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Characterization and complexity of Thin Strip Graphs

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Acknowledgment

I want to thank ...

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Chapter 1

Introduction

> *There's no such thing in the world as absolute reality. Most of what they call real is actually fiction; what you think you see is only as real as your brain tells you it is. It's not whether you were right or wrong, but how much faith you were willing to have, that decides the future.*
>
> — **Solid Snake**
> (Metal Gear Solid 2)

This work is mainly focused on the characterization and complexity of variants of unit disk graphs, where the domain of possible locations for the disks is limited. We are also going to see their close relation to a certain family of interval graphs. In this chapter, we will overview the open questions we will focus on and our main results. Further details about the results discussed in this chapter will be introduced later in the thesis as well as a background in Chapter 2.

Interval graphs

In Chapter 3 we introduce the concept of interval graphs and some of their use cases. An **interval graph** is a graph in which each one of its vertices is a closed interval on the real line and they are adjacent if they overlap; interval graphs where the length of its intervals is the same is called **unit interval graphs (UIG)**.

Moreover, we introduce two new subclasses of graphs. **Mixed unit interval graphs (MUIG)** [Joo13] can be seen as unit interval graphs but the endpoints of each interval can be open or closed. Another variant are **unfettered unit interval graphs (UUIG)** [HKO+17], where we can chose whether two touching intervals (so that one of their endpoints are in the same position) are adjacent or not.

Joos describes the class MUIG [Joo13] with a list of graphs that cannot be MUIGs. Also, Hayashi et al. describe the class UUIG with the next theorem.

Theorem. *A graph is an UUIG if and only if it has a level structure such that each level is a clique.*

Finally, we take an algorithmic approach to study these classes of graphs. A graph recognition problem for a class of graphs is the problem to guess whether a given graph is of a certain class. The recognition of MUIG is of $\mathcal{O}(n^2)$ [TK14] and the recognition of UUIG is only overviewed. For the moment, we know that recognition of UUIG is in $\mathcal{NP}$.

Strip graphs

In Chapter 4 we introduce the main class of graphs of this thesis. **Unit disk graphs (UDG)** are intersection graphs of disks on a plane when the diameter of the disks are unitary. **c-strip graphs (SG(c))** [Bre96] is a subclass of UDG, where the center of the disks can only be located between two horizontal lines with a separation of c. More formally, for each disk v in the graph G, $v_y \in [0, c]$. Breu [Bre96] defined this class of graphs and studied early phases of its charaterization and recognition. However, this is not complete as there is still no answer to the complexity of TSG recognition.

Thin strip graphs

Thin strip graphs (TSG) is a subclass of UDG that that can be defined as the intersection of every SG(c) with $c > 0$. This is equivalent to say that $TSG = SG(\varepsilon)$ with ε and arbitrarily small number. Hayashi et al. [HKO$^+$17] present this class of graphs in their work and found some interesting properties about them.

Theorem. *There is no constant t such that $TSG = SG(t)$.*

More importantly, TSG is well located in the hierarchy of the graphs seen until now. We know that $MUIG \subsetneq TSG \subsetneq UUIG$. This helps us to find a characterization for TSG because we know that the characterization of MUIG is complete. We also see that every forbidden graph for MUIG is also forbidden in TSG except for one of them.

Two-level graphs

to add or not? we'll see in the end

Chapter 2

Background

In this chapter we review some definitions and notations used in this thesis. We limit ourselves to the basic notations used during the work. However, the bibliography of each subject will be referenced for further details about the topic.

2.1   Graph theory

A **graph** is defined as a tuple $G = (V, E)$ where V is the set of **vertices** and E is a set of **edges** where $E = \binom{V}{2}$. An **orientation** of a graph G is an assignment of a direction to each edge, we denote the orientation of the edges by $\overrightarrow{E}$. An orientation is **transitive** if $uv \in \overrightarrow{E}$ and $vw in \overrightarrow{E}$, then $uw \in \overrightarrow{E}$. If two vertices are share the same edge e they are called **adjacent** and also the **endpoints** of e. A **subgraph** $H = (V', E')$ of a graph G is a graph such that $V' \subseteq V$ and $E' \subseteq E$. An **induced subgraph** of a graph is a subgraph H of a graph G such that for every edge of G is also in H if its two endpoints are in V'. A **clique** is a subgraph such that every vertex is adjacent to each other. A graph that is also a clique is called a **complete graph** and it is denoted as K_n. A graph is **bipartite** if there exist two disjoint subsets of the vertex set $A \cup B = V$ such that two vertices of the same subset are not adjacent. A **complete bipartite graph** $K_{n,m}$ is a bipartite graph such that $v \in A$ and $w \in B$ implies $vw \in E$ where n and m are the size of each bipartition.
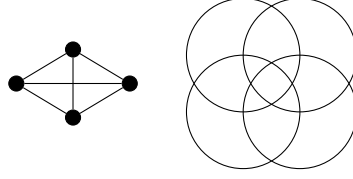
Figure 2.1: Realization of a UDG (unit disk graph).

A **path** $P_n = v_1 \ldots v_{n+1}$ of a graph is a sequence of pairwise distinct n vertices such that two consequent vertices are adjacent. A **cycle** is a path $C_n = v_1 \ldots v_n v_{n+1}$ such that $v_1 = v_{n+1}$. A graph is **connected** if there exists a path between every pair of vertices. A **chord** of a cycle C_n with $n \geqslant 4$ is an edge that connects two non adjacent vertices of the cycle. A graph is **chordal** if there is a chord in every cycle bigger than four.

Some graphs can be characterized with properties. An **isomorphism** between two graphs $G = (V, E)$ and $H = (V', E')$ is a bijection $f : V \to V'$ between the two vertex sets such that u, v are adjacent in G if and only if $f(u), f(v)$ are adjacent in H. A graph **property** is a property of the graph that is preserved in all its isomophisms; this will help us to set properties that are based on the abstraction of the graph and not only its drawings. A property is **hereditary** if it is also preserved under all taking subgraphs.

For notation in this thesis, sometimes the class of a certain type of graphs is denoted by its initials (*e.g.* the class of unit interval graphs is denoted by **UIG**) to avoid extreme repetition.

2.1.1   Intersection graphs

An **intersection graph** is a graph $G = (\zeta, E)$ of a collection of objects ζ is a graph such that $v, w \in \zeta$ and $v \cup w \neq \varnothing$ implies that $vw \in E$. An **interval graph** is an intersection graph of intervals on the plane; when the size of the intervals is equal they are called **unit interval graphs**. A **unit disk graph** is an intersection graph of disks on a plane that have the same diameter - you can find an example in Figure 2.1.
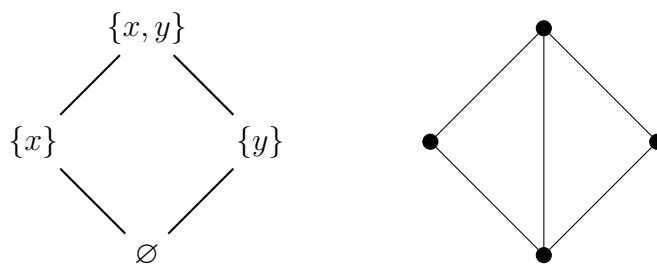
Figure 2.2: On the left, Hasse diagram of a poset of the power set of 2 elements ordered by inclusion. On the right, the comparability graph of this poset.

For more details about graph theory we recommend

2.2 Order and set theory

The **powerset** $\wp(S)$ of a set S is the set of subsets of S. A **partial order** is a binary relation $\leqslant$ over a set A satisfying three axioms:

- if $a \leqslant b$ and $b \leqslant a$ then $a = b$ (**antisymmetry**).

