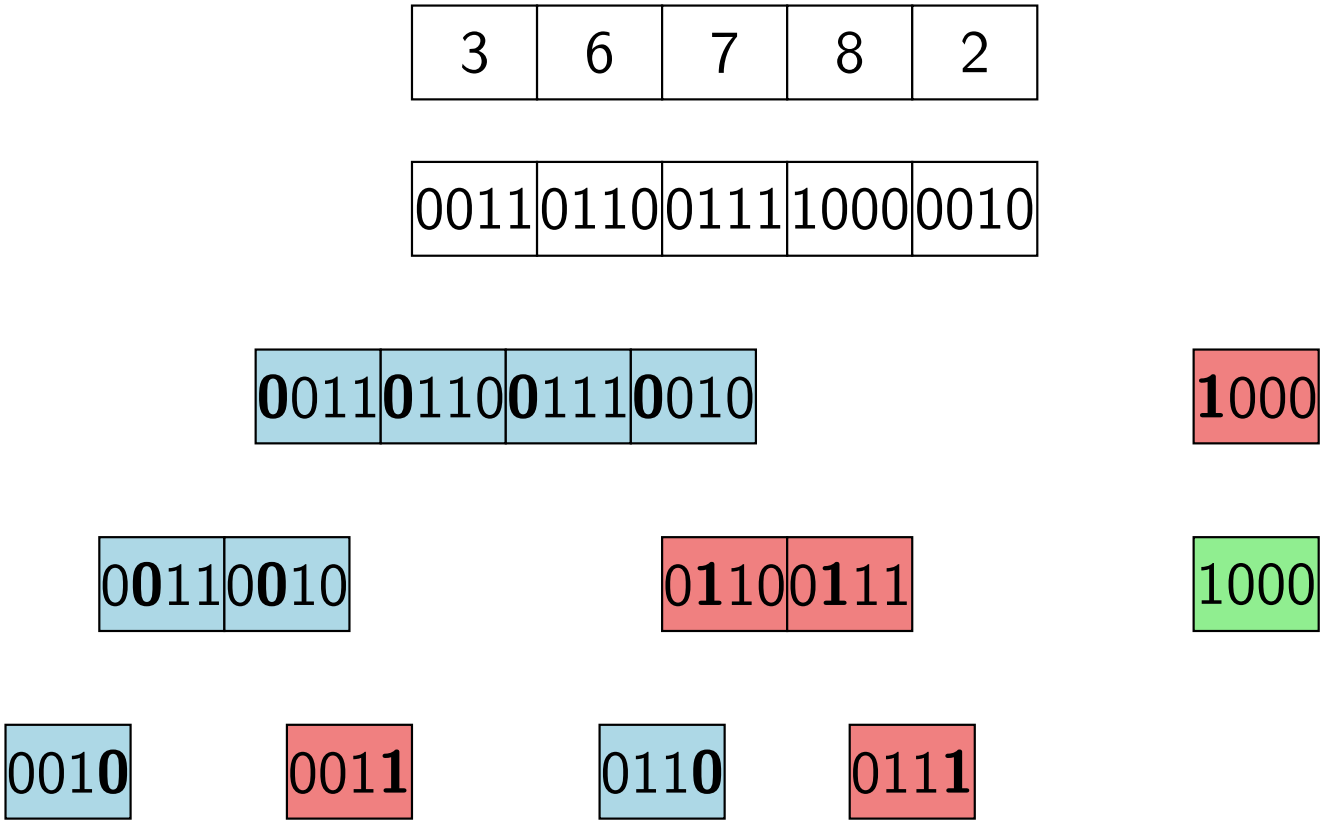

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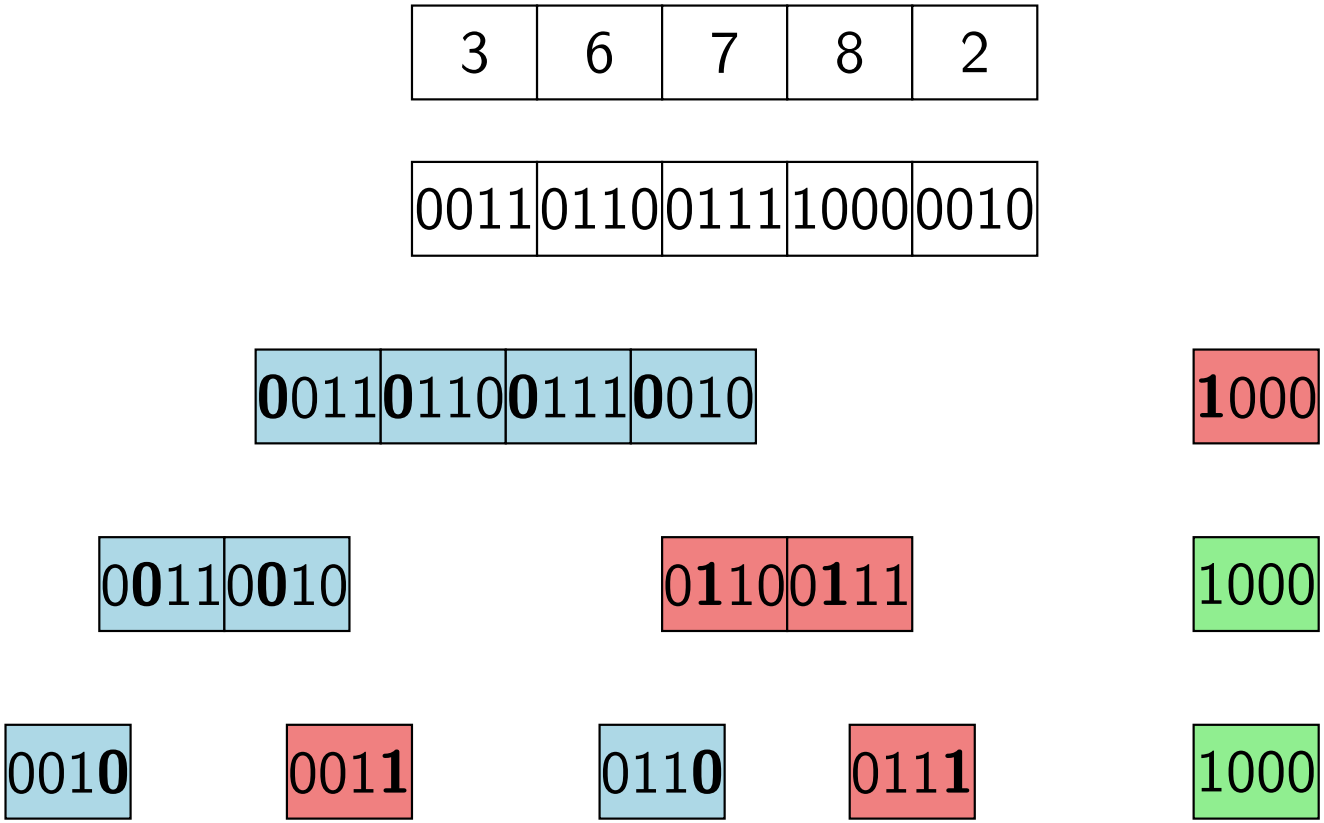

