MATH-F-427 students

Coxeter groups

Course notes

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ULB

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Part I

Coxeter groups

1

Introduction

This chapter is based on the first chapter of [MKS04]. This chapter will be an introduction of what groups are and how they are generated.

We recall in group theory that a group $(G, \cdot)$ is a non-empty set G of elements with a binary operation $\cdot$ for which the next axioms are satisfied:

- **Closure:** For all $a, b \in G$, c such that $a \cdot b = c$ implies that $c \in G$.
- **Associativity:** The operation $\cdot$ is associative, which means that for any elements $a, b, c \in G$:
$$(ab)c = a(bc)$$
- **Identity element:** There exists an element of G noted 1 for which:
$$a \cdot 1 = 1 \cdot a = a$$
- **Inverse element:** For any $a \in G$ there exists an *element* a^{-1} for which:
$$a \cdot a^{-1} = a^{-1} \cdot a = 1$$

We know two ways of defining a group; defining a *symmetry* of a set and if it is presented by generators and relators.

1.1 Symmetric groups

Definition 1.1 (Symmetric group). *The symmetric group on the set G is the group whose elements are permutations of the elements of G and its operation is the permutation composition. If $G = \{1, \ldots, n$ we call it S_n. [Sag01].*

Proposition 1.2. *S_n has order $n!$ and every group G of order n is a subgroup of S_n.*

1.1.1 Permutations

Now that we know what symmetric groups are, we know that it's mainly based in permutations. In this subsection we define every operation on permutations used in these groups.

Definition 1.3 (Two-line notation). *Given $i \in \{1, \ldots, n\}$ and π the permutation function we represent a permutation listing every elements of the set in two lines where in the first line we have the elements and the second one its image in the π function:*

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & i \\ \pi(1) & \pi(2) & \pi(3) & \pi(4) & \pi(5) & & \pi(i) \end{pmatrix} \tag{1.1}$$

Definition 1.4 (Cycle notation). *Given $i \in \{1, \ldots, n\}$ and π the permutation function, the elements of the sequence $i, \pi(i), \pi^2(i), \ldots$ cannot be distinct. Taking the power p such that $\pi^p(i) = i$, we can note the permutation as the cycle:*

$$(i, \pi(i), \pi^2(i), \ldots, \pi^{p-1}(i)) \tag{1.2}$$

Which means that given a cycle (i, j, k), the element i is send to j, j is sent to k and k is sent to i, cyclically, e.g. the permutation 23145 of $n = \{1, 2, 3, 4, 5\}$ can be written with cycle notation as $(1, 2, 3)(4)(5)$. Remark that every element of the set has to be used.

1.2 Presentation of groups

In this section we show how a group can be defined by generators and relators:

Definition 1.5. *A group can be defined by:*

$$Gr \cong \langle G|R\rangle \tag{1.3}$$

being $G = \{a, b, c, \ldots\}$ the set of generators *and $R = \{A, B, C, \ldots\}$ the set of* relators *or* relations *such that every $X \in R$ is a word in $(G \cup G^{-1})^*$ such that $X = 1$.*

For example, the dihedral group D_n has the next presentation where q is a rotation and r a reflection:

$$D_n \cong \langle \{q, r\}|\{r^2, q^n, rqrq\}\rangle$$

2

Coxeter groups

In this chapter, we explain the interest and motivation of coxeter groups by analysing a symmetric group. The main example of this chapter will be the symmetric group S_3. This group can be defined in three ways.

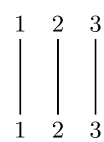


Fig. 2.1. The wiring diagram for the permutation 123.

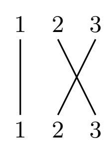


Fig. 2.2. The wiring diagram for the permutation 132.

- Combinatorial: Given a cyclic noted permutation we can represent it as a *wiring diagram.*
  For example, for the permutation 123 we have the following wire diagram:
- Algebraic: Presentation of groups as seen in section 1.2.

$$\begin{aligned} S_3 \cong \langle \{s_1, s_2\} | s_1 s_2 s_1 = s_2 s_1 s_2, s_1^2 = e, s_2^2 = e, \\ (s_1 s_2)^3 = s_1 s_2 s_1 s_2 s_1 s_2 = e\} \rangle \end{aligned} \tag{2.1}$$

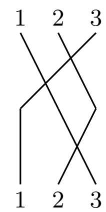


Fig. 2.3. The wiring diagram for the permutation composition 132 ∘ 231.

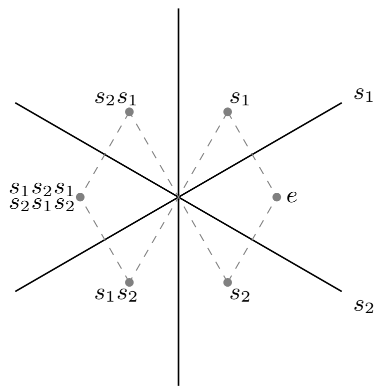


Fig. 2.4. Geometrical representation of S_3. The grey dotted lines represent the reflection applied between two points.

- Geometrical: We can represent geometrically a coxeter group by reflections. In figure 2.4 you can see a geometrical representation of S_3 where s_1 and s_2 are the generators of S_3 and also reflections on the plane.
  If you see the figure you can remark that $s_1s_2s_1 = s_2s_1s_2$, $s_1^2 = e$ and s_2^2; S_3 is a coxeter group.

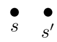
2.1 Coxeter groups

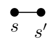
In this section we define the mathematical objects that represent coxeter groups. S being a finite set:

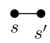
Definition 2.1. *A* coxeter matrix $m : S \times S \to \{1, 2, \ldots, \infty\}$ *such that:*

1. $m(s, s') = 1 \;\; \forall s \in S$
2. $m(s, s') = m(s', s) \;\; \forall s \in S$
3. $m(s, s') > 1$ *if* $s \neq s'$

Definition 2.2. *A* coxeter diagram *is a graph where the set of vertices are the elements of* S *and the labelled edges are such that:*

1. *if* $m(s,s') = 2$*:* 

2. *if* $m(s,s') = 3$*:* 

3. *if* $m(s,s') \geq 4$*:* 

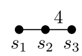
where the label of the edge is $m(s,s')$*.*

where m *is the coxeter matrix.*

Example 2.3. If we have a coxeter matrix m:

$$m = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \\ 2 & 4 & 1 \end{bmatrix} \tag{2.2}$$

where the line or column i corresponds to the generator s_i of the coxeter group. Then the related coxeter diagram can be drawn like this:



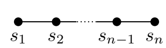
Given the definition of m, we can now define formally a coxeter group.

