MATH-F-427 students

Coxeter groups

Course notes

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ULB

Contents

Warnig: hypothesis reduced in the exchange property!

Theorem 0.1 (Matsumoto). *Let W be a group and $S \subset W$ a finite subset of generators of W of order 2. Then the following assertions are equivalent:*

- (i) *(W, S) is a Coxeter system.*
- (ii) *(W, S) satisfies the exchange property.*
- (iii) *(W, S) satisfies the deletion property.*

Proof. $(i) \Rightarrow (ii)$. This implication has already been shown above.

$(ii) \Rightarrow (iii)$. Let $w = s_1 \dots s_k$ such that $\ell(w) < k$. Let i be maximal such that $s_i s_{i+1} \dots s_k$ is not reduced (i.e. $s_{i+1} \dots s_k$ is reduced). We have $\ell(s_i s_{i+1} \dots s_k) \leq k - i = \ell(s_{i+1} \dots s_k)$. Now, using exchange property, we obtain $s_i s_{i+1} \dots s_k = s_{i+1} \dots \hat{s}_j \dots s_k$ for some $i + 1 \leq j \leq k$. Therefore, $w = s_1 \dots s_{i-1} s_i s_{i+1} \dots s_k = s_1 \dots s_{i-1} \hat{s}_i s_{i+1} \dots \hat{s}_j \dots s_k$ and we have the result (let us note that this implication remains true for weaker hypothesis since we did not use the fact that S is of order 2).

$(iii) \Rightarrow (ii)$. Let $w = s_1 \dots s_k$, $k = \ell(w)$, $s \in S$, such that $\ell(sw) = \ell(ss_1 \dots s_k) \leq \ell(w) = \ell(s_1 \dots s_k) = k$. So $ss_1 \dots s_k$ is not reduced. We can apply the deletion property. Suppose that $sw = ss_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$ (but $\ell(sw) \leq k - 1 < l(w)$). So $ssw = sss_i \dots \hat{s}_i \dots \hat{s}_j \dots s_k$. This leads to $\ell(s_1 \dots ...\hat{s}_i \dots \hat{s}_j \dots s_k) < k$, which is a contradiction, so this case has to be excluded. Hence, we have $sw = \hat{s}s_1 \dots \hat{s}_i \dots s_k$.

$(ii) \Rightarrow (i)$. Using $(ii) \Rightarrow (iii)$, we can assume both (ii) and (iii). Define $m(s, s') =$ order of ss' in W, for all $s, s' \in S$. Let $(\tilde{W}, S)$ be the Coxeter group associated to m. Clearly, $\phi : \tilde{W} \mapsto W, s \to s$ is a surjective homomorphism. We need to show that ϕ is also injective. Let $s_1 \dots s_m = e$ in W. By the deletion property, m is even, say $m = 2k$. So we can write our relation on the form

$$s_1 \dots s_k = s'_1 \dots s'_k \tag{0.1}$$

where $s'_1 = s_{2k}, \dots s'_k = s_{k+1}$. We must now prove that (0.1) is a consequence of the pairwise relations $(ss')^{m(s,s')} = e$. The proof is done by induction on k, the case $k = 1$ being trivially correct.

- Case 1: Suppose $w := s_1 \dots s_k$ is not reduced. By deletion property, there exists a position $1 \leq i < k$ such that $s_{i+1} s_{i+2} \dots s_k$ is reduced but $s_i s_{i+1} s_{i+2} ... s_k$ is not. By the exchange property, we then have that $s_{i+1} s_{i+2} \dots s_k = s_i s_{i+1} \dots \hat{s}_j \dots s_k$ for some $i < j \leq k$. This relation is of length $< 2k$ and hence fine. Plugging this result into (0.1) gives $s_1 \dots s_i s_i s_{i+1} \dots \hat{s}_j \dots s_k = s'_1 s'_2 \dots s'_k$. The factor $s_i s_i$ can be deleted, leaving a relation of length $< 2k$. Hence the relation (0.1) is fine.
- Case 2: Suppose $w = s_1 \dots s_k$ is reduced, $k = \ell(w)$. We can assume that $s_1 \neq s'_1$ (otherwise the relation (0.1) is equivalent to a shorter relation). We have $\ell(s'_1 s_1 s_2 \dots s_k) = \ell(s'_1 s'_1 s'_2 \dots s'_k) = \ell(s'_2 \dots s'_k) \leq k - 1 < \ell(s_1 \dots s_k)$. Using exchange property, we have $s'_1 s_1 \dots s_k = s_1 \dots \hat{s}_i \dots s_k$ for some i. Hence, $s_1 \dots \hat{s}_i \dots s_k = s'_2 \dots s'_k$.

