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# Characterization and complexity of Thin Strip Graphs

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Master Thesis in Computer Sciences

**Academic year 2018 - 2019**

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# Acknowledgment

I want to thank ...

# Contents

# Introduction

Talk about what this is etc...

# Chapter 1

# Background

Text about background intro

## 1.1 Graphs and intersections

### 1.1.1 Graphs

A graph $G$ is defined as $G = (V, E)$, where $V$ is the set of vertices and $E$ the set of edges, where $E \subseteq \binom{V}{2}$. The vertices $v, w \in V$ such that $e = vw \in E$ links are called the *endpoints* of $e$.

**Definition 1.1.1.** An embedding of a graph $G$ into a surface $\Sigma$ is a mapping of $G$ in $\Sigma$ where the vertices correspond to distinct points and the edges correspond to simple arcs connecting the images of their endpoints. [GF17].

A graph $G$ is planar if there is an embedding of this graph that does not have any crossing between the edges.

**Definition 1.1.2.** Let $G = (V, E)$ and $S \subset V$, an induced subgraph is a graph $H$ of $G$ whose vertex set is $S$ and its edge set $F = \{vw : v, w \in S, vw \in E\}$.

**Definition 1.1.3.** Let $G = (V, E)$ its complement graph $\overline{G}$ is the graph such that its edge set is defined as: $\{vw : v, w \in V, vw \notin E\}$.

**Definition 1.1.4.** $H$ is called a *minor* of $G$ if $H$ can be constructed by deleting edges and vertices, or contracting edges.

**Theorem 1.1.5** (Kuratowski)**.** *A graph $G$ is planar if and only if it does not contain $K_5$ or $K_{3,3}$ as a minor or a induced subgraph.*

**Definition 1.1.6.** A path $P_n$ in a graph $G$ is a sequence of vertices $v_1 v_2 v_3 \dots v_n$ such that $v_i v_{i+1} \in E$.

**Definition 1.1.7.** A cycle $C_n$ in a graph $G = (V, E)$ is a path $v_1 \dots v_n$ such that $v_1 = v_n$.

**Definition 1.1.8.** A chord of a cycle $C_n$ with $n \geq 4$ is an edge that connects two non consecutive vertices of $C_n$.

**Definition 1.1.9.** A triangular chord of a cycle is a chord that will create a new triangle ($C_3$).

**Definition 1.1.10.** A graph $G = (V, E)$ is complete if every pair of distinct $v_1, v_2 \in V$ are adjacent. This is denoted $K_n$ with $n$ the size of the graph. If $G$ is an induced graph of $H$ then $G$ is a clique of $H$.

**Definition 1.1.11.** A graph $G$ is bipartite if there exist two disjoint subsets $A, B \subset V$ such that $A \cup B = V$ and each edge $e \in E$ has an endpoint on $A$ and the another on $B$.

**Definition 1.1.12.** A bipartite graph $G$ with bipartitions $A$ and $B$ is complete bipartite if every pair of vertices $v \in A, w \in B$ are adjacent. It is denoted as $K_{n,m}$, being $n$ and $m$ the size of each bipartition.

Some graphs can be characterized with properties. A property of a graph is a property that is preserved under all its isomophisms. These properties are called *hereditary* if they are also preserved under all its induced subgraphs; they are called *minor-hereditary* if they are also preserved under its minors (e.g. Kuratowski's planar graph characterization [1.1.5]).

**Definition 1.1.13.** An forbidden induced subgraph (minor) of a graph class $X$ is a graph such that if it is the induced subgraph (minor) of a graph $G$, we know that $G \notin X$.

The coloration of a graph is a color assignment to each vertex such that the color of the two endpoints of every edge of the graph is different.

**Definition 1.1.14.** The chromatic number of a graph $\chi(G)$ is the smallest number of colors needed to have an acceptable coloration of $G$.

**Definition 1.1.15.** The clique number of a graph $\omega(G)$ is the size of the biggest clique of $G$. We can observe that for every graph: $\chi(G) \geq \omega(G)$.

**Definition 1.1.16.** A perfect graph is a graph that respects this condition for every induced subgraph:

$$\omega(G) = \chi(G)$$

**Theorem 1.1.17** (Lovasz)**.** *$G$ is perfect if and only if $\overline{G}$ is perfect.*

### 1.1.2 Intersection graphs

**Definition 1.1.18.** The *intersection graph* of a collection $\zeta$ of objects is the graph $(\zeta, E)$ such that $c_1c_2 \in E \Leftrightarrow c_1 \cap c_2 \neq \varnothing$.

An intersection can be seen as a relationship between two objects. In this thesis, it will be important to define these relations more formally to characterize intersection graphs.

**Definition 1.1.19.** A partial order is a binary relation $\leq$ over a set $A$ satisfying these axioms:

- if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry).
- if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).
- $a \leq a$ (reflexivity).

**Definition 1.1.20.** A total order is a partial order where the reflexivity order is replaced by the connex property:

$$a \leq b \text{ or } b \leq a$$

**Definition 1.1.21.** A partially ordered set or poset $(S, \leq)$ where $S$ a set and $\leq$ a partial order on $S$.

**Definition 1.1.22.** A spanning order $(V, <)$ of a graph $G = (V, E)$ is a total order on $V$ such that for any three vertices $u < v < w$:

$$uw \in E \to uv \in E \text{ or } vw \in E$$

**Definition 1.1.23.** A graph $G = (V, E)$ is a comparibility graph if there exists a partial order $\leq$ such that $vw \in E \Leftrightarrow v \leq w$ or $w \leq v$. Equivalently, $G$ is a comparability graph if it is the comparability graph of a poset. For example, the Hasse diagram (figure 1.1) is a comparability graph where the relation is inclusion.

