MATH-F-427 students

Coxeter groups

Course notes

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ULB

Contents

Proposition 0.1. *Take $\varphi \in GL(V)$ with $\langle \varphi(u), \varphi(v)\rangle = \langle u, v\rangle$, $\forall u, v \in V$. Then, we have the following properties:*

- *1) $\varphi\sigma_\alpha\varphi^{-1} = \sigma_{\varphi(\alpha)}$,*
- *2) $\sigma_\alpha = \sigma_\beta$ if and only if $\alpha = c\beta$ with $c \neq 0$,*
- *3) σ_α and σ_β commute if and only if $\langle \alpha, \beta\rangle = 0$ or $\alpha = c\beta$ with $c \neq 0$.*

Proof. 1) This is a simple computation let to the reader.

2) It is obvious to see that the condition is sufficient. Let us show that it is necessary. If $\sigma_\alpha = \sigma_\beta$, then

$$\begin{aligned} \sigma_\alpha(v) &= v - 2\frac{\langle v, \alpha\rangle\alpha}{\langle \alpha, \alpha\rangle} \\ &= v - 2\frac{\langle v, \alpha\rangle}{\langle \alpha, \alpha\rangle} \\ &= v - 2\frac{\langle v, \beta\rangle}{\langle \beta, \beta\rangle} \\ &= \sigma_\beta(v) \end{aligned} \tag{0.1}$$

$\forall v \in V$. This implies that

$$\frac{\langle v, \alpha\rangle}{\langle \alpha, \alpha\rangle} = \frac{\langle v, \beta\rangle}{\langle \beta, \beta\rangle} \tag{0.2}$$

$\forall v \in V$, which leads to $\alpha = c\beta$, with $c \neq 0$.

3) We have

$$(\sigma_\alpha\sigma_\beta - \sigma_\beta\sigma_\alpha)v = (\langle v, \beta\rangle\alpha - \langle v, \alpha\rangle\beta)\frac{4\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle\langle \beta, \beta\rangle} \tag{0.3}$$

$\forall v \in V$. The right-hand side gives 0 if and only if $\langle \alpha, \beta\rangle = 0$ or $\alpha = c\beta$ with $c \neq 0$.

Proposition 0.2. *Let (Φ, Δ) be a root system. We have the following properties:*

- *1) If $\beta \in \Phi$, then $\langle \beta, \beta\rangle > 0$, $\sigma_\beta \in W$, and $-\beta \in \Phi$.*
- *2) If $\alpha, \beta \in \Delta$, $\alpha \neq \beta$, then $\langle \alpha, \beta\rangle \leq 0$.*

Proof. 1) Let $\beta \in \Phi$, $\beta = w\alpha$, where $w \in W$ and $\alpha \in \Delta$ ("$w\alpha$" designates the action of w on α). We have

$$\langle \beta, \beta\rangle = \langle w\alpha, w\alpha\rangle = \langle \alpha, \alpha\rangle > 0 \tag{0.4}$$

Furthermore,

$$\sigma_\beta = \sigma_{w\alpha} = w\sigma_\alpha w^{-1} \in W \tag{0.5}$$

and

$$\sigma_\beta(\beta) = -\beta \in \Phi \tag{0.6}$$

2) We have

$$\sigma_\alpha(\beta) = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha \tag{0.7}$$

Since $\langle \alpha, \alpha \rangle \leq 0$, using the point R.5 of the definition of root system, we obtain $\langle \alpha, \beta \rangle \leq 0$.

Lemma 0.3. *If $\alpha \in \Delta$, then σ_α permutes with $\Phi^+ \backslash \{\alpha\}$.*

Proof. Let $\beta \in \Phi^+ \backslash \{\alpha\}$. We have

$$\sigma_\alpha(\beta) = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha \in \Phi^+ \backslash \{\alpha\} \tag{0.8}$$

Theorem 0.4. *If (Φ, Δ) is a root system, then (W, S) with $S = \{\sigma_\alpha | \alpha \in \Delta\}$ is a Coxeter system.*

This theorem follows directly using Matsumoto theorem and the following result:

Theorem 0.5. *Let $\alpha \in \Delta$ and $w = \sigma_1\sigma_2 \dots \sigma_n$ a reduced word, where $\sigma_i \in S$. The following assertions are equivalent:*

- *1) $w\alpha < 0$,*
- *2) $l(w\sigma_\alpha) < l(w)$,*
- *3) $w\sigma_\alpha = \sigma_1 \dots \hat{\sigma}_i \dots \sigma_n$.*