- if $a \leqslant b$ and $b \leqslant c$ then $a \leqslant c$ (**transitivity**).

- $a \leqslant a$ (**reflexivity**).

On the other side, a **total order** is a partial order where the reflexivity order is replaced by the **connexity** property – $a \leq b$ or $b \leq a$. A **partially ordered set** (or **poset**) $(S, \leqslant)$ is a set such that the elements of S are partially ordered by the relation $\leqslant$. A good way to represent a poset is the **Hasse diagram** (Figure 2.2).

2.2.1 Comparability graphs

A **spanning order** $(V, <)$ on a graph $G = (V, E)$ is a total order on V such that for any three vertices $u < v < w$:

$$uw \in E \rightarrow uv \in E \text{ or } vw \in E$$

The class of comparability graphs are built on the ideas of order theory. A graph G is a **comparability graph** if there exists a partial order $\leqslant$ such that $uv \in E \Leftrightarrow v \leqslant w$ or $w \leqslant v$. The complement of comparability graphs are called **co-comparability graphs**.

2.3 Complexity

Complexity theory has the objective to establish lower bounds on how efficient an algorithm can be for a given problem. This approach let us have a reference point to establish the difficulty of a problem. A **decision problem** is a problem where we have to decide if a statement is true or false. A **decider** of a decision problem is defined as the deterministic machine that solves this problem. The problem is **polinomially decidable** if it has a polynomial time decider. A **verifier** of a decision problem is a deterministic machine that verifies whether an answer to the decision problem is true or false. Equally, a problem is **polinomially verifiable** if it has a polynomial time verifier. We denote by $\mathcal{P}$ the class of polinomially decidable problems. On the other hand, $\mathcal{NP}$ denotes the class of polinomially verifiable problems. We can see that $\mathcal{P} \subseteq \mathcal{NP}$.

A **reduction** of a problem L to a problem M is a mapping of an instance of L (I_L) to an instance of M (I_M) such that I_L is true for the problem L if and only if I_M is true for the problem M. This is denoted by $L \leq M$ and $L \leq_P M$ if the reduction is done in polynomial time. We usually prove bounds of complexity for an unknown problem L by reducing it to another problem with an already known complexity. Thus, we can define the class $\mathcal{NP}$-**hard** as the set of problems such that we can reduce every $\mathcal{NP}$ problem to one of them. The set of problems that are both $\mathcal{NP}$ and $\mathcal{NP}$-hard are called $\mathcal{NP}$-**complete**.

2.4   Geometry

We must recall some really basic definitions of geometry. Every geometrical object of this thesis is located in $\mathbb{R}^2$ if it is not otherwise specified. The **distance** between two points as $\text{dist}(a, b)$. An object S is **convex** if for every point p, q the segment between the two points is also contained in S. More formally:

$$\forall \lambda \in [0, 1] : (1 - \lambda)p + \lambda q \in S$$

A **stabbing** is a point that traverses a set of intersecting objects. A lot of research has been done [Sch13] on the minimal amount of stabbings to cover every object in a set. If instead of points we use more complex object, we denote it by a **covering**. The **Helly** theorem says that:

Theorem (Helly ([Hel23])). *Given a set S of objects in $\mathbb{R}^d$, if for each subset of S of size $d + 1$ their intersection is non empty, then $\bigcap_{s \in S} \neq \varnothing$.*

We say that a set S satisfies the **Helly property** if every subfamily of S composed of pairwise intersecting objects has also a non-empty intersection. For more details about algorithmic geometry, we recommend the reading of Berg *et al.* [dB08].

Chapter 3

Interval graphs

> *There's no such thing in the world as absolute reality. Most of what they call real is actually fiction; what you think you see is only as real as your brain tells you it is. It's not whether you were right or wrong, but how much faith you were willing to have, that decides the future.*
>
> — **Solid Snake**
> (Metal Gear Solid 2)

The goal of this chapter is to present the family of classes of interval graphs that are related to the class of thin strip graphs. We introduce the class of interval graphs, which is one of the most used classes of intersection graphs. There are multiple types of interval graphs and those that are the most relevant for the thesis are going to be defined below.

First, we recall the basic definition of an interval graphs and their multiple characterizations. Also, we present unit interval graphs, where we see their characterization and complexity such as Robert's characterization [Rob68]. Then, we see some characterizations such as Joos's paper about mixed unit interval graphs [Joo13] and the paper from Hayashi *et al.* [HKO+17] where

the unfettered unit interval graphs are defined and also characterized as well as some equivalences with **unit disk graphs** are presented.

3.1 Interval graphs

First we present the main characterizations of interval graphs. In the next sections we present some other subclasses of interval graphs that will help us characterize the thin strip graphs on Chapter 4. There are multiple characterizations of interval graphs that are equivalent, in this thesis we present Gilmore and Hoffman's characterization described in Theorem 3.1.1. From this theorem it is clear that *IG* class is a subclass of the *CO-CO* class.

Theorem 3.1.1 (Gilmore and Hoffman [GH64]). *G is an interval graph if and only if G does not contain C_4 as an induced subgraph and $\overline{G}$ can be ordered partially, in other words, $\overline{G}$ is a comparability graph.*

The first interesting subclass of IG is the class of **unit interval graphs** which is defined by the interval graphs that have intervals with the same length (or equal to one). This class of graphs is equivalent to the class of **proper interval graph** which is the class of intervals where no interval is a strict subset of another. This statement is powerful because the study of unit interval graphs can be more confortable because of the simplicity of its definition and characterization as seen in Theorem 3.1.2.

Theorem 3.1.2 (Roberts [Rob68]). *An interval graph is a unit interval graph if and only if it has no induced subgraph $K_{1,3}$ (or* **claw***).*

3.2 Mixed unit interval graphs

We can define a new class of graphs that is related to UIG by its definition. This class is closely related to thin strip graphs as we will see in Chapter 4. **Mixed unit interval graphs** are graphs where the intervals have the same size as the unit interval graphs. However, in this class, the endpoints of the intervals can be open or closed - or one of each.
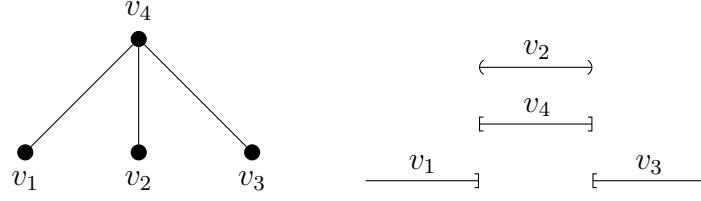
Figure 3.1: Representation of $K_{1,3}$ as a MUIG.

Formally, MUIG is defined by using the next classes of graphs:

$$\mathcal{U}^{++} = \{[x, y] : x, y \in \mathbb{R}, x \le y\}$$

$$\mathcal{U}^{--} = \{(x, y) : x, y \in \mathbb{R}, x \le y\}$$

$$\mathcal{U}^{+-} = \{[x, y) : x, y \in \mathbb{R}, x \le y\}$$

$$\mathcal{U}^{-+} = \{(x, y] : x, y \in \mathbb{R}, x \le y\}$$

where $\mathcal{U}^{++} = \text{UIG}$.

Dourado, by defining these classes of unit interval graphs with open/closed intervals also found that, for unit interval graphs, it does not matter if the endpoints are open, closed, or closed open (Theorem 3.2.1).