Definition 2.4. *A* coxeter group *is a group with the following presentation:*

$$\langle S \mid (s,s')^{m(s,s')} = e, \forall s, s' \in S\} \rangle \tag{2.3}$$

where S *si a group and* m *the coxeter matrix with* $m(s,s') < \infty$*.*

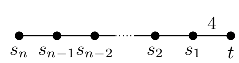
Example 2.5. The symmetric group S_n can be represented with the following coxeter diagram:



because for every $i < n$, we have that $s_i s_{i+1} s_i s_{i+1} = e$, so $(s_i s_{i+1})^2 = e$, which means that $m(s_i, s_{i+1}) = 2$ for every $i < n$.

Definition 2.6. *The tuple* (W,S) *is a coxeter system where* W *is a coxeter group and* S *a group of generators.*

We introduce the hyperoctahedral group S_n^B being the group of signed permutations of $[n] := \{1, 2, \ldots, n\}$. The group set is $\{1, 2, \ldots, m, \bar{1}, \bar{2}, \ldots, \bar{n}\}$. We have a new generator t, that will change the sign of an element. In figure 2.1 you can see an example of permutations withing this group. This is a coxeter group and its diagram is:



Given a coxeter system (W,S), we can then define $T := \{wsw^{-1} \mid s \in S, w \in W\} \subseteq W$ where s are simple reflections. Now we can define π.

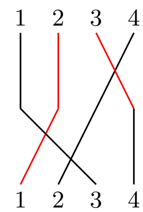


Fig. 2.5. The wiring diagram for the permutation $3\bar{1}24 \circ 1\bar{2}\bar{4}3$. You can see that there is a permutation that changes its sign two times.

Definition 2.7. *Given $W \to S_T^B$ and $w \to \pi_w$ with $t \in T$ and $s \in S$:*

$$\pi_s(t) := \begin{cases} -s & \textit{if } t = s \\ sts & \textit{if } t \neq s \end{cases} \tag{2.4}$$

This application is a bijection because its inverse is itself ($\pi_s(\pi_s(s)) = s$):

$$\pi(\pi_s(t)) = \begin{cases} \pi_s(-s) & \text{if } t = s \\ \pi_s(tsts) & \text{if } t \neq s \end{cases} \tag{2.5}$$

We clearly see that $\pi_s(sts) = s(sts)s = t$ and $\pi_s(-s) = -\pi_s(s) = -s$.

Theorem 2.8. *The application π that we defined on the set of generators S of the coxeter System (W, S), extend uniquely to an injective homomorphism $\pi : W \to S_T^B$*

Proof. First of all, we need to show that the extension of π is well defined. It was clear, due to the definition of π on S that for every $s \in S$, the application $\pi_s \in S_T^B$. Indeed, for every $t \in T$ we had that $\pi_s(t) \in T \cup \overline{T}$ and we had that π_s defined a bijection on $T \cup \overline{T}$. Now, we need to check that its extension on all of W is still well defined. We need to check 2 things. First, we need to check that $\forall w \in W$ the application $\pi_w \in S_T^B$. But, since we extended π from S to W to be a group morphism, we know that π_w is by definition the composition of π_s for some $s \in S$ and thus is an element of S_T^B. Secondly, we need to check that this application π_w doesn't depend on the writing of $w \in W$. To this aim, let's take $t \in T$ and let $w = s_1 s_2 .. s_k$ for some $s_i \in S$ (this is the form of every element of W since $s_i = s_i^{-1}$ for all i). Since, we want π to be a homomorphism, we have that :

$$\begin{aligned}
\pi_w(t) &= \pi_{s_1} \circ \pi_{s_2} \circ \ldots \circ \pi_{s_k}(t) \\
&= \pi_{s_1} \circ \pi_{s_2} \circ \ldots \circ \pi_{s_{k-1}}(\pm s_k t s_k) \\
&\quad \text{(with } - \text{ iff } s_k t s_k = s_k \iff t = s_k) \\
&= \pi_{s_1} \circ \pi_{s_2} \circ \ldots \circ \pi_{s_{k-2}}(\pm \pm s_{k-1} s_k t s_k s_{k-1}) \\
&\quad \text{(with } - \text{ iff } s_{k-1} s_k t s_k s_{k-1} = s_{k-1} \iff t = s_k s_{k-1} s_k) \\
&= \pi_{s_1} \circ \pi_{s_2} \circ \ldots \circ \pi_{s_{k-3}}(\pm \pm \pm s_{k-2} s_{k-1} s_k t s_k s_{k-1} s_{k-2}) \\
&\quad \text{(with } - \text{ iff } s_{k-1} s_k t s_k s_{k-1} = s_{k-1} s_{k-2} \iff t = s_k s_{k-1} s_{k-2} s_{k-1} s_k) \\
&= \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
&= \pm \pm \cdots \pm s_1 s_2 \ldots s_k t s_k s_{k-1} \ldots s_1 \\
&\quad \text{(with } - \text{ iff } s_1 \ldots s_{k-1} s_k t s_k s_{k-1} \ldots s_1 = s_1 \iff t = s_k \ldots s_2 s_1 s_2 \ldots s_k) \\
&= \operatorname{sgn}_w(t)\ w t w^{-1}
\end{aligned} \tag{2.6}$$

Where the function $\operatorname{sgn}_w(t)$ is a sign function counting the number of times we and an index $l \in \{1, 2, \ldots k\}$ such that $t = s_k \ldots s_{l-1} s_l s_{l-1} \ldots s_k$. Namely :

$$\operatorname{sgn}_w(t) \;=\; (-1)^{\#\{1 \leq l \leq k \,:\, t = s_k \ldots s_{l-1} s_l s_{l-1} \ldots s_k\}} \tag{2.7}$$

As we will show just after this sign function doesn't depend of the writing of $w \in W$ in the coxeter system (W, S). But first, let us get some intuition about what this sign function is counting, by looking to the case of S_n : aaaaaaaaaaa

