Exercise: Symbolic Computation

In *symbolic differentiation* we are given an arithmetic expression e and the task is to compute the derivative of this expression with respect to a given variable x. The task is called *symbolic* differentiation because the result is not a numerical value, but rather an arithmetic expression. For example, if the expression e is given as

$$e = x \cdot \exp(x),$$

then the derivative of e with respect to x can be calculated using the *product rule* and is seen to be

$$\frac{d}{dx}\Big(x \cdot \exp(x)\Big) = 1 \cdot \exp(x) + x \cdot \exp(x).$$

In order to be able to develop an algorithm for symbolic differentiation, we first define the set $\mathcal{E}$ of arithmetic expressions.

1. The strings `x`, `y` and `z` are arithmetical expressions, we have

   $$\mathtt{x} \in \mathcal{E}, \quad \mathtt{y} \in \mathcal{E}, \quad \text{and} \quad \mathtt{z} \in \mathcal{E}.$$

   Of course, these strings are interpreted as variables.

2. All natural numbers are arithmetical expressions:

   $n \in \mathcal{E}$ for all $n \in \mathbb{N}$.

3. If s and t are arithmetical expressions, then we have:

   (a) $s + t \in \mathcal{E}$,
   (b) $s - t \in \mathcal{E}$,
   (c) $s \cdot t \in \mathcal{E}$,
   (d) $s \;/\; t \in \mathcal{E}$.

4. If $s \in \mathcal{E}$ and $n \in \mathbb{N}$, then $s^n \in \mathcal{E}$.

In order to be able to manipulate arithmetical expressions with a SETL program, we have to define how these expressions are represented in SETL2. There, we define a function

$$\mathtt{rep} : \mathcal{E} \to \text{SETL2}.$$

This function takes an arithmetical expression as its argument and transforms it into a *Setl* data structure. The value $\mathtt{rep}(e)$ is defined by induction on e:

1. The representation of a variable is the corresponding string. Therefore we have

   $\mathtt{rep}(v) = v$ for all variables $v \in \{\mathtt{x}, \mathtt{y}, \mathtt{z}\}$.

2. A number is represented by itself:

   $\mathtt{rep}(n) = n$ for all $n \in \mathbb{N}$.

3. $\mathtt{rep}(s + t) := [\mathtt{rep}(s), \text{“}+\text{”}, \mathtt{rep}(t)]$.
4. $\mathtt{rep}(s - t) := [\mathtt{rep}(s), \text{“}-\text{”}, \mathtt{rep}(t)]$.
5. $\mathtt{rep}(s * t) := [\mathtt{rep}(s), \text{“}*\text{”}, \mathtt{rep}(t)]$.
6. $\mathtt{rep}(s \;/\; t) := [\mathtt{rep}(s), \text{“}/\text{”}, \mathtt{rep}(t)]$.
7. $\mathtt{rep}(s^n) := [\mathtt{rep}(s), \text{“}**\text{”}, n]$ for all $n \in \mathbb{N}$.

Exercise 1: Implement a SETL2 procedure `diff` such that `diff`(e, v) computes the derivate of the arithmetical expression E with respect to the variable v.

1. You will find a program skeleton for this task at the following location:

   `http://www.ba-stuttgart.de/~stroetma/SETL2/diff-frame.stl`

   This program skeleton already contains a parser, a pretty printer and some test cases.
2. The representation of arithmetical expressions given above does not account for the unary minus operator "`-`". Therefore, an expression of the form $-e$ has to be represented as $0 - e$.
3. The parser is not able to parse negative numbers and the unary minus operator is not supported either. However, instead of writing, for example, $-5 \cdot x$ use the expression $0 - 5 \cdot x$.
4. Remember the following rules:
   (a) $\frac{d}{dx}(g + h) = \frac{dg}{dx} + \frac{dh}{dx}$,
   (b) $\frac{d}{dx}(g - h) = \frac{dg}{dx} - \frac{dh}{dx}$,
   (c) $\frac{d}{dx}(g \cdot h) = \frac{dg}{dx} \cdot h + g \cdot \frac{dh}{dx}$
   (d) $\frac{d}{dx}\left(\frac{g}{h}\right) = \dfrac{\frac{dg}{dx} \cdot h - g \cdot \frac{dh}{dx}}{h \cdot h}$
   (e) $\frac{d}{dx} g^n = n \cdot g^{n-1} \cdot \frac{dg}{dx}$ for all $n \in \mathbb{N}$
5. For testing types, SETL provides the procedure `is_integer()`.

Exercise 2: Extend the program so that you can handle the functions exp(), `ln`(), sqrt(), sin(), cos(), tan(), and arctan().

Remark: The derivatives of these functions are as follows:

$f(x)$	$\frac{d}{dx} f$	$f(x)$	$\frac{d}{dx} f$
$\mathtt{exp}(x)$	$\mathtt{exp}(x)$	$\mathtt{ln}(x)$	$\frac{1}{x}$
$\mathtt{sin}(x)$	$\mathtt{cos}(x)$	$\mathtt{cos}(x)$	$-\mathtt{sin}(x)$
$\mathtt{tan}(x)$	$\dfrac{1}{\mathtt{cos}^2(x)}$	$\mathtt{arctan}(x)$	$\dfrac{1}{1 + x^2}$
$\sqrt{x}$	$\dfrac{1}{2\sqrt{x}}$		

Hint: Don't forget to apply the chain rule!

Exercise 3: Implement a procedure `simplify` that takes an arithmetical expression and simplifies it using the following identities:

$$1 \cdot x = x \cdot 1 = x, \quad 0 \cdot x = x \cdot 0 = 0, \quad 0 + x = x + 0 = x.$$

Hint: Use recursion!

Exercise 4*: Extend your program so that it can calculate the derivative of an expression of the form s^t where s and t are arbitrary arithmetical expressions. Test your implementation by computing

$$\frac{d}{dx}(x^x).$$