**Definition 1.1.24.** A graph $G = (V, E)$ is a co-comparability graph if its complement is a comparability graph.

There are multiple characterizations for the co-comparability graph class; we will see one that uses a poset to characterize it:

**Theorem 1.1.25** (Damaschke [Dam92])**.** *A graph $G$ is a co-comparability graph if and only if it has a spanning order.*

### Disks

A disk graph $G$ is a graph that is an intersection graph of disks on the plane, when the size of the disk is unitary, we talk about unit disk graphs. This class of graphs is important for this thesis, as thin strip graphs are a sub-class of unit disk graphs.

We will refer to the unit disk graph class as UDG and an example of a realization can be found in the figure 1.2.

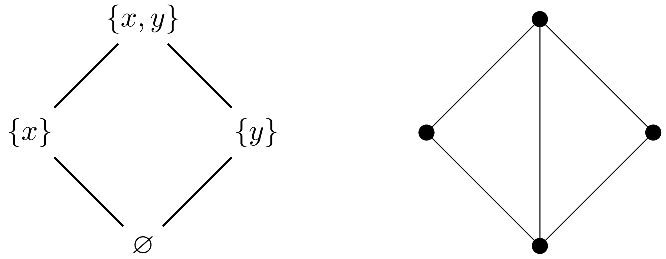



Figure 1.1: On the left, Hasse diagram of a poset of the power set of 2 elements ordered by inclusion. On the right, the comparability graph of this poset.

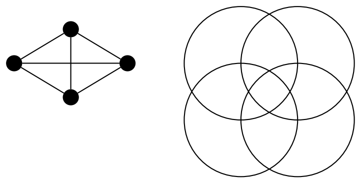


Figure 1.2: Realization of a UDG (Unit Disk Graph).

**Induced forbidden subgraphs** The characterization of this class with respect to its induced forbidden subgraphs has been studied [AZ16].

**Theorem 1.1.26** (Atminas-Zamaraev)**.** *For every integer $k > 1$, $\overline{K_2 + C_{2k+1}}$ is a minimal induced subgraph of UDG.*

**Theorem 1.1.27** (Atminas-Zamaraev)**.** *For every integer $k > 4$, $\overline{C_{2k}}$ is a minimal induced subgraph of UDG.*

## 1.2 Complexity

Complexity theory has the objective to establish lower bounds on how efficient an algorithm can be for a given problem [Sip06]. This approach let us have a reference point to establish the difficulty of a problem.

**Definition 1.2.1.** Let $\Sigma$ be a finite alphabet, $\Sigma^*$ every word derived from $\Sigma$, $L \subseteq \Sigma^*$ is a decision problem.

**Definition 1.2.2.** A decider for a decision problem $A$ is an deterministic algorithm $V$ where

$$A = \{w | V \, accepts \; w\}$$

$A$ is polynomially decidable if it has a polynomial time decider [Sip06].

**Definition 1.2.3.** A verifier for a decision problem $A$ is an deterministic algorithm $V$ where

$$A = \{w|V\,accepts\ \langle w, c\rangle\ for\ some\ string\ c\}$$

$A$ is polynomially verifiable if it has a polynomial time verifier [Sip06].

### 1.2.1 P vs NP

**Definition 1.2.4.** A problem $L \in \mathcal{P}$ if $L$ is polynomially decidable.

**Definition 1.2.5.** A problem $L \in \mathcal{NP}$ if $L$ is polynomially verifiable. Thus, $\mathcal{P} \subseteq \mathcal{NP}$.

To prove a bound of complexity on an unknown problem $L$ we have to find another problem with already known complexity and find equivalences between those two. This can be achieved through *reductions*.

**Definition 1.2.6.** A reduction of a problem $L$ to a problem $M$ is a mapping of an instance of $L$ ($I_L$) to an isntance of $M$ ($I_M$) such that $I_L$ is true for the problem $L$ if and only if $I_M$ is true for the problem $M$. This is noted $L \leq M$ and $L \leq_P M$ if the reduction is done in polynomial time.

With this concept we can define new complexity classes. $\mathcal{NP}$-hard is the set of problems so that we can reduce every $\mathcal{NP}$ problem to. The set of problems that are both $\mathcal{NP}$-hard and $\mathcal{NP}$ are called $\mathcal{NP}$-complete. This is generalized to every complexity class ($\mathcal{P}$, $\exists\mathbb{R}$, RP, etc...)

**Satisfiability problem** The satisfiability problem (SAT) is to decide the satisfiability of a CNF formula $\phi$. A CNF formula is a boolean formula that is a conjunction of multiple clauses $c_k$. A clause is a disjunction of multiple literals. A literal may be a variable or a negation of a variable.

**Theorem 1.2.7** (Cook-Levin)**.** *SAT is $\mathcal{NP}$-complete.*

**Clique problem** The clique problem is to find a maximum clique of a graph $G$.

**Theorem 1.2.8.** *CLIQUE is $\mathcal{NP}$-complete. [Kar72]*

**Theorem 1.2.9.** *CLIQUE is QPTAS when applied to disk graphs. [BGK+17]*

**Theorem 1.2.10** (Clark-Colbourn)**.** *CLIQUE is $\mathcal{P}$ when applied to unit disk graphs. [CCJ90]*

### 1.2.2 $\exists\mathbb{R}$ complexity class

$\exists\mathbb{R}$ is the class that describes the problems that can be reduced to *the existential theory of the reals*[Exi06a]. The existential theory of the reals is the problem of deciding if a sentence of this form is true:

$$(\exists X_1 \dots \exists X_n) : F(\exists X_1, \dots, \exists X_n)$$

where $F$ is a quantifier-free formula in the reals. In other words, it is a conjuntion of clauses where each clause is a real polynomial inequality where each variable $X_k$ is a real number. We can see that ETR is NP-hard because SAT can be reduced to it.