We are now going to use equation 2.6 to prove that π is a well defined homomorphism which is equivalent to show that the sign function doesn't depend on the writing of $w \in W$ in the Coxeter system (W, S). It suffice to show that all the relations we had in (W, S) are satisfied by their image in S_T^B. I.e. let's take $s, s' \in S$ we want to show that :

$$(\pi_s \circ \pi_{s'})^{m(s,s')} \;=\; \mathrm{Id}_{S_T^B} \tag{2.8}$$

Since $(ss')^{-1} = s's$, equation 2.6 gives for all $t \in T$:

$$(\pi_s \circ \pi_{s'})^{m(s,s')}(t) \;=\; \pm\ (ss')^{m(s,s')} t (s's)^{m(s,s')} \;=\; \pm\ ete \;=\; \pm\ t \tag{2.9}$$

But the sign must be $+$ as here, $w = (ss')^{m(s,s')}$ and therefore we look at :

$$\#\{1 \leq l \leq m(s,s') \;:\; t = \underbrace{s'ss' \ldots s'ss'}_{2l-1 \text{ characters}}\ \} \tag{2.10}$$

Which must be even. Indeed, if for some $l \leq m(s,s')/2$ we have :

- if $m(s,s')$ is even :

$$t \;=\; \underbrace{s'ss' \ldots s'ss'}_{2l-1 \text{ characters}} \;=\; \underbrace{s'ss' \ldots s'ss'}_{2l-1+m(s,s') \text{ characters}} \;=\; \underbrace{s'ss' \ldots s'ss'}_{2(l+m(s,s')/2)-1 \text{ characters}} \tag{2.11}$$

- if $m(s,s')$ is odd :

$$t \;=\; \underbrace{s'ss'...s'ss'}_{2l-1 \text{ characters}} \;=\; \underbrace{s'ss'...s'ss'}_{2l-1+m(s,s') \text{ characters}} \;=\; \underbrace{s'ss'...s'ss'}_{2((m(s,s')-1)/2+l)+1 \text{ characters}} \tag{2.12}$$

Which shows that if one index is counted below $m(s,s')/2$ then there exists an other index counted strictly upper than $m(s,s')/2$ and vis versa. Thus the set must be even and the sign must be $+$. Therefore, equation (2.8), is proved and we π is a well defined morphism.

It last to show that the extension of π is injective. Let $u, v \in W$ be such that $\pi_u = \pi_v$ then, we have that :

$$\pi_{uv^{-1}} \;=\; \pi_u \circ \pi_{v^{-1}} \;=\; \mathrm{Id}_{S_T^B} \;=\; \pi_e \tag{2.13}$$

Thus, we just need to show that if $w \in W$ is such that $\pi_w = \pi_e$ then $w = e$ to prove the injectivity of π. Now, let's take $w \in W$ such that $\pi_w = \pi_e$ and let's suppose absurdly that $w \neq e$ then, there exists $k \geq 1$ such that $w = s_1...s_k$ is the shorter way possible to write $w \in W$ then (meaning that k is the smallest possible) :

$$\begin{aligned} s_k \;=\; \pi_e(s_k) \;=\; \pi_w(s_k) \;&=\; \mathrm{sgn}_w(s_k)\; s_1...s_{k-1}s_k s_k s_k s_{k-1}...s_1 \\ &=\; \mathrm{sgn}_w(s_k)\; s_1...s_{k-1}s_k s_{k-1}...s_1 \end{aligned} \tag{2.14}$$

But $\mathrm{sgn}_w(s_k) = -1$ because :

$$\{1 \leq l \leq k \;:\; t = s_k...s_{l-1}s_l s_{l-1}...s_k\} \;=\; \{k\} \tag{2.15}$$

Indeed, for $l = k$ we have $s_k = s_k$. But if $l \neq k$ and if we had :

$$s_k = \; s_k..s_l..s_k \tag{2.16}$$

Then we would have :

$$s_{l-1}...s_k s_k \;=\; s_l...s_k \tag{2.17}$$

And therefore we would have a contradiction with the minimality of k since :

$$\begin{aligned} w &= s_1...s_l s_{l-1} s_l...s_k \\ &=\; s_1...s_{l-1}s_{l-1}...s_k s_k \\ &=\; s_1..s_{l-2}s_{l+1}...s_{k-1} \\ &=\; s_1...s_{l-2}s_{l+11}...s_{k-1} \end{aligned} \tag{2.18}$$

which is a shorter way to write w. Therefore, we have that $\mathrm{sgn}_w(s_k) = -1$ and thus equation 2.14 gives :

$$s_k \;=\; -\; s_1...s_{k-1}s_k s_{k-1}...s_1 \tag{2.19}$$

Which is a contradiction due to the presence of a sign. $\square$

We are now going to define the notions of **parity** and **length** of an element in a Coxeter group.

Definition 2.9. *Let (W, S) be a coxeter system, and let $w \in W$, then we say that $w = s_1 ... s_k$ $(s_l \in S)$ is :*

- ***even** when k is even.*
- ***odd** when k is odd.*

*This is what we call the **parity** of $w \in W$.*

Remark 2.10. As all the relations of a Coxeter group involve a pair number of $s \in S$ we see that the parity of an element $w \in W$ doesn't depend on its writing in W.

The set of even elements of a Coxeter system (W, S) is a subgroup of W called the **alternating** subgroup.

Remark 2.11. When S_n is seen as a Coxeter group with $S = \{s_1 ... s_{n-1}\}$ and the Coxeter matrix $m(s_i, s_{i+1}) = 3$ and $m(s, s') = 2$ for every other couple $()s, s') \neq (s, s)$ see that the two notions of alternating group does coincide.

Definition 2.12. *Let (W, S) be a Coxeter system, the **length** $l(w)$ of an element $w \in W$ is defined as the smallest integer $k \in \mathbb{N}$ such that there exists $s_1, ..., s_k \in S$ with $w = s_1 ... s_k$.*

The purpose of what follows is to prove the following theorem :

Theorem 2.13. *Let (W, S) be a Coxeter system, and let $w \in W$ then :*

$$l(w) \ = \ \#\{t \in T \ : \ sgn_{w^{-1}}(t) = -1\} \tag{2.20}$$

Example 2.14. In the case where $W = S_n$ with the common representation, $l(w)$ is exactly the number of inversion of w^{-1} which is exactly the same as the number of inversion of w itself.

