Characterizatio and complexity of thin strip graphs

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ABSTRACT

Disk graphs are well studied in graph theory, complexity theory and geometry because of their several practical approaches. This work will present Thin Strip Graphs, a subclass of unit disk graphs and relevant bibliography and state of the art of related subjects.

1 Introduction

Disk graphs are graphs that represent the intersection of disks on the plane. They are currently used to give a graph-theoretic model to several problems [Sei], mainly in the area of network broadcasting [MPR] where the model is implicit: a set of antennas (senders and receivers) are placed in a terrain and there has not to be a frequency collision between pretty close antennas to avoid interferences. This can be modelized by taking a disk for each antenna and their range would be the radius of its respective disk.

The main idea of this thesis is to study a more specific and recently presented class of disk graphs: Thin Strip Graphs [HKO$^+$]. The properties of this class of graphs will be detailed in section 5. State of the art and notations of related topics will be detailed in sections 2 (graph theory), 3 (complexity theory) and 4 (geometry). Open questions about this class of graphs are proposed at the end of the article.

2 Graphs and intersections

2.1 Graphs

A graph G is defined as $G = (V, E)$, where V is the set of vertices and E the set of edges, where $E \subseteq \binom{V}{2}$. The vertices $v, w \in V$ such that $e = vw \in E$ links are called the *endpoints* of e.

Definition 1 *An embedding of a graph G into a surface Σ is a mapping of G in Σ where the vertices correspond to distinct points and the edges correspond to simple arcs connecting the images of their endpoints. [GF].*

A graph G is planar if there is an embedding of this graph that does not have any crossing between the edges.

Definition 2 *Let $G = (V, E)$ and $S \subset V$, an induced subgraph is a graph H of G whose vertex set is S and its edge set $F = \{vw : v, w \in S, vw \in E\}$.*

Definition 3 *Let $G = (V, E)$ its complement graph $\overline{G}$ is the graph such that its edge set is defined as: $\{vw : v, w \in V, vw \notin E\}$.*

Definition 4 *H is called a minor of G if H can be constructed by deleting edges and vertices, or contracting edges.*

Theorem 5 (Kuratowski) *A graph G is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor or a induced subgraph.*

Definition 6 *A path P_n in a graph G is a sequence of vertices $v_1v_2v_3 \ldots v_n$ such that $v_iv_{i+1} \in E$.*

Definition 7 *A cycle C_n in a graph $G = (V, E)$ is a path $v_1 \ldots v_n$ such that $v_1 = v_n$.*

Definition 8 *A chord of a cycle C_n is an edge that connects two non consecutive vertices of C_n.*

Definition 9 *A graph $G = (V, E)$ is complete if every pair of distinct $v_1, v_2 \in V$ are adjacent. This is denoted K_n with n the size of the graph. If G is an induced graph of H then G is a clique of H.*

Definition 10 *A graph G is bipartite if there exist two disjoint subsets $A, B \subset V$ such that $A \cup B = V$ and each edge $e \in E$ has an endpoint on A and the another on B.*

Definition 11 *A bipartite graph G with bipartitions A and B is complete bipartite if every pair of vertices $v \in A, w \in B$ are adjacent. It is denoted as $K_{n,m}$, being n and m the size of each bipartition.*

Definition 12 *An induced forbidden subgraph of a graph class X is a graph such that if it is the induced subgraph of a graph G, we know that $G \notin X$.*

The coloration of a graph is a color assignment to each vertex such that the color of the two endpoints of every edge of the graph is different.