*Proof.* Let's take an instance of SAT $\phi_{SAT}$ with clauses $c_k$ and variables $x_k$, we can construct an instance of ETR $\phi_{ETR}$ where we can construct variables in the domain $\{0, 1\}$ with this equality, so for each variable $X_k$:

$$X_k - X_k^2 = 0$$

Each literal of each clause will be positive or negative depending if the literal is cancelled in $\phi_{SAT}$:

$$x_k \to l = X_k$$
$$\neg x_k \to l = (1 - X_k)$$

Then for each clause we can have a polynomial for which the sum of the values of every literal in the clause must be greater than one, so that at least one literal is true:

$$\sum_{l \in c_k} l \geq 1$$

With this proof, it is easy to see that $\phi_{ETR}$ is valid if and only if $\phi_{SAT}$ is also valid. □

□

This result can show us that $P \subseteq NP \subseteq \exists\mathbb{R}$.

#### Problems in $\exists\mathbb{R}$

In this section we will describe some problems that are $\exists\mathbb{R}$-complete and will give an overview of the proof.

**The art gallery problem** Given a simple polygon $P$ (without crossings between every side), we introduce *guards*. A guard $g$ is a point such that every point of the polygon is watched by a guard. A point $p$ is watched by a point $q$ if the segment $pq$ is contained in $P$. The subset $G$, being $G$ the set of guards and $G \subseteq P$, is optimum if it has the minimal cardinality covering the whole polygon.

The art gallery problem is to decide, given a polygon $P$ and a number of guards $k$, whether there exists a configuration of $k$ guards in $G$ guarding the whole polygon. The art gallery problem is $\exists\mathbb{R}$-complete [AAM17].

**Proof idea** First of all, we can see that the art gallery problem is in $\exists\mathbb{R}$ if we reduce this problem to ETR. If we have an instance $(P, k)$ of the art gallery problem we can have a formula [EH06] like this:

$$\phi = \{\exists x_1 y_1, \dots x_k y_k \forall p_x p_y : \text{INSIDE-POLYGON}(p_x, p_y) \to \bigvee_{1 \leq i \leq k} \text{SEES}(x_i, y_i, p_x, p_y)\}$$

Where INSIDE-POLYGON returns 1 if $(p_x, p_y) \in P$ and SEES returns 1 if the segment $(x, y)(p_x, p_y) \in P$. $\phi$ is not a ETR formula, so we would like to construct a quantifier-free formula with the idea of $\phi$. To achieve this, the main idea is to have a small set of points $Q \subseteq P$ such that if these points are watched, the whole polygon is watched. This subset $Q$ is called the *witness set*. The only thing is now to create a polynomial for each point that ensures that the point is watched by a guard.

To finish the proof we have to prove that the art gallery problem is $\exists\mathbb{R}$-hard. For this part an $\exists\mathbb{R}$-complete problem has been deducted from ETR. For the problem ETR-INV we have a set of variables $\{x_1, \dots, x_n\}$ and a set of equations of this form:

$$x = 1, \quad x + y = z, \quad x \cdot y = 1$$

and the problem decides if it exists a solution to this set of equations such that the value of each variable is real in $[\frac{1}{2}, 2]$.

A reduction of ETR-INV is found to the art gallery problem by constructing a polygon $P$ and finding a number $g$ for that polygon such that the instance of ETR-INT is true if and only if $P$ is covered by at most $g$ guards.

**Stretchability** A pseudoline is a simple closed curve in the plane. The stretchability problem is to decide if given a pseudoline arrangement, it is equivalent to an arrangement of straight lines.

**Proof idea** ETR can be reduced to STRETCHABILITY due to Mnev's universality theorem. [Sch10]

**Unit disk graph recognition** The unit disk graph recognition is the problem that decides if a graph $G$ is a unit disk graph. Unit disk graph recognition is $\exists\mathbb{R}$-complete. [Sch13a]

**Proof idea** UDG recognition is a corollary of deciding whether a graph with a given length is realizable. This problem is $\exists\mathbb{R}$-complete.

The reduction is done from STRETCHABILITY [Sch13a]. The reduction is done by adding a vertex to $V$ for each pseudoline intersection. For each three consecutive points $u_1, u_2, u_3$ along a pseudoline a widget will be added that will be only realizable if and only if the pseudoline can be stretched with the same arrangement.

## 1.3 Geometry

**Definition 1.3.1.** $\operatorname{dist}(a, b)$ denotes the distance between the points $a$ and $b$ and is calculated with:

$$\operatorname{dist}(a, b) = \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2}$$

The intersection of convex objects is a matter well studied for multiple subjects. In our case, it is interesting to know some properties about the intersection of disks, those ones being convex objects.

A set $S$ is convex if:

$$\forall p, q \in S \ \forall \lambda \in [0, 1] : (1 - \lambda)p + \lambda q \in S$$

### 1.3.1 Stabbing

A *stabbing* is a point that traverses a set of intersecting objects. A lot of research has been done [Sch13b] on the minimal amount of stabbings to cover every object in a set. Stabbings can also be done with more complex structures than points, in that case we are talking about *coverings*.

