MATH-F-427 students

Coxeter groups

Course notes

May 30, 2019

ULB

Contents

The following developments aim to prove the theorem 0.7.

Definition 0.1. *Let $V = \bigoplus_{n\geq 0} V_n$ a graded vector space. The Hilbert series of V is defined as*

$$Hilb_V(t) = \sum_{n\geq 0} (\dim V_n) t^n \in \mathbb{Q}[t] \tag{0.1}$$

Example 0.2. Consider $\mathbb{C}[\theta_1, \ldots, \theta_n] \subset \mathbb{C}[\bar{x}]$. We have

$$\begin{aligned} \mathrm{Hilb}_{\mathbb{C}[\theta_1,\ldots,\theta_n]}(t) &= \frac{1}{(1-t^{d^1})(1-t^{d_2})\ldots(1-t^{d^n})} \\ &= (1+t^{d_1}+t^{2d_1}+t^{3d_1}+\ldots)(1+t^{d_2}+t^{2d_2}+\ldots)\ldots \\ &= \sum_{d\geq 0}\left(\sum_{(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n:\alpha_1 d_1+\ldots+\alpha_n d_n=d} \underbrace{t^{\alpha_1 d_1} t^{\alpha_2 d_2}\ldots t^{\alpha_n d_n}}_{t^d}\right) \end{aligned} \tag{0.2}$$

Furthermore,

$$\mathbb{C}[\theta_1,\ldots,\theta_n]_d = \mathrm{Span}_{\mathbb{C}}\{\theta_1^{\alpha_1}\theta_2^{\alpha_2}\ldots\theta_n^{\alpha_n} | \alpha_1 d_1+\ldots+\alpha_n d_n = d\} \tag{0.3}$$

and

$$\dim \mathbb{C}[\theta_1,\ldots,\theta_n]_d = \sharp\{(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n | \alpha_1 d_1+\ldots+\alpha_n d_n = d\} \tag{0.4}$$

Theorem 0.3. *[Molien] Let $G \subset GL(\mathbb{C}^n)$ be a finite group. We have*

$$Hilb_{\mathbb{C}[\bar{x}]^G}(t) = \frac{1}{|G|}\sum_{\pi\in G}\frac{1}{\det(I-\pi t)} \tag{0.5}$$

Proof. Consider $(\mathbb{C}^n)^G = \{v \in \mathbb{C}^n | \pi v = v, \forall \pi \in G\}$. We define

$$P_G := \frac{1}{|G|} \sum_{\pi \in G} \pi \in \mathrm{End}(\mathbb{C}^n) \tag{0.6}$$

This operator is the projection of $\mathbb{C}^n$ onto $(\mathbb{C}^n)^G$. We have $P_G^2 = P_G$ and

$$\begin{aligned} \dim(\mathbb{C}^n)^G &= \mathrm{rank} P_G \\ &= \mathrm{Tr}(P_G) \\ &= \frac{1}{|G|} \sum_{\pi \in G} \mathrm{Tr}\pi \end{aligned} \tag{0.7}$$

Recall that

$$\mathbb{C}[\bar{x}] = \bigoplus_{d \geq 0} \mathbb{C}[\bar{x}]_d \tag{0.8}$$

For each $\pi \in G$, we write $\pi^{(d)} \in GL(\mathbb{C}[\bar{x}]_d)$ its restriction to $\mathbb{C}[\bar{x}]_d$. Not that

$$\mathbb{C}[\bar{x}]^G = \bigoplus_{d \geq 0} \mathbb{C}[\bar{x}]_d^G \tag{0.9}$$

Now, we can identify $\mathbb{C}^n$ with $\mathbb{C}[\bar{x}]_1$. Let $\pi \in G$ and $\ell_{\pi,1}, \ldots, \ell_{\pi,n} \in \mathbb{C}[\bar{x}]_1$ a basis of eigenvectors associated with eigenvalues $\lambda_{\pi,1}, \ldots, \lambda_{\pi,n}$, respectively. So $\{\ell_{\pi,1}^{d_1}, \ldots, \ell_{\pi,n}^{d_n} | d_1 + \ldots + d_n = d\}$ is a basis of $\mathbb{C}[\bar{x}]_d$. Here, $\ell_{\pi,1}^{d_1}, \ldots, \ell_{\pi,n}^{d_n}$ are eigenvectors of $\pi^{(d)}$ associated with eigenvalues $\lambda_{\pi,1}^{d_1}, \ldots, \lambda_{\pi,n}^{d_n}$. Therefore,

$$\begin{aligned} \dim \mathbb{C}_d^G &= \frac{1}{|G|} \sum_{\pi \in G} \mathrm{Tr}\pi^{(d)} \\ &= \frac{1}{|G|} \sum_{\pi \in G} \left(\sum_{(d_1,\ldots,d_n) \in \mathbb{N}^n : d_1 + \ldots + d_n = d} \lambda_{\pi,1}^{d_1} \ldots \lambda_{\pi,n}^{d_n} \right) \end{aligned} \tag{0.10}$$