Definition 13 *The chromatic number of a graph $\chi(G)$ is the smallest number of colors needed to have an acceptable coloration of G.*

Definition 14 *The clique number of a graph $\omega(G)$ is the size of the biggest clique of G. We can observe that for every graph: $\chi(G) \geq \omega(G)$.*

Definition 15 *A perfect graph is a graph that respects this condition for every induced subgraph:*

$$\omega(G) = \chi(G)$$

Theorem 16 (Lovasz) *G is perfect if and only if $\overline{G}$ is perfect.*

2.2 Intersection graphs

Definition 17 *The intersection graph of a collection ζ of objects is the graph (ζ, E) such that $c_1c_2 \in E \Leftrightarrow c_1 \cap c_2 \neq \varnothing$.*

Definition 18 *A partial order is a binary relation $\leq$ over a set A satisfying these axioms:*

- *$a \leq a$ (reflexivity).*
- *if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry).*
- *if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).*

Definition 19 *A partially ordered set or poset $(S, \leq)$ where S a set and $\leq$ a partial order on S.*

Definition 20 *A graph $G = (V, E)$ is a comparibility graph if there exists a partial order $\leq$ such that $vw \in E \Leftrightarrow v \leq w$ or $w \leq v$. Equivalently, G is a comparability graph if it is the comparability graph of a poset. For example, the Hasse diagram (figure 1) is a comparability graph where the relation is inclusion.*

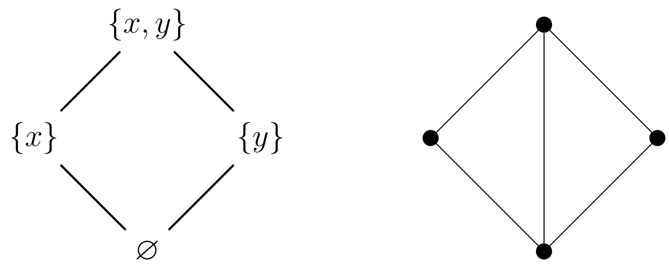

Figure 1: On the left, Hasse diagram of a poset of the power set of 2 elements ordered by inclusion. On the right, the comparability graph of this poset.

2.2.1 Interval graphs

An interval graph is a graph G that is the intersection graph of a collection of closed intervals in $\mathbb{R}$. If the length of each interval is unitary, then G is a unit interval graph (UIG).

Theorem 21 *G is an interval graph if and only if every simple cycle of four or more points has a chord. [Fis]*

Theorem 22 *An interval graph is a unit interval graph if and only if it has no induced subgraph $K_{1,3}$ [Rob].*

Another interesting class of interval graphs are mixed unit interval graphs, where each interval can be closed, open, open-closed or closed-open. In this paper we will denote those four classes like this:

$$\mathcal{I}^{++} = \{[x, y] : x, y \in \mathbb{R}, x \leq y\}$$
$$\mathcal{I}^{--} = \{(x, y) : x, y \in \mathbb{R}, x \leq y\}$$
$$\mathcal{I}^{+-} = \{[x, y) : x, y \in \mathbb{R}, x \leq y\}$$
$$\mathcal{I}^{-+} = \{(x, y] : x, y \in \mathbb{R}, x \leq y\}$$

$\mathcal{I}$ will be replaced by $\mathcal{U}$ when we are talking about unit mixed interval graphs and their class is denoted MUIG.

Theorem 23 *The classes of the graphs $\mathcal{U}^{--}$, $\mathcal{U}^{++}$, $\mathcal{U}^{-+}$, $\mathcal{U}^{+-}$, and $\mathcal{U}^{-+} \cup \mathcal{U}^{+-}$ are the same (equivalent for $\mathcal{I}$). [DLP+]*

Unlike for UIG class, $K_{1,3}$ is a MUIG as seen in figure 2. Some characterizations have been already found for these classes of graphs [SSTW] [Joo].

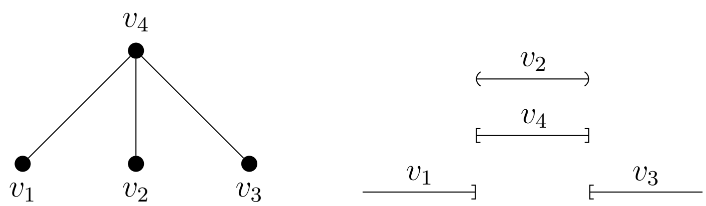

Figure 2: Representation of $K_{1,3}$ as a MUIG.

