Example: exchangeability

$Y_1 = \begin{cases} 1 & \text{if ball 1 is black} \\ 0 & \text{otherwise} \end{cases}$

$Y_2 = \begin{cases} 1 & \text{if ball 2 is black} \\ 0 & \text{otherwise} \end{cases}$

1. Consider a bag of 1 white and 1 black ball. Let the random variables be

  - Y_1 the outcome of drawing one ball from the bag
  - Y_2 the outcome of drawing second ball from the bag after replacing the first ball ←

- Are Y_1 and Y_2 exchangeable? Yes
- Are Y_1 and Y_2 independent? No

$P(Y_1, Y_2 \mid \theta_1, \theta_2) = \theta_1^{Y_1} \theta_2^{Y_2}$

$= P(Y_1, Y_2 \mid I)$

$\overset{?}{=} P(Y_1 \mid I) P(Y_2 \mid I)$

$I: \theta_1 = \dfrac{\#\text{ black in draw 1}}{\#\text{ all in draw 1}}$

$\theta_2 = \dfrac{\#\text{ black in draw 2}}{\#\text{ all in draw 2}}$

$\theta_1 = \theta_2 \Rightarrow \theta_1^{Y_1} \theta_2^{Y_2} = \theta_1^{Y_1} \theta_1^{Y_2}$

- Y_1 and Y_2 are not independent
- Y_1 and Y_2 are exchangeable

Example: exchangeability

2. Consider a bag of 1 white and 1 black ball. Let the random variables be
    - Y_1 the outcome of drawing one ball from the bag
    - Y_2 the outcome of drawing second ball from the bag <u>without replacing the first ball</u>

- Are Y_1 and Y_2 exchangeable? Yes
- Are Y_1 and Y_2 independent? No

$$I: \theta_1 = \frac{\text{\# black balls in draw 1}}{\text{\# all in draw 1}} = \frac{1}{2}$$

$$\theta_2 = ?$$

$$\underline{P(Y_1, Y_2 \mid I)} = \underbrace{\boxed{P(Y_1 \mid I)}}_{\theta^{Y_1}(1-\theta)^{1-Y_1}} \boxed{P(Y_2 \mid Y_1, I)}$$

$$P(Y_2 \mid Y_1, I) = \int P(Y_2 \mid \theta_2) \overbrace{P(\theta_2 \mid Y_1, I)}^{\text{posterior for } \theta_2} d\theta_2$$

$$\theta_2 = \{0, 1\} \quad Pr(\theta_2 = 0) = \begin{cases} 1 & \text{if } Y_1 = 1 \\ 0 & \text{if } Y_1 = 0 \end{cases}$$

Example: exchangeability

3. Consider a bag of 10000 white and 10000 black balls. Let the random variables be

- Y_1 the outcome of drawing one ball from the bag
- Y_2 the outcome of drawing second ball from the bag <u>without replacing the first ball</u>

- Are Y_1 and Y_2 exchangeable? Yes
- Are Y_1 and Y_2 independent? No
- Can we act as if Y_1 and Y_2 were independent? Yes

$$\underbrace{P(Y_2 \mid Y_1)}_{} = \theta_2(Y_1)^{Y_2}(1-\theta_2(Y_1))^{1-Y_2}$$

$$\approx P(Y_2 \mid I)$$

$I: \theta_1 = \frac{1}{2}$

$$\theta_2(Y_1) = \frac{9999}{19999} \approx \frac{1}{2} \text{ if } Y_1 = 1$$

$$= \frac{10000}{19999} \approx \frac{1}{2} \text{ if } Y_1 = 0$$

Example: exchangeability

4. Consider a bag of white and black balls (n is known but the proportion is unknown). Let the random variables be

- Y_1 the outcome of drawing one ball from the bag
- Y_2 the outcome of drawing second ball from the bag after replacing the first ball

- Are Y_1 and Y_2 exchangeable? Yes
- Are Y_1 and Y_2 independent? No
- Can we act as if Y_1 and Y_2 were independent? Yes if n is large

$$P(y_1, y_2 \mid I) \stackrel{?}{=} P(y_1 \mid I) P(y_2 \mid I)$$

$$P(y_1 \mid I) = \int P(y_1 \mid \theta, I) P(\theta \mid I) d\theta$$

$I: \theta_1 = ?$

$\theta_1 = \theta_2 \equiv \theta$

$$P(y_2 \mid y_1, I) = \int \underbrace{P(y_2 \mid \theta, I)}_{\theta^{y_2}(1-\theta)^{1-y_2}} \underbrace{P(\theta \mid y_1, I)}_{\neq P(\theta \mid I)} d\theta \neq P(y_2 \mid I)$$

Example: exchangeability

5. Consider a bag of white and black balls (<u>n is known but the proportion is unknown</u>). Let the random variables be

- Y_1 the outcome of drawing one ball from the bag
- Y_2 the outcome of drawing second ball from the bag <u>without replacing the first ball</u>

- Are Y_1 and Y_2 exchangeable? Yes
- Are Y_1 and Y_2 independent? No
- Can we act as if Y_1 and Y_2 were independent?

Yes, if n is very large
(compare to case 3)

Example: exchangeability

- 6. Consider a bag of white and black balls (n is unknown and the proportion is unknown). Let the random variables be
  - Y_1 the outcome of drawing one ball from the bag
  - Y_2 the outcome of drawing second ball from the bag without replacing the first ball
- Are Y_1 and Y_2 exchangeable? Yes
- Are Y_1 and Y_2 independent? No
- Can we act as if Y_1 and Y_2 were independent?

Depends on your prior assumptions on n.