**Theorem 1.3.2** (Helly). *Given a set $Q$ of objects in $\mathbb{R}^d$, if for each subset of $Q$ of size $d + 1$ their intersection is non empty, then $\bigcap_{q \in Q} \neq \varnothing$. [Hel23]*

**Theorem 1.3.3.** *The problem that for a set of $n$ disks whether there exists a regular $n$-gon whose vertices stab every disk of the set can be decided in $O(n^{10.5}/\sqrt{\log(n)})$ [Sch13b]*

### 1.3.2 Coin graphs

Penny graphs can be defined as disk graphs where the disks can just touch each other without overlapping. A famous theorem is derived from this class of graphs: the circle packing theorem.

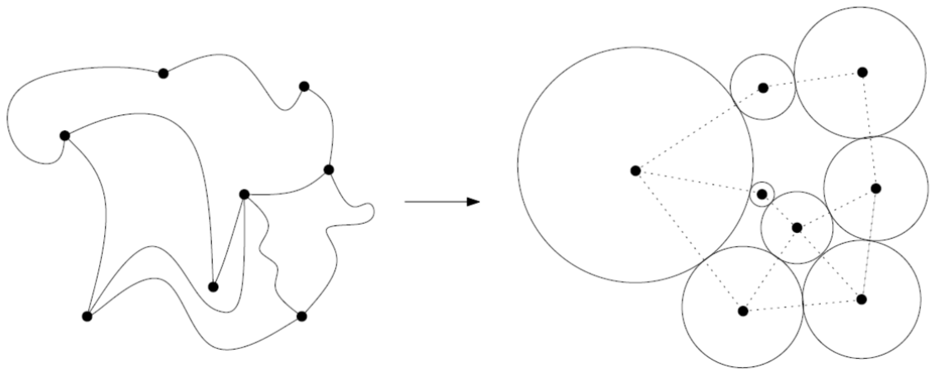


Figure 1.3: Circle packing of a planar graph. [Nac16]

**Theorem 1.3.4** (Circle packing theorem)**.** *The circle packing theorem states that every simple connected planar graph $G$ is a penny graph. [BS93]*

**Corollary 1.3.5.** *Planar graphs $\subseteq$ disk graphs [Spi12].*

# Chapter 2

# Interval graphs

An interval graph is a graph $G$ that is the intersection graph of a collection of closed intervals in $\mathbb{R}$.

First we present the main characterizations of interval graphs. In the next sections we present some other subclasses of interval graphs that will help us characterize the thin strip graphs on chapter 3.

## 2.1 Interval graphs

Interval graphs

**Theorem 2.1.1** (Fishburn [Fis85]). *$G$ is an interval graph if and only if every simple cycle of four or more points has a chord and any three independent vertices can be ordered ($u < v < w$) such that every path from $u$ to $w$ passes through a neighbour of $v$.*

## 2.2 Unit interval graphs

If the length of each interval is unitary, then $G$ is a unit interval graph (UIG). UIG is equivalent to the proper interval graphs class, where no interval can be properly included in another one [Rob68].

**Theorem 2.2.1** (Roberts [Rob68]). *An interval graph is a unit interval graph if and only if it has no induced subgraph $K_{1,3}$.*

## 2.3 Mixed unit interval graphs

Another interesting class of interval graphs are mixed unit interval graphs, where each interval is unitary and can be closed, open, open-closed or closed-open. In this paper we will denote those four classes like this:

$$\mathcal{I}^{++} = \{[x, y] : x, y \in \mathbb{R}, x \leq y\}$$

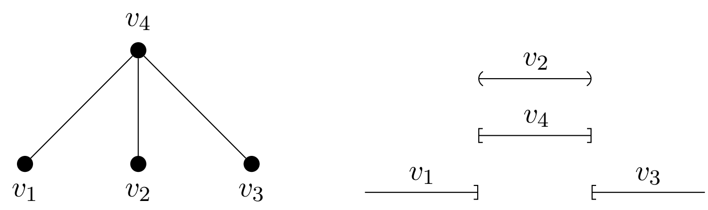



Figure 2.1: Representation of $K_{1,3}$ as a MUIG.

$$\mathcal{I}^{--} = \{(x, y) : x, y \in \mathbb{R}, x \leq y\}$$
$$\mathcal{I}^{+-} = \{[x, y) : x, y \in \mathbb{R}, x \leq y\}$$
$$\mathcal{I}^{-+} = \{(x, y] : x, y \in \mathbb{R}, x \leq y\}$$

$\mathcal{I}$ will be replaced by $\mathcal{U}$ when we are talking about unit mixed interval graphs and their class is denoted MUIG.

**Theorem 2.3.1.** *The classes of the graphs $\mathcal{U}^{--}$, $\mathcal{U}^{++}$, $\mathcal{U}^{-+}$, $\mathcal{U}^{+-}$, and $\mathcal{U}^{-+} \cup \mathcal{U}^{+-}$ are the same (equivalent for $\mathcal{I}$). [DLP+12]*

Unlike for UIG class, $K_{1,3}$ is a MUIG as seen in figure 2.1. A complete characterization by induced forbidden subgraphs have been found independently by F. Joos [SSTW14a] and A. Schuchat et al. [Joo13]. In the next subsection the characterization of F. Joos will be reviewed by adding some remarks about graph inclusions.