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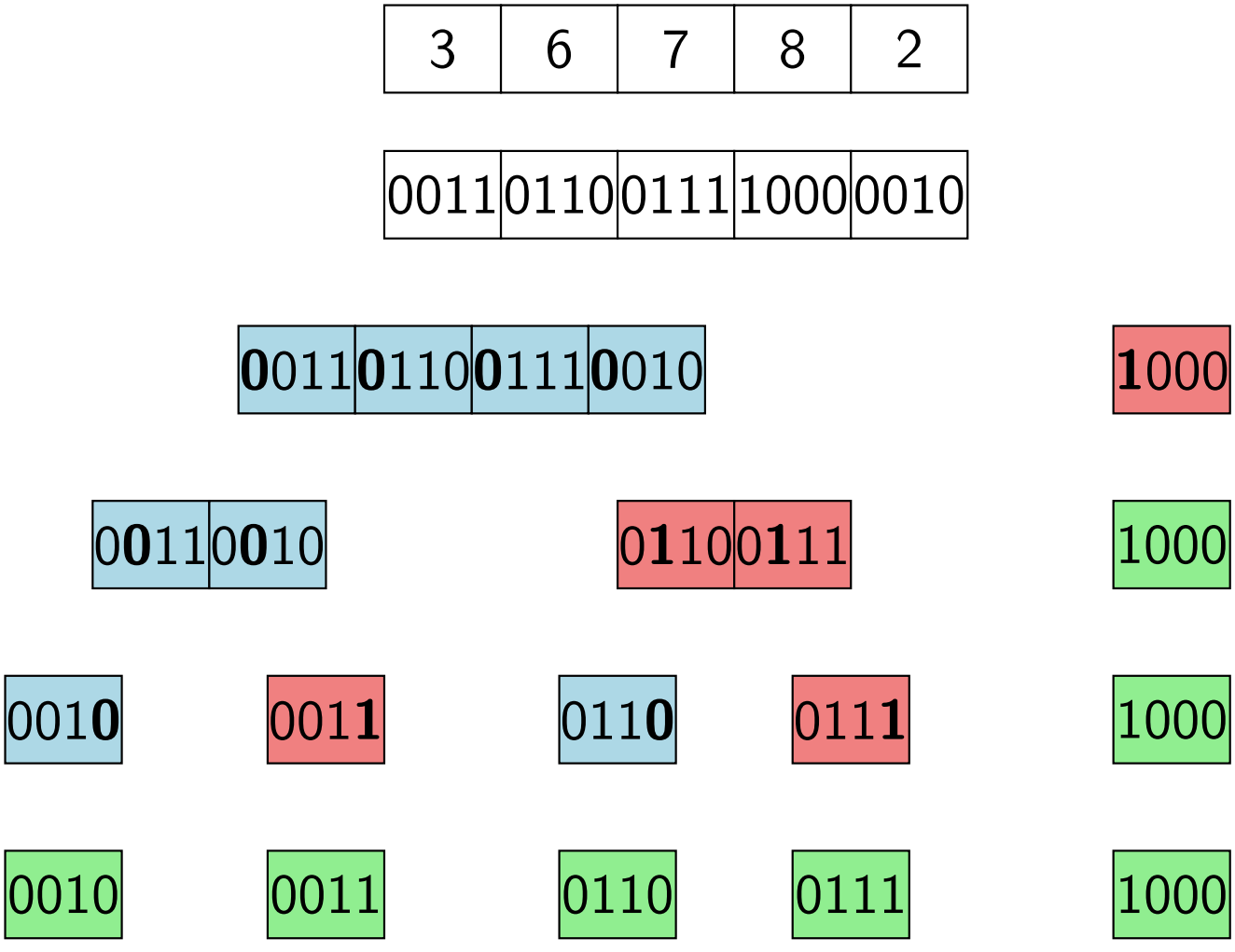