Before proving this thorem, we focus our attention on some lemma :

Lemma 2.15. *Let (W, S) be a Coxeter system and let $w \in W$, $t \in T$ then :*

$$sgn_{w^{-1}}(t) = -1 \quad \Longleftrightarrow \quad l(tw) \ < \ l(w) \tag{2.21}$$

Proof. Let's suppose that $\mathrm{sgn}_{w^{-1}}(t) = -1$ and let $w = s_1 ... s_k$ with $k = l(w)$ then $w^{-1} = s_k ... s_1$. We know that there must exists some $1 \leq l \leq k$ such that $t = s_1 ... s_l .. s_1$ but then :

$$\begin{aligned} tw &= s_1 s_2 ... s_l ... s_1 \ q_1 s_2 ... s_l s_{l+1} ... s_k \\ &= s_1 s_2 ... s_{l-1} s_{l+1} ... s_k \\ &= s_1 s_2 ... \hat{s}_l ... s_k \end{aligned} \tag{2.22}$$

From which we conclude that $l(tw) \leq k-1 \ < \ k = l(w)$ and the first implication is proven.
Conversely, let's suppose that $l(tw) < l(w)$ then, as $tt = e$ we have that :

$$l(tw) \ < \ l(ttw) \ \Rightarrow l(ttw) \not< l(tw) \tag{2.23}$$

Therefore, the first implication that we already proved, gives us by taking $w \tilde{=} tw$ that :

$$\mathrm{sgn}_{w^{-1}t}(t) = +1 \tag{2.24}$$

Thus,

$$\pi_{(tw)^{-1}}(t) \ = \ +1 \ (tw)^{-1} \ t \ (tw) \ = \ w^{-1}tw \tag{2.25}$$

But, by the fact that π is a morphism we have that :

$$\pi_{(tw)-1} \ = \ \pi_{w^{-1}t} \ = \ = \pi_{w^{-1}} \circ \pi_t \tag{2.26}$$

Now let's remark that $\forall t \in T$ we have that :

$$\pi_t(t) \ = \ \mathrm{sgn}_t(t) \ ttt \ = \ -t \tag{2.27}$$

Indeed, let's write $t = s_1...s_k s s_k...s_1$ for a k that is minimal. then it is clear that :

$$\{1 \leq l \leq 2k+1 \ : \ t = s_1...s_{l-1}s_l s_{l-1}...s_1\} \ = \ \{k+1\} \tag{2.28}$$

as by the minimality, it can't be true for $l \leq k$ that $t = s_1...s_{l-1}s_l s_{l-1}...s_1$ and as if it is true for some $l = k+1+l'$ with $l' > 0$ we have that

$$t \ = \ s_1 s_2...s_k s s_k...s_{k-l'+1}s_{k-l'}s_{k-l'+1}...s_k s s_k...s_2 s_1 \tag{2.29}$$

Therefore, by multiplying both sides by $s_1 s_2...s_k s$ from the right and by $s s_k...s_2 s_1$ from the left, we obtain that :

$$s \ = \ s_k...s_{k-l'+1}s_{k-l'}s_{k-l'+1}...s_k \tag{2.30}$$

Therefore, by replacing s in t we have that :

$$t \ = \ s_1...s_k s s_k...s_1 \ = \ s_1...s_k s_k...s_{k-l'+1}s_{k-l'}s_{k-l'+1}...s_k s_k...s_1 \ = \ s_1...s_{k-l'}...s_1 \tag{2.31}$$

Which again contradicts the minimality of k. Therefore, the equality (2.28) is verified and we have that :

$$\pi_t(t) \ = \ -t \tag{2.32}$$

And by computing equality (2.26) on t we obtain that :

$$\begin{aligned} \pi_{(tw)^{-1}}(t) \ &= \ \pi_{w^{-1}}\pi_t(t) \\ &= \ \pi_{w^{-1}}(-t) \\ &= \ - \ \pi_{w^{-1}}(t) \\ &= \ -\,\mathrm{sgn}_{w^{-1}}(t) \ w^{-1}tw \end{aligned} \tag{2.33}$$

And we finally conclude that $\mathrm{sgn}_{w^{-1}}(t) = -1$. $\square$

As a Corollary we have the following lemma :

Lemma 2.16. *The exchange property*
Let (W, S) be a Coxeter system and let $w = s_1 s - 2 ... s_k \in W$ and $t \in T$, then, if $l(tw) < l(w)$ then, there exists some $1 \leq l \leq k$ such that :

$$tw \ = \ s_1 s_2 ... \hat{s}_l ... s_k \tag{2.34}$$

Proof. By the previous lemma, we know that $\mathrm{sgn}_{w^{-1}} t \ = \ -1$. Therefore, we know there exists a $1 \leq l \leq k$ such that $tw \ = \ s_1 s_2 ... \hat{s}_l ... s_k$. □

Lemma 2.17. *Let (W, S) be a Coxeter system and let $w = s_1 s_2 ... s_k \in W$, with $k = l(w)$ and let's take $t \in T$. Then, the following are equivalent :*

1. $l(tw) < l(w)$
2. $tw \ = \ s_1 ... \hat{s}_l ... s_1$ for some $1 \leq l \leq k$
3. $t = s_1 ... s_l ... s_1$ for some $1 \leq l \leq k$

Moreover, such an l is uniquely determined.

Proof. By Lemma 2.15 we already know that (1) implie (2). Furthermore, the equivalence between (2) and (3) is a tautology. Let us prove that (2) implies (1). Indeed, if $tw \ = \ s_1 ... \hat{s}_l ... s_1$ for some $1 \leq l \leq k$ then :

$$l(tw) \ \leq \ k + 1 \ < \ k \ = \ l(w) \tag{2.35}$$

which is (1). It last to show that this l appearing in property (2) and (3) is unique under the hypothesis that $k = l(w)$. Let us define $t_i = s_1 s_2 ... s_i ... s_1$ for all $1 \leq i \leq k$. Then, we want to show that $t_i \neq t_j$ for every $i \neq j$. Let's reason by absurd and suppose the contrary. Therefore, there exists $i < j$ such that $t_i = t_j$. Then,

$$\begin{aligned} w \ &= \ t_i t_j \ w \\ &= \ t_i s_1 ... \hat{s_j} ... s_k \\ &= \ s_1 ... \hat{s_i} ... \hat{s_j} ... s_k \end{aligned} \tag{2.36}$$

As i was less than j. But this is a contradiction with the exchange property applied for $t = t_i t_j$. Therefore we needed that $t_i \neq t_j$ for every $i \neq j$. In particular l must be unique. □

With all those lemma, we are now ready to prove theorem 2.13.