### 2.3.1 Characterization

Joos proves in his paper the following theorem:

**Theorem 2.3.2** (Joos [Joo13])**.** *$G$ is a MUIG if and only if it is a $\{F\} \cup \mathcal{R} \cup \mathcal{S} \cup \mathcal{S}'' \cup \mathcal{T}$-free interval graph.*

We will present each family of forbidden induced subgraphs and analyze whether they are included in other relevant classes of graphs (e.g. unit disk graphs). Only properties that will be used in next chapters will be presented,

**Lemma 2.3.3.** *$\mathcal{R}$ is a family of co-comparability graphs.*

*Proof.* If we recall Theorem 1.1.25, in order to prove that $\mathcal{R}$ is a family of co-comparability graphs we will have to find a spanning order for every $R_i$ with $i \geq 0$. We will proceed to label our vertices with a mapping function $f : V \to \mathbb{N}$ such that $f(v) \in [1, |V|]$. This mapping will give us a spanning order by induction:

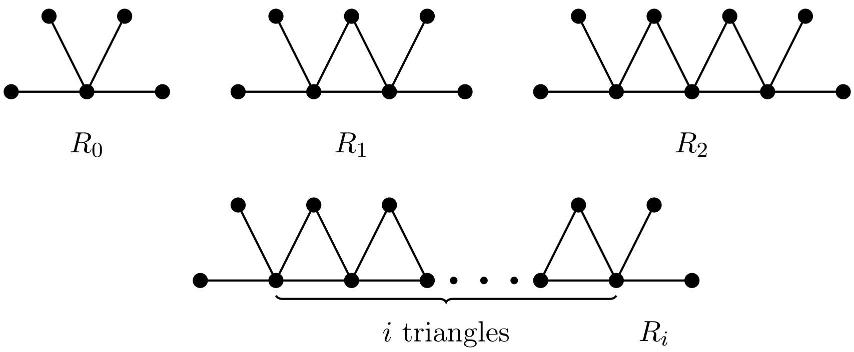



Figure 2.2: The class $\mathcal{R}$. [Joo13]

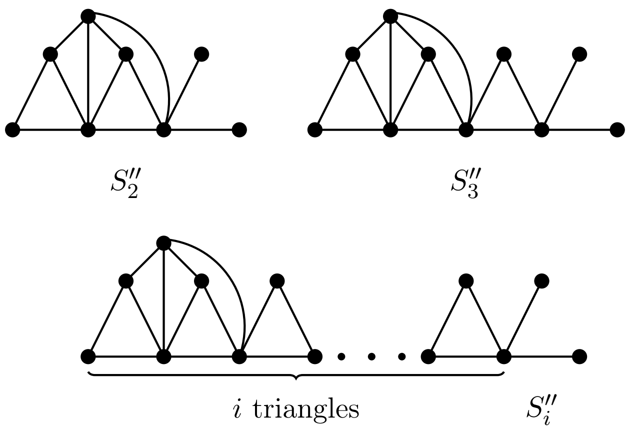



Figure 2.3: The class $\mathcal{S}''$.

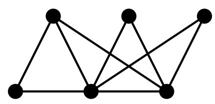
- $i = 0$: We assign the number 1 to the vertex with maximum degree $v_1$. We assign then the rest of the numbers to the other vertices. We see then that $\forall u < v < w : uw \in E \to uv \in E$ because every vertex is adjacent to $v_1$.

- $i = i + 1$: We define $\lambda_i = 5 + 2i$ where $\lambda_i = |V(R_i)|$. We have two on each graph, where their labels are $\lambda_i + 1$ and $\lambda_i + 2$ and there are three new edges: $v_{\lambda_i}v_{\lambda_i - 1}, v_{\lambda_i}v_{\lambda_i+1}, v_{\lambda_i}v_{\lambda_i+2} \in E$.

  By induction we only have to see if it holds with the new edges. We can say that it still holds with $v_{\lambda_i}v_{\lambda_i-1}$ and $v_{\lambda_i}v_{\lambda_i+1}$ because:

  $$\nexists k \in \mathbb{N} : i < k < i + 1$$

  Finally, we see that $v_{\lambda_i}v_{\lambda_i+2}$ is a valid edge because $v_{\lambda_i}v_{\lambda_i+1} \in E$. □



$F$

Figure 2.4: The graph $F$. [Joo13]

## 2.4 Unfettered unit interval graphs

An unfettered unit interval graph can be defined as an interval graph such that for every touching

Hayashi has characterized this class of graphs by levels. A *level structure* of a graph $G = (V, E)$ is a partition $L = \{L_i : i \in [1, t]\}$ of $V$ such that

$$v \in L_k \to N(v) \subseteq L_{k-1} \cup L_k \cup L_{k+1}$$

where $L_0 = L_t + 1 = \varnothing$.

**Theorem 2.4.1** (Hayashi et al. [HKO+17])**.** *A graph $G$ is an unfettered unit interval graph if and only if it has a level structure where each level is a clique.*

We can clearly see that MUIG $\in$ UUIG. However, we still have to see what is the location of UUIG in the higher graph classes hierarchy:

**Proposition 2.4.2.** *UUIG $\subset$ co-comparability.*

*Proof.* For each vertex of a partition $L_k$ of UUIG (Theorem 2.4.1) we assign arbitrarily a number $i \in [\max(V(L_{k-1})) + 1, \max(V(L_{k-1})) + |V(L_k)| + 1]$; intuitively, we assign every available number from the beginning in order ($|V(L_1)|$ first numbers on the first partition and consecutively).

