Characterization and complexity of Thin Strip Graphs

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ABSTRACT

Abstract

1 Graphs and disks

1.1 Graphs

A graph G is defined as $G = (V, E)$, where V is the set of vertices and E the set of edges, where $E \subseteq \binom{V}{2}$. The vertices $v, w \in V$ such that $e = vw \in E$ links are called the *endpoints* of e.

Definition 1 *An embedding of a graph G into a surface Σ is a mapping of G in Σ where the vertices correspond to distinct points and the edges correspond to simple arcs connecting the images of their endpoints. [4].*

A graph G is planar if there is an embedding of this graph that does not have any crossing between the edges.

Definition 2 *Let $G = (V, E)$ and $S \subset V$, an induced subgraph is a graph H of G whose vertex set is S and its edge set $F = \{vw : v, w \in S, vw \in E\}$.*

Definition 3 *H is called a minor of G if H can be constructed by deleting edges and vertices, or contracting edges.*

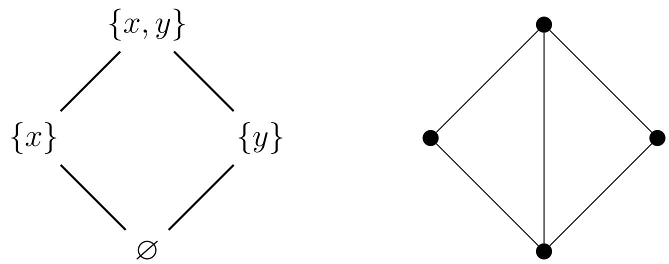

Figure 1: On the left, Hasse diagram of a poset of the power set of 2 elements ordered by inclusion. On the right, the comparability graph of this poset.

In graph theory, (forbidden graph characterization)...

Theorem 4 (Kuratowski) *A graph G is planar if and only if it doesn't contain K_5 or $K_{3,3}$ as a minor or a induced subgraph.*

1.2 Intersection graphs

Definition 5 *The intersection graph of a collection ζ of objects is the graph (ζ, E) such that $c_1c_2 \in E \Leftrightarrow c_1 \cap c_2 \neq \varnothing$.*

Definition 6 *A partial order set is a binary relation $\leq$ over a set A satisfying these axioms:*

- *$a \leq a$ (reflexivity).*
- *if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry).*
- *if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).*

Definition 7 *A partially ordered set or poset $(S, \leq)$ where S a set and $\leq$ a partial order on S.*

Definition 8 *A graph G is a comparibility graph if for each edge $\{u, v\} \in E$ there is a binary relation R such that $u \leq v$ or $v \leq u$. Equivalently, G is a comparability graph if it is the comparability graph of a poset. For example, the Hasse diagram (figure 1) is a comparability graph where the relation is inclusion.*

1.2.1 Interval graphs

Definition of interval Graphs

Properties

Definition of MIXED interval graphs

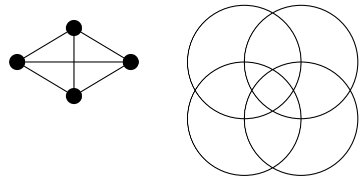

Figure 2: Realization of a UDG (Unit Disk Graph).

1.3 Realizations

Definition 9 *A graph G is said realizable if*

The *graph realizability problem* is the problem that finds a realization of a given length $l(e)$ for a graph G (this means that the edge e has to be represented by a straight line of length $l(e)$ in $\mathbb{R}^2$).

A unit distance graph G is a graph that has a realization where 2 points u, v have $\text{dist}(u, v) = 1$ if and only if their respective vertices are adjacent. This problem will be shown at chapter 2 to be $\exists\mathbb{R}$-complete. If this realization doesn't have any crossing then G is a *matchstick graph*.