If $i < k$, then $s_1' s_1 s_2 \dots s_{k-1} = s_1 \dots \hat{s}_i \dots s_{k-1}$. So $s_1' s_1 s_2 \dots s_{k-1} s_k = s_1 \dots \hat{s}_i \dots s_{k-1} s_k$. Hence, $s_1' s_1 \dots s_k = s_2' \dots s_k$ is a consequence of Coxeter relations.
If $i = k$, we have to work a little bit harder. We have $s_1' s_1 \dots s_{k-1} = s_1' s_2' \dots s_k'$. Thus it will suffice to show that $s_1 s_1 \dots s_{k-1} = s_1 s_2 \dots s_k$ is a consequence of Coxeter relations. Applying recursively the rule, we have $s_1 s_1' s_1 \dots s_{k-2} = s_1' s_1 \dots s_{k-1} \Rightarrow s_1' s_1 s_1' s_1 \dots s_{k-3} = s_1 s_1' s_1 \dots s_{k-2} \Rightarrow \dots$ Thus in the end, the question will be reduced to the relation $s_1 s_1' s_1 s_1' \dots = s_1' s_1 s_1' s_1 \dots$, which is of course a consequence of the Coxeter relation $(s_1 s_1')^{m(s,s')} = e$.

Example 0.2. The group S_n can be generated by transpositions, which are order 2 elements. Using the above theorem, we conclude that S_n is actually a Coxeter group.

0.1 Geometric representation

Let (W, S) be a Coxeter system, $S = \{s_1, \dots s_n\}$, m the associated Coxeter matrix. We write $m_{ij} = m(s_i, s_j)$. Let V be a $\mathbb{R}$-vector space of dimension n, with a basis $\alpha_1, \dots, \alpha_n$. We consider the symmetric bilinear map

$$\langle \cdot, \cdot \rangle : V \times V \mapsto \mathbb{R} \tag{0.2}$$

defined through

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} -\cos\left(\frac{\pi}{m_{ij}}\right) & \text{if } m_{ij} < +\infty \\ -1 & \text{if } m_{ij} = +\infty \end{cases} \tag{0.3}$$

Not that $\langle \cdot, \cdot \rangle$ is not positive definite in general.

Proposition 0.3. *The following map extends to a homomorphism:*

$$W \mapsto GL(V), s_i \to \sigma_i \tag{0.4}$$

where $\sigma_i : v \to v - 2\langle v, \alpha_i \rangle \alpha_i$.

Remark 0.4. We have $\sigma_i(\alpha_i) = \alpha_i - 2\langle \alpha_i, \alpha_i \rangle \alpha_i = -\alpha_i$. Thus, if $v \in V$ is such that $\langle v, \alpha_i \rangle = 0$, then $\sigma_i(v) = v$. Therefore, if $\langle \cdot, \cdot \rangle$ was positive definite, σ_i would be interpreted as a reflexion through the hyperplane orthogonal to α_i.

Proof. First, let us show that σ_i is invertible for all i. We have $\sigma_i^2(v) = \sigma_i(v) - 2\langle v, \alpha_i \rangle \sigma_i(\alpha_i) = v - 2\langle v, \alpha_i \rangle \alpha_i + 2\langle v, \alpha_i \rangle \alpha_i = v$.