Proof. Let $w = s_1 s_2 ... s_k$ with $k = l(w)$, then $w^{-1} = s_k ... s_1$ and due to the previous lemma, we know that :

$$\begin{aligned} &\#\{t \in T \ : \ \mathrm{sgn}_{w^{-1}}(t) = -1\} \\ = \ &\#\{t \in T \ : \ t = s_1 ... s_i ... s_k \text{ for some } 1 \leq l \leq k\} \ = \ k \ = l(w) \end{aligned} \tag{2.37}$$

As every of the $t_i = s_1 ... s_i ... s_1$ are different from each other. □

The following theorem, describe the writing reduction of a world in a Coxeter group when it's not written in one of its minimal writings.

Theorem 2.18. *Deletion property*

Let (W, S) be a Coxeter system and let $w = s_1 s_2 ... s_k$ for some k with $l(w) < k$ then there exists two different indices $1 \leq i < j \leq k$ such that :

$$w = s_1 ... \hat{s_i} ... \hat{s_j} ... s_k \tag{2.38}$$

a consequence is the following proposition :

Proposition 2.19. *Let (W, S) be a Coxeter system and let $w = s_1 ... s_k$ for some $s_i \in S$ then, if $l(w) < k$ there exists a sub-word $s_{i_1} ... s_{i_{l(w)}}$ of $s_1 ... s_k$ such that $w = s_{i_1} ... s_{i_{l(w)}}$.*

This proposition is used in the following :

Proposition 2.20. *Let (W, S) be a Coxeter system, and let's suppose that $w = s_1 s_2 ... s_k = s'_1 s'_2 ... s'_k$ for some $s_i, s'_i \in S$ with $k = l(w)$. Then,*

$$\{s_1, s_2 ... s_k\} = \{s'_1, s'_2 ... s'_k\} \tag{2.39}$$

Remark 2.21. To be precise, the upper equality is an equality of sets an not of multi-sets. Indeed, as a simple example that the multi-sets can be different, we take the Coxeter group S_3 and the permutation $(2,3)(1,2)(2,3) = (1,3) = (1,2)(2,3)(1,2)$ therefore, we have the multi-sets :

$$\{(2,3),(1,2),(2,3)\} \quad \text{and} \quad \{(1,2),(2,3),(1,2)\} \tag{2.40}$$

Proof. Suppose that the two sets are not equal. Therefore, there exists an $1 \leq i \leq k$ minimal such that $s_i \notin \{s'_1, s'_2 ... s'_k\}$. Furthermore, by lemma **??** we know that :

$$\begin{aligned} \{s'_1 ... s'_j ... s'_1 : j = 1,2,...,k\} &= \{t \in T \ : \ l(tw) < l(w)\} \\ &= \{s_1 ... s_j ... s_1 : j = 1,2,...,k\} \end{aligned} \tag{2.41}$$

As those sets are equal, there must be an index $1 \leq j \leq k$ such that for our previous i we have :

$$s_1 ... s_i ... s_1 = s'_1 ... s'_j ... s'_1 \tag{2.42}$$

In particular, by previous proposition, there exists a sub-word of the right hand side which is of size 1 and which is equal to $s_i \in W$. Therefore, either s_i is one of the previous $s_1 ... s_{i-1}$ which would be a contradiction with the minimality of i, or s_i is one of the $s'_1, ..., s'_j$ which is a contradiction with our choice of i. Therefore, the two sets must be the same.

Warnig: hypothesis reduced in the exchange property!

Theorem 2.22 (Matsumoto). *Let W be a group and $S \subset W$ a finite subset of generators of W of order 2. Then the following assertions are equivalent:*

- *(i) (W,S) is a Coxeter system.*
- *(ii) (W,S) satisfies the exchange property.*
- *(iii) (W,S) satisfies the deletion property.*

Proof. $(i) \Rightarrow (ii)$. This implication has already been shown above.

$(ii) \Rightarrow (iii)$. Let $w = s_1 \dots s_k$ such that $\ell(w) < k$. Let i be maximal such that $s_i s_{i+1} \dots s_k$ is not reduced (i.e. $s_{i+1} \dots s_k$ is reduced). We have $\ell(s_i s_{i+1} \dots s_k) \leq k - i = \ell(s_{i+1} \dots s_k)$. Now, using exchange property, we obtain $s_i s_{i+1} \dots s_k = s_{i+1} \dots \hat{s}_j \dots s_k$ for some $i+1 \leq j \leq k$. Therefore, $w = s_1 \dots s_{i-1} s_i s_{i+1} \dots s_k = s_1 \dots s_{i-1} \hat{s}_i s_{i+1} \dots \hat{s}_j \dots s_k$ and we have the result (let us note that this implication remains true for weaker hypothesis since we did not use the fact that S is of order 2).

$(iii) \Rightarrow (ii)$. Let $w = s_1 \dots s_k$, $k = \ell(w)$, $s \in S$, such that $\ell(sw) = \ell(ss_1 \dots s_k) \leq \ell(w) = \ell(s_1 \dots s_k) = k$. So $ss_1 \dots s_k$ is not reduced. We can apply the deletion property. Suppose that $sw = ss_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$ (but $\ell(sw) \leq k-1 < l(w)$). So $ssw = sss_i \dots \hat{s}_i \dots \hat{s}_j \dots s_k$. This leads to $\ell(s_1 \dots ..\hat{s}_i \dots \hat{s}_j \dots s_k) < k$, which is a contradiction, so this case has to be excluded. Hence, we have $sw = \hat{s} s_1 \dots \hat{s}_i \dots s_k$.