Because we know that each partition $L_k$ is a clique, we can say that for each three vertices $u < v < w$, if $vw \in E \to uv \in E$ or $vw \in E$. We know this because given $u \in L_i$ and $w \in L_j$: if $uw \in E$ it means that levels $L_i$ and $L_j$ are adjacent, which means that $v \in L_i$ or $v \in L_j$ so $v$ will be adjacent either to $u$ or $w$. $\square$

# Chapter 3

# Thin strip graphs

Introduction of the chapter.

Explain Breu's characterization of strip graphs

$c$-strip graphs are unit disk graphs such that the centers of the disks belong to $\{(x,y) : -\infty < x < \infty, 0 \leq y \leq c\}$. The class is denoted by SG($c$). We have SG(0) = UIG and SG($\infty$) = UDG. [HKO$^+$17]

**Definition 3.0.1.** Thin strip graphs are defined as TSG $= \bigcap_{c>0}$ SG($c$).

*Remark* 3.0.2. SG(0) $\neq$ TSG. We can construct a $K_{1,3}$ such that we have 3 vertices with the coordinates $(1,0)$, $(0,0)$, $(1,0)$ and a last one $(0,\varepsilon)$ with $\varepsilon > 0$ and arbitrarily small as seen in Figure 3.1.

**Theorem 3.0.3** (Hayashi et al. [HKO$^+$17])**.** *There is no constant t such that SG(t) = TSG.*

Since this class is newly defined we have to characterize it. For this purpose, some relations have been found between this class and interval graphs.

### 3.0.1 Interval graphs

**Theorem 3.0.4** (Hayashi et al. [HKO$^+$17])**.** *MUIG $\subsetneq$ TSG.*

We can define a new class of graphs: unfettered unit interval graphs. These graphs are unit interval graphs where if two intersections touch, we can decide whether they intersect or not. We denote this class UUIG.

**Theorem 3.0.5** (Hayashi et al. [HKO$^+$17])**.** *TSG $\subsetneq$ UUIG.*

## 3.1 Characterization of thin strip graphs

One of the main goals of this thesis is to characterize TSG. by forbidden induced subgraphs. To approach this, we will see how many induced forbidden subgraphs are also forbidden for TSG. We have described the familes

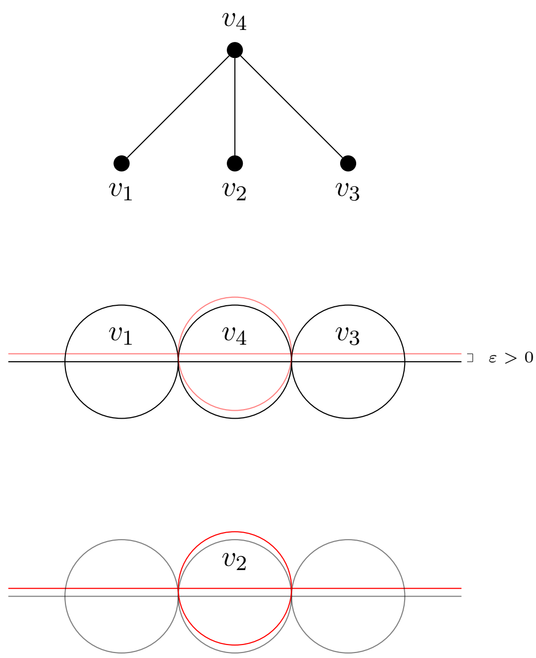



Figure 3.1: A construction of $K_{1,3}$ with a disk realization, being this graph a TSG.

of forbidden induced subgraphs for MUIG in section 2.3 and one of these familes has been proven to be a forbidden induced subgraph for TSG:

**Theorem 3.1.1** (Hayashi et al. [HKO+17])**.** *$\mathcal{R}$ is a forbidden induced subgraph family of TSG.*

With the properties presented in this chapter, we can begin to state our first hypothesis:

**Hypothesis 3.1.2.** *$F \in (UDG \cap UUIG) \setminus TSG$.*

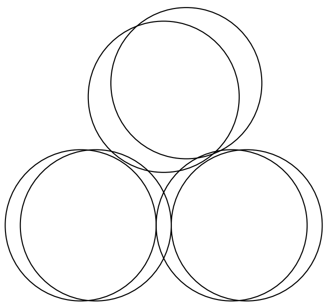


Figure 3.2: A construction of $F$ with a unit disk realization.

**Claim 3.1.3.** $F \in (UDG \cap UUIG)$

*Proof.* We can see that $F \in$ UDG because we can have a unit disk realization (Figure 3.2) and also has a level structure $L = \{K_2, K_3, K_1\}$. □

**Theorem 3.1.4.** *$F \notin TSG$.*

To prove this theorem we will first state some claims and observations on behalf of the characteristics of this graph.

Symsolver SO HARD.

**Definition 3.1.5.** The *critical region* of two points is noted as $\alpha_{ab} = \sqrt{1 - (a_y - b_y)^2}$. For a $c$-strip graph we have $\alpha_{ab} \leq \sqrt{1 - c^2}$.

**Observation 3.1.6.** *If a graph $G = (V, E)$ is a diamond with $V = \{a, b, c, d\}$ where $ad \notin E$, then:*

$$dist(a, d) \leq 2$$

*which also means that*

$$|a_x, d_x| \leq 2\alpha_{ad}$$

*and also, $b_x, c_x \in [a_x, d_x]$.*

*Proof.* The distance between $a$ and $d$ has to be at most 2, because there is at least one vertex that is adjacent to both. And clearly, the other points have to be between them because they are adjacent to both, so if they were not.

