as2_q5

2024-10-26

Question 5

The continuous random variable X has the following probability density function (PDF), for some positive constant c:

$$f(x) = \frac{3}{(1+x)^3}, \quad 0 \leq x \leq c$$

To determine the constant c such that $f(x)$ is a legitimate PDF, we need to ensure that the total area under the curve of $f(x)$ over the given domain is equal to 1.

Step 1: Set Up the Integral

To find the value of c:

$$\int_0^c f(x)\,dx = 1$$

Substituting $f(x)$:

$$\int_0^c \frac{3}{(1+x)^3}\,dx = 1$$

Step 2: Solve the Integral

To evaluate this integral, we use substitution. Let:

$$u = 1 + x, \quad \text{then } du = dx$$

Rewrite the limits in terms of u:

- When $x = 0$, $u = 1$
- When $x = c$, $u = 1 + c$

The integral becomes:

$$\int_1^{1+c} \frac{3}{u^3}\,du$$

Now, evaluate the antiderivative:

$$\int \frac{3}{u^3}\,du = 3\int u^{-3}\,du = 3 \cdot \left(\frac{u^{-2}}{-2}\right) = -\frac{3}{2}u^{-2}$$

Step 3: Substitute the Limits

Evaluate the definite integral from $u = 1$ to $u = 1 + c$:

$$\left[-\frac{3}{2} \cdot \frac{1}{u^2}\right]_1^{1+c} = -\frac{3}{2(1+c)^2} + \frac{3}{2}$$

Simplify:

$$\frac{3}{2} - \frac{3}{2(1+c)^2} = 1$$

Step 4: Solve for c

To find c:

1. Move the constant to the other side:

$$\frac{3}{2} - 1 = \frac{3}{2(1+c)^2}$$

$$\frac{1}{2} = \frac{3}{2(1+c)^2}$$

2. Cross multiply to solve for $(1+c)^2$:

$$1 \cdot 2(1+c)^2 = 2 \cdot 3$$

$$2(1+c)^2 = 6$$

3. Divide both sides by 2:

$$(1+c)^2 = 3$$

4. Take the square root of both sides:

$$1 + c = \sqrt{3}$$

5. Solve for c:

$$c = \sqrt{3} - 1$$

Conclusion

The value of c that makes $f(x)$ a legitimate PDF is:

$$c = \sqrt{3} - 1$$

Now that we have the value of c, we can plot the PDF in R:

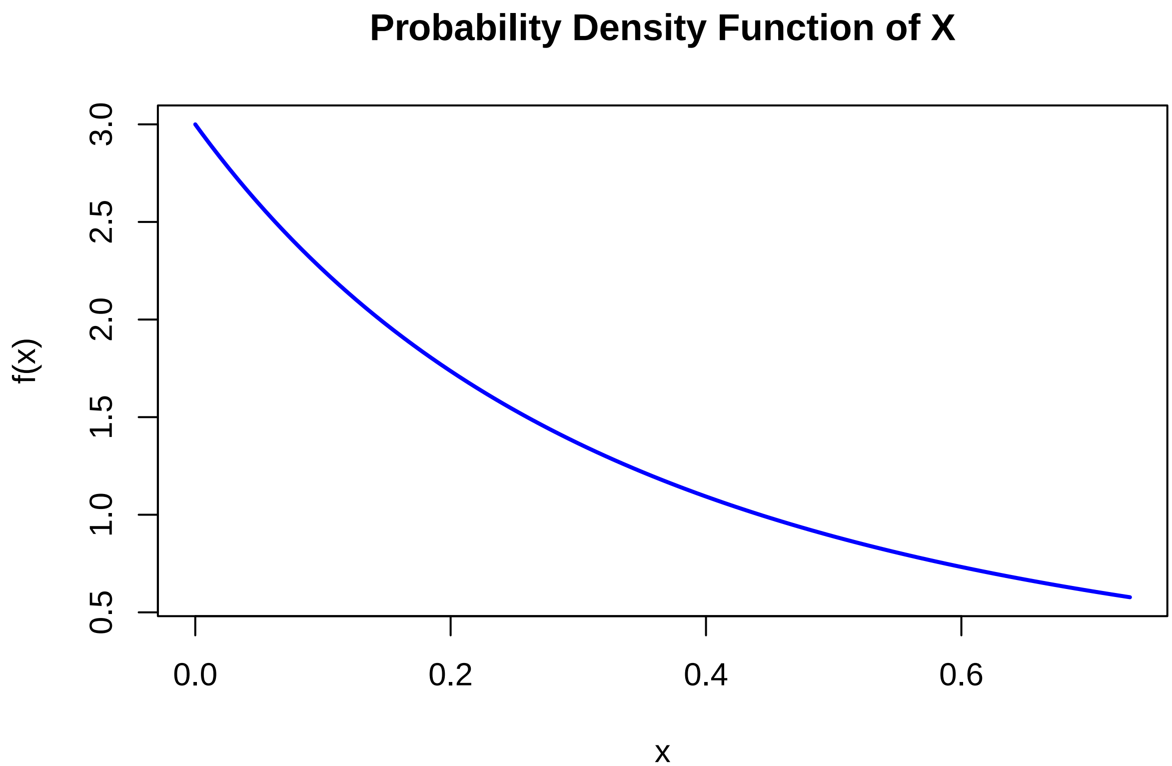
```r
# (b)
# Define the value of c
c <- sqrt(3) - 1

# Define the PDF function
f <- function(x) {
  3 / (1 + x)^3
}

# Define the range for plotting
x_vals <- seq(0, c, length.out = 1000)

# Calculate the y-values for each x
y_vals <- f(x_vals)

# Plot the PDF
plot(x_vals, y_vals, type = "l", col = "blue", lwd = 2,
     main = "Probability Density Function of X",
     xlab = "x", ylab = "f(x)")
```