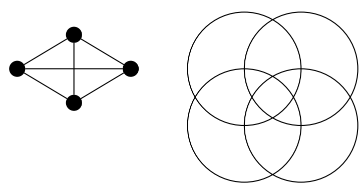

Figure 3: Realization of a UDG (Unit Disk Graph).

2.2.2 Disks

A disk graph G is a graph that is an intersection graph of disks on the plane, when the size of the disk is unitary, we talk about unit disk graphs. This class of graphs is important for this thesis, as thin strip graphs are a sub-class of unit disk graphs (section 5).

We will refer to the unit disk graph class as UDG and an example of a realization can be found in the figure 3.

Induced forbidden subgraphs The characterization of this class with respect to its induced forbidden subgraphs have been studied [AZ].

Theorem 24 (Atminas-Zamaraev) *For every integer* $k > 1$, $\overline{K_2 + C_{2k+1}}$ *is a minimal induced subgraph.*

Theorem 25 (Atminas-Zamaraev) *For every integer* $k > 4$, $\overline{C_{2k}}$ *is a minimal induced subgraph.*

3 Complexity

Complexity theory has the objective to establish lower bounds on how efficient an algorithm can be for a given problem [Sip]. This approach let us have a reference point to establish the difficulty of a problem.

Definition 26 *Let Σ be a finite alphabet, Σ^* every word derived from Σ, $L \subseteq \Sigma^*$ is a decision problem.*

Definition 27 *A decider for a decision problem A is an deterministic algorithm V where*

$$A = \{w | V \, accepts \; w\}$$

A is polynomially decidable if it has a polynomial time decider [Sip].

Definition 28 *A verifier for a decision problem A is an deterministic algorithm V where*

$$A = \{w | V \, accepts \; \langle w, c \rangle \; for \; some \; string \; c\}$$

A is polynomially verifiable if it has a polynomial time verifier [Sip].

3.1 P vs NP

Definition 29 *A problem $L \in \mathcal{P}$ if L is polynomially decidable.*

Definition 30 *A problem $L \in \mathcal{NP}$ if L is polynomially verifiable. Thus, $\mathcal{P} \subseteq \mathcal{NP}$.*

To prove a bound of complexity on an unknown problem L we have to find another problem with already known complexity and find equivalences between those two. This can be achieved through *reductions*.

Definition 31 *A reduction of a problem L to a problem M is a mapping of an instance of L (I_L) to an isntance of M (I_M) such that I_L is true for the problem L if and only if I_M is true for the problem M. This is noted $L \leq M$ and $L \leq_P M$ if the reduction is done in polynomial time.*

With this concept we can define new complexity classes. $\mathcal{NP}$-hard is the set of problems so that we can reduce every $\mathcal{NP}$ problem to. The set of problems that are both $\mathcal{NP}$-hard and $\mathcal{NP}$ are called $\mathcal{NP}$-complete. This is generalized to every complexity class ($\mathcal{P}$, $\exists\mathbb{R}$, RP, etc...)

Satisfiability problem The satisfiability problem (SAT) is to decide the satisfiability of a CNF formula ϕ. A CNF formula is a boolean formula that is a conjunction of multiple clauses c_k. A clause is a disjunction of multiple literals. A literal may be a variable or a negation of a variable.

Theorem 32 (Cook-Levin) *SAT is $\mathcal{NP}$-complete.*

Clique problem The clique problem is to find a maximum clique of a graph G.

Theorem 33 *CLIQUE is $\mathcal{NP}$-complete. [Kar]*

Theorem 34 *CLIQUE is QPTAS when applied to disk graphs. [BGK$^+$]*

Theorem 35 (Clark-Colbourn) *CLIQUE is $\mathcal{P}$ when applied to unit disk graphs. [CCJ]*

3.2 $\exists\mathbb{R}$ complexity class

$\exists\mathbb{R}$ is the class that describes the problems that can be reduced to *the existential theory of the reals*[Exi]. The existential theory of the reals is the problem of deciding if a sentence of this form is true:

$$(\exists X_1 \ldots \exists X_n) : F(\exists X_1, \ldots, \exists X_n)$$

where F is a quantifier-free formula in the reals. In other words, it is a conjuntion of clauses where each clause is a real polynomial inequality where each variable X_k is a real number. We can see that ETR is NP-hard because SAT can be reduced to it.