A unit disk graph G is a graph that has a realization where 2 points have $\text{dist}(u, v) \leq 1$ if and only if their respective vertices are adjacent. Each point can be represented as the center of a disk of unit diameter and the edges can be represented as the intersection of 2 disks. This class of graphs is important for this thesis, as the Thin Strip Graphs are a sub-class of Unit Disk Graphs (section 4). Unit Disk Graph realizability is $\exists\mathbb{R}$-complete. We will refer to the Unit Disk Graph class as UDG and an example of a realization can be found in the figure 2.

2 Complexity

Complexity theory has the objective to establish lower bounds on how efficient an algorithm can be for a given problem [9]. This approach let us have a reference point to establish the difficulty of a problem.

Definition 10 *Let Σ be a finite alphabet, Σ^* every word derived from Σ, $L \subseteq \Sigma^*$ is a decision problem.*

Definition 11 *A decider for a decision problem A is an deterministic algorithm V where*

$$A = \{w | V \, accepts \; w\}$$

A is polynomially decidable if it has a polynomial time decider [9].

Definition 12 *A verifier for a decision problem A is an deterministic algorithm V where*

$$A = \{w | V \, accepts \, \langle w, c \rangle \text{ for some string } c\}$$

A is polynomially verifiable if it has a polynomial time verifier [9].

2.1 P vs NP

Definition 13 *A problem $L \in \mathcal{P}$ if L is polynomially decidable.*

Definition 14 *A problem $L \in \mathcal{NP}$ if L is polynomially verifiable. Thus, $\mathcal{P} \subseteq \mathcal{NP}$.*

To prove a bound of complexity on an unknown problem L we have to find other problems with already known complexity and find equivalences between those two. This can be achieved through *reductions*.

Definition 15 *A reduction of a problem L to a problem M is a mapping of an instance of L (I_L) to an isntance of M (I_M) such that I_L is true for the problem L if and only if I_M is true for the problem M. This is noted $L \leq M$ and $L \leq_P M$ if the reduction is done in polynomial time.*

With this concept we can define new complexity classes. $\mathcal{NP}$-hard is the set of problems so that we can reduce every $\mathcal{NP}$ problem to. The set of problems that are $\mathcal{NP}$-hard and $\mathcal{NP}$ are called $\mathcal{NP}$-complete. This is generalized to every complexity class ($\mathcal{P}$, $\exists\mathbb{R}$, RP, etc...)

Satisfiability problem The satisfiability problem (SAT) is to decide the satisfiability of a CNF formula ϕ. A CNF formula is a boolean formula that is a conjunction of multiple clauses c_k. A clause is a disjunction of multiple literals. A literal may be a variable or a negation of a variable.

Theorem 16 (Cook-Levin) *SAT is $\mathcal{NP}$-complete.*

2.2 $\exists\mathbb{R}$ complexity class

$\exists\mathbb{R}$ is the class that describes the problems that can be reduced to *the existential theory of the reals*[1]. The existential theory of the reals is the problem of deciding if a sentence of this form is true:

$$(\exists X_1 \ldots \exists X_n) : F(\exists X_1, \ldots, \exists X_n)$$

where F is a quantifier-free formula in the reals. In other words, it is a conjuntion of clauses where each clause is a real polynomial inequality where each variable X_k is a real number. We can see that ETR is NP-hard because SAT can be reduced to it.

Proof. Let's take an instance of SAT ϕ_{SAT} with clauses c_k and variables x_k, we can construct an instance of ETR ϕ_{ETR} where we can construct variables in the domain $\{0, 1\}$ with this equality, so for each variable X_k:

$$X_k - X_k^2 = 0$$

Each literal of each clause will be positive or negative depending if the literal is cancelled in ϕ_{SAT}:

$$x_k \to l = X_k$$

$$\neg x_k \to l = -X_k$$

Then for each clause we can have a polynomial that will sum the value of every literal in the clause must be greater that one, so that at least one literal is true:

$$\sum_{l \in c_k} l \geq 1$$

With this proof, it is easy to see that ϕ_{ETR} is valid if and only if ϕ_{SAT} is also valid. □

This result can show us that $P \subseteq NP \subseteq \exists\mathbb{R}$.