$(ii) \Rightarrow (i)$. Using $(ii) \Rightarrow (iii)$, we can assume both (ii) and (iii). Define $m(s,s') =$ order of ss' in W, for all $s, s' \in S$. Let $(\tilde{W}, S)$ be the Coxeter group associated to m. Clearly, $\phi : \tilde{W} \mapsto W, s \to s$ is a surjective homomorphism. We need to show that ϕ is also injective. Let $s_1 \dots s_m = e$ in W. By the deletion property, m is even, say $m = 2k$. So we can write our relation on the form

$$s_1 \dots s_k = s'_1 \dots s'_k \tag{2.43}$$

where $s'_1 = s_{2k}, \dots s'_k = s_{k+1}$. We must now prove that (2.43) is a consequence of the pairwise relations $(ss')^{m(s,s')} = e$. The proof is done by induction on k, the case $k = 1$ being trivially correct.

- <u>Case 1:</u> Suppose $w := s_1 \dots s_k$ is not reduced. By deletion property, there exists a position $1 \leq i < k$ such that $s_{i+1} s_{i+2} \dots s_k$ is reduced but $s_i s_{i+1} s_{i+2} ... s_k$ is not. By the exchange property, we then have that $s_{i+1} s_{i+2} \dots s_k = s_i s_{i+1} \dots \hat{s}_j \dots s_k$ for some $i < j \leq k$. This relation is of length $< 2k$ and hence fine. Plugging this result into (2.43) gives $s_1 \dots s_i s_i s_{i+1} \dots \hat{s}_j \dots s_k = s'_1 s'_2 \dots s'_k$. The factor $s_i s_i$ can be deleted, leaving a relation of length $< 2k$. Hence the relation (2.43) is fine.
- <u>Case 2:</u> Suppose $w = s_1 \dots s_k$ is reduced, $k = \ell(w)$. We can assume that $s_1 \neq s'_1$ (otherwise the relation (2.43) is equivalent to a shorter relation). We have $\ell(s'_1 s_1 s_2 \dots s_k) = \ell(s'_1 s'_1 s'_2 \dots s'_k) = \ell(s'_2 \dots s'_k) \leq k - 1 < \ell(s_1 \dots s_k)$. Using exchange property, we have $s'_1 s_1 \dots s_k = s_1 \dots \hat{s}_i \dots s_k$ for some i. Hence, $s_1 \dots \hat{s}_i \dots s_k = s'_2 \dots s'_k$.
  If $i < k$, then $s'_1 s_1 s_2 \dots s_{k-1} = s_1 \dots \hat{s}_i \dots s_{k-1}$. So $s'_1 s_1 s_2 \dots s_{k-1} s_k = s_1 \dots \hat{s}_i \dots s_{k-1} s_k$. Hence, $s'_1 s_1 \dots s_k = s'_2 \dots s_k$ is a consequence of Coxeter relations.

If $i = k$, we have to work a little bit harder. We have $s'_1 s_1 \dots s_{k-1} = s'_1 s'_2 \dots s'_k$. Thus it will suffice to show that $s_1 s_1 \dots s_{k-1} = s_1 s_2 \dots s_k$ is a consequence of Coxeter relations. Applying recursively the rule, we have $s_1 s'_1 s_1 \dots s_{k-2} = s'_1 s_1 \dots s_{k-1} \Rightarrow s'_1 s_1 s'_1 s_1 \dots s_{k-3} = s_1 s'_1 s_1 \dots s_{k-2}$ $\Rightarrow \dots$ Thus in the end, the question will be reduced to the relation $s_1 s'_1 s_1 s'_1 \dots = s'_1 s_1 s'_1 s_1 \dots$, which is of course a consequence of the Coxeter relation $(s_1 s'_1)^{m(s,s')} = e$.

Example 2.23. The group S_n can be generated by transpositions, which are order 2 elements. Using the above theorem, we conclude that S_n is actually a Coxeter group.

2.2 Geometric representation

Let (W, S) be a Coxeter system, $S = \{s_1, \dots s_n\}$, m the associated Coxeter matrix. We write $m_{ij} = m(s_i, s_j)$. Let V be a $\mathbb{R}$-vector space of dimension n, with a basis $\alpha_1, \dots, \alpha_n$. We consider the symmetric bilinear map

$$\langle \cdot, \cdot \rangle : V \times V \mapsto \mathbb{R} \tag{2.44}$$

defined through

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} -\cos\left(\frac{\pi}{m_{ij}}\right) & \text{if } m_{ij} < +\infty \\ -1 & \text{if } m_{ij} = +\infty \end{cases} \tag{2.45}$$

Not that $\langle \cdot, \cdot \rangle$ is not positive definite in general.

Proposition 2.24. *The following map extends to a homomorphism:*

$$W \mapsto GL(V), s_i \to \sigma_i \tag{2.46}$$

where $\sigma_i : v \to v - 2\langle v, \alpha_i \rangle \alpha_i$.

Remark 2.25. We have $\sigma_i(\alpha_i) = \alpha_i - 2\langle \alpha_i, \alpha_i \rangle \alpha_i = -\alpha_i$. Thus, if $v \in V$ is such that $\langle v, \alpha_i \rangle = 0$, then $\sigma_i(v) = v$. Therefore, if $\langle \cdot, \cdot \rangle$ was positive definite, σ_i would be interpreted as a reflexion through the hyperplane orthogonal to α_i.

Proof. First, let us show that σ_i is invertible for all i. We have $\sigma_i^2(v) = \sigma_i(v) - 2\langle v, \alpha_i \rangle \sigma_i(\alpha_i) = v - 2\langle v, \alpha_i \rangle \alpha_i + 2\langle v, \alpha_i \rangle \alpha_i = v$.

Now, let us show that $(\sigma_i \sigma_j)^{m_{ij}} = Id_V$. For $i \neq j$, define $V_{ij} = \mathrm{Span}_{\mathbb{R}}(\{\alpha_i, \alpha_j\})$. Furthermore, $V_{ij}^{\perp} = \{v \in V | \langle v, \alpha_i \rangle = 0, \langle v, \alpha_j \rangle = 0\}$. Before proceeding, we show the following lemma:

Lemma 2.26. $V = V_{ij} \oplus V_{ij}^{\perp}$ *if* $m_{ij} < +\infty$.