Proof. Let's take an instance of SAT ϕ_{SAT} with clauses c_k and variables x_k, we can construct an instance of ETR ϕ_{ETR} where we can construct variables in the domain $\{0, 1\}$ with this equality, so for each variable X_k:

$$X_k - X_k^2 = 0$$

Each literal of each clause will be positive or negative depending if the literal is cancelled in ϕ_{SAT}:

$$x_k \to l = X_k$$
$$\neg x_k \to l = (1 - X_k)$$

Then for each clause we can have a polynomial for which the sum of the values of every literal in the clause must be greater than one, so that at least one literal is true:

$$\sum_{l \in c_k} l \geq 1$$

With this proof, it is easy to see that ϕ_{ETR} is valid if and only if ϕ_{SAT} is also valid. $\square$

This result can show us that $P \subseteq NP \subseteq \exists\mathbb{R}$.

3.2.1 Problems in $\exists\mathbb{R}$

In this section we will describe some problems that are $\exists\mathbb{R}$-complete and will give an overview of the proof.

The art gallery problem Given a simple polygon P (without crossings between every side), we introduce *guards*. A guard g is a point such that every point of the polygon is watched by a guard. A point p is watched by a point q if the segment pq is contained in P. The subset G, being G the set of guards and $G \subseteq P$, is optimum if it has the minimal cardinality covering the whole polygon.

The art gallery problem is to decide, given a polygon P and a number of guards k, whether there exists a configuration of k guards in G guarding the whole polygon. The art gallery problem is $\exists\mathbb{R}$-complete [AAM].

Proof idea First of all, we can see that the art gallery problem is in $\exists\mathbb{R}$ if we reduce this problem to ETR. If we have an instance (P, k) of the art gallery problem we can have a formula [EHP] like this:

$$\phi = \{\exists x_1 y_1, \ldots x_k y_k \forall p_x p_y : \text{INSIDE-POLYGON}(p_x, p_y) \rightarrow \bigvee_{1 \leq i \leq k} \text{SEES}(x_i, y_i, p_x, p_y)\}$$

Where INSIDE-POLYGON returns 1 if $(p_x, p_y) \in P$ and SEES returns 1 if the segment $(x, y)(p_x, p_y) \in P$. ϕ is not a ETR formula, so we would like to construct a quantifier-free formula with the idea of ϕ. To achieve this, the main idea is to have a small set of points $Q \subseteq P$ such that if these points are watched, the whole polygon is watched. This subset Q is called the *witness set*. The only thing is now to create a polynomial for each point that ensures that the point is watched by a guard.

To finish the proof we have to prove that the art gallery problem is $\exists\mathbb{R}$-hard. For this part an $\exists\mathbb{R}$-complete problem has been deducted from ETR. For the problem ETR-INV we have a set of variables $\{x_1, \ldots, x_n\}$ and a set of equations of this form:

$$x = 1, \quad x + y = z, \quad x \cdot y = 1$$

and the problem decides if it exists a solution to this set of equations such that the value of each variable is real in $[\frac{1}{2}, 2]$.

A reduction of ETR-INV is found to the art gallery problem by constructing a polygon P and finding a number g for that polygon such that the instance of ETR-INT is true if and only if P is covered by at most g guards.

Stretchability A pseudoline is a simple closed curve in the plane. The stretchability problem is to decide if given a pseudoline arrangement, it is equivalent to an arrangement of straight lines.