Theorem 3.2.1 (Dourado et al. [DLP$^+$12a]). *The classes of the graphs $\mathcal{U}^{--}$, $\mathcal{U}^{++}$, $\mathcal{U}^{-+}$, $\mathcal{U}^{+-}$, and $\mathcal{U}^{-+} \cup \mathcal{U}^{+-}$ are the same.*

However, MUIG is defined as $\mathcal{U}^{++} \cup \mathcal{U}^{--} \cup \mathcal{U}^{+-} \cup \mathcal{U}^{-+}$ which is also denoted as $\mathcal{U}$. In this case it is clear that this class is not equivalent to UIG. As we have seen in Theorem 3.1.2, a UIG can be seen as a $K_{1,3}$-free IG. Nevertheless, MUIG can accept this graph as seen in Proposition 3.2.2

Proposition 3.2.2 (Dourado *et al.* [DLP$^+$12b]). *MUIG has a $K_{1,3}$ representation. Also, for every MUIG representation $\phi : V(K_{1,3}) \to \mathcal{U}$ such that $\phi(V(K_{1,3}))$ contains:*

- $a = [x, x + 1]$

- $b = (x, x + 1)$

- $c = [x+1, x+2]$ *or* $[x+1, x+2)$

- $d = [x-1, x]$ *or* $[x-1, x)$

Proof. Let $\phi : V(K_{1,3}) \to \mathcal{U}$ be the representation of $K_{1,3}$ as a mixed unit interval intersection diagram as illustrated in Figure 3.1. Let $V(K_{1,3}) = \{v_1, v_2, v_3, v_4\}$ and $E(K_{1,3}) = \{v_1v4, v_2v_4, v_3v_4\}$. Let $x(v_1) = I(v_1) \cap I(v_4)$, $x(v_2) = I(v_2) \cap I(v_4)$ and $x(v_3) = I(v_3) \cap I(v_4)$. Because v_1, v_2 and v_3 are not adjacent, we can assume that $x(v_1) < x(v_2) < x(v_3)$. Since $x(v_1) \in I(v_4)$ and $x(v_3) \in I(v_4)$, then $x(v_3) - x(v_1) \leqslant 1$. Since $I(v_1), I(v_2)$ and $I(v_3)$ are disjoint, $I(v_2)$ must be a proper subset of $(x(v_1), x(v_3))$. Since $I(v_2)$ is a mixed unit interval, then it implies that $x(v_3) = x(v_1) + 1$, $I(v_2) = (x(v_1), x(v_1 + 1))$, $I(v_4) = [x(v_1), x(v_1) + 1]$, $I(v_1) = \{(x(v_1) - 1, x(v_1)], [x(v_1) - 1, x(v_1)]\}$ and $I(v_3) = \{[x(v_3) - 1, x(v_3)), [x(v_3) - 1, x(v_3)]\}$. $\square$

continue HERE

Theorem 3.2.3 (Dourado *et al.* [DLP+12b]). *UIG $\subsetneq$ MUIG.*

Proof. The strict inclusion is straightforward: we know that UIG $= \mathcal{U}^{++} \subset$ *MUIG* by definition. For the inequality, we prove it by Proposition 3.2.2, as $K_{1,3}$ is not realizable in UIG. $\square$

A complete characterization by induced forbidden subgraphs have been found independently by A. Schuchat et al. [SSTW14a] and F. Joos [Joo13]. However, the Schuchat paper gives a polynomial algorithm to recognize MUIGs:

Theorem 3.2.4 (Schuchat et al. [SSTW14b]). *The MUIG recognition problem is in $\mathcal{P}$. Moreover, there is an algorithm that solves it in $O(|V|^2)$ for V the vertex set of a graph.*

3.2.1 Characterization

In this section we will go over the characterization of MUIG given by Joos with forbidden subgraphs. We will also review each one of these forbidden subgraphs and discuss them:

Theorem 3.2.5 (Joos [Joo13])**.** *G is a MUIG if and only if it is a $\{F\} \cup \mathcal{R} \cup \mathcal{S} \cup \mathcal{S}'' \cup \mathcal{T}$-free interval graph.*
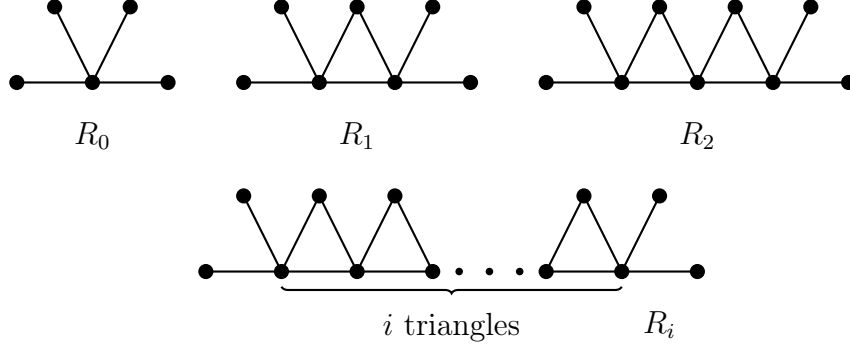

Figure 3.2: The class $\mathcal{R}$. [Joo13]

Without including F, every family of forbidden graphs of MUIG is infinite, and is defined recursively by its precedent: then every property of these graphs has to be proved recursively. We begin first with $\mathcal{R}$.

Lemma 3.2.6. *$\mathcal{R}$ is a family of co-comparability graphs.*

Proof. If we recall Theorem **??**, in order to prove that $\mathcal{R}$ is a family of co-comparability graphs we will have to find a spanning order for every R_i with $i \geq 0$. We will proceed to label our vertices with a mapping function $f : V \to \mathbb{N}$ such that $f(v) \in [1, |V|]$. This mapping will give us a spanning order by induction:

- $i = 0$: We assign the number 1 to the vertex with maximum degree v_1. We assign then the rest of the numbers to the other vertices. We see then that $\forall u < v < w : uw \in E \to uv \in E$ because every vertex is adjacent to v_1.

- $i = i + 1$: We define $\lambda_i = 5 + 2i$ where $\lambda_i = |V(R_i)|$. We add two vertices on each graph, where their labels are $\lambda_i + 1$ and $\lambda_i + 2$ and we also add three new edges: $v_{\lambda_i} v_{\lambda_i - 1}, v_{\lambda_i} v_{\lambda_i + 1}, v_{\lambda_i} v_{\lambda_i + 2} \in E$.
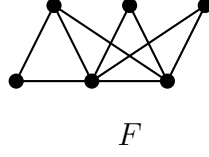
Figure 3.4: The graph F. [Joo13]