(c) The Expected Value $E(X)$

The expected value $E(X)$ is defined as:

$$E(X) = \int_0^c x \cdot f(x)\, dx = \int_0^c x \cdot \frac{3}{(1+x)^3}\, dx$$

Step 2.1: Set Up and Simplify the Integral

Substituting the value of $f(x)$:

$$E(X) = \int_0^c \frac{3x}{(1+x)^3}\, dx$$

We use substitution again:

- Let $u = 1 + x$, then $du = dx$, and $x = u - 1$.
- When $x = 0$, $u = 1$.
- When $x = c$, $u = 1 + c$.

The integral becomes:

$$E(X) = \int_1^{1+c} \frac{3(u-1)}{u^3}\, du = 3 \int_1^{1+c} \left(\frac{u-1}{u^3} \right)\, du$$

Expanding:

$$E(X) = 3\int_1^{1+c} \left(\frac{1}{u^2} - \frac{1}{u^3}\right) du$$

Step 2.2: Integrate Each Term

Integrate each term:

$$\int u^{-2}\, du = -\frac{1}{u}$$

$$\int u^{-3}\, du = -\frac{1}{2u^2}$$

Evaluating the definite integral:

$$E(X) = 3\left(\left[-\frac{1}{u}\right]_1^{1+c} - \left[-\frac{1}{2u^2}\right]_1^{1+c}\right)$$

Substituting the limits:

1. **For** $-\frac{1}{u}$:

$$\left[-\frac{1}{u}\right]_1^{1+c} = -\frac{1}{1+c} + \frac{1}{1} = 1 - \frac{1}{1+c}$$

2. **For** $-\frac{1}{2u^2}$:

$$\left[-\frac{1}{2u^2}\right]_1^{1+c} = -\frac{1}{2(1+c)^2} + \frac{1}{2}$$

Putting it all together:

$$E(X) = 3\left(1 - \frac{1}{1+c} + \frac{1}{2} - \frac{1}{2(1+c)^2}\right)$$

Step 2.3: Substitute $c = \sqrt{3} - 1$

Substituting $c = \sqrt{3} - 1$:

1. $1 + c = \sqrt{3}$

$$E(X) = 3\left(1 + \frac{1}{2} - \frac{1}{\sqrt{3}} - \frac{1}{2(\sqrt{3})^2}\right)$$

Simplify:

- $1 + \frac{1}{2} = \frac{3}{2}$
- $\frac{1}{2(\sqrt{3})^2} = \frac{1}{6}$

Combining:

$$E(X) = 3\left(\frac{3}{2} - \frac{1}{\sqrt{3}} - \frac{1}{6}\right)$$

Common denominator for $\frac{3}{2}$ and $\frac{1}{6}$:

$$\frac{3}{2} - \frac{1}{6} = \frac{8}{6} = \frac{4}{3}$$

So:

$$E(X) = 3\left(\frac{4}{3} - \frac{1}{\sqrt{3}}\right)$$

Multiply by 3:

$$E(X) = 4 - \frac{3}{\sqrt{3}}$$

Simplify:

$$E(X) = 4 - \sqrt{3}$$

Final Answer

The expected value $E(X)$ is:

$$E(X) = 4 - \sqrt{3}$$

$$E(X) = 2.26$$

```r
#(d) Define the function for the cumulative distribution (inverse transformation method)

func_inv <- function(u) {
  (2 * (1 - u))^(-1/2) - 1
}

# Simulate 1000 random observations
set.seed(123)  # Set a seed for reproducibility
u_vals <- runif(1000)
simulated_data <- func_inv(u_vals)

# (e) Estimate mean and variance
estimated_mean <- mean(simulated_data)
estimated_variance <- var(simulated_data)

# Print the results
cat("Estimated Mean:", estimated_mean, "\n")
```

```
## Estimated Mean: 0.3725972
```

```r
cat("Estimated Variance:", estimated_variance, "\n")
```

```
## Estimated Variance: 1.944269
```