Proof idea ETR can be reduced to STRETCHABILITY due to Mnev's universality theorem. [Scha]

Unit disk graph recognition The unit disk graph recognition is the problem that decides if a graph G is a unit disk graph. Unit disk graph recognition is $\exists\mathbb{R}$-complete. [Schb]

Proof idea UDG recognition is a corollary of deciding whether a graph with a given length is realizable. This problem is $\exists\mathbb{R}$-complete.

The reduction is done from STRETCHABILITY [Schb]. The reduction is done by adding a vertex to V for each pseudoline intersection. For each three consecutive points u_1, u_2, u_3 along a pseudoline a widget will be added that will be only realizable if and only if the pseudoline can be stretched with the same arrangement.

4 Geometry

The intersection of convex objects is a matter well studied for multiple subjects. In our case, it is interesting to know some properties about the intersection of disks, those ones being convex objects.

A set S is convex if:

$$\forall p, q \in S \; \forall \lambda \in [0,1] : (1-\lambda)p + \lambda q \in S$$

4.1 Stabbing

A *stabbing* is a point that traverses a set of intersecting objects. A lot of research has been done [Schc] on the minimal amount of stabbings to cover every object in a set. Stabbings can also be done with more complex structures than points, in that case we are talking about *coverings.*

Theorem 36 (Helly) *Given a set Q of objects in $\mathbb{R}^d$, if for each subset of Q of size $d+1$ their intersection is non empty, then $\bigcap_{q \in Q} \neq \varnothing$. [Hel]*

Theorem 37 *The problem that for a set of n disks whether there exists a regular n-gon whose vertices stab every disk of the set can be decided in $O(n^{10.5}/\sqrt{\log(n)})$ [Schc]*

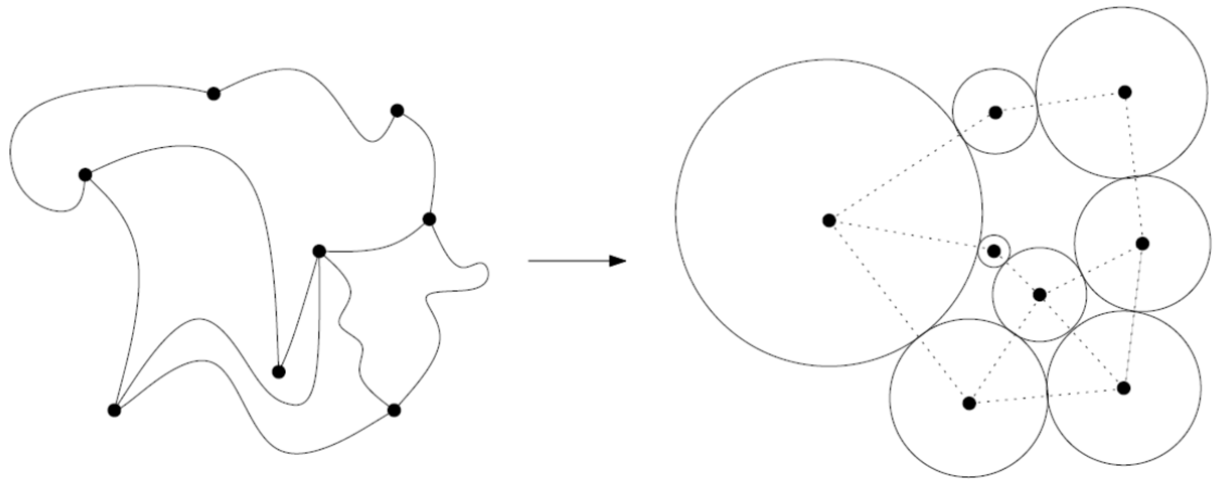

Figure 4: Circle packing of a planar graph. [Nac]

4.2 Coin graphs

Penny graphs can be defined as disk graphs where the disks can just touch each other without overlapping. A famous theorem is derived from this class of graphs: the circle packing theorem.