By induction we only have to see if it holds with the new edges. We can say that it still holds with $v_{\lambda_i}v_{\lambda_i-1}$ and $v_{\lambda_i}v_{\lambda_i+1}$ because:

$$\nexists k \in \mathbb{N} : i < k < i+1$$
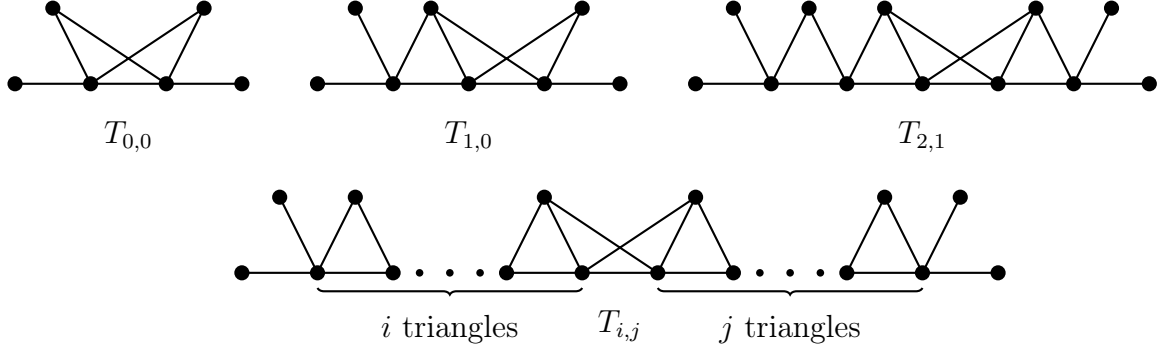
Finally, we see that $v_{\lambda_i}v_{\lambda_i+2}$ is a valid edge because $v_{\lambda_i}v_{\lambda_i+1} \in E$. $\qquad\square$
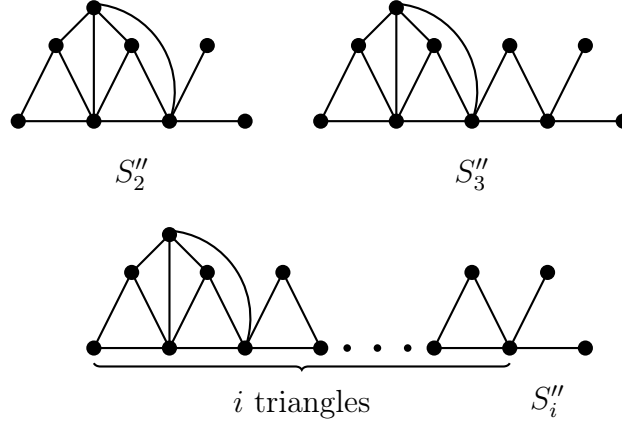
Figure 3.3: The class $\mathcal{S}''$. [Joo13]

3.3 Unfettered unit interval graphs

An unfettered unit interval graph can be defined as an unit interval graph such that for every touching endpoints we can chose either if they are adjacent or not.

Figure 3.5: The class $\mathcal{T}$. [Joo13]

Hayashi has characterized this class of graphs by levels. A ***level structure*** of a graph $G = (V, E)$ is a partition $L = \{L_i : i \in [1, t]\}$ of V such that

$$v \in L_k \to N(v) \subseteq L_{k-1} \cup L_k \cup L_{k+1}$$

where $L_0 = L_t + 1 = \varnothing$.

Theorem 3.3.1 (Hayashi et al. [HKO$^+$17])**.** *A graph G is an unfettered unit interval graph if and only if it has a level structure where each level is a clique.*

We can clearly see that MUIG $\in$ UUIG. However, we still have to see what is the location of UUIG in the higher graph classes hierarchy:

Proposition 3.3.2. *UUIG $\subset$ co-comparability.*

Proof. This proposition is equivalent to say that a graph G is a UUIG if and only if it has a spanning order.

For each vertex of a partition L_k of UUIG (Theorem 3.3.1) we assign arbitrarily a number $i \in [\max(V(L_{k-1})) + 1, \max(V(L_{k-1})) + |V(L_k)| + 1]$; intuitively, we assign every available number from the beginning in increasing order ($|V(L_1)|$ first numbers on the first partition and consecutively).

Because we know that each partition L_k is a clique, we can say that for each three vertices $u < v < w$, if $vw \in E \to uv \in E$ or $vw \in E$. We know this because given $u \in L_i$ and $w \in L_j$: if $uw \in E$ it means that levels L_i and

L_j are adjacent, which means that $v \in L_i$ or $v \in L_j$ so v will be adjacent either to u or w. This is a spanning order. $\qquad\square$

If we recall the characterization of MUIG in section 3.2.1, we can see that every forbidden graph of MUIG is an UUIG (except for $\mathcal{R}$); which means that they are also co-comparability graphs.

In the other hand, we can find a graph in UUIG that is not an UDG. This theorem will be used in Chapter 4.

Theorem 3.3.3 (Hayashi et al. [HKO$^+$17]). *UUIG $\neq$ UDG.*

Proof. We can define $G = (L_1 \cup L_2, E)$ a UUIG with two levels $L_1 = \{v_1, v_2, v_3, v_4\}$ and $L_2 = \wp(L_1)$ and $E = \binom{L_1}{2} \cup \binom{L_2}{2} \cup \{vw : w \in L_2, v \in w\}$.

We can see L_1 as a Venn diagram with four sets. We know by instance that a Venn diagram cannot be constructed if the number of sets is bigger than four [Ven80]. Thus, $G \notin$ UDG. $\qquad\square$

3.3.1 Recognition

As we mentioned in the previous section, UUIG is a class of graphs very relevant to define TSG and that is why we are interested in knowing how this class of graphs is recognized.

Lemma 3.3.4. *Let G be a connected UUIG with a level structure with levels $L_1, \ldots, L_n$. $G \setminus L_i$ is a graph where each connected component is also an UUIG and the number of connected components is not bigger than two.*

Proof. By definition for a graph with a level structure, if $v \in L_i$, $N(v) = L_{i-1} \cup L_i \cup L_{i+1}$. This said, if we delete a level L_i, L_{i-1} and L_{i+1} are disconnected, but they are still connected to the other consecutive levels (L_{i-1} is connected to L_{i-2}, which is connected to L_{i-3}... and viceversa with L_{i+1}).

And because a level is only adjacent to two other levels, we only have two connected components, only one if $L_i = L_1$ or $L_i = L_n$. $\qquad\square$

By this lemma we can suppose that the input graph G is a connected graph. Given an input graph G, we can compute a level structure where each level is a clique in exponential time.

Theorem 3.3.5. *UUIG recognition is in $\mathcal{NP}$.*

Proof. We can design a deterministic algorithm that solves UUIG recognition in exponential time when G is a connected graph. We begin by taking an arbitrary vertex $v \in G$. By instance, this vertex is included in the maximal clique $K \subseteq G$.

We have $P(K \setminus \{v\})$ the powerset of the clique K excluding v. For each subset $s \in P(K \setminus \{v\})$, we have a subgraph $H = G \setminus (s \cup \{v\})$. We recall that $q(G)$ denotes the set of connected components of G. We can have three cases:

1. $|q(H)| > 2$: If the number of components of H is bigger than 2, the current chosen clique is connected to more than two different cliques (or different levels); this is an invalid level, and we choose another clique from $s \in P(K \setminus \{v\})$.

2. $0 < |q(H)| \leq 2$: The current chosen clique is connected to one or two levels, which still respects our definition of level structure for the current chosen level. We check recursively if those connected components are also UUIG.

3. $|q(H)| = 0$: The current clique is an UUIG with only one level. This is a valid valid.

To prove that UUIG recognition is in UUIG, we have to prove that it is in $\mathcal{NP}$. We have an upper bound on the complexity of this algorithm that is given by:

$$T(n) \leq 2^{\omega(G)-1}T(n-1)$$

Which gives us:

$$T(n) \leq (2^{\omega(G)-1})^n = 2^{O(n\omega(G))}$$

with $\omega(G)$ the size of the maximum clique of G. $\qquad\square$

Future work on the recognition of unit unfettered interval graphs would be to adapt this algorithm to avoid combinatorial complexity. In our case we are interested in seeing the recognition of UUIG for unit disk graphs. We know that the CLIQUE problem is in $\mathcal{P}$ for unit disk graphs and the first hypothesis was that given an UUIG G, at least one level of G is a maximal clique of the graph. Nevertheless, we have a counterexample in $T_{0,0}$ (Fig. 3.5) where the levels of the graph are $\{K_1, K_2, K_2, K_1\}$ while $\omega(T_{0,0}) = 3$.