Hence,

$$\begin{aligned} \mathrm{Hilb}_{\mathbb{C}[\bar{x}]^d}(t) &= \sum_{d \geq 0} t^d \frac{1}{|G|} \sum_{\pi \in G} \left(\sum_{(d_1,\ldots,d_n) \in \mathbb{N}^n : d_1 + \ldots + d_n = d} \lambda_{\pi,1}^{d_1} \ldots \lambda_{\pi,n}^{d_n} \right) \\ &= \frac{1}{|G|} \sum_{\pi \in G} \left(\sum_{(d_1,\ldots,d_n) \in \mathbb{N}^n} (\lambda_{\pi,1} t)^{d_1} \ldots (\lambda_{\pi,n} t)^{d_n} \right) \\ &= \frac{1}{|G|} \sum_{\pi \in G} \frac{1}{(1 - \lambda_{\pi,1} t) \ldots (1 - \lambda_{\pi,n} t)} \\ &= \frac{1}{|G|} \sum_{\pi \in G} \frac{1}{\det(I - t\pi)} \end{aligned} \tag{0.11}$$

Lemma 0.4. *Let $G \subset GL(\mathbb{C}^n)$ be a finite group. Let r be be the number of reflections in G. Then the Laurent expansion of $Hilb_{\mathbb{C}[\bar{x}]^G}(t)$ at $t = 1$ begins as*

$$Hilb_{\mathbb{C}[\bar{x}]^G}(t) = \frac{1}{|G|}(1-t)^{-n} + \frac{r}{2|G|}(1-t)^{-n+1} + \mathcal{O}((1-t)^{-n+2}) \qquad (0.12)$$

Proof. By Molien's theorem 0.3, we have

$$\begin{aligned}
\mathrm{Hilb}_{\mathbb{C}[\bar{x}]^G}(t) &= \frac{1}{|G|}\sum_{\pi\in G}\frac{1}{\det(I-\pi t)} \\
&= \frac{1}{|G|}(1-t)^{-n} + \sum_{\sigma \text{ reflections}} \frac{1}{(1-t)^{n-1}(1-\det\sigma t)} \\
&= \frac{1}{|G|}(1-t)^{-n} + \frac{(1-t)^{-n+1}}{|G|}\sum_{\sigma \text{ reflections}}\frac{1}{(1-\det\sigma)} + \mathcal{O}((1-t)^{-n+2})
\end{aligned} \qquad (0.13)$$

Furthermore,

$$\begin{aligned}
2\sum_{\sigma \text{ reflections}}\frac{1}{(1-\det\sigma)} &= \sum_{\sigma \text{ reflections}}\left(\frac{1}{(1-\det\sigma)} + \frac{1}{(1-\det\sigma^{-1})}\right) \\
&= \sum_{\sigma \text{ reflections}}\frac{1-\det\sigma^{-1}+1-\det\sigma}{(1-\det\sigma)(1-\det\sigma^{-1})} \\
&= \sum_{\sigma \text{ reflections}} 1 \\
&= r
\end{aligned} \qquad (0.14)$$

This concludes the proof of the lemma.

Corollary 0.5. *Let $G \subset GL(\mathbb{C}^n)$ be a finite group and $\mathbb{C}[\bar{x}]^G = \mathbb{C}[\theta_1, \ldots, \theta_n]$ with θ_i algebraically independent and homogeneous of degrees d_i. Then,*

$$|G| = d_1 d_2 \ldots d_n \quad \textit{and} \quad \sum_{i=1}^{n}(d_i-1) = r = \sharp \textit{ of reflections} \qquad (0.15)$$

Proof. We have

$$\begin{aligned}
\mathrm{Hilb}_{\mathbb{C}[\theta_1,\ldots,\theta_n]}(t) &= \frac{1}{(1-t^{d_1})\ldots(1-t^{d_n})} \\
&= \frac{1}{(1-t)^n}\frac{1}{(1+t+\ldots+t^{d_1-1})\ldots(1+t+\ldots+t^{d_n-1})} \\
&= \frac{1}{(1-t)^n}\left(\frac{1}{d_1\ldots d_n} + \frac{\binom{d_1}{2}}{d_1^2 d_2\ldots d_n} + \ldots + \frac{\binom{d_n}{n}}{d_1\ldots d_{n-1}d_n^2}\right)(1-t) + \mathcal{O}((1-t)^{-n+2}) \\
&= \frac{1}{d_1\ldots d_n}(1-t)^{-n} + \frac{\sum_{d=1}^{n}(d_i-1)}{2d_1\ldots d_n}(1-t)^{-n+1} + \mathcal{O}((1-t)^{-n+2})
\end{aligned} \qquad (0.16)$$

From lemma 0.4, we obtain the results.

