MATH-F-427 students

Coxeter groups

Course notes

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ULB

Contents

Part I

Coxeter groups

Theorem 0.1. *The application π, defined on the set of generators S of the coxeter system (W,S), extend uniquely to an injective homomorphism :*

$$\pi \ : \ W \ \rightarrow \ S_T^B \tag{0.1}$$

Proof. First of all, we need to show that the extension of π is well defined. It was clear, due to the definition of π on S that for every $s \in S$, the application $\pi_s \in S_T^B$. Indeed, for every $t \in T$ we had that $\pi_s(t) \in T \cup \overline{T}$ and π_s defined a bijection on $T \cup \overline{T}$. In order to check that its extension on all of W is well defined we need to check 2 things. First, we need to check that $\forall w \in W$ the application $\pi_w \in S_T^B$. However, since we extended π from S to W to be a group morphism, we know that π_w is by definition the composition of π_s for some $s \in S$ and thus is an element of S_T^B. Secondly, we need to check that this application π_w does not depend on the writing of $w \in W$. In order to show this, let us take some element $t \in T$ and let $w = s_1 s_2 .. s_k$ for some $s_i \in S$ (this is the form of every element of W since $s_i = s_i^{-1}$ for all i). Since, we want π to be a homomorphism, we have that :

$$\begin{aligned}
\pi_w(t) \ &= \ \pi_{s_1} \circ \pi_{s_2} \circ ... \circ \pi_{s_k}(t) \\
&= \ \pi_{s_1} \circ \pi_{s_2} \circ ... \circ \pi_{s_{k-1}}(\pm s_k t s_k) \\
& \quad (\text{with } - \text{ iff } s_k t s_k = s_k \iff t = s_k) \\
&= \ \pi_{s_1} \circ \pi_{s_2} \circ ... \circ \pi_{s_{k-2}}(\pm \pm \, s_{k-1} s_k t s_k s_{k-1}) \\
& \quad (\text{with } - \text{ iff } s_{k-1} s_k t s_k s_{k-1} = s_{k-1} \iff t = s_k s_{k-1} s_k) \\
&= \ \pi_{s_1} \circ \pi_{s_2} \circ ... \circ \pi_{s_{k-3}}(\pm \pm \pm s_{k-2} s_{k-1} s_k t s_k s_{k-1} s_{k-2}) \\
& \quad (\text{with } - \text{ iff } s_{k-1} s_k t s_k s_{k-1} = s_{k-1} s_{k-2} \iff t = s_k s_{k-1} s_{k-2} s_{k-1} s_k) \\
&\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \quad \vdots \\
&= \ \pm \pm \cdots \pm \, s_1 s_2 ... s_k t s_k s_{k-1} ... s_1 \\
& \quad (\text{with } - \text{ iff } s_1 ... s_{k-1} s_k t s_k s_{k-1} ... s_1 = s_1 \iff t = s_k ... s_2 s_1 s_2 ... s_k) \\
&= \operatorname{sgn}_w(t) \ w t w^{-1}
\end{aligned} \tag{0.2}$$

Where the function $\operatorname{sgn}_w(t)$ is a sign function counting the number of indices $l \in \{1, 2, ...k\}$ such that $t = s_k ... s_{l-1} s_l s_{l-1} ... s_k$. Namely :

$$\operatorname{sgn}_w(t) \ = \ (-1)^{\#\{1 \leq l \leq k \ : \ t = s_k ... s_{l-1} s_l s_{l-1} ... s_k\}} \tag{0.3}$$

As we will show just after this sign function does not depend on the writing of $w \in W$ in the coxeter system (W,S). But first, let us get some intuition about what this sign function is counting, by looking to the case of S_n : aaaaaaaaaaa

