MATH-F-427 students

Coxeter groups

Course notes

June 1, 2019

ULB

Contents

0.1 Classification of the Coxeter groups

Let (W, S) be a Coxeter system and (m_{ij}) the associated Coxeter matrix. As discussed above, if V is a real vector space with basis $\{\alpha_1, \ldots, \alpha_n\}$, we can define a symmetric bilinear form as

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} -\cos\left(\frac{\pi}{m_{ij}}\right) & \text{if } m_{ij} < +\infty \\ -1 & \text{if } m_{ij} = +\infty \end{cases} \tag{0.1}$$

We proved above that W is finite if and only if $\langle ., . \rangle$ is positive definite.

Definition 0.1. *W is irreducible if its Coxeter graph is connected.*

This definition can be understood from the following observation: if $\Gamma = \Gamma_1 \sqcup \Gamma_2$, then $W_\Gamma = W_{\Gamma_1} \times W_{\Gamma_2}$.

Let us now introduce some basic notations. Take $\{e_1, \ldots, e_n\}$ be a basis of a vector space V. Take $v = \sum_i x_i e_i$ and $w = \sum_i y_i e_i$ in V. Writing $g_{ij} = \langle e_i, e_j \rangle$, we have

$$\begin{aligned} \langle v, w \rangle &= \sum_i \sum_j x_i y_j \langle e_i, e_j \rangle \\ &= \sum_i \sum_j x_i y_j g_{ij} \\ &= x^T g y \end{aligned} \tag{0.2}$$

where, in the last equality, we used matrix notation.

Lemma 0.2 (Sylvester). *$\langle ., . \rangle$ is positive definite if and only if all the principal minors of g are positive.*

Proof. The condition is necessary. In fact, taking $v = (v_1, \dots, v_r, 0, \dots, 0)$, we have $\langle v, v\rangle = v^T g v$.

The condition is also sufficient. In order to show it, let us proceed by induction on the size of g. If g is not positive definite, there must be 2 negative eigenvalues since the determinant is positive. Let $x \neq 0 \neq y$ be 2 orthogonal eigenvectors associated to these eigenvalues. Let $\alpha, \beta \in \mathbb{R}$ such that $(\alpha, \beta) \neq (0, 0)$. We write

$$v := \alpha x + \beta y = (*, *, \dots, *0) \neq 0 \tag{0.3}$$

We have

$$v^T g v = \alpha^2 x^T g x + \beta^2 y^T g y \leq 0 \tag{0.4}$$

This implies that that the determinant is non-positive.

Theorem 0.3. *The irreducible finite Coxeter groups are given in figure 0.3.*

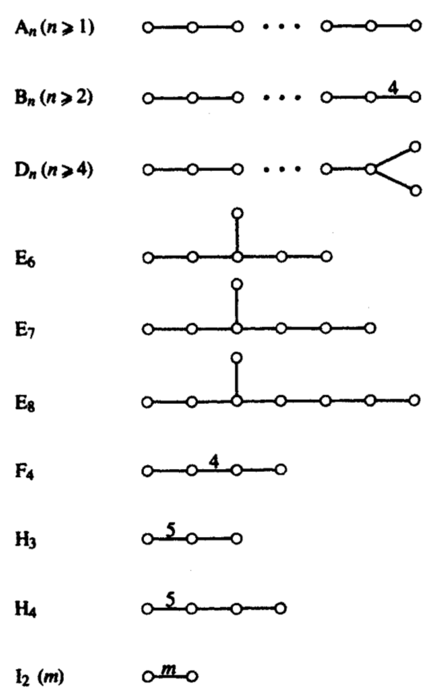

Fig. 0.1.

Remark 0.4. Define the matrix A_Γ by $[A_\Gamma]_{ij} = 2\langle \alpha_i, \alpha_j\rangle$ and set $d(\Gamma) = \det(A_\Gamma)$. We make the following observations:

- Removing one node in a diagram is equivalent to remove the associated line and column.

- If $\Gamma = \Gamma_1 \sqcup \Gamma_2$, then $d(\Gamma) = d(\Gamma_1).d(\Gamma_2)$.