Observation 3.3.6. *Given an* **UUIG** *G, no level of G has to be a maximal clique.*

Chapter 4

Thin strip graphs

The goal of this chapter is to introduce you to the main subject of this thesis. Thin strip graphs is a class of graphs that lay between unit disk graphs and mixed interval graphs. We can define formally a c-strip graph as a unit disk graph such that the centers of the disks belong to $\{(x, y) : -\infty < x < \infty, 0 \leq y \leq c\}$, more intuitively we can see this as a unit disk graph where the centers of the disks lay between two parallel horizontal lines with a distance of c between them. We denote this class by $\mathrm{SG}(c)$. We have then that $\mathrm{SG}(0)$ = UIG and $\mathrm{SG}(\infty)$ = UDG.

The definition and main work for this class comes from Breu in his thesis [Bre96]. However, Hayashi et al. [HKO+17] expand his work by defining the class of **thin strip graphs**.

4.1 Thin strip graphs

A thin strip graph can be intuitively defined as a c-strip graph where c is an arbitrarily little ε. Also, we can see that $\mathrm{SG}(k) \subseteq \mathrm{SG}(l)$ with $k < l$. A more strict definition emerges from this observation:

Definition 4.1.1. Thin strip graphs are defined as $\mathrm{TSG} = \bigcap_{c>0} \mathrm{SG}(c)$.

Remark 4.1.2. $\mathrm{SG}(0) \neq \mathrm{TSG}$. We can construct a $K_{1,3}$ such that we have 3 vertices with the coordinates $(1, 0)$, $(0, 0)$, $(1, 0)$ and a last one $(0, \varepsilon)$ with $\varepsilon > 0$ and arbitrarily small as seen in Figure 4.1.
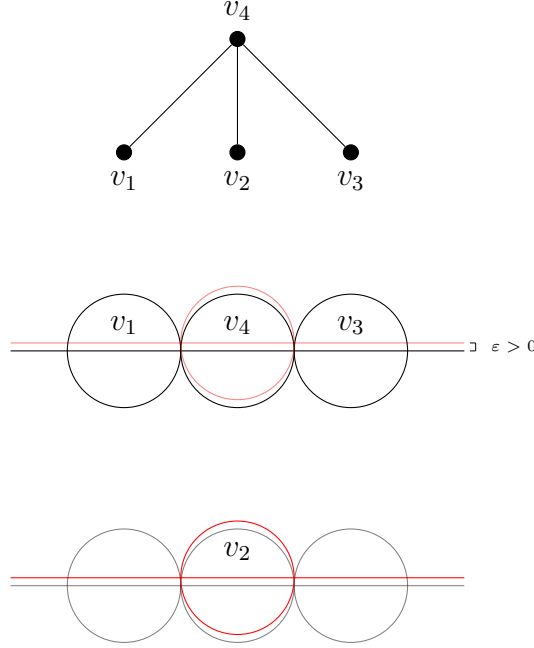
Figure 4.1: A construction of $K_{1,3}$ with a disk realization, being this graph a TSG.

Theorem 4.1.3 (Hayashi et al. [HKO$^+$17])**.** *There is no constant t such that SG(t) = TSG.*

Theorem 4.1.4 (Hayashi et al. [HKO$^+$17])**.** *There is no constant t such that SG(t) = UDG.*

Hayashi et al. left some open problems. We try to expand the knowledge around some of these problems to understand them better, largely for the recognition of this class of graphs. Before that, we see where this class lays in the hierarchy of classes. We know by definition that TSG $\subsetneq$ UDG.

4.1.1 Interval graphs

Thin strip graphs shares their geometrical structure with interval graphs (remember SG(0) = UIG). In this subsection, we overview the results of Hayashi et al. [HKO$^+$17] where they find maximal and minimal superclasses for TSG in the interval graphs presented in chapter 3. The following theorem

will be proven by taking the proof written by Hayashi et al. in order to use their mapping in other theorems (e.g. 5).

Theorem 4.1.5 (Hayashi et al. [HKO$^+$17]). *MUIG $\subsetneq$ TSG.*

Proof. First, we prove that MUIG $\neq$ TSG. This can be proven because $C_4 \in$ TSG if we take as points $(0,0), (0,\varepsilon), (1,0), (1,\varepsilon)$ with $1 > \varepsilon > 0$ and $C_4 \notin$ MUIG because it is a chordal graph.

Then, we have to prove that MUIG $\subseteq$ TSG. Let $G = (V,E) \in$ MUIG where each vertex is a unit mixed interval denoted as I_v. We define $t = \min\{|I_u \cap I_v| : |I_u \cap I_v| > 0, \{I_u, I_v\} \subseteq V\}$ and $s = \min\{\ell(I_v) - r(I_u) : \ell(I_v) > r(I_u), \{I_u, I_v\} \subseteq V\}$. We have then t being the minimum length of an intersection bigger than zero (that is, not endpoint-adjacent) and s is the minimum distance between non-adjacent vertices (also not endpoint-adjacent). We also define $c(I_v) = \frac{\ell(I_v)+r(I_v)}{2}$ as the center of the interval and $p(I_v) = (-1)^{\lfloor c(I_v) \rfloor}$.

Let d be a real such that $0 < d < \frac{2}{3}$, $d \leq \frac{t}{4}$, $d < \frac{s}{2}$ and $\varepsilon \geq 2\sqrt{d - d^2}$. If we let $h = \sqrt{d - d^2}$, then we can create a $2h$-realization of G with the following mapping:

$$
\phi(v) = \begin{cases}
(c(I_v), 0) & \text{if } I_v \text{ is a closed interval} \\
(c(I_v), hp(I_v)) & \text{if } I_v \text{ is an open interval} \\
(c(I_v) - d, hp(I_v)) & \text{if } I_v \text{ is a closed-open interval} \\
(c(I_v) + d, hp(I_v)) & \text{if } I_v \text{ is an open-closed interval}
\end{cases}
$$

For two vertices u and v of G such that $u \leq v$, we have the three following cases:

1. $r(I_u) < \ell(I_v)$:

   I_u and I_v are not adjacents, which means that $\text{dist}(\phi(u), \phi(v)) > 1$. If we minimize the distance between them we have $\phi(u) = (c(I_u) + d, hp(I_u))$ and $\phi(v) = (c(I_v) - d, hp(I_v))$ with $p(I_u) = p(I_v)$. Therefore, we only have to compare their x-coordinates:

$$\text{dist}(\phi(u), \phi(v)) \geq (c(I_v) - d) - (c(I_u) + d) = c(I_v) - c(I_u) - 2d$$

By definition, $s \leq l(I_v) - r(I_u)$. If we take the centers, then $s \leq c(I_v) - c(I_u) - 1$, which means finally that $s + 1 \leq c(I_v) - c(I_u)$

$$\text{dist}(\phi(u), \phi(v)) \geq s + 1 - 2d > 1$$

2. $r(I_u) > \ell(I_v)$: In this case u and v are adjacent. We maximize $\text{dist}(\phi(u), \phi(v))$ when $\phi(u) = (c(I_u) - d, hp(I_u))$ and $\phi(v) = (c(I_v) + d, hp(I_v))$ with $p(I_u) \neq p(I_v)$. Therefore,

$$\text{dist}(\phi(u), \phi(v)) \leq \sqrt{((c(I_v) + d) - (c(I_u) - d))^2 + (h + h)^2}$$
$$\text{with the same reasoning as before } c(I_v) - c(I_u) \leq 1 - t$$
$$\leq \sqrt{(1 - t + 2d)^2 + 4h^2}$$
$$\leq \sqrt{(1 - 4d + 2d)^2 + 4(d - d^2)}$$
$$= \sqrt{1 - 4d + 4d^2 + 4d - 4d^2} = 1$$

3. $r(I_u) = \ell(I_v)$:

   In this case, u and v are adjacent only if $r(I_u)$ and I_v are closed. We know that $c(I_v) = c(I_u) + 1$ and $p(I_u) \neq p(I_v)$. Without loss of generality, we suppose that $p(I_u) = 1$ and $p(I_v) = -1$. We have two cases:

   (a) Both ends are closed. So we have this set of possible assignments for each one of the vertices:

   $$\phi(u) \in \{(c(I_u), 0), (c(I_u) + d, h)\}$$
   $$\phi(v) \in \{(c(I_u) + 1, 0), (c(I_u) + 1 - d, -h)\}$$

   This gives us a rectangle with its diagonal smaller than one.