We are now going to use equation 0.2 to prove that the sign function does not depend on the writing of $w \in W$ in the Coxeter system (W,S) and therefore that π is a well defined homomorphism. In order to show this, it suffices to show that every relations we had in (W, S) are satisfied by their image in S_T^B. In other words, we want to show that taking two elements $s, s' \in S$ we have that :

$$(\pi_s \circ \pi_{s'})^{m(s,s')} \;=\; \mathrm{Id}_{S_T^B} \tag{0.4}$$

Since $(ss')^{-1} = s's$, equation 0.2 gives us for every $t \in T$:

$$(\pi_s \circ \pi_{s'})^{m(s,s')}(t) \;=\; \pm\; (ss')^{m(s,s')} t (s's)^{m(s,s')} \;=\; \pm\; ete \;=\; \pm\; t \tag{0.5}$$

The sign must be $+$ as here, $w = (ss')^{m(s,s')}$ and therefore we look at :

$$\#\{1 \leq l \leq m(s,s') \;:\; t = \underbrace{s'ss'...s'ss'}_{2l-1 \text{ characters}} \;\} \tag{0.6}$$

which is even since for every $l \leq m(s,s')/2$ we have :

- if $m(s,s')$ is even :

$$t \;=\; \underbrace{s'ss'...s'ss'}_{2l-1 \text{ characters}} \;=\; \underbrace{s'ss'...s'ss'}_{2l-1+m(s,s') \text{ characters}} \;=\; \underbrace{s'ss'...s'ss'}_{2(l+m(s,s')/2)-1 \text{ characters}} \tag{0.7}$$

- if $m(s,s')$ is odd :

$$t \;=\; \underbrace{s'ss'...s'ss'}_{2l-1 \text{ characters}} \;=\; \underbrace{s'ss'...s'ss'}_{2l-1+m(s,s') \text{ characters}} \;=\; \underbrace{s'ss'...s'ss'}_{2((m(s,s')-1)/2+l)+1 \text{ characters}} \tag{0.8}$$

In particular, this implies that if one index is counted below $m(s,s')/2$ then there exists an other index counted strictly bigger than $m(s,s')/2$ and vis versa. Thus the set must be even and the sign must be $+$. In particular, this proves equation (0.4) and π is a well defined morphism.

It last to show that the extension of π is injective. Let $u, v \in W$ be such that $\pi_u = \pi_v$ then, we have that :

$$\pi_{uv^{-1}} \;=\; \pi_u \circ \pi_{v^{-1}} \;=\; \mathrm{Id}_{S_T^B} \;=\; \pi_e \tag{0.9}$$

Thus, in order to prove the injectivity of π we just need to show that if $w \in W$ is such that $\pi_w = \pi_e$ then $w = e$. Now, let's take $w \in W$ such that $\pi_w = \pi_e$ and let us suppose absurdly that $w \neq e$ then, there exists $k \geq 1$ such that $w = s_1...s_k$ is the shorter way possible to write $w \in W$ (meaning that k is the smallest possible) then :

$$\begin{aligned} s_k \;=\; \pi_e(s_k) \;=\; \pi_w(s_k) \;&=\; \mathrm{sgn}_w(s_k)\; s_1...s_{k-1}s_ks_ks_ks_{k-1}...s_1 \\ &=\; \mathrm{sgn}_w(s_k)\; s_1...s_{k-1}s_ks_{k-1}...s_1 \end{aligned} \tag{0.10}$$

On the other hand, $\mathrm{sgn}_w(s_k) = -1$ because :

$$\{1 \leq l \leq k \ : \ t = s_k ... s_{l-1} s_l s_{l-1} ... s_k\} \ = \ \{k\} \tag{0.11}$$

Indeed, for $l = k$ we have $s_k = s_k$. But if $l \neq k$ and if we had :

$$s_k = \ s_k .. s_l .. s_k \tag{0.12}$$

Then we would have :

$$s_{l-1} ... s_k s_k \ = \ s_l ... s_k \tag{0.13}$$

And therefore we would have a contradiction with the minimality of k since :

$$\begin{aligned} w &= s_1 ... s_l s_{l-1} s_l ... s_k \\ &= \ s_1 ... s_{l-1} s_{l-1} ... s_k s_k \\ &= \ s_1 .. s_{l-2} s_{l+1} ... s_{k-1} \\ &= \ s_1 ... s_{l-2} s_{l+1} ... s_{k-1} \end{aligned} \tag{0.14}$$

which is a shorter way to write w. Therefore, we have that $\mathrm{sgn}_w(s_k) = -1$ and thus equation 0.10 gives :

$$s_k \ = \ - \ s_1 ... s_{k-1} s_k s_{k-1} ... s_1 \tag{0.15}$$

Which is a contradiction due to the presence of a sign. □

We are now going to define the notions of **parity** and **length** of an element in a Coxeter group.