## 3.2 Recognition

The recognition of this class of graphs is stated by Breu in his thesis .

explain everything about dangerous cycles and complement oriented graphs in Breu's paper

# Chapter 4

# Thin two-level graphs

Breu [Bre96] has presented in his thesis a similar class of constrained unit disk graphs where the disks are placed on $k$ horizontal parallel lines. More formally: a disk $(x, y)$ can be placed in $x \in (-\infty, \infty)$ and $y \in L$ with $|L| = k$.

In this chapter I define thin two-level graphs as a two-level graph where $L = \{0, \epsilon\}$ and $\epsilon$ is an arbitrarily small real number.

## 4.1 Characterization of TTL

**Proposition 4.1.1.** *An $\sigma$-SG(c) graph $G$ (with $c < 1$) can be characterized by computing $\delta : A \times B \to E$ where $A, B \subseteq G$, $A$ and $B$ are UIG, and $A \cup B = \varnothing$:*

$$\delta(x, y) = \begin{cases} xy & \textit{if } \mathit{dist}(x, y) \leq 1 \\ \varnothing, & \textit{otherwise} \end{cases}$$

*This class of graphs is close to our main class TSG. But we have to know at what point we can rely in this class of graphs to study TSG:*

this proposition is ugly, redo this with Breu's notations.

**Lemma 4.1.2** (Breu [Bre96])**.** *Let abcda be a chordless 4-cycle in a two-level graph $G = (V, E)$. Then ad and bc are level edges (they are adjacent in the same level), and the others are cross edges for every realization $\phi$ of $G$ for which $\phi(a)_x < \phi(c)_x$ and $\phi(b)_x < \phi(d)_x$.*

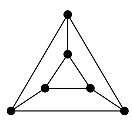


Figure 4.1: Forbidden graph in TTL

*With this preliminary result, we proceed to one of our main results:*

**Theorem 4.1.3.** *TTL* $\subsetneq$ *TSG*

*Proof.* By definition, we know that TTL $\subset$ TSG because the area where the disks can be placed in TTL is included in the area in TSG.

We can prove that TTL $\neq$ TSG because we can construct a graph $G$ such that $G \in$ TSG and $G \notin$ TTL. This graph $D$ is a net* graph as described in Figure 4.1.

**Part 1.** *$D$ is a TSG because we can realize it as a TSG if we take as center of disks $(0,0)$, $(0,z)$, $(0,\epsilon)$, $(1,0)$, $(1,z)$, $(1,\epsilon)$ such that $0 < z < \epsilon$.*

**Part 2.** *Now we have to prove that $D$ is a forbidden induced subgraph of TTL. We will try to construct it by taking a induced subgraph that is realizable: we take $D_{-1} = D - x$ with $x \in V(D)$. We notice that $V(D_{-1})$ is a chordless $C_4$ (abcd) with a vertex $e$ adjacent to any two consecutive vertices $x, y \in V(C_4)$ creating the triangle xye.*

*By Lemma 4.1.2 we know that abcda is a cycle if ab and cd are level edges. We can classify these vertices in two sets: $\ell(V) = a, d$ and $r(V) = b, c$ where $\forall u \in \ell(V) v \in r(V) : \phi(u)_x < \phi(v)_x$.*

*To realize $D_{-1}$ we have to add a vertex $e$ to $C_4$. In our case, $e$ cannot be added in two*

Finish proof here

### 4.1.1 Relation with MUIG

A big question that was asked during this research is: What is the relationship between TTL and MUIG? We know that the net*-graph is forbidden for every two-level graph, this graph is also forbidden for MUIG because it includes an induced $C_4$; we can also see that every forbidden induced subgraph for MUIG also is for TTL.

However, $C_4$ is realizable for two-level graphs with $c = 1$ in general, which means TTL $\neq$ interval = MUIG. We then know that TTL $\cup$ MUIG $\subseteq$ TSG with TTL $\neq$ MUIG.

Rewrite this better.

## 4.2 Induced forbidden subgraphs

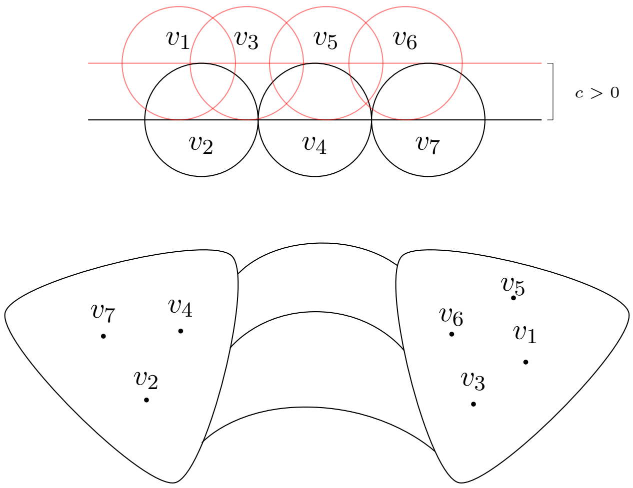



Figure 4.2: A representation of a TL($c$)

# Chapter 5

# Complexity

## 5.1 Recognizing Thin Strip Graphs

# Conclusions

The conclusions are to be written with care, because it will be sometimes the part that could convince a potential reader to read the whole document.