(b) One of the ends is closed, we suppose $r(I_u)$ is open. In this case, we have these solutions:

$$\phi(u) \in \{(c(I_u) - d, h), (c(I_u), h)\}$$

$$\phi(v) \in \{(c(I_u) + 1, 0), (c(I_u) + 1, -h), (c(I_u) + 1 \pm d, -h)\}$$

Every distance between every points is greater than 1 if we take into consideration the domain of d. □

From this theorem, UIG $\subsetneq$ TSG. Actually, there exists a stronger connection between these two classes:

Theorem 4.1.6 (Breu [Bre96]). *Let G a c-strip graph with $c \in \mathbb{R}_0^+$. G has an induced $K_{1,3}$ or C_4 if and only if G is not an unit interval graph.*

Thin strip graphs can also be seen as unfettered unit interval graphs, which means that if a graph is a thin strip graph, then we can partition this graph with a level structure where each level is a clique. This information will be relevant in the next section.

Theorem 4.1.7 (Hayashi et al. [HKO+17]). *TSG $\subsetneq$ UUIG.*

Proof. See [HKO+17].

4.2 Characterization of thin strip graphs

One of the main goals of this thesis is to characterize thin strip graphs by forbidden induced subgraphs. We know that TSG is an hereditary class, then a way to characterize this class of graphs is by looking for its forbidden subgraphs the same way as MUIG has been characterized by Joos. Furthermore, MUIG $\subsetneq$ TSG by Theorem 4.1.5, so the first we can do is to check if the forbidden subgraphs of MUIG are also for TSG.

One of the main goals of this thesis is to characterize TSG. by forbidden induced subgraphs. To approach this, we will see how many induced forbidden subgraphs are also forbidden for TSG. We have described the families

of forbidden induced subgraphs for MUIG in section 3.2 and one of these familes has been proven to be a forbidden induced subgraph for TSG.

4.2.1 Mixed unit interval graph forbidden subgraphs

In the previous section we have shown that MUIG $\subsetneq$ TSG 4.1.5. We have even shown every forbidden induced subgraph of MUIG in Chapter 3. Here we are going to overview these forbidden induced subgraphs and we their inclusion in TSG.

In this subsection, we are going to see the relationship between thin strip graphs and mixed unit interval graphs.

Theorem 4.2.1 (Hayashi et al. [HKO$^+$17]). $\mathcal{R}$ *is a forbidden induced subgraph family of TSG.*

Proof. A way to prove this theorem is to prove that $\mathcal{R} \notin$ UUIG because TSG $\subsetneq$ UUIG. We can prove this by taking into consideration the embedding of the graphs in Figure 3.2.

Let v be the leftmost vertex of R_k with $k \in \mathbb{N}$ and L_i the ith level of the level structure of the graph. We have two choices:

- $v \in L_1 = K_1$: we have $H = R_k \setminus L_1$. H has only one connected component, which means that it is a valid level. The next step is to take $N(L_1 \cap H) = L_2$, then $N(L_2 \cap H') = L_3$ where $H' = H \setminus L_2$. We repeat this until we arrive to the end of our graph. The last one will divide the graph in two components of K_1, which does not respect our condition because L_n has already one adjacent level (L_{n-1}).

- $v \in L_1' = K_2$: in this case H has two connected components, K_1 and $H \setminus K_1$. This level is valid, however, because H has two components L_1' cannot be the first level of our level structure (see definition), so we take the neighbour K_1 as the first level L_1. We can observe that we are in the same case as before, where $L_1 = K_1$.

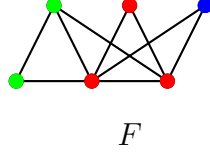Another (more extended) proof can be found in [HKO$^+$17]. $\square$

F

Figure 4.2: The graph F where each level is represented by a different color.

We see that $\mathcal{R}$ is a family of forbidden subgraphs of TSG. Nevertheless, the rest of the forbidden subgraphs for MUIG are thin strip graphs. The main reason is because they are unfettered unit interval graphs. We see our first example with the forbidden graph for MUIG F.

Theorem 4.2.2. $F \in TSG$.

Proof. To prove this we have to find an ϵ-realization for our graph $F = (V, E)$ with ε arbitrarily small. Let $\phi(v)$ be the mapping of our vertices on the plane. We know that the level structure of F is $L = \{L_1 = K_2, L_2 = K_3, L_3 = K_1\}$ as seen in Figure 4.2. For each $v_k \in L_2$ with $k \in [0, 2]$ as follows:

$$\phi(v_k) = \left(0, \varepsilon \frac{k}{2}\right)$$

Then, for each $u_k \in L_1$ with $l \in [1, 2]$ as follows if we take into consideration that $v_0 u_0, v_1 u_0, v_0 u_1 \in E$:

$$\phi(u_1) = \left(\left(\frac{\varepsilon}{4}\right)^2 - 1, \varepsilon \frac{1}{4}\right)$$

$$\phi(u_2) = (-1, 0)$$

If you can see L_1 at the left of L_2. Finally, we have $w \in L_3$. We can see that w and u_1 share the same neighbours, so they can be put in the same y-coordinate we put it at the right side of L_3.

$$\phi(w) = \left(1 - \left(\frac{\varepsilon}{4}\right)^2, \varepsilon \frac{1}{4}\right)$$

□

We can also prove the same for $\mathcal{T}$ and $\mathcal{S}''$. In this case these graphs have induced $K_{1,3}$ (before we had an induced C_4). There is a property about $K_{1,3}$ and TSGs that will help us embed those graphs in the plane.

Lemma 4.2.3. *The only way to represent $K_{1,3}$ as a thin strip graph is for three points u, v, w such that $u_x = v_x - 1$ and $w_x = v_x + 1$ with the same y-coordinate and the fourth point t is placed such $t_x = v_x$ and $t_y \neq v_y$ (see Figure 4.1).*

Proof. We proceed to prove this lemma by contradiction. We begin constructing the realization of $K_{1,3}$ by taking its induced $P_3 \in K_{1,3}$. The middle point of P_3 has to be between the other two points horizontally, so we know then that $u_x < v_x < w_x$ with v the middle point.

Now we introduce t, the vertex that is adjacent to the middle point of P_3. We know by fact that $u_x < t_x < w_x$: if we take $t_x \leq u_x$, t has to be adjacent to u in order to intersect v which is not the case; viceversa for w.

Let $\alpha_{u,v} = \sqrt{1 - (u_y - v_y)^2}$ be a real number that represents the ***critical region*** between two points. Note that if $|u_x - v_x| \leq \alpha_{u,v}$ then u and v are touching.