Proof. Let $v \in V$. We want to find $\lambda_i, \lambda_j \in \mathbb{R}$ such that $\tilde{v} = \lambda_i \alpha_i + \lambda_j \alpha_j \in V_{ij}$ and $v - \tilde{v} \in V_{ij}^{\perp}$. We have

$$\begin{aligned} \langle \tilde{v}, \alpha_i \rangle &= \lambda_i \langle \alpha_i, \alpha_i \rangle + \lambda_j \langle \alpha_i, \alpha_j \rangle \\ &= \lambda_i + C\lambda_j \end{aligned} \tag{2.47}$$

where $C = \langle \alpha_i, \alpha_j \rangle = -\cos\left(\frac{\pi}{m_{ij}}\right)$. Furthermore,

$$\langle \tilde{v}, \alpha_j \rangle = \lambda_i C + \lambda_j \tag{2.48}$$

Since

$$\det \begin{pmatrix} 1 & C \\ C & 1 \end{pmatrix} = 1 - C^2 = 1 - \cos^2\left(\frac{\pi}{m_{ij}}\right) \neq 0, \tag{2.49}$$

if $m_{ij} < +\infty$. Therefore, we can find unique λ_i and λ_j such that

$$\langle \tilde{v}, \alpha_i \rangle = \langle v, \alpha_i \rangle \quad \text{and} \quad \langle \tilde{v}, \alpha_j \rangle = \langle v, \alpha_j \rangle \tag{2.50}$$

Now let us come back to the proof of the proposition. Using the lemma, we have $v = \tilde{v} + (v - \tilde{v})$ such that $\langle v - \tilde{v}, \alpha_i \rangle = 0 = \langle v - \tilde{v}, \alpha_j \rangle = 0$. Hence $\sigma_i(v - \tilde{v}) = v - \tilde{v} - 2\langle v - \tilde{v}, \alpha_i \rangle \alpha_i = v - \tilde{v}$ and $\sigma_j(v - \tilde{v}) = v - \tilde{v}$. In the basis $\{\alpha_i, \alpha_j\}$ of V_{ij}, the matrix associated to σ_i is given by

$$\begin{pmatrix} -1 & -2C \\ 0 & 1 \end{pmatrix} \tag{2.51}$$

In fact, we have $\sigma_i(\alpha_i) = -\alpha_i$ and $\sigma_j(\alpha_j) = \alpha_j - 2C\alpha_i$. Similarly, the matrix associated to σ_j is given by

$$\begin{pmatrix} 1 & 0 \\ -2C & -1 \end{pmatrix} \tag{2.52}$$

Therefore,

$$(\sigma_i)(\sigma_j) = \begin{pmatrix} -1 & -2C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2C & -1 \end{pmatrix} = \begin{pmatrix} -1 + 4C^2 & 2C \\ -2C & -1 \end{pmatrix} \tag{2.53}$$

The characteristic polynomial P of this matrix is given by $P(t) = t^2 - (-2 + 4C^2)t + 1$.The roots are given by $t_{\pm} = \cos\left(\frac{2\pi}{m_{ij}}\right) \pm i \sin\left(\frac{2\pi}{m_{ij}}\right)$. This characterizes a rotation of $2\pi/m_{ij}$. So, the order of $\sigma_i\sigma_j$ is given by m_{ij}.

2.3 Combinatorial representation

As explained in the first part of this chapter, coxeter groups can also be viewed as combinatorial objects. In this section, we recall some concepts of combinatorics.

Definition 2.27. *A poset is a tuple $(P, \leq)$ where P is a set and $\leq$ is a binary relation such that:*

- $x \leq y \quad \forall x, y \in P$
- *if* $x \leq y$ *and* $y \leq x$ *then* $x = y$
- *if* $x \leq y$ *and* $y \leq z$ *then* $x \leq z$

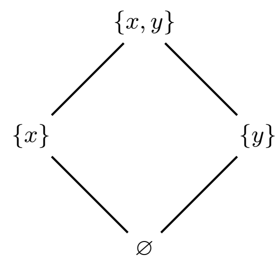


Fig. 2.6. The Hasse Diagram of the poset $(P(x, y), \subseteq)$.

A poset can be represented as a hasse diagram, as viewed in figure 2.6.

Definition 2.28. *Let (W, S) be a coxeter system and $T = \{wsw^{-1} | w \in W, s \in S\}$ a set of reflections. $u, w \in W$ define a function $u \to v$ if $l(v) > l(u)$ and $v = ut$ for some $t \in T$.*

We define $w \leq v$ if there exist:

$$u \to u_1 \to u_2 \to \cdots \to u_k = v \tag{2.54}$$

A Bruhat order is a partial order on the elements of a Coxeter group. By using the relation defined in the previous definition, we can have a bruhat order as shown in figure 2.7.

Theorem 2.29. *Let $u, v \in S_n$, $u \leq v$ if and only if the intersection degree of u is $\leq$ then the one of v.*

Example 2.30. If we have $35124 \leq 45213$:

$$\begin{aligned} 3 &\leq 4 \\ 35 &\leq 45 \\ 135 &\leq 245 \\ 1235 &\leq 1245 \\ 12345 &\leq 12345 \end{aligned} \tag{2.55}$$

Theorem 2.31 (Subword property). *Let $v = s_1 s_{2q}$ reduced word for $v \in W$. Then $u \leq v$ if and only if $u = s_{i_1} s_{i_2} \ldots s_{i_k}$ reduced for some $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq q$.*

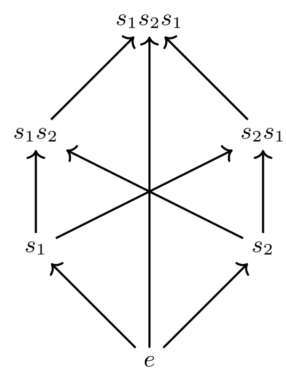


Fig. 2.7. The Bruhat order $(S_3, \{s_1, s_2\}), T = \{s_1, s_2, s_3, s_4, s_5\}$

Proof. We first prove the *if* part of the theorem. Assume $u \to v$, so $v = ut$ and $l(v) > l(u) = l(utt) = l(vt)$ for some $t \in T$.

