MATH-F-427 students

# Coxeter groups

## Course notes

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ULB

## Contents

**Definition 0.1.** *Let $A$ be an algebra over $\mathbb{C}$. Then, the **transcendence degree** of $A$ over $\mathbb{C}$ is the maximal number of algebraically independent elements of $A$. Recall that a subset $\{a_1, ..., a_n\} \subset A$ is **algebraically independent** if and only if there exists no polynomial $P \in \mathbb{C}[Y_1, ..., Y_n] \backslash \{0\}$ satisfying that $P(a_1, ..., a_n) = 0$.*

**Proposition 0.2.** *Let $G \subset GL(\mathbb{C}^n)$ then, the transcendence degree of $\mathbb{C}[x]^G$ over $G$ is $n$.*

*Proof.* First of all, let us remember that $\mathbb{C}[x]^G$ is a sub algebra of $\mathbb{C}[x]$. In particular, this implies that the transcendence degree $\mathbb{C}[x]^G$ over $\mathbb{C}$ is at most $n$. Indeed, if $\{a_1, ..., a_{n+1}\} \subset \mathbb{C}[x]^G$ were algebraically independent over $\mathbb{C}$, this would imply that $\{a_1, ..., a_{n+1}\}$ is an algebraically independent set of $\mathbb{C}[x]$ and therefore contradicts the fact that the transcendence degree of $\mathbb{C}[x]$ is $n$. On the other hand, let us remark every of the $x_i$ is algebraic over $\mathbb{C}[x]^G$. Indeed, it is not hard to realise that for every $i \in \{1, ..., n\}$, the polynomial :

$$P_i(t) \;=\; \prod_{A \in G} (Ax_i \;-\; t) \tag{0.1}$$

is in $\mathbb{C}[x]^G$. Furthermore, since $\mathrm{Id}_{\mathbb{C}^n} \in G$, we know that $P_i(x_i) = 0$. In particular, this proves that the $x_i$ are algebraic over $\mathbb{C}[x]^G$. However, since those are algebraically independent elements in $C[x]$, this is only possible if there exists at least $n$ algebraically independent elements of $\mathbb{C}[x]$. This proves that the transcendence degree of $\mathbb{C}[x]$ is at least $n$ and therefore, as a consequence of our previous discussion, this proves that it is exactly $n$.

**Definition 0.3.** *An element $A \in GL(\mathbb{C}^n)$ is a **pseudo-reflection** if $dim(Ker(A) - Id_{\mathbb{C}^n}) = n - 1$ and $A$ is of finite order in $G$.*

**Definition 0.4.** *A finite subgroup $G$ of $GL(\mathbb{C}^n)$ is a **complex reflection group** if it is generated by reflections.*

*Example 0.5.* The dihedral group $D^6$ can be seen as a group generated by reflection when it is considered as a subgroup of $\mathrm{GL}(\mathbb{C}^2)$. In this case, it is nothing more than the group of symmetries of a regular hexagon in the plane. However, when we consider $D^6$ as the subgroup of $\mathrm{GL}(\mathbb{C}^3)$ generated by :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

can not be generated by reflections. In particular, it is therefore not a complex reflection group in $\mathrm{GL}(\mathbb{C}^3)$. This example is of special interest since it shows that the definition depends on the dimension considered.

Now, let $\sigma \in \mathrm{GL}(\mathbb{C}^n)$ be reflection and let $H_\sigma = \mathrm{Ker}(\sigma - \mathrm{Id}_{\mathbb{C}^n})$. Then, we know that $H_\sigma$ is the solution of a linear equation given by a polynomial

$L_\sigma \in \mathbb{C}[x]$. Moreover, this linear polynomial is unique up to multiplication by a non-zero complex number. The following lemma gives describe an interesting property of this polynomial.

**Lemma 0.6.** *For every function $f \in \mathbb{C}[x]$ and for every reflection $\sigma \in GL(\mathbb{C}^n)$ we the polynomial $L_\sigma$ divides the polynomial $\sigma f - f$.*

*Proof.* Let $v \in H_\sigma$. Then, by definition $\sigma(v) = v$. In particular, this implies for every $f \in \mathbb{C}[x]$ that $\sigma f(v) = f(v)$ and therefore that $\big(\sigma f - f\big)(v) = 0$ for every $v \in H_\sigma$. Furthermore, since $L_\sigma$ is of degree one, it is irreducible. In particular, the Nullstellensatz theorem implies that $L_\sigma$ divides $\sigma f - f$ since $H_\pi$ is the ideal generated by the irreducible polynomial $L_\sigma$.

Now, let us define $I_G$ as the ideal of $\mathbb{C}[x]$ generated by homogeneous invariant polynomial of positive degree.