Now that we know that $u_x < t_x < w_x$, if we set $t_x < v_x$ and maximize the distance between u and v (so $u_x = v_x - 1$) we should have:

$$t_x > \alpha_{u,t} + u_x$$

for every t_y. We assume that $u_y = 0$ and $u_x = k$ with $k \in \mathbb{R}$ without loss of generality.

$$t_x > \alpha_{u,t} + v_x - 1$$

$$t_x + 1 > \alpha_{u,t} + v_x$$

We know that $t_x < v_x$, which means that $\alpha_{u,v}$ has to be bigger than one, which is impossible given the definition of $\alpha_{u,v}$. The same occurs with w_x and $t_x > v_x$.

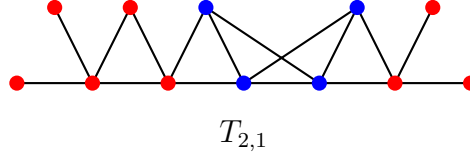The only case that is left is with $t_x = v_x$. If u and t touch then:

$T_{2,1}$

Figure 4.3: The graph $T_{2,1}$ with the diamond in blue and the arms in red.

$$t_x \leq \alpha_{u,t} + u_x$$

$$t_x + 1 \leq \alpha_{u,t} + v_x$$

we know that $t_x = v_x$:

$$v_x + 1 \leq \alpha_{u,t} + v_x$$

$$1 \leq \alpha_{u,t}$$

$$1 \leq \sqrt{1 - t_y^2}$$

which is impossible except when $t_y = 0$, which means that the only solutions are when $t_y \neq v_y$, $t_x = v_x$, $u_x = v_x - 1$ and $w_x = v_x + 1$. $\qquad \square$

We can see that $\mathcal{S}''$ and $\mathcal{T}$ both have induced $K_{1,3}$ while F has an induced C_4.

Main idea is that we construct first the induced $K_{1,3}$, then the rest follows, just have to finish writing.

4.3 Recognition

The recognition of this class of graphs is approached by Breu in his thesis [Bre96]. He gives a polynomial-time algorithm to recognise strip graphs for a given input with an assignment of y-coordinates for each vertex of the graph and an orientation of the edges of its complement.

Theorem 4.3.1 (Breu [Bre96]). *Let $G = (V, E, \gamma, \overrightarrow{E})$ a graph where $\gamma : V \to [0, c]$ is a function associating a y-coordinate (or a level) to each vertex*

and $\overrightarrow{E}$ an orientation of the complement of the graph. The recognition of c-strip graphs with this input is in $\mathcal{P}$.

Observation 4.3.2. *Recognition of c-strip graphs without a given $\overrightarrow{E}$ is in $\mathcal{NP}$.*

Proof. Given an polynomial-time algorithm with a complexity of $O(f(n))$ to solve recognition of $G = (V, E, \gamma, \overrightarrow{E})$, we can run again this algorithm by testing every possible orientation of its complement. This algorithm would take $O(f(n))2^{|E|-1} = O(f(n)2^{|E|})$ time to execute. $\qquad\square$

We would like to have an algorithm that solves this problem without knowing the y-coordinates of the vertices. Nevertheless, further research would concentrate on recognition of UUIGs. We know that TSG $\subsetneq$ UUIG, and recognition of UUIGs is $\mathcal{NP}$. If we the problem of recognising TSG given a UUIG and is solved in polynomial time, then TSG recognition would be $\mathcal{NP}$. However, given the observations in the end of chapter 3, there may be a polynomial-time algorithm for UUIG.

Chapter 5

Thin two-level graphs

Breu [Bre96] has presented in his thesis a similar class of constrained unit disk graphs where the disks are placed on k horizontal parallel lines. More formally: a disk (x, y) can be placed in $x \in (-\infty, \infty)$ and $y \in L$ with $|L| = k$.

In this chapter we define thin two-level graphs as a two-level graph where $L = \{0, \varepsilon\}$ and ε is an arbitrarily small real number.

5.1 Thin two-level graph

A two-level graph can be defined intuitively as a strip graph such that the disks are placed only on the horizontal lines. In the same way, we can define also

This class of graphs is close to our main class TSG. But we have to know at what point we can rely in this class of graphs to study TSG:

Lemma 5.1.1 (Breu [Bre96]). *Let abcda be a chordless 4-cycle in a two-level graph $G = (V, E)$. Then ad and bc are level edges (they are adjacent in the*
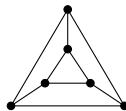

Figure 5.1: Forbidden graph in TTLG

same level), and the others are cross edges for every realization ϕ of G for which $\phi(a)_x < \phi(c)_x$ and $\phi(b)_x < \phi(d)_x$.

With this preliminary result, we proceed to one of our main result:

Theorem 5.1.2. *TTLG $\subsetneq$ TSG*

Proof. By definition, we know that TTLG $\subset$ TSG because the area where the disks can be placed in TTLG is included in the area in TSG.

We can prove that TTLG $\neq$ TSG because we can construct a graph G such that $G \in$ TSG and $G \notin$ TTLG. This graph D is a net* graph as described in Figure 5.1.

Part 1. *D is a TSG because we can realize it as a TSG if we take as center of disks $(0,0)$, $(0,z)$, $(0,\epsilon)$, $(1,0)$, $(1,z)$, $(1,\epsilon)$ such that $0 < z < \epsilon$.*

Part 2. *Now we have to prove that D is a forbidden induced subgraph of TTLG. We will try to construct it by taking an induced subgraph that is realizable: we take $D_{-1} = D - x$ with $x \in V(D)$. We notice that $V(D_{-1})$ is a chordless C_4 (abcd) with a vertex e adjacent to any two consecutive vertices $x, y \in V(C_4)$ creating the triangle xye.*

By Lemma 5.1.1 we know that abcda is a cycle if ab and cd are level edges. We can classify these vertices in two sets: $\ell(V) = a, d$ and $r(V) = b, c$ where $\forall u \in \ell(V) v \in r(V) : u_x < v_x$.

To realize D_{-1} we have to add a vertex i to C_4. We can either put it between two line-vertices or put it between two vertices with different level. In the case where we want to put it between two line vertices a and b we have:

$$b_x < d_x < c_x$$

In this case, we have d_x that is adjacent to at least one vertex of the other level. The only way to do this is to put it adjacent to two different level vertices (a and b). Now that we have a realization of D_{-1}, we should add a last vertex j adjacent to i, c and d. If we put j on the right of the cycle, then i has to be on the left to be able to touch c and d. However, it is impossible for j to reach i because between because at each level there is a region $a \cap d$ and $b \cap c$ that neither of these disks can breach, so they will always be disjoint. $\square$

5.1.1 Relation with interval graphs

The relationship between two-level graphs and interval graphs is clear: a k-two-level graph with $k > 1$ gives us a disconnected graph where every connected component is a unit interval graph. Here we could say that a two-level graph $G = F \cup H$ being F and H unit interval graphs.

Furthermore, this relationship between unit interval graphs and two-level graphs is even stronger:

Theorem 5.1.3 (Breu [Bre96]). *For any value of k, a k-two-level-graph is an union of two unit interval graphs.*

Before proving this, we have to define:

Definition 5.1.4. A short edge τ-two-level graph is a τ-two-level graph $G(V, E)$ such that given $vw \in E$, then $|v_x - w_x| \leq \sqrt{1 - \tau^2}$.

Claim 5.1.5. *A short edge two-level graph is a unit interval graph.*

Proof. See [Bre96].

Proof of theorem. Let $G = (V, E)$ be a two-level graph. Let have $G_S = (V, E_S)$ the graph induced on the short edges and $G_L = (V, E_L)$ a graph induced on the line-edges (between points in the same line). Both of these graphs are unit interval graphs.

