

Characterization and complexity of Thin Strip Graphs

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ABSTRACT

Abstract

1 Graphs and disks

1.1 Graphs

A graph G is defined as $G = (V, E)$, where V is the set of vertices and E the set of edges. A vertex $v \in V$ is the fundamental unit of a graph. An edge $e \in E$ is a structures that links two vertices. The vertices $vw \in V$ that $e \in E$ links are called the *endpoints*.

Definition 1 *An embedding of a graph G is a representation of this graph on the plane.*

A graph G is planar if there is an embedding of this graph that doesn't have any crossing between the edges.

Theorem 2 (Kuratowski) *A graph G is planar iff it doesn't contain K_5 or $K_{3,3}$ as a minor.*

1.2 Intersection graphs

Given a geometric construction with multiple objects, an intersection graph is a graph that maps the objects into vertices and every intersection between objects is an edge between the corresponding vertices.

Definition 3 *A graph G is a comparability graph if for each edge $\{u, v\} \in E$ there is a partial order \leq such that $u \leq v$ or $v \leq u$.*

1.2.1 Interval graphs

Definition of interval Graphs

Properties

Definition of MIXED interval graphs

1.2.2 Unit disk graphs

Definition of UDG.

Definition of a realization.

2 Complexity

Problem solving is based on the complexity of a problem and not only a particular algorithm that solves it [6].

Definition 4 Let Σ be a finite alphabet, Σ^* every word derived from Σ , $L \subseteq \Sigma^*$ is a decision problem.

Definition 5 The algorithm A decides problem $L \subseteq \Sigma^*$ if for all word $w \in \Sigma^*$:

- A finishes and returns *TRUE* if $w \in L$.
- A finishes and returns *FALSE* if $w \notin L$.

Definition 6 A problem is decidable if there's an algorithm that decides it.

Definition 7 A problem is decidable if there's an algorithm that decides it.

2.1 P vs NP

Definition 8 A problem $L \in \mathcal{P}$ if L can be decided in polynomial time $\mathcal{O}(n^k)$.

Definition 9 A problem $L \in \mathcal{NP}$ if L can be verified in polynomial time $\mathcal{O}(n^k)$. Thus, $\mathcal{P} \subseteq \mathcal{NP}$.

2.2 $\exists\mathbb{R}$ complexity class

$\exists\mathbb{R}$ is the class that describes the problems such that they can be reduced to *the existential theory of the reals*[1]. The existential theory of the reals

2.2.1 Problems in $\exists\mathbb{R}$

The art gallery problem is $\exists\mathbb{R}$ -complete.[2]

Recognition of Unit Disk Graphs is $\exists\mathbb{R}$ -complete. (corollary of graph realizability problem)[4]

Stretchability is $\exists\mathbb{R}$ -complete.

3 Geometry

4 Thin Strip Graphs

4.1 Stabbing disks

Definition of stabbing.

Stabbing geometric structures.[5]

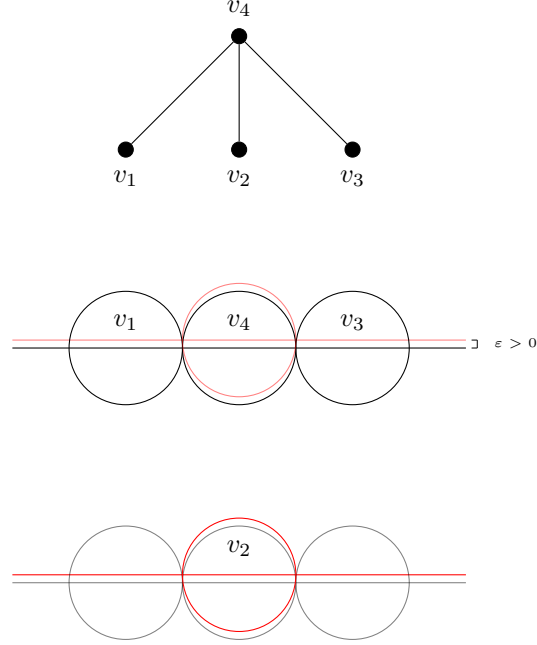


Figure 1: A construction of $K_{1,3}$ with a disk realization, being this graph a TSG.

4.2 Thin Strip Graphs

c -strip graphs are unit disk graphs such that the centers of the disks are delimited on the area $\{(x, y) : -\infty < x < \infty, 0 < y \leq c\}$ and its class noted $SG(c)$. We can say that $SG(0) = \text{UIG}$ and $SG(\infty) = \text{UDG}$. [3]

Definition 10 *Thin strip graphs are defined as $TSG = \bigcap_{c>0} SG(c)$.*

Remark 11 $SG(0) \neq TSG$. We can construct a $K_{1,3}$ such that we have 3 vertices with the coordinates $(1, 0)$, $(0, 0)$, $(1, 0)$ and a last one $(0, \varepsilon)$ with $\varepsilon > 0$ as seen in Figure 1.

It has been proven that $\text{MUIG} \subsetneq \text{TSG}$.

Denote that there's not constant t such that $SG(t) = \text{TSG}$.

Unfettered unit interval graphs = UIIG

$\text{MUIG} \subsetneq \text{TSG} \subsetneq \text{UIIG}$

$\text{UIIG} \subseteq \text{co-comparability graphs}$ (to prove).

In the following sections we state the problems that are being studied for the thesis.

4.2.1 Forbidden subgraphs of Thin Strip Graphs

We've proven that $\text{MUIG} \subsetneq \text{TSG} \subsetneq \text{UIIG}$. Knowing the (Why F_k is a co-comparability unit disk graph?)

4.2.2 Complexity class of TSG recognition

We've shown in section 2 that some intersection geometric problems are in $\exists\mathbb{R}$ (unit disk graph recognition problem or the art gallery problem) and we'd like to know if TSG recognition or even $\text{SG}(c)$ recognition is in NP knowing that $\text{TSG} \subseteq \text{UDG}$.

References

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- [5] L.M. Schlipf. *Stabbing and Covering Geometric Objects in the Plane*.
- [6] Michael Sipser. *Introduction to the Theory of Computation*. Course Technology, second edition.