2.2.1 Problems in $\exists\mathbb{R}$

In this section we will describe some problems that are $\exists\mathbb{R}$-complete and will give an overview of the proof.

The art gallery problem Given a simple polygon P (without crossings between every side), we introduce *guards*. A guard g is a point such that every point of the polygon is watched by a guard. A point p is watched by a point q if the segment pq is contained in P. The subset G, being G the set of guards and $G \subseteq P$, is optimum if it has the minimal cardinality covering the whole polygon.

The art gallery problem is to decide, given a polygon P and a number of guards k, whether there exists a configuration of k guards in G guarding the whole polygon. The art gallery problem is $\exists\mathbb{R}$-complete [2].

Proof idea First of all, we can see that the art gallery problem is in $\exists\mathbb{R}$ if we reduce this problem to ETR. If we have an instance (P, k) of the art gallery problem we can have a formula [3] like this:

$$\phi = \{\exists x_1 y_1, \dots x_k y_k \forall p_x p_y : \text{INSIDE-POLYGON}(p_x, p_y) \rightarrow \bigvee_{1 \leq i \leq k} \text{SEES}(x_i, y_i, p_x, p_y)\}$$

Where INSIDE-POLYGON returns 1 if $(p_x, p_y) \in P$ and SEES returns 1 if the segment $(x, y)(p_x, p_y) \in P$. ϕ is not a ETR formula, so we would like to construct a quantifier-free formula with the idea of ϕ. To achieve this, the main idea is to have a small set of points $Q \subseteq P$ such that if these points are watched, the whole polygon is watched. This subset Q is called the *witness set*. The only thing is now to create a polynomial for each point that ensures that the point is watched by a guard.

To finish the proof we have to prove that the art gallery problem is $\exists\mathbb{R}$-hard. For this part an $\exists\mathbb{R}$-complete problem has been deducted from ETR. For the problem ETR-INV we have a set of variables $\{x_1, \dots, x_n\}$ and a set of equations of this form:

$$x = 1, \quad x + y = z, \quad x \cdot y = 1$$

and the problem decides if it exists a solution to this set of equations such that the value of each variable is real in $[\frac{1}{2}, 2]$.

A reduction of ETR-INV is found to the art gallery problem by constructing a polygon P and finding a number g for that polygon such that the instance of ETR-INT is true if and only if P is covered by at most g guards.

Unit Disk Graph recognition The Unit Disk Graph recognition is the problem that decides if a graph G has a realization ϕ as a Unit Disk Graph. Unit Disk Graph recognition is $\exists\mathbb{R}$-complete.

Recognition of Unit Disk Graphs is $\exists\mathbb{R}$-complete. (corollary of graph realizability problem)[7]

Stretchability is $\exists\mathbb{R}$-complete.

3 Geometry

The intersection of convex objects is a matter well studied for multiple subjects. In our case, it is interesting to know some properties about the intersection of disks, those being convex objects.

A set S is convex if:

$$\forall p, q \in S \ \forall \lambda \in [0, 1] : (1 - \lambda)p + \lambda q \in S$$

.

3.1 Stabbing

A *stabbing* is a point that traverses a set of intersecting objects. A lot of research has been done [8] on the minimal amount of stabbings to cover every object in a set. Stabbings can also be done with more complex structures than points, in that case we're talking about *coverings*.

Theorem 17 (Helly) *Given a set Q of objects in $\mathbb{R}^d$, if for each subset of Q of size $d+1$ their intersection is non empty, then $\bigcap_{q \in Q} \neq \varnothing$. [6]*

Koebe's planar $\subseteq$ disk = Planar graph duality

4 Thin Strip Graphs

c-strip graphs are unit disk graphs such that the centers of the disks belong to $\{(x, y) : -\infty < x < \infty, 0 < y \leq c\}$. The class is denoted by SG(c). We have SG(0) = UIG and SG(∞) = UDG. [5]

Definition 18 *Thin strip graphs are defined as TSG $= \bigcap_{c>0}$ SG(c).*

Remark 19 *SG(0) $\neq$ TSG. We can construct a $K_{1,3}$ such that we have 3 vertices with the coordinates $(1, 0)$, $(0, 0)$, $(1, 0)$ and a last one $(0, \varepsilon)$ with $\varepsilon > 0$ as seen in Figure 3.*

It has been proven that MUIG $\subsetneq$ TSG.