Lemma 0.6. *If $f_1, \dots, f_n \in \mathbb{C}[\bar{x}]$ are algebraically independent over G, then* $\det\left(\frac{\partial f_i}{\partial x_j}\right)_{1\le i,j\le n} \neq 0$.

Proof. We know that $\mathbb{C}[x_1, \dots, x_n]$ has transcendent degree n. Hence, $x_i, f_1, \dots, f_n$ are algebraically dependent. Let $h_i(y_0, y_1, \dots, y_n)$ be a polynomial of maximal degree such that $h_i(x_i, f_1, \dots, f_n) = 0$. For $k \in \{1, 2, \dots, n\}$, we have

$$\frac{\partial h_i(x_i, f_1, \dots, f_n)}{\partial x_k} = \sum_{j=1}^{n} \frac{\partial h_i}{\partial y_j}(x_i, f_1, \dots, f_n)\frac{\partial f_j}{\partial x_k} + \delta_{ik}\frac{\partial h_i}{\partial y_0}(x_i, f_1, \dots, f_n) = 0 \tag{0.17}$$

Since the f_i are algebraically independent, h_i has positive degree in y_0. Hence,

$$\frac{\partial h_i(y_0, y_1, \dots, y_n)}{\partial y_0} \neq 0 \tag{0.18}$$

Since it has smaller degree, we have

$$\frac{\partial h_i(x_i, f_1, \dots, f_n)}{\partial y_0} \neq 0 \tag{0.19}$$

From (0.17), we find

$$\frac{\partial h_i}{\partial y_j}(x_i, f_1, \dots, f_n)\frac{\partial f_j}{\partial x_k} = -\delta_{ik}\frac{\partial h_i}{\partial y_0}(x_i, f_1, \dots, f_n) \tag{0.20}$$

Since

$$\det\left(-\delta_{ik}\frac{\partial h_i}{\partial y_0}(x_i, f_1, \dots, f_n)\right) \neq 0 \tag{0.21}$$

we find the desired result

$$\det\left(\frac{\partial f_j}{\partial x_k}\right) \neq 0 \tag{0.22}$$

Theorem 0.7. *Let $G \subset GL(\mathbb{C}^n)$ be a finite group and $\mathbb{C}^G = \mathbb{C}[\theta_1, \dots, \theta_n]$ where θ_i are algebraically independent homogeneous of degrees d_i. Then G is a reflection group.*

Proof. Let H be the subgroup generated by the reflections of G. Using previous result, we know that $\mathbb{C}[\bar{x}]^H = \mathbb{C}[\Psi_1, \dots, \Psi_n]$, with Ψ_i algebraically independent homogeneous of degree e_i. Clearly,

$$\mathbb{C}[\bar{x}]^G \subset \mathbb{C}[\bar{x}]^H = \mathbb{C}[\Psi_1, \dots, \Psi_n] \tag{0.23}$$

Hence, θ_i is a polynomial in $\Psi_1, \dots, \Psi_n$ for all i. By lemma 0.6, we have

$$\det\left(\frac{\partial \theta_i}{\partial \Psi_j}\right) \neq 0 \tag{0.24}$$

Thus, for some permutation π, we have

$$\frac{\partial\theta_{\pi(1)}}{\partial\Psi_1}\frac{\partial\theta_{\pi(2)}}{\partial\Psi_2}\cdots\frac{\partial\theta_{\pi(n)}}{\partial\Psi_n} \neq 0 \tag{0.25}$$

This means that Ψ_i actually occurs in $\theta_{\pi(i)}$. Hence,

$$e_i = \deg \Psi_i \leq \det \theta_{\pi(i)} = d_{\pi(i)} \tag{0.26}$$

Let r be the number of reflections in G, and so also in H. By corollary 0.5, we obtain

$$r = \sum_{i=1}^{n}(d_i - 1) = \sum_{i=1}^{n}(e_i - 1) \tag{0.27}$$

Therefore, we must have $e_i = d_{\pi(i)}$ for all i. Using corollary 0.5 again, we have

$$|G| = d_1 \ldots d_n = e_1 \ldots e_n = |H| \tag{0.28}$$

We deduce $H = G$.