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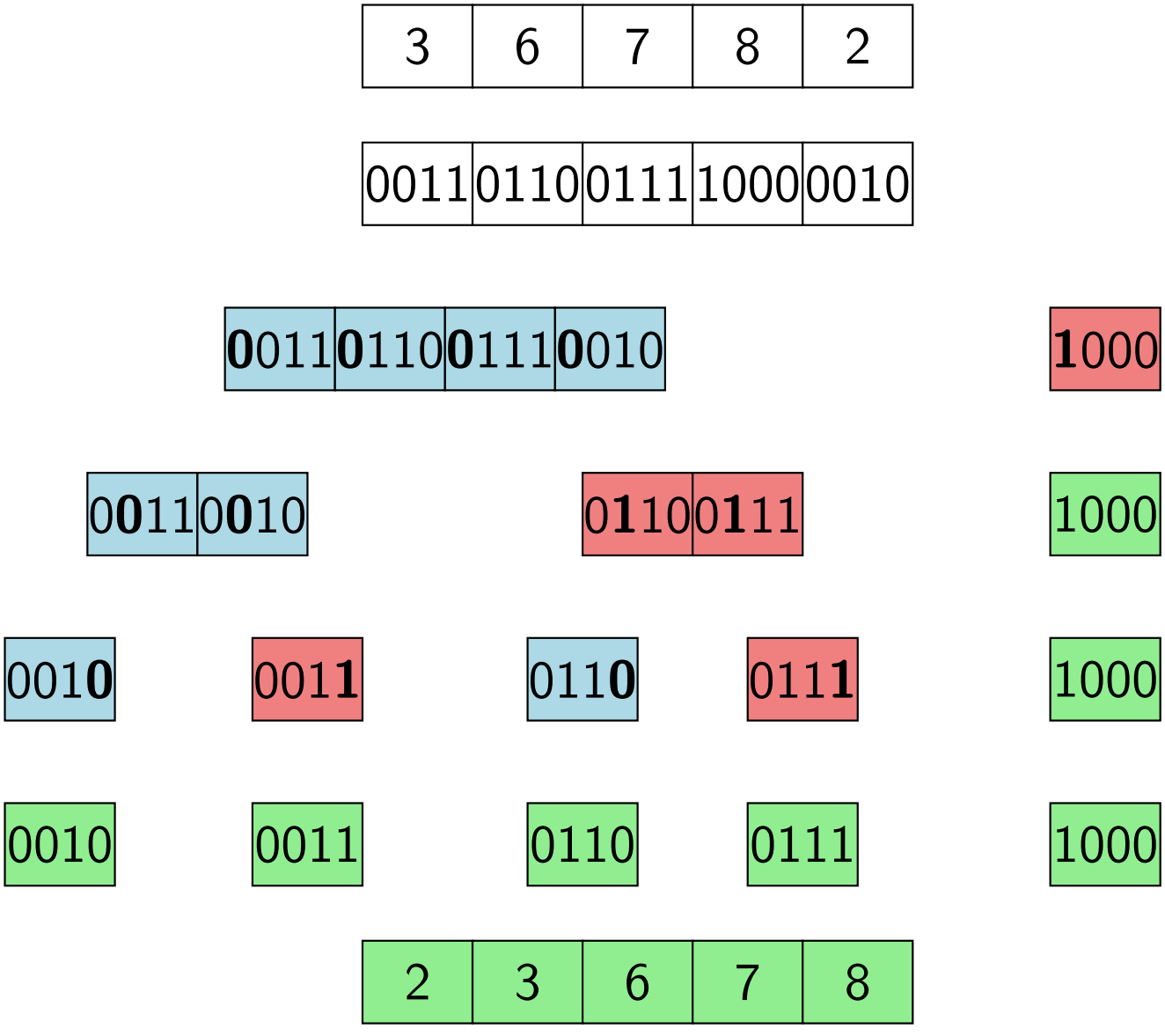

Ref

- *Introduction to Algorithms*. T. Cormen, C. Leiserson, R. Rivest, C. Stein
- *The Art of Computer Programming*, Vol. 3. D. Knuth
- for fun
  - `https://www.toptal.com/developers/sorting-algorithms`
  - `https://www.youtube.com/watch?v=kPRA0W1kECg`