Denote that there's not constant t such that SG(t) = TSG.

Unfettered unit interval graphs = UUIG

MUIG $\subsetneq$ TSG $\subsetneq$ UUIG

UUIG $\subseteq$ co-comparability graphs (to prove).

Proof Let's have a relation of non-increasing order $\leq$ between the left endpoints of each interval v ($l(v)$). This relation will (nope, find another proof...).

4.1 Open questions about Thin Strip Graphs

In this section we state the problems that are being studied for the thesis.

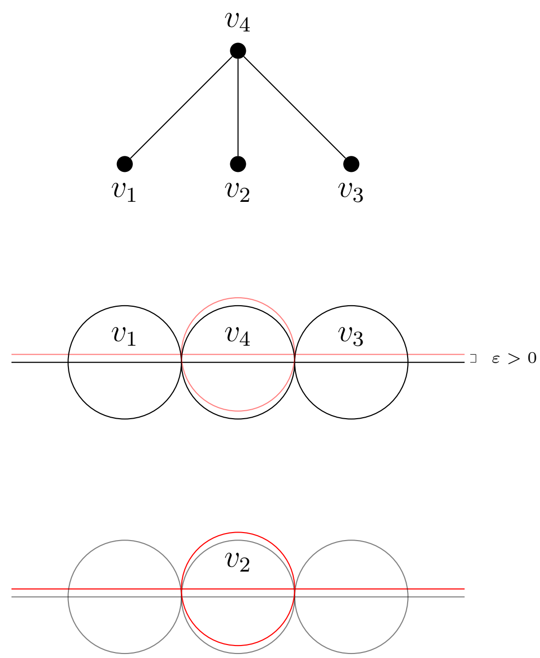

Figure 3: A construction of $K_{1,3}$ with a disk realization, being this graph a TSG.

Forbidden subgraphs of Thin Strip Graphs We've proven that MUIG $\subsetneq$ TSG $\subsetneq$ UUIG. Knowing the
(Why F_k is a co-comparability unit disk graph?)

Complexity class of TSG recognition We've shown in section 2 that some intersection geometric problems are in $\exists\mathbb{R}$ (unit disk graph recognition problem or the stretchability problem) and we'd like to know if TSG recognition or even SG(c) recognition is in NP knowing that TSG $\subseteq$ UDG.

References

[1] Existential Theory of the Reals. In *Algorithms in Real Algebraic Geometry*, volume 10, pages 505–532. Springer Berlin Heidelberg.

[2] Mikkel Abrahamsen, Anna Adamaszek, and Tillmann Miltzow. The Art Gallery Problem is $\exists \mathbb{R}$-complete.

[3] Alon Efrat and Sariel Har-Peled. Guarding galleries and terrains. 100(6):238–245.

[4] Palash Goyal and Emilio Ferrara. Graph Embedding Techniques, Applications, and Performance: A Survey.

[5] Takashi Hayashi, Akitoshi Kawamura, Yota Otachi, Hidehiro Shinohara, and Koichi Yamazaki. Thin Strip Graphs. 216:203–210.

[6] E. Helly. ber Mengen konvexer Krper mit gemeinschaftlichen Punkten. 32:175–176. cited By 187.

[7] Marcus Schaefer. Realizability of Graphs and Linkages. In Jnos Pach, editor, *Thirty Essays on Geometric Graph Theory*, pages 461–482. Springer New York.

[8] L.M. Schlipf. *Stabbing and Covering Geometric Objects in the Plane.*

[9] Michael Sipser. *Introduction to the Theory of Computation.* Course Technology, second edition.