Definition 0.2. *Let (W, S) be a Coxeter system, and let $w \in W$, then we say that $w = s_1 ... s_k$ $(s_l \in S)$ is :*

- ***even** when k is even.*
- ***odd** when k is odd.*

*This is what we call the **parity** of $w \in W$.*

Remark 0.3. As every relations in a Coxeter group involve an even number of $s \in S$ we see that the parity of an element $w \in W$ does not depend on its writing in W.

The set of even elements of a Coxeter system (W, S) is a subgroup of W called the **alternating** subgroup.

Remark 0.4. When S_n is seen as a Coxeter group with $S = \{s_1 ... s_{n-1}\}$ and the Coxeter matrix $m(s_i, s_{i+1}) = 3$ and $m(s, s') = 2$ for every other couple of the type $(s, s') \neq (s, s)$, it is quite easy to remark that the two notions of alternating group does coincide and therefore that this appellation is well chosen.

Definition 0.5. *Let (W, S) be a Coxeter system, the **length** $l(w)$ of an element $w \in W$ is defined as the smallest integer $k \in \mathbb{N}$ such that there exists simple reflections $s_1, ..., s_k \in S$ satisfying $w = s_1 ... s_k$.*

The purpose of what follows is to prove the following theorem :

Theorem 0.6. *Let (W, S) be a Coxeter system, and let $w \in W$ then :*

$$l(w) \;=\; \#\{t \in T \;:\; sgn_{w^{-1}}(t) = -1\} \tag{0.16}$$

Example 0.7. In the case where $W = S_n$ with the common representation, $l(w)$ is exactly the number of inversion of w^{-1} which is exactly the same as the number of inversion of w itself.

Before proving this thorem, we focus our attention on some lemma :

Lemma 0.8. *Let (W, S) be a Coxeter system and let $w \in W$, $t \in T$ then :*

$$sgn_{w^{-1}}(t) = -1 \quad\iff\quad l(tw) \;<\; l(w) \tag{0.17}$$

Proof. Let's suppose that $\mathrm{sgn}_{w^{-1}}(t) = -1$ and let $w = s_1...s_k$ with $k = l(w)$ then $w^{-1} = s_k...s_1$. We know that there must exists some $1 \leq l \leq k$ such that $t = s_1...s_l..s_1$ but then :

$$\begin{aligned} tw &= s_1 s_2 ... s_l ... s_1 \; s_1 s_2 ... s_l s_{l+1} ... s_k \\ &= s_1 s_2 ... s_{l-1} s_{l+1} ... s_k \\ &= s_1 s_2 ... \hat{s}_l ... s_k \end{aligned} \tag{0.18}$$

From which we conclude that $l(tw) \leq k - 1 \;<\; k = l(w)$ and the first implication is proved.

Conversely, let's suppose that $l(tw) < l(w)$ then, as $tt = e$ we have that :

$$l(tw) \;<\; l(ttw) \;\Rightarrow l(ttw) \not< l(tw) \tag{0.19}$$

Therefore, using the first implication of the Lemma we obtain by taking $\tilde{w} = tw$ that :

$$\mathrm{sgn}_{\tilde{w}^{-1}}(t) \;=\; \mathrm{sgn}_{w^{-1}t}(t) \;=\; +1 \tag{0.20}$$

Thus,

$$\pi_{(tw)^{-1}}(t) \;=\; +1 \; (tw)^{-1} \; t \; (tw) \;=\; w^{-1}tw \tag{0.21}$$

