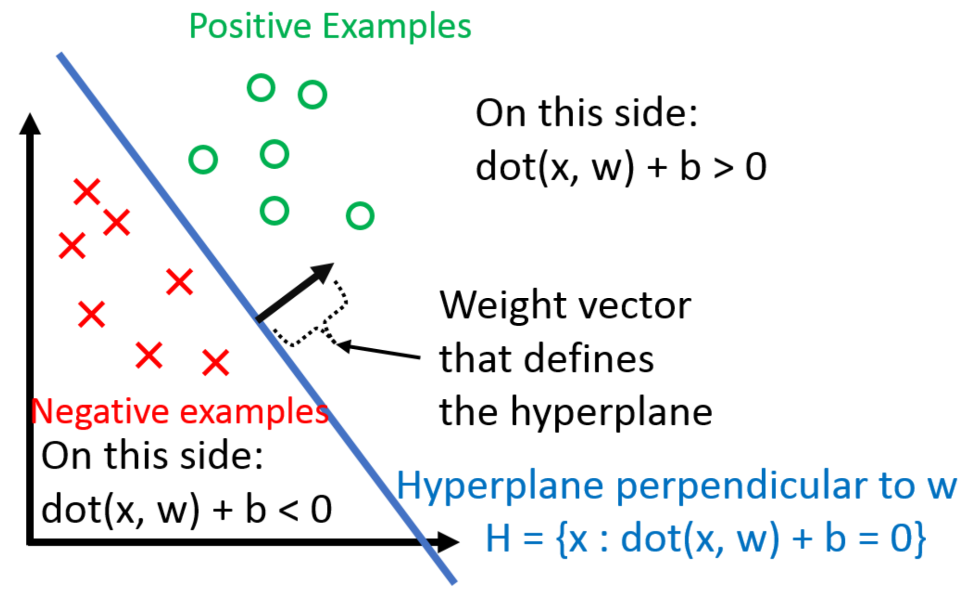
The Perceptron

Assumptions

1. Binary classification (i.e. $y_i \in \{-1, +1\}$)
2. Data is linearly separable

Classifier

$$h(x_i) = \textrm{sign}(\vec{w} \cdot \vec{x_i} + b)$$



b is the bias term (without the bias term, the hyperplane that $\vec{w}$ defines would always have to go through the origin). Dealing with b can be a pain, so we 'absorb' it into the feature vector $\vec{w}$ by adding one additional *constant* dimension. Under this convention,

$$\vec{x_i} \quad \text{becomes} \quad \begin{bmatrix} \vec{x_i} \\ 1 \end{bmatrix}$$
$$\vec{w} \quad \text{becomes} \quad \begin{bmatrix} \vec{w} \\ b \end{bmatrix}$$

We can verify that

$$\begin{bmatrix} \vec{x_i} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \vec{w} \\ b \end{bmatrix} = \vec{w} \cdot \vec{x_i} + b$$

Using this, we can simplify the above formulation of $h(x_i)$ to

$$h(x_i) = \textrm{sign}(\vec{w} \cdot \vec{x})$$

<u>Observation:</u> Note that

$$y_i(\vec{w} \cdot \vec{x_i}) > 0 \Longleftrightarrow x_i \quad \text{is classified correctly}$$

where 'classified correctly' means that x_i is on the correct side of the hyperplane defined by $\vec{w}$. Also, note that the left side depends on $y_i \in \{-1, +1\}$ (it wouldn't work if, for example $y_i \in \{0, +1\}$).

Perceptron Algorithm

Now that we know what the $\vec{w}$ is supposed to do (defining a hyperplane the separates the data), let's look at how we can get such $\vec{w}$.

Perceptron Algorithm

```
Initialize w⃗ = 0⃗                      // Initialize w⃗. w⃗ = 0⃗ misclassifies everything.
while TRUE do                           // Keep looping
    m = 0                               // Count the number of misclassifications, m
    for (x_i, y_i) ∈ D do               // Loop over each (data, label) pair in the dataset, D
        if y_i(w⃗^T · x⃗_i) ≤ 0 then      // If the pair (x⃗_i, y_i) is misclassified
            w⃗ ← w⃗ + y x⃗                  // Update the weight vector w⃗
            m ← m + 1                   // Counter the number of misclassification
        end if
    end for
    if m = 0 then                       // If the most recent w⃗ gave 0 misclassifications
        break                           // Break out of the while-loop
    end if
end while                               // Otherwise, keep looping!
```

Geometric Intuition

Quiz#1: Can you draw a visualization of a Perceptron update?
Quiz#2: How often can a Perceptron misclassify a point $\vec{x}$ repeatedly?

Perceptron Convergence

Suppose that $\exists \vec{w}^*$ such that $y_i(\vec{w}^* \cdot \vec{x}) > 0 \, \forall (\vec{x}_i, y_i) \in D$.