**Proposition 0.7.** *Let $G$ be a finite reflection group, let $h_1, ..., h_m$ be homogeneous polynomial polynomial of $\mathbb{C}[x]$, let $g_1, ..., g_m \in \mathbb{C}[x]^G$ be homogeneous invariant polynomials and let us suppose that :*

$$g_1 h_1 \ + \ g_2 h_2 \ + \ \cdots \ + \ g_m h_m \ = \ 0 \ \in \mathbb{C}[x] \quad . \tag{0.2}$$

*Then, either $h_1 \in I_G$ or $g_1$ belongs to the ideal of $\mathbb{C}[x]$ generated by $\{g_2, ..., g_m\}$.*

*Proof.* The proof is done by induction on the degree of $h_1$.

- When the degree of $h_1 = 0$ we make two cases. If $h_1 = 0$, we know that $h_1 \in I_G$ and the claim follows. On the other hand, when $h_1$ is a non zero constant, Equation 0.2 implies that $g_1$ is in the ideal generated by $\{g_2, ..., g_m\}$.
- Now, let us suppose that the degree of $h_1$ is bigger than 1 and that the claim is true for every $h_1'$ less than the degree of $h_1$. Now, let us suppose that $g_1$ is not in the ideal generated by $\{g_2, ..., g_m\}$. Then, for every reflection $\sigma$ and since $g_i \in \mathbb{C}[x]^G$ for every $i = 1, ..., m$, we know that :

$$\begin{aligned} 0 \ = \ \sigma(0) \ &= \ \sigma \left( \ \sum_{i=1}^m \ g_i h_i \ \right) \\ &= \ \sum_{i=1}^m \ g_i \ \sigma h_i. \end{aligned} \tag{0.3}$$

  On the other hand, as a consequence of previous lemma, we know that for every $i = 1, ..., m$ there exists a polynomial $\tilde{h}_i$ such that :

$$\sigma(h_i) \ = \ h_i \ + \ L_\sigma \ \tilde{h}_i. \tag{0.4}$$

  Further more, since $h_i$, $\sigma h_i$ and $L_\sigma$ are homogeneous, this polynomial $\tilde{h}_\sigma$ is also homogeneous in $\mathbb{C}[x]$. Furthermore, the degree of this polynomial $\tilde{h}_i$

is by definition of degree of $\deg(h_i)-1$ since $L_\sigma$ is of degree 1.In particular, we obtain that :

$$0 \;=\; \sum_{i=1}^{m} g_i(h_i + L\sigma\tilde{h}_i) \;=\; \sum_{i=1}^{m} g_i h_i \;+\; L_\sigma \sum_{i=1}^{m} g_i\tilde{h}_i \;=\; L_\sigma \sum_{i=1}^{m} g_i\tilde{h}_i \;. \tag{0.5}$$

In particular, using the induction hypotheses, this implies that $\tilde{h}_1 \in I_G$ and therefore that $\sigma h_1 - h_1 \;=\; L_\sigma \tilde{h}_1 \in I_G$. However, we know that $G$ is generated by reflection. In particular, this implies that for every $\pi \in G$ there exists reflections $\sigma_1, ..., \sigma_k$ such that $\pi = \sigma_1...\sigma_k$. Now, using a telescopic sum, this implies that :

$$\begin{aligned} \pi h_1 \;-\; h_1 \;&=\; \sigma_1...\sigma_k h_1 \;-\; h_1 \\ &=\; \sum_{i=1}^{k-1} \sigma_1...\sigma_{i+1} h_1 \;-\; \sigma_1...\sigma_i h_1 \\ &=\; \sum_{i=1}^{k-1} \sigma_1...\sigma_i \;(\sigma_{i+1}h_1 - h_1) \;\in\; I_G \;. \end{aligned} \tag{0.6}$$

In particular, since :

$$\frac{1}{|G|} \sum_{\pi\in G} \;(\pi h_1 - h_1) \;=\; R_G(h_1) \;-\; h_1 \tag{0.7}$$

and since $R_G(h_1) \in I_G$ this implies that $h_1 \in I_G$.

**Theorem 0.8 (Shepard - Todd - Chevalley).** *Let $G \subset GL(\mathbb{C}^n)$ be a finite group. Then, $\mathbb{C}[x]^G$ is generated by $n$ algebraically independent homogeneous invariant if and only if $G$ is a reflection group.*