Lemma 0.5. *Let Γ, Γ' and Γ'' be as in figure 0.2.*

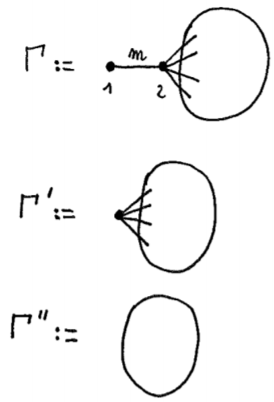

Fig. 0.2.

Then $d(\Gamma) = 2d(\Gamma') - 4\cos^2\left(\frac{\pi}{m}\right) d(\Gamma'')$.

Proof. Consider

$$A = \begin{pmatrix} 2 & -\cos\left(\frac{\pi}{m}\right) & 0 & \cdots \\ -\cos\left(\frac{\pi}{m}\right) & 2 & \cdots & \\ 0 & \vdots & 2 & \cdots \\ \vdots & & \vdots & \end{pmatrix} \tag{0.5}$$

We have

$$d(\Gamma) = 2d(\Gamma') - (-2)\cos\left(\frac{\pi}{m}\right)(-2)\cos\left(\frac{\pi}{m}\right) d(\Gamma'') \tag{0.6}$$

We now compute $d(\Gamma)$ for all the cases in the classification 0.3, using the previous lemma.

- For the case A_n, we have

$$d(A_n) = d(A_{n-1}) - 4\cos^2\left(\frac{\pi}{3}\right) d(A_{n-2}) = n + 1 \tag{0.7}$$

where, to obtain the last equality, we used

$$d(\circ) = 2 \quad \text{and} \quad d(\circ - \circ) = \det\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 3 \tag{0.8}$$

- For B_n, we have
$$d(B_n) = 2d(B_{n-1}) - d(B_{n-2}) \tag{0.9}$$
for $n \geq 4$,
$$d(B_3) = 2d(B_2) - d\circ) = 4 - 2 = 2 \tag{0.10}$$
and
$$d(B_2) = \det \begin{pmatrix} 2 & -2\cos\left(\frac{\pi}{4}\right) \\ -2\cos\left(\frac{\pi}{4}\right) & 2 \end{pmatrix} = 4 - 4\cos^2\left(\frac{\pi}{4}\right) = 2 \tag{0.11}$$
Therefore, $d(B_n) = 2$.
- For D_n, we have
$$d(D_n) = 2d(D_{n-1}) - d(D_{n-2}) \tag{0.12}$$
Furthermore,
$$\begin{aligned} d(\substack{\circ \\ \circ} > \circ - \circ) &= 2d(\substack{\circ \\ \circ} >) - d(\substack{\circ \\ \circ}) \\ &= 2d(A_3) - d(A_1)^2 \\ &= 2.4 - 2^2 \\ &= 4 \end{aligned} \tag{0.13}$$
and
$$\begin{aligned} d(\circ - \circ - \circ <\substack{\circ \\ \circ}) &= 2d(\circ - \circ <\substack{\circ \\ \circ}) - d(\circ <\substack{\circ \\ \circ}) \\ &= 2.4 - 4 \\ &= 4 \end{aligned} \tag{0.14}$$
We deduce that $d(D_n) = 4$.
- We proceed in the same way for the other elements the list 0.3. We get $d(E_6) = 3$, $d(E_7) = 2$, $d(E_8) = 1$, $d(F_4) = 1$, $d(H_3) = 3 - \sqrt{5} > 0$, $d(H_4) = \frac{7-3\sqrt{5}}{2} > 0$ and $d(I_2(m)) = 4\sin^2\left(\frac{\pi}{m}\right) > 0$, for $m \geq 3$ ($m = 2$ is disconnected).

Now, we have to show that there are no other diagrams. In order to do that, le us make some observations.

- We cannot have diagrams of the type given in figure 0.3, because there are no relations between these generators. They generate an infinite group.

Fig. 0.3.

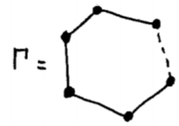

Fig. 0.4.