We can see that $E_S \cup E_L \subseteq E$, we only have to prove that $E \subseteq E_S \cup E_L$. Given an edge $vw \in E$, if $|v_x - w_x| > \alpha$, then $vw \in E_L$ because two graphs on different levels cannot touch. In the other case, when $|v_x - w_x| \leq \alpha$, v and w can be either in the same level or in different levels.

This shows that every edge of E is in either E_L or E_S, so $E \subseteq E_L \cup E_S$. Which means that $G = G_L \cup G_S$.
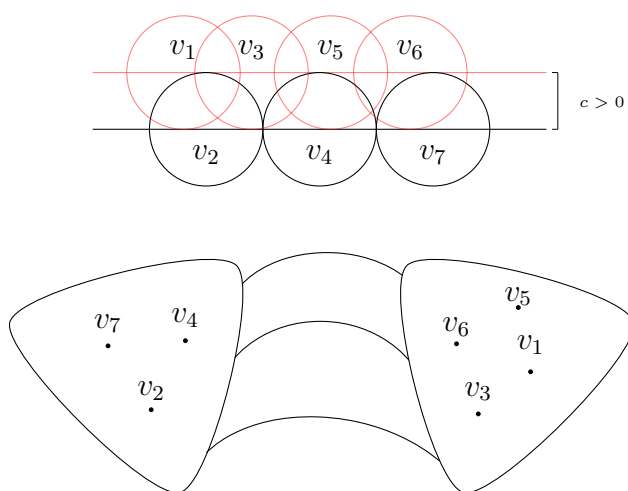
Figure 5.2: A representation of a TL(c)

Conclusions

The conclusions are to be written with care, because it will be sometimes the part that could convince a potential reader to read the whole document.

Appendices
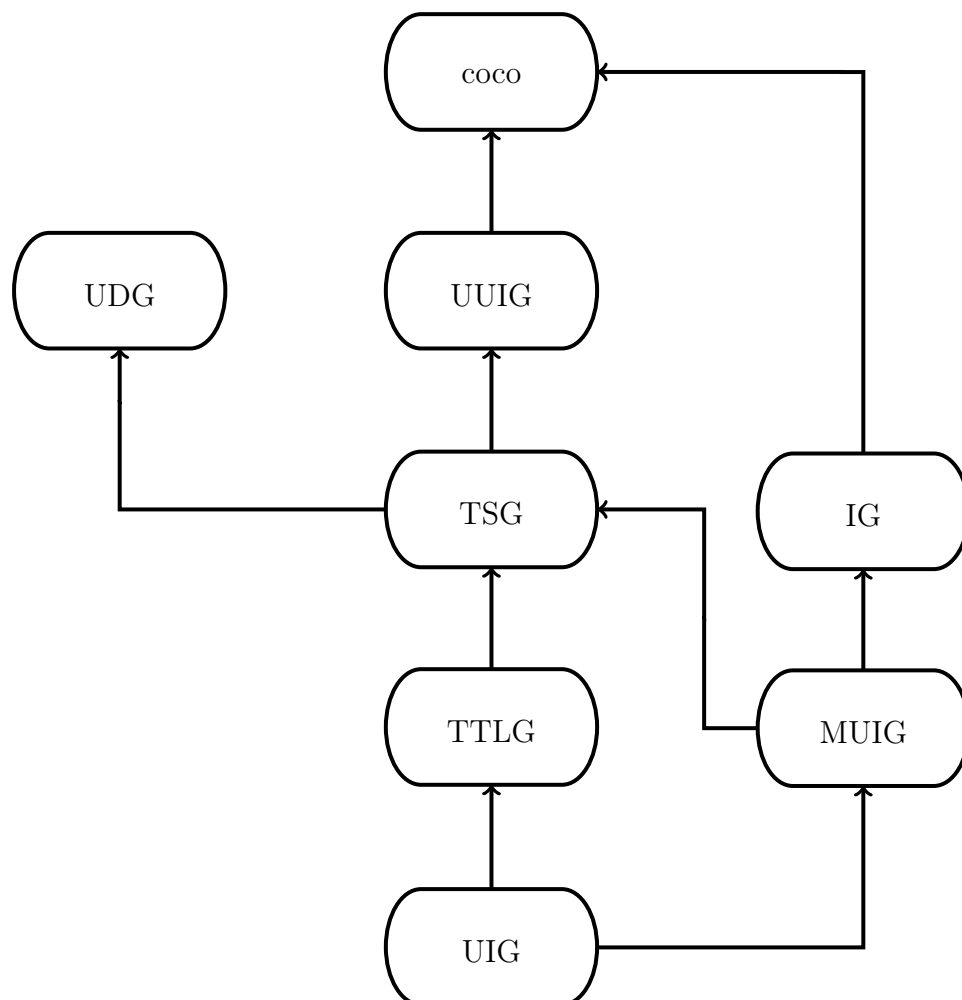
Appendix A

Graph classes hierarchy

Figure A.1: A hierarchy of every relevant graph of this document. The relation $\text{class}_1 \rightarrow \text{class}_2$ means that $\text{class}_1 \subset \text{class}_2$.

Appendix B

Problems in inclusion

- **MUIG $\subsetneq$ TSG $\subsetneq$ UUIG** : Hayashi [HKO$^+$17]

- **MUIG $\neq$ TTLG (Open)**: To prove that MUIG $\subsetneq$ TSG, Hayashi [HKO$^+$17] could simulate MUIGs with 4 different levels. Having only two levels, I conjecture that this is not possible. However, MUIG can have C_4, so an inclusion between these two classes is impossible (it has to be rewritten).

- **TTLG $\subsetneq$ TSG (Open)**: This problem has been solved in my thesis by finding a forbidden graph for TTLG, theorem 4.1.3.

- **TLG $\subset$ TSG (Open)**: This is a plausible stronger statement than the one before. However, this result could make the study of TTLG less relevant. Thus, this result would imply:

$$G \in \mathrm{TLG}(j) \rightarrow G \in \mathrm{SG}(k) : j, k \in \mathbb{R}$$

Appendix C

Problems in forbidden induced subgraph characterization

- **MUIG**: Joos [Joo13] gives us a complete characterization of forbidden graphs.

- **TSG (Open)**: Hayashi [HKO$^+$17] says that MUIG's forbidden induced subgraphs also are in TSG. He claims that finding a graph $F \in (\text{UDG} \cap \text{UUIG}) \setminus \text{TSG}$ could be a good starting point. In my thesis I show that a forbidden induced subgraph for MUIG is in UDG$\cap$UUIG.

- **TTLG (Open)**: There are many properties about these graphs in Breu's thesis [Bre96].

- **UDG (Open)**: There is no complete characterization of UDG. Can the results of this thesis help find new ones?U

Appendix D

Problems in complexity

- **UIG/IG recognition**: Both of these problems are polynomial.

- **MUIG recognition**: Schuchat et al. give a linear algorithm ($O(|V|^2)$) to recognise MUIGs [SSTW14b].

- **UDG recognition**: $\exists\mathbb{R}$-complete [Exi06].

- **SG(c) recognition (Open)**: Breu [Bre96] states that SG(c) recognition is polynomial if a complement edge orientation and a mapping $\phi : V \rightarrow [0, c]$ is polynomial as an input of the decision problem.

- **TSG recognition (Open)**: Can we get rid of the mapping as input to recognise TSGs? In that case the problem would be at least NP.

- **UUIG recognition (Open)**: Informally the recognition of this class of graphs **cannot** be polynomial because we have to find all the cliques of the graph; the CLIQUE problem is NP-complete.

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