# Appendices

# Appendix A

# Graph classes hierarchy

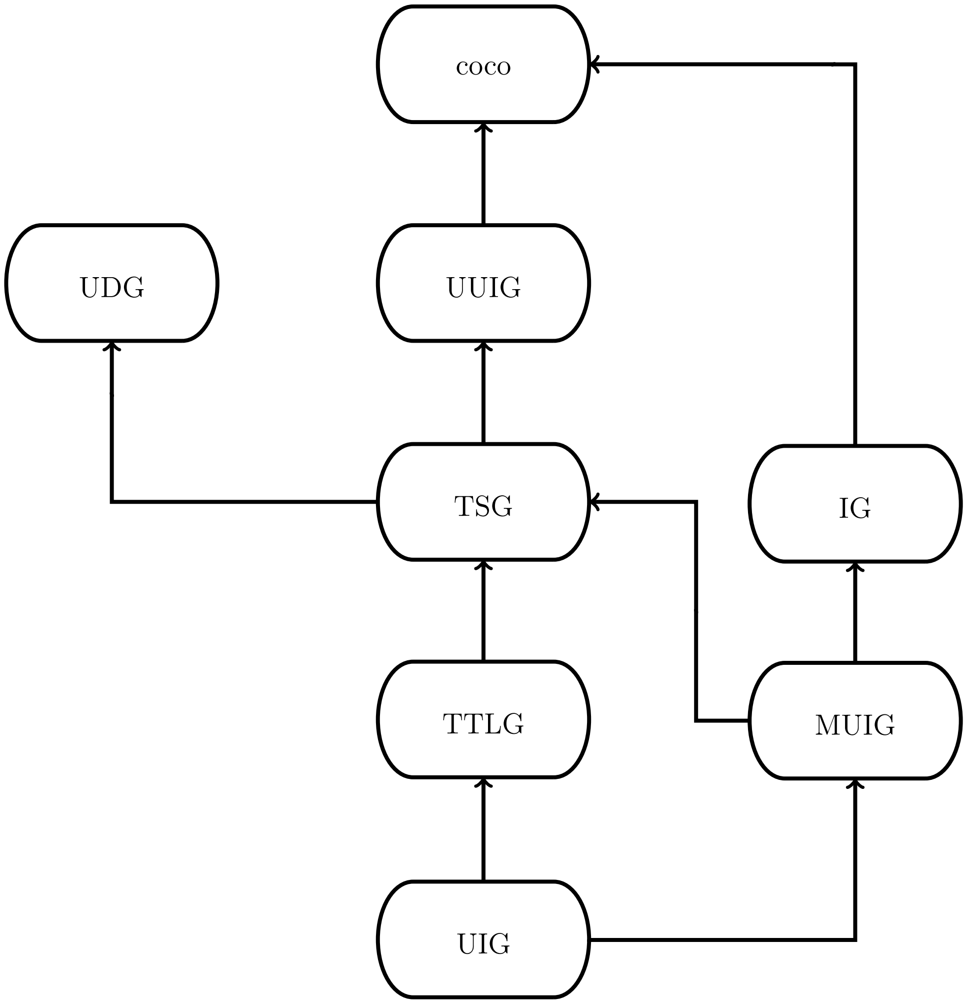



Figure A.1: A hierarchy of every relevant graph of this document. The relation $\text{class}_1 \to \text{class}_2$ means that $\text{class}_1 \subset \text{class}_2$.

# Appendix B

# Problems in inclusion

- **MUIG $\subsetneq$ TSG $\subsetneq$ UUIG** : Hayashi [HKO+17]
- **MUIG $\neq$ TTLG (Open)**: To prove that MUIG $\subsetneq$ TSG, Hayashi [HKO+17] could simulate MUIGs with 4 different levels. Having only two levels, I conjecture that this is not possible. However, MUIG can have $C_4$, so an inclusion between these two classes is impossible (it has to be rewritten).
- **TTLG $\subsetneq$ TSG (Open)**: This problem has been solved in my thesis by finding a forbidden graph for TTLG, theorem 4.1.3.
- **TLG $\subset$ TSG (Open)**: This is a plausible stronger statement than the one before. However, this result could make the study of TTLG less relevant. Thus, this result would imply:

$$G \in \mathrm{TLG}(j) \to G \in \mathrm{SG}(k) : j, k \in \mathbb{R}$$

# Appendix C

# Problems in forbidden induced subgraph characterization

- **MUIG**: Joos [Joo13] gives us a complete characterization of forbidden graphs.

- **TSG (Open)**: Hayashi [HKO$^+$17] says that MUIG's forbidden induced subgraphs also are in TSG. He claims that finding a graph $F \in (\mathrm{UDG} \cap \mathrm{UUIG}) \setminus \mathrm{TSG}$ could be a good starting point. In my thesis I show that a forbidden induced subgraph for MUIG is in UDG∩UUIG.

- **TTLG (Open)**: There are many properties about these graphs in Breu's thesis [Bre96].

- **UDG (Open)**: There is no complete characterization of UDG. Can the results of this thesis help find new ones?U

# Appendix D

# Problems in complexity

- **UIG/IG recognition**: Both of these problems are polynomial.
- **MUIG recognition**: Schuchat et al. give a linear algorithm ($O(|V|^2)$) to recognise MUIGs [SSTW14b].
- **UDG recognition**: $\exists\mathbb{R}$-complete [Exi06b].
- **SG($c$) recognition (Open)**: Breu [Bre96] states that SG($c$) recognition is polynomial if a complement edge orientation and a mapping $\phi : V \to [0, c]$ is polynomial as an input of the decision problem.
- **TSG recognition (Open)**: Can we get rid of the mapping as input to recognise TSGs? In that case the problem would be at least NP.
- **UUIG recognition (Open)**: Informally the recognition of this class of graphs **cannot** be polynomial because we have to find all the cliques of the graph; the CLIQUE problem is NP-complete.

# Index

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# Todo list