However, since π is a morphism we have that :

$$\pi_{(tw)-1} \;=\; \pi_{w^{-1}t} \;=\; = \pi_{w^{-1}} \circ \pi_t \tag{0.22}$$

Now let's remark that $\forall t \in T$ we have that :

$$\pi_t(t) \;=\; \mathrm{sgn}_t(t) \; ttt \;=\; -t \tag{0.23}$$

Indeed, let us write $t = s_1...s_k s s_k...s_1$ for k minimal. Then it is clear that :

$$\{1 \leq l \leq 2k+1 \;:\; t = s_1...s_{l-1} s_l s_{l-1}...s_1\} \;=\; \{k+1\} \tag{0.24}$$

as by the minimality, it can not be true for some index $l \leq k$ that $t = s_1...s_{l-1}s_ls_{l-1}...s_1$ and as if it was true for some index $l = k+1+l'$ with $l' > 0$ we would have that :

$$t = s_1s_2...s_kss_k...s_{k-l'+1}s_{k-l'}s_{k-l'+1}...s_kss_k...s_2s_1 \tag{0.25}$$

Therefore, by multiplying both sides by $s_1s_2...s_ks$ from the right and by $ss_k...s_2s_1$ from the left, we would obtain that :

$$s = s_k...s_{k-l'+1}s_{k-l'}s_{k-l'+1}...s_k \tag{0.26}$$

Therefore, by replacing s in t we would have that :

$$t = s_1...s_kss_k...s_1 = s_1...s_ks_k...s_{k-l'+1}s_{k-l'}s_{k-l'+1}...s_ks_k...s_1 = s_1...s_{k-l'}...s_1 \tag{0.27}$$

which would contradict the minimality of k. In particular, this proves that the equality (0.24) is verified and we have that :

$$\pi_t(t) = -t \tag{0.28}$$

Further more, by computing equality (0.22) on t we obtain that :

$$\begin{aligned} \pi_{(tw)^{-1}}(t) &= \pi_{w^{-1}}\pi_t(t) \\ &= \pi_{w^{-1}}(-t) \\ &= -\ \pi_{w^{-1}}(t) \\ &= -\operatorname{sgn}_{w^{-1}}(t)\ w^{-1}tw \end{aligned} \tag{0.29}$$

And we finally conclude that $\operatorname{sgn}_{w^{-1}}(t) = -1$. □

As a Corollary we have the following lemma :

Lemma 0.9 (The exchange property). *Let (W,S) be a Coxeter system, let $w = s_1s_2...s_k \in W$ and $t \in T$, then, if $l(tw) < l(w)$, there exists some $1 \leq l \leq k$ such that :*

$$tw = s_1s_2...\hat{s}_l...s_k \tag{0.30}$$

Proof. By the previous lemma, we know that $\operatorname{sgn}_{w^{-1}}(t) = -1$. Therefore, we know there exists an index $1 \leq l \leq k$ such that $tw = s_1s_2...\hat{s}_l...s_k$. □

Lemma 0.10. *Let (W,S) be a Coxeter system and let $w = s_1s_2...s_k \in W$, with $k = l(w)$ and let us take some $t \in T$. Then, the following are equivalent :*

1. *$l(tw) < l(w)$*
2. *$tw = s_1...\hat{s}_l...s_1$ for some $1 \leq l \leq k$*
3. *$t = s_1...s_l...s_1$ for some $1 \leq l \leq k$*

Moreover, such an index l is uniquely determined.

Proof. By Lemma 0.8 we already know that (1) implies (2). Furthermore, the equivalence between (2) and (3) is a tautology. Let us prove that (2) implies (1). Indeed, if $tw \; = \; s_1...\hat{s_l}...s_1$ for some $1 \leq l \leq k$ then :

$$l(tw) \;\leq\; k+1 \;<\; k \;=\; l(w) \tag{0.31}$$

which is (1). It last to show that this l appearing in property (2) and (3) is unique under the hypothesis that $k = l(w)$. Let us define $t_i = s_1 s_2 ... s_i ... s_1$ for all $1 \leq i \leq k$. Then, we want to show that $t_i \neq t_j$ for every $i \neq j$. Let us reason by contradiction and suppose the contrary. In other words, let us suppose that there exists some indices $i < j$ such that $t_i = t_j$. Then,

$$\begin{aligned} w \;&=\; t_i t_j \; w \\ &=\; t_i s_1 ... \hat{s_j} ... s_k \\ &=\; s_1 ... \hat{s_i} ... \hat{s_j} ... s_k \end{aligned} \tag{0.32}$$

As i was less than j. But this is a contradiction with the exchange property applied to $t = t_i t_j$. Therefore we needed that $t_i \neq t_j$ for every $i \neq j$. In particular l must be unique. $\square$

We are now ready to prove theorem 0.6.