- We cannot have circuits, i.e. diagrams of the type of figure 0.4
  In fact,

$$A_\Gamma = \begin{pmatrix} 2 & * & 0 & \cdots & 0 & * \\ * & 2 & * & 0 & \cdots & 0 \\ 0 & * & 2 & * & 0 & \cdots \\ \vdots & & & \ddots & & \\ 0 & & & & & * \\ * & 0 & \cdots & 0 & * & 2 \end{pmatrix} \tag{0.15}$$

  where

$$\begin{aligned} * &= -2\cos\left(\frac{\pi}{m}\right) \quad (m \geq 3) \\ &\leq -2\cos\left(\frac{\pi}{3}\right) \\ &-1 \end{aligned} \tag{0.16}$$

  Therefore,

$$\begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} A_\Gamma \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 2n + \sum_{\sharp(*)=2n} (*) \leq 2n - 2n = 0 \tag{0.17}$$

  and we conclude that A_Γ cannot be positive definite.
- If Γ has at most one edge > 3. In fact, consider a diagram of the type of figure 0.5.

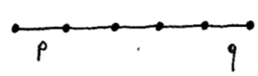

Fig. 0.5.

$$A_\Gamma = \begin{pmatrix} 2 & -2\cos\left(\frac{\pi}{p}\right) & & & & & \\ -2\cos\left(\frac{\pi}{p}\right) & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & & & -2\cos\left(\frac{\pi}{p}\right) \\ & & & & & -2\cos\left(\frac{\pi}{q}\right) & 2 \end{pmatrix} \quad (0.18)$$

We have

$$\begin{aligned} d(A_\Gamma) &= 2d(B_{n-1}) - 4\cos^2\left(\frac{\pi}{q}\right) d(B_{n-2}) \\ &= 4 - 8\cos^2\left(\frac{\pi}{q}\right) \\ &\leq 0 \end{aligned} \quad (0.19)$$

for $q \geq 4$. The case $q \leq 4$ can be done using the same strategy as in (0.17) and is let as an exercise.

- If Γ has one edge > 3, then Γ is a straight line. Consider a diagram of the type of figure 0.6.

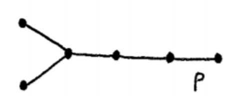

Fig. 0.6.

$$\begin{aligned} d(\Gamma) &= 2d(\Gamma_{n-1}) - 4\cos^2\left(\frac{\pi}{p}\right) d(D_{n-2}) \\ &= 8 - 16\cos^2\left(\frac{\pi}{p}\right) \leq 0 \\ &\leq 0 \end{aligned} \quad (0.20)$$

- Γ has at most one branching point. In fact consider a diagram of the type of figure 0.7.
  We have

$$d(\Gamma) = 2d(\mathrm{d}_{n-1})) - d(D_{n-3} \cup A_1) = 8 - 2.4 = 0 \quad (0.21)$$

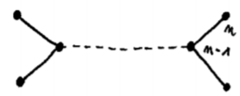

Fig. 0.7.

- Γ has no branching point with 4 or more branches. In fact,

$$\begin{aligned} d(> \circ <) &= 2d(> \circ -) - d({}^{\circ}_{\circ}\circ) \\ &= 2.4 - 2^3 \\ &= 0 \end{aligned} \tag{0.22}$$

- It is let as an exercise to show the relations in figure 0.8.

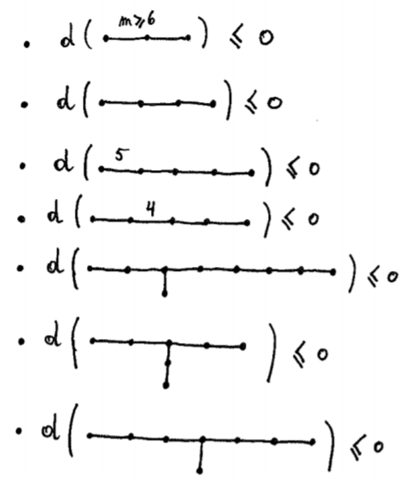

Fig. 0.8.

Therefore, we conclude that no other diagrams than those identified in theorem 0.3 are valid.