Question 1:

a). Find constant c such that $f(x)$ is a legitimate probability function (pdf).

$$\int_0^\infty f(x)\, d(x) = \int_0^\infty \frac{1}{c(1+x)^{3/2}}\, dx = 1$$

$$\int_0^\infty (1+x)^{-3/2}\, dx.$$

for $u = 1+x$; $du = dx$

we have $\int_1^\infty u^{-3/2}\, du.$

so: $\int u^{-3/2}\, du = \int u^{-1.5}\, du = \frac{u^{-1/2}}{-1/2}$

$= -2u^{-1/2} = -\frac{2}{\sqrt{u}}.$

For $u = 1 \to u = \infty$:

$$\int_1^\infty u^{-3/2}\, du = \lim_{a \to \infty} \left[-\frac{2}{\sqrt{u}}\right]_1^b = \lim_{a\to\infty}\left(-\frac{2}{\sqrt{a}} + 2\right)$$

as $a \to \infty$, $\frac{2}{\sqrt{a}} \to 0$. thus:

$$\int_1^\infty u^{-3/2}\, du = 2.$$

From the initial equation:

$$\int_0^\infty (1+x)^{-3/2}\, dx = 2.$$

$$\int_0^\infty f(x)\, dx = \int_0^\infty \frac{1}{c(1+x)^{3/2}}\, dx =$$

$$= \frac{1}{c} \cdot 2 = \frac{2}{c}.$$

$$\frac{2}{c} = 1 \Rightarrow \underline{c = 2}$$

b). Find the (cdf) $F(x)$

With $f(x) = \dfrac{1}{2(1+x)^{3/2}}$;

$x > 0$.

The cdf $f(x)$ is given by:

$$f(x) = \int_0^x f(t)\,dt = \int_0^x \frac{1}{2(1+t)^{3/2}}\,dt.$$

$$\int_0^x (1+t)^{-3/2}dt = \left[-2(1+t)^{-1/2}\right]_0^x$$

$$= -2\frac{1}{\sqrt{1+x}} + 2\frac{1}{\sqrt{1+0}}$$

$$\int_0^x (1+t)^{-3/2}dt = -\frac{2}{\sqrt{1+x}} + 2.$$

with $f(t) = \frac{1}{2}(1+t)^{-3/2}$.

$$F(x)\ \frac{1}{2}\left(-\frac{2}{\sqrt{1+x}} + 2\right)$$

$$= \frac{2 - \frac{2}{\sqrt{1+x}}}{2}$$

$$F(x) = 1 - \frac{1}{\sqrt{1+x}};\ x > 0$$

c) use the inverse (cdf) method to generate data

For the inverse; we have:

If $U \sim (0,1)$: then $U = F(X)$

So: $U = F(X) \iff U = 1 - \frac{1}{\sqrt{1+X}}$

$$1 - U = \frac{1}{\sqrt{1+X}}.$$

$$\sqrt{1+X} = \frac{1}{1-U}$$

$$1+X = \frac{1}{(1-U)^2} \Rightarrow X = \frac{1}{(1-U)^2} - 1$$

$$X = \frac{1}{(1-U)^2} - 1$$