*Proof.* ($\Leftarrow$) Using the Hilbert basis theorem, we know that $I_G$ is finitely generated. In particular, since each of those generating polynomials is invariant, it can be generated by finitely many homogeneous invariant polynomials and we obtain that :

$$I_G \;=\; <\; f_1, ..., f_m \;> \tag{0.8}$$

with $f_1, ..., f_m$ homogeneous invariant polynomials. Let us remark that this implies that $\mathbb{C}[x]^G = \mathbb{C}[f_1, ..., f_m]$. To understand why, let us suppose the opposite. Let $h \in \mathbb{C}[x]^G \backslash [f_1, ..., f_m]$ be an homogeneous polynomial of minimal degree for this property. Then, $h \;=\; \sum_{i=1}^{m} g_i f_i$ for some homogeneous polynomials $g_i$. In particular, because of the $G$ invariance of $h$ this implies that :

$$h \;=\; R_G(h) \;=\; \sum_{i=1}^{m} R_G(g_i)\; h_i \;. \tag{0.9}$$

However, $R_G(g_i)$ is an homogeneous polynomial of degree smaller than $h$. In particular, this implies by definition of $h$ that $R_G(g_i) \in \mathbb{C}[f_1, ..., f_m]$ and we conclude that $h \in \mathbb{C}[f_1, ..., f_m]$. This leads to some contradiction and proves that $\mathbb{C}[x]^G = \mathbb{C}[f_1, ..., f_m]$.

Now, let $m$ be some minimal positive integer satisfying the property that :

$$I_G \ = \ < \ f_1, ..., f_m \ > \tag{0.10}$$

with $f_1, ..., f_m$ homogeneous invariant polynomials. We want to show that $m = n$ or equivalently that $\{f_1, ..., f_m\}$ are algebraically independent since the transcendence degree of $\mathbb{C}[x]^G$ is $n$. In order to prove this independence, let us reason by contradiction. Let us consider a polynomial $g(Y_1, ..., Y_m) \in \mathbb{C}[Y_1, ..., Y_m] \backslash \{0\}$ be such that :

$$g(f_1, ..., f_m) \ = \ 0 \tag{0.11}$$

and assume that $g$ has minimal degree and that every monomials of $g(f_1, ..., f_m)$ before cancellation have the same degree.

For every $i = 1, ..., m$ let us consider the polynomial :

$$g_i \ = \ \left(\frac{\partial g}{\partial Y_i}\right)(f_1, ..., f_m) \ \in \ \mathbb{C}[x]^G \ . \tag{0.12}$$

We know that each of the $g_i$ is either 0 or homogeneous of degree $d - \deg(f_i)$. Since $g(Y_1, ..., Y_m)$ is not constant, there exists some index $i$ such that $\frac{\partial g}{\partial Y_i} \neq 0$ and therefore, by minimality assumption, $g_i \neq 0$. Now, let $I =< g_1, ..., g_m >$. Up to renaming those polynomials, we can assume that $I =< g_1, ..., g_k >$, that no proper subset of $\{g_1, ..., g_k\}$ generates $I$ and that $k$ is minimal for this property. Then, for every $i = k+1, ..., m$ there must exists homogeneous polynomials $h_{ij}$ equal to 0 or of degree $\deg(g_i) - \deg(g_j) = \deg(f_i) - \deg(f_j)$ such that :

$$g_i \ = \ \sum_{i=1}^{k} \ g_{ij} \ g_j \ . \tag{0.13}$$

In particular, we see that :

$$\begin{aligned} 0 \ &= \ \frac{\partial}{\partial x_s} g(f_1, ..., f_m) \ = \ \sum_{i=1}^{m} \ g_i \ \frac{\partial f_i}{\partial x_s} \\ &= \ \sum_{i=1}^{k} \ g_i \ \frac{\partial f_i}{\partial x_s} \ + \ \sum_{i=1}^{m} \ \left( \sum_{j=1}^{m} \ h_{ij} g_j \right) \ \frac{\partial f_i}{\partial x_s} \\ &= \ \sum_{i=1}^{k} \ g_i \ \left( \frac{\partial f_i}{\partial x_s} \ + \ \sum_{j=1}^{m} \ h_{ij} \frac{\partial f_i}{\partial x_s} \right) \ . \end{aligned} \tag{0.14}$$

As $g_1 \notin < g_2, ..., g_m >$, the last proposition implies that :

$$\frac{\partial f_1}{\partial x_s} \ + \ \sum_{j=k+1}^{m} h_{ij} \ \frac{\partial f_i}{\partial x_j} \ \in \ I_G \ . \tag{0.15}$$

In particular, this implies that :

$$\begin{aligned} \tilde{f} \ &= \ \sum_{s=1}^{n} x_s \left( \frac{\partial f_1}{\partial x_s} \ + \ \sum_{j=k+1}^{m} h_{ij} \ \frac{\partial f_i}{\partial x_j} \right) \\ &= \ \deg(f_1) f_1 \ + \ \sum_{j=1}^{m} \deg(f_i) \ h_j \ f_j \\ &\in I_G \ < x_1, ..., x_n > \ \subset \ < x_1 f_1, ..., x_n f_m > \ + \ < f_2, ..., f_m > . \end{aligned} \tag{0.16}$$

Ib particular, since every of the polynomial $x_1 f_1, ..., x_n f_m$ is of degree strictly bigger than $\tilde{f}$, this implies that $\tilde{f} \in < f_2, ..., f_m >$. In particular, $f_1 \in < f_2, ..., f_m >$ which leads to some contradiction with the minimality of $m$.