Theorem 38 (Circle packing theorem) *The circle packing theorem states that for every simple connected planar graph G is a penny graph. [BS]*

Corollary 39 *Planar graphs $\subseteq$ disk graphs [Spi].*

5 Thin Strip Graphs

c-strip graphs are unit disk graphs such that the centers of the disks belong to $\{(x, y) : -\infty < x < \infty, 0 < y \leq c\}$. The class is denoted by SG(c). We have SG(0) = UIG and SG(∞) = UDG. [HKO+]

Definition 40 *Thin strip graphs are defined as $TSG = \bigcap_{c>0} SG(c)$.*

Remark 41 *SG(0) $\neq$ TSG. We can construct a $K_{1,3}$ such that we have 3 vertices with the coordinates $(1, 0)$, $(0, 0)$, $(1, 0)$ and a last one $(0, \varepsilon)$ with $\varepsilon > 0$ and arbitrarily small as seen in Figure 5.*

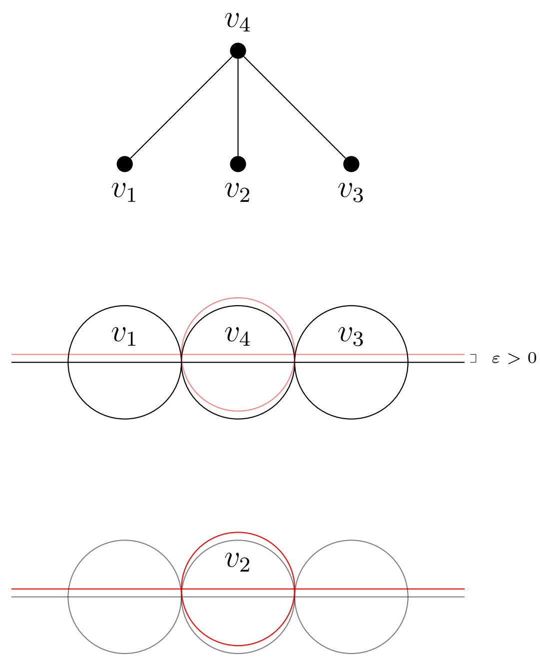

Figure 5: A construction of $K_{1,3}$ with a disk realization, being this graph a TSG.

Theorem 42 *There is no constant t such that SG(t) = TSG.*

Since this class is newly defined we have to characterize it. For this purpose, some relations have been found between this class and interval graphs.

5.1 Interval graphs

Theorem 43 *MUIG $\subsetneq$ TSG.*

We can define a new class of graphs: unfettered unit interval graphs. These graphs are unit interval graphs where if two intersections touch, we can decide whether they intersect or not. We denote this class UUIG.

Theorem 44 *TSG $\subsetneq$ UUIG.*

5.2 Open questions about Thin Strip Graphs

In this section we state the problems that are being studied for the thesis.

Forbidden subgraphs of Thin Strip Graphs We have proven that MUIG $\subsetneq$ TSG $\subsetneq$ UUIG. Knowing some characterizations of MUIG, we can have an approach to this question by exploring the already known forbidden subgraphs of MUIG.

From the results shown in this section, we can say that TSG $\subseteq$ UDG $\cap$ UUIG, being UUIG a subclass of co-comparability graphs.

Complexity class for TSG recognition We have shown in section 3 that some intersection geometric problems are in $\exists\mathbb{R}$ (unit disk graph recognition problem or the stretchability problem) and we would like to know if TSG recognition or even SG(c) recognition is in NP knowing that TSG $\subseteq$ UDG $\cap$ UUIG.

Complexity class of graph problems applied to TSG We have shown in section 3 that CLIQUE is in $\mathcal{P}$ for unit disk graphs. Knowing that TSG $\subseteq$ UDG $\cap$ UUIG we know that CLIQUE is also polynomial for TSG. The question is, is there any problem that for the superclasses UUIG (and also co-comparability graphs) and UDG whose complexity is reduced for TSG or TSG(c)?

We can also analyze the domains of c for which some problems are easier. This could help us bound the complexity of those problems.

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