Proof. Let $w = s_1 s_2 ... s_k$ with $k = l(w)$, then $w^{-1} = s_k ... s_1$ and due to the previous lemma, we know that :

$$\begin{aligned} &\#\{t \in T \;:\; \mathrm{sgn}_{w^{-1}}(t) = -1\} \\ =\; &\#\{t \in T \;:\; t = s_1 ... s_i ... s_k \text{ for some } 1 \leq i \leq k\} \;=\; k \;= l(w) \end{aligned} \tag{0.33}$$

As every of the $t_i = s_1 ... s_i ... s_1$ are different from each other. $\square$

The following theorem, describe the writing reduction of a word in a Coxeter group when this one is not written in a minimal way.

Theorem 0.11 (Deletion property). *Let (W, S) be a Coxeter system and let $w = s_1 s_2 ... s_k$ for some k with $l(w) < k$ then there exists two different indices $1 \leq i < j \leq k$ such that :*

$$w \;=\; s_1 ... \hat{s_i} ... \hat{s_j} ... s_k \tag{0.34}$$

As a simple consequence of this theorem , we obtain the following :

Proposition 0.12. *Let (W, S) be a Coxeter system and let $w = s_1 ... s_k$ for some $s_i \in S$ then, if $l(w) < k$ there exists a subword $s_{i_1} ... s_{i_{l(w)}}$ of $s_1 ... s_k$ such that $w = s_{i_1} ... s_{i_{l(w)}}$.*

This proposition is used in the following :

Proposition 0.13. *Let (W, S) be a Coxeter system, and let's suppose that $w = s_1 s_2 ... s_k = s'_1 s'_2 ... s'_k$ for some $s_i, s'_i \in S$ with $k = l(w)$. Then,*

$$\{s_1, s_2, ..., s_k\} \;=\; \{s'_1, s'_2, ..., s'_k\} \tag{0.35}$$

Remark 0.14. To be precise, the upper equality is an equality of sets an not of multi-sets. Indeed, as a simple example that the multi-sets can be different, we take the Coxeter group S_3 and the permutation $(2,3)(1,2)(2,3) = (1,3) = (1,2)(2,3)(1,2)$. In particular, in this example, even if the sets are equal, we have different multi sets associated to $(1,3)$. Namely :

$$\{(2,3),(1,2),(2,3)\} \quad \text{and} \quad \{(1,2),(2,3),(1,2)\} \tag{0.36}$$

Proof. Suppose that the two sets are not equal. Therefore, there exists an $1 \leq i \leq k$ minimal such that $s_i \notin \{s'_1, s'_2 ... s'_k\}$. Furthermore, by lemma 0.10 we know that :

$$\begin{aligned} \{s'_1...s'_j...s'_1 : j = 1,2,...,k\} &= \{t \in T \ : \ l(tw) < l(w)\} \\ &= \{s_1...s_j...s_1 : j = 1,2,...,k\} \end{aligned} \tag{0.37}$$

As those sets are equal, there must be an index $1 \leq j \leq k$ such that for our minimal index i we have :

$$s_1...s_i...s_1 \ = \ s'_1...s'_j...s'_1 \tag{0.38}$$

In particular, by previous proposition, there exists a subword of the right hand side which is of size 1 and which is equal to $s_i \in W$. Therefore, either s_i is one of the previous $s_1...s_{i-1}$ which would be a contradiction with the minimality of i, or s_i is one of the $s'_1, ..., s'_j$ which is a contradiction with our choice of i. Therefore, the two sets must be the same.