By the exchange property:

$$u = vt = s_1 s_2 \dots \hat{s_i} \dots s_q \tag{2.56}$$

We omit factors iteratively until we have what we want.

Second, we prove the *only if* part of the theorem by induction on $l(v) - l(u)$. If $l(v) - l(u) = 1$ and we have

$$u = s_1 \dots \hat{s_i} \dots s_q \tag{2.57}$$

with

$$t = s_q \dots s_i \dots s_q \tag{2.58}$$

then $ut = v$ which means that $u \to v$.

If $l(v) - l(u) > 1$ and we have

$$u = s_1 \dots s_{\hat{a_1}} \dots s_{\hat{a_2}} \dots s_{\hat{a_k}} \dots s_q \tag{2.59}$$

with a minimal k such that $l(v) - l(u) = k$, we take

$$t = s_q \dots s_{a_k} \dots s_q \tag{2.60}$$

and

$$ut = s_1 \dots s_{\hat{a_1}} \dots s_{\hat{a_2}} \dots s_{a_k} \dots s_q \tag{2.61}$$

We could have two cases:

- Case $l(ut) > l(u)$:

  then $l(ut) = l(u) + 1$,. so $u \to ut$. $l(v) - l(ut) - 1$. By induction $ut \leq v$, so $u \leq v$.

- Case $l(ut) < l(u)$:

  this will be proved impossible by contradiction. By exchange we have

$$ut = s_1 \dots \hat{s_{a_1}} \dots \hat{s_i} \dots \hat{s_{a_{k-1}}} \dots s_{a_k} \dots s_q \tag{2.62}$$

If $i > a_k$ and we have

$$t = s_q \dots s_1 \dots s_q \tag{2.63}$$

then:

$$\begin{aligned} vtt &= s_1 \dots s_q(s_q \dots s_{a_k} \dots s_q)(s_q \dots s_i \dots s_q) \\ &= s_q \dots \hat{s_{a_k}} \dots \hat{s_i} \dots s_q \end{aligned} \tag{2.64}$$

which is a contradiction.

If $i < a_k$,

$$\begin{aligned} u = utt &= s_1 \dots \hat{s_{a_1}} \dots s_q(s_q \dots \hat{s_{a_k}} \dots s_i \dots \hat{s_k} \dots s_q)(s_q \dots s_{a_k} \dots s_q) \\ &= s_1 \dots \hat{s_{a_1}} \dots \hat{s_i} \dots s_{a_k} \dots s_q \end{aligned} \tag{2.65}$$

You can see that we omit elements to the left of a_k, which contradicts the minimality of a_k. This proves that the case when $l(ut) < l(u)$ is impossible. Thus, we have proved the first case true which finishes the proof. □

From this theorem we can deduct some corollaries.

Corollary 2.32. *$u \leq v$ if and only if $u^{-1} \leq v^{-1}$.*

Corollary 2.33. *If $u \leq v$ and $l(v) - l(u) = k$, then*

$$u \leq u_1 \leq u_2 \leq \dots \leq u_k \tag{2.66}$$

with $l(u_i) = l(u) + i$ for every i.

Proof. Let $v = s_1 \dots s_q$ and $l(v) = q$, $u \leq v$, $l(v) - l(u) = k$ so $l(u) = q - k$. By subword property

$$u_1 = s_i \dots \hat{s_{a_k}} \dots s_q = u\underbrace{s_q \dots s_{a_k} \dots s_q}_{t} \tag{2.67}$$

We have then $l(u_1) = l(ut) = l(u)+1$, so $u \to u_1$ which means that $u \leq u_1$.
□

Theorem 2.34 (Lifting property). *If $u, v \in W$ and $s \in S$ with $u \leq w$, $u \leq sw$ and $su \leq w$.*

Proof. Take $sw = s_1 \dots s_q$ in a reduced form.

As $sw \leq w$, $w = ss_1 \dots s_q$ is also reduced.

As $u \leq w$, u is a subword of $ss_1 \dots s_q$. If u is a subword of $s_1 \dots s_q$, which corresponds to the theorem. If not,

$$u = ss_{i_1} \dots s_{i_r} \tag{2.68}$$

is reduced with

$$1 \leq i_1 \leq i_2 \leq \dots \leq i_r \leq q \tag{2.69}$$

So $su = s_{i_1} \dots s_{i_2}$, which is a contradiction as $u \leq su$. The proof of $su \leq w$ is left to the reader. □

Lemma 2.35. *If $u, v \in W$ then it exists a $w \in W$ such that $u \leq w$ and $v \leq w$.*

Proof. We can prove this by induction on $l(u) + l(v)$. Say $l(u) > 0$:

We take $s \in S$ such that $l(su) = l(u) - 1$. By induction $l(su) + l(v) < l(u) + l(v)$ so it exists indeed a $w \in W$ such that $su \leq w$ and $v \leq w$. We may have two different cases: if $sw \leq w$ or $w \leq sw$:

- Case $w \leq sw$: by the lifting property, $v \leq sw$ and $u \leq sw$.
- Case $sw \leq w$: again by the lifting property, $v \leq w$ and $u \leq w$.

This proves that there exists always an element that respects the lemma, either w or sw. □

Corollary 2.36. *If W is finite there is an unique maximal $w_0 \in W$.*

Example 2.37. In S_n we have $w_0 = n(n-1)\dots 1$.

Corollary 2.38. *If there is a $w_0 \in W$ such that $sw_0 \leq w_0$ for all $s \in S$, then $w \leq w_0$ for all $w \in W$ and W is finite.*

Proof. The proof is left to the reader. You may prove this by induction on $l(w)$, by showing that $w \leq w_0$ for all $w \in W$. Then show finitely many choices of w.

This maximal element is very important because it has some properties that relates it with the set of reflections T:

Proposition 2.39. *Let $w_0 \in W$ be the maximal element of W and $T_L(w) = \{t \in T |\ l(tw) < l(w)\}$. The following statements are true:*

- $w_0^2 = e$
- $l(ww_0) = l(w_0) - l(w)$
- $T_L(ww_0) = T \setminus T_L(w_0)$
- $l(w_0) = |T|$

Corollary 2.40. *$u \leq v$ if and only if $vw_0 \leq uw_0$.*

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