Proof. 1) $\Rightarrow$ 3). We have $\alpha > 0$, $\sigma_n\alpha > 0$, $\sigma_{n-1}\sigma_n\alpha > 0$, …, $\sigma_1 \dots \sigma_n\alpha < 0$. There is a $i \in \{1, \dots, n\}$ such that $\sigma_{i+1} \dots \sigma_n\alpha > 0$ and $\sigma_i\sigma_{i+1} \dots \sigma_n\alpha < 0$. Using the lemma, we have that $\sigma_{i+1} \dots \sigma_n\alpha = \alpha_i$ is a simple root, and, writing

$$\alpha = \underbrace{\sigma_n\sigma_{n-1} \dots \sigma_{i+1}}_{u}\alpha_i \tag{0.9}$$

we have $\sigma_\alpha = \sigma_{u\alpha_i} = u\sigma_{\alpha_i}u^{-1}$. Hence,

$$w\sigma_\alpha = \sigma_1 \dots \sigma_n\sigma_n \dots \sigma_{i+1}\sigma_i\sigma_{i+1} \dots \sigma_n = \sigma_1 \dots \hat{\sigma}_i \dots \sigma_n \tag{0.10}$$

3) $\Rightarrow$ 2). This implication is obvious.

2) $\Rightarrow$ 1). We prove the contrapositive. By the point R.5 of the definition of root system, we have $w\alpha > 0$ and so $w(-\alpha) = w\sigma_\alpha(\alpha) < 0$. From 1) $\Rightarrow$ 3) $\Rightarrow$ 2), we get

$$\ell(w) = \ell(w\sigma_\alpha\sigma_\alpha) < \ell(w\sigma_\alpha) \tag{0.11}$$

which achieves the proof.

Now, we are going to show that a root system can be associated to every Coxeter group. First, recall that given (W, S) a Coxeter system with $S = \{s_1, \dots, s_n\}$, and V a $\mathbb{R}$-vector space with basis $\{\alpha_1, \dots, \alpha_n\}$, we can define a symmetric bilinear form as

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} -\cos\left(\frac{\pi}{m_{ij}}\right) & \text{if } m_{ij} < +\infty \\ -1 & \text{if } m_{ij} = +\infty \end{cases} \tag{0.12}$$

Writing $\sigma_i = \sigma_{\alpha_i}$, we showed that the map

$$W \mapsto GL(V), s_i \to \sigma_i \tag{0.13}$$

is a homomorphism. Let us define

$$\begin{aligned} \Delta &:= \{\alpha_1, \ldots, \alpha_n\} \\ \Phi &:= \{w\alpha_i | w \in W, \alpha_i \in \Delta\} \end{aligned} \tag{0.14}$$

Theorem 0.6. (Φ, Δ) *is a root system.*

The points R.1, R.2, R.3 and R.4 of the definition of a root system are clearly satisfied. The point R.5 is a consequence of the following theorem:

Theorem 0.7. *Let* $w \in W$ *and* $s_i \in S$*. We have:*

- *1)* $\ell(ws_i) > \ell(w) \Rightarrow w\alpha_i > 0$.
- *2)* $\ell(ws_i) < \ell(w) \Rightarrow w\alpha_i < 0$.

Proof. First, we show that 1) $\Rightarrow$ 2). We have $\ell(ws_i) < \ell(w) = \ell(ws_is_i)$. Hence $0 < ws_i\alpha_i = w(-\alpha_i) = -w\alpha_i$, which leads to $w\alpha_i < 0$.

Now, we prove 1). First, it can be shown for the dihedral group (exercise[1]). We now consider the general case. Let $s_i \in S$, $w \in W$, such that $\ell(ws_i) > \ell(w)$. We proceed by induction on $\ell(w)$. For $\ell(w) = 0$, this is obvious. For $\ell(w) \geq 1$, let $s \in S$ be such that $\ell(ws) < \ell(w)$. Let $J := \{s_i, s\} \subset S$. We write

$$w = w^J w_J, \quad w^J \in W^J, w_J \in W_J \tag{0.15}$$

with $\ell(w) = \ell(w^J) + \ell(w_J)$, and w^J is the minimum of wW_J. Note that $\ell(w_J) \geq 1$: if $w = w^J$, we have $\ell(w^J s) < \ell(w^J)$, which leads to a contradiction since W^J is the minimum of $w^J W_J$.

Claim: $w_J\alpha_i > 0$.