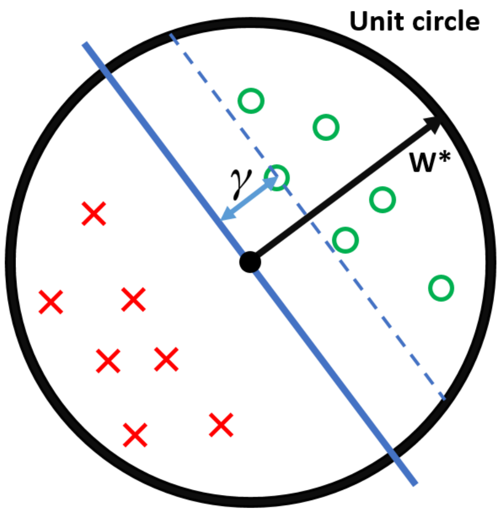
Now, suppose that we rescale each data point and the $\vec{w}^*$ such that

$$||\vec{w}^*|| = 1 \qquad \text{and} \qquad ||\overrightarrow{x_i}|| \leq 1 \;\; \forall \overrightarrow{x_i} \in D$$

The Margin of a hyperplane, γ, is defined as

$$\gamma = \min_{(\overrightarrow{x_i}, y_i) \in D} |\vec{w}^* \cdot \overrightarrow{x_i}|$$

We can visualize this as follows



- All inputs $\overrightarrow{x_i}$ live within the unit sphere
- γ is the distance from the hyperplane (blue) to the closest data point
- $\vec{w}^*$ lies on the unit sphere

Theorem: If all of the above holds, then the perceptron algorithm makes at most $1/\gamma^2$ mistakes.

Proof:
Keeping what we defined above, consider the effect of an update ($\vec{w}$ becomes $\vec{w} + y\vec{x}$) on the two terms $\vec{w} \cdot \vec{w}^*$ and $\vec{w} \cdot \vec{w}$. We will use two facts:

- $y(\vec{x} \cdot \vec{w}) \leq 0$: This holds because $\vec{x}$ is misclassified by $\vec{w}$ - otherwise we wouldn't make the update.
- $y(\vec{x} \cdot \vec{w}^*) > 0$: This holds because $\vec{w}^*$ is a separating hyper-plane and classifies all points correctly.

1. Consider the effect of an update on $\vec{w} \cdot \vec{w}^*$:

$$(\vec{w} + y\vec{x}) \cdot \vec{w}^* = \vec{w} \cdot \vec{w}^* + y(\vec{x} \cdot \vec{w}^*) \geq \vec{w} \cdot \vec{w}^* + \gamma$$

   The inequality follows from the fact that, for $\vec{w}^*$, the distance from the hyperplane defined by $\vec{w}^*$ to $\vec{x}$ must be at least γ (i.e. $y(\vec{x} \cdot \vec{w}^*) = |\vec{x} \cdot \vec{w}^*| \geq \gamma$).

   <u>This means that for each update, $\vec{w} \cdot \vec{w}^*$ grows by **at least** γ.</u>
2. Consider the effect of an update on $\vec{w} \cdot \vec{w}$:

$$(\vec{w} + y\vec{x}) \cdot (\vec{w} + y\vec{x}) = \vec{w} \cdot \vec{w} + 2y(\vec{w} \cdot \vec{x}) + y^2(\vec{x} \cdot \vec{x}) \leq \vec{w} \cdot \vec{w} + 1$$

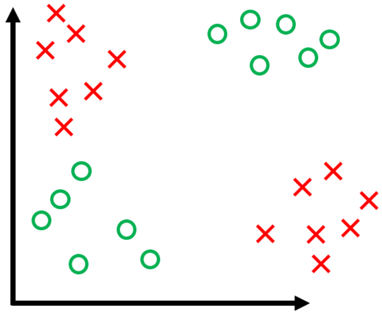
   The inequality follows from the fact that
   - $2y(\vec{w} \cdot \vec{x}) < 0$ as we had to make an update, meaning $\vec{x}$ was misclassified
   - $y^2(\vec{x} \cdot \vec{x}) \leq 1$ as $y^2 = 1$ and all $\vec{x} \cdot \vec{x} \leq 1$ (because $\|\vec{x}\| \leq 1$).

   <u>This means that for each update, $\vec{w} \cdot \vec{w}$ grows by **at most** 1.</u>
3. Now we can put together the above findings. Suppose we had M updates.

$$\begin{aligned}
M\gamma &\leq \vec{w} \cdot \vec{w}^* && \text{By (1)} \\
&= |\vec{w} \cdot \vec{w}^*| && \text{By (1) again (the dot-product must be non-negative because the initialization is 0 and each update increases it by at leas} \\
&\leq ||\vec{w}||\ ||\vec{w}^*|| && \text{By Cauchy-Schwartz inequality} \\
&= ||\vec{w}|| && \text{As } ||\vec{w}^*|| = 1 \\
&= \sqrt{\vec{w} \cdot \vec{w}} && \text{by definition of } \|\vec{w}\| \\
&\leq \sqrt{M} && \text{By (2)} \\
&\Rightarrow M\gamma \leq \sqrt{M} \\
&\Rightarrow M^2\gamma^2 \leq M \\
&\Rightarrow M \leq \frac{1}{\gamma^2}
\end{aligned}$$

History

- Initially, huge wave of excitement ("Digital brains")
- Then, contributed to the A.I. Winter. Famous counter-example XOR problem (Minsky 1969):



- If data is not linearly separable, it loops forver.