Now, let us show that $(\sigma_i \sigma_j)^{m_{ij}} = Id_V$. For $i \neq j$, define $V_{ij} = \mathrm{Span}_{\mathbb{R}}(\{\alpha_i, \alpha_j\})$. Furthermore, $V_{ij}^{\perp} = \{v \in V | \langle v, \alpha_i \rangle = 0, \langle v, \alpha_j \rangle = 0\}$. Before proceeding, we show the following lemma:

Lemma 0.5. *$V = V_{ij} \oplus V_{ij}^{\perp}$ if $m_{ij} < +\infty$.*

Proof. Let $v \in V$. We want to find $\lambda_i, \lambda_j \in \mathbb{R}$ such that $\tilde{v} = \lambda_i\alpha_i + \lambda_j\alpha_j \in V_{ij}$ and $v - \tilde{v} \in V_{ij}^{\perp}$. We have

$$\begin{aligned} \langle \tilde{v}, \alpha_i \rangle &= \lambda_i \langle \alpha_i, \alpha_i \rangle + \lambda_j \langle \alpha_i, \alpha_j \rangle \\ &= \lambda_i + C\lambda_j \end{aligned} \tag{0.5}$$

where $C = \langle \alpha_i, \alpha_j \rangle = -\cos\left(\frac{\pi}{m_{ij}}\right)$. Furthermore,

$$\langle \tilde{v}, \alpha_j \rangle = \lambda_i C + \lambda_j \tag{0.6}$$

Since

$$\det \begin{pmatrix} 1 & C \\ C & 1 \end{pmatrix} = 1 - C^2 = 1 - \cos^2\left(\frac{\pi}{m_{ij}}\right) \neq 0, \tag{0.7}$$

if $m_{ij} < +\infty$. Therefore, we can find unique λ_i and λ_j such that

$$\langle \tilde{v}, \alpha_i \rangle = \langle v, \alpha_i \rangle \quad \text{and} \quad \langle \tilde{v}, \alpha_j \rangle = \langle v, \alpha_j \rangle \tag{0.8}$$

Now let us come back to the proof of the proposition. Using the lemma, we have $v = \tilde{v} + (v - \tilde{v})$ such that $\langle v - \tilde{v}, \alpha_i \rangle = 0 = \langle v - \tilde{v}, \alpha_j \rangle = 0$. Hence $\sigma_i(v - \tilde{v}) = v - \tilde{v} - 2\langle v - \tilde{v}, \alpha_i \rangle \alpha_i = v - \tilde{v}$ and $\sigma_j(v - \tilde{v}) = v - \tilde{v}$. In the basis $\{\alpha_i, \alpha_j\}$ of V_{ij}, the matrix associated to σ_i is given by

$$\begin{pmatrix} -1 & -2C \\ 0 & 1 \end{pmatrix} \tag{0.9}$$

In fact, we have $\sigma_i(\alpha_i) = -\alpha_i$ and $\sigma_j(\alpha_j) = \alpha_j - 2C\alpha_i$. Similarly, the matrix associated to σ_j is given by

$$\begin{pmatrix} 1 & 0 \\ -2C & -1 \end{pmatrix} \tag{0.10}$$

Therefore,

$$(\sigma_i)(\sigma_j) = \begin{pmatrix} -1 & -2C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2C & -1 \end{pmatrix} = \begin{pmatrix} -1 + 4C^2 & 2C \\ -2C & -1 \end{pmatrix} \tag{0.11}$$

The characteristic polynomial P of this matrix is given by $P(t) = t^2 - (-2 + 4C^2)t + 1$.The roots are given by $t_{\pm} = \cos\left(\frac{2\pi}{m_{ij}}\right) \pm i \sin\left(\frac{2\pi}{m_{ij}}\right)$. This characterizes a rotation of $2\pi/m_{ij}$. So, the order of $\sigma_i\sigma_j$ is given by m_{ij}.