Let us prove this claim. We have $ws_i = w^J w_J s_i$ and $wJs_i \in W$. Hence, using the hypothesis, we obtain

$$\ell(w) + 1 = \ell(ws_i) = \ell(w^J) + \ell(w_J s_i) \tag{0.16}$$

This leads to $\ell(w_J s_i) = \ell(w_J) + 1 > \ell(w)$. Therefore, $w_J\alpha_i > 0$, which proves the claim.

Now, we have

$$w_J\alpha_i = a\alpha_i + b\alpha_s \tag{0.17}$$

with $a, b \geq 0$ and

[1] To show the dihedral case, drawing a picture may be useful.

$$w\alpha_i = w^J(a\alpha_i + b\alpha_s) = aw^J\alpha_i + bw^J\alpha_s \tag{0.18}$$

Since $\ell(w_J) \geq 1$, we have $\ell(w^J) < \ell(w)$. Furthermore, since w^J is the minimum of wW_J, we have

$$\ell(w^J s) > \ell(w^J) \quad \text{and} \quad \ell(w^J s_i) > \ell(w^J) \tag{0.19}$$

So, by induction, $w^J\alpha_s > 0$, and $w^J\alpha_i > 0$, which leads to $w\alpha_i > 0$.

Corollary 0.8. *We have $\Phi = \Phi^+ \sqcup \Phi^-$.*

Corollary 0.9. *The geometric representation $W \mapsto GL(V), s_i \to \sigma_i$ is faithful.*

Proof. Let $w \in W$ such that $\sigma(w) = id$. We have $w\alpha_i = \alpha_i > 0$, $\forall i$. Therefore, using theorem 0.7, we obtain $\ell(ws_i) > \ell(w)$ $\forall i$, which implies $w = e$.

Proposition 0.10. *For $w \in W$, $\ell(w) =$number of positive roots that are sent to negative roots by w.*

Proof. Let us proceed by induction on $\ell(w)$. For $\ell(w)0$, this is obvious. The case $\ell(w) = 1$ also holds using the above lemma. Now, take $\ell(w) > 1$, and let $s_i \in S$ be such that $\ell(xs_i) < \ell(w)$. By induction, there are $\ell(ws_i)$ many $\alpha > 0$ such that $ws_i\alpha_i < 0$. By the above lemma, $s_i\alpha > 0$, unless $\alpha = \alpha_i$. On the other hand, using theorem 0.7, we have $ws_i\alpha_i = -w\alpha_i > 0$. Hence, $\alpha \neq \alpha_i$. So we found $\ell(ws_i)$ many $s_i\alpha > 0$ such that $ws_i\alpha < 0$. Adding α_i, we get the right number. If $\beta > 0$, $\beta \neq \alpha_i$, such that $w\beta < 0$, $(ws_i)(s_i\beta) < 0$ and $s_i\beta > 0$. Hence, $s_i\beta$ is one of the many α, so β is one of many $s_i\alpha$.

Remark 0.11. Let (W, S) a Coxeter system and $T = \{wsw^{-1} | s \in S, w \in W\}$. Taking (0.13) into account and noting that $w\sigma_{\alpha_i}w^{-1} = \sigma_{w\alpha_i}$, we get the following correspondence:

$$\{\text{Algebraic reflexions } t \in T\} \leftrightarrow \{\text{Geometric relfection } \sigma_\beta\} \leftrightarrow \{\text{Positive roots}\} \tag{0.20}$$

Proposition 0.12. *Let $w \in W$, $t \in T$ and α the corresponding positive root. We have:*

- *1) $\ell(wt) > \ell(w) \Rightarrow w\alpha > 0$,*
- *2) $\ell(wt) < \ell(w) \Rightarrow w\alpha < 0$.*

Proof. First, we show that 2) $\Rightarrow$ 1). We have $\ell(wt) < \ell(w) = \ell(wtt)$. Hence $0 < xt\alpha = w(\alpha)$, and so $w\alpha < 0$.

Now we prove 2). Suppose that $\ell(wt) < \ell(w)$. Let $w = s_1 \dots s_k$ reduced, with $s_i \in S$. By the exchange property, we have

$$t = \underbrace{s_k s_{k-1} \dots s_{i+1}}_{u} s_i \underbrace{s_{i+1} \dots s_k}_{u^{-1}} \tag{0.21}$$

Therefore, $\alpha = u\alpha_i$, where we use the correspondence $\alpha_i \leftrightarrow s_i$. Now, $w\alpha = uu\alpha_i = s_1 s_2 \dots s_i\alpha_i$. But $\ell(s_1 s_2 \dots s_i s_i) < \ell(s_1 s_2 \dots s_{i-1})$. By theorem 0.7, we deduce $s_1 s_2 \dots s_i\alpha_i < 0$.