UNIVERSITÉ LIBRE DE BRUXELLES
Faculté des Sciences
Département d'Informatique

Characterization and complexity of Thin Strip Graphs

Abdeselam El-Haman Abdeselam

Promotor : Prof. Jean Cardinal
Master Thesis in Computer Sciences

Academic year 2018 - 2019

You may want
to write a dedication here

"*You may also include one or more general quotes related to your topic.*"

Name of the author, date

"*Another quote.*"

Name of the author, date

Acknowledgment

I want to thank ...

Contents

Introduction

Talk about what this is etc...

Chapter 1

Background

Text about background intro

1.1 Graphs and intersections

1.1.1 Graphs

A G is defined as $G = (V, E)$, where V is the set of vertices and E the set of edges, where $E \subseteq \binom{V}{2}$. The vertices $v, w \in V$ such that $e = vw \in E$ links are called the *endpoints* of e.

Definition 1.1.1. An embedding of a graph G into a surface Σ is a mapping of G in Σ where the vertices correspond to distinct points and the edges correspond to simple arcs connecting the images of their endpoints. [GF17].

A graph G is planar if there is an embedding of this graph that does not have any crossing between the edges.

Definition 1.1.2. Let $G = (V, E)$ and $S \subset V$, an induced subgraph is a graph H of G whose vertex set is S and its edge set $F = \{vw : v, w \in S, vw \in E\}$.

Definition 1.1.3. Let $G = (V, E)$ its complement graph $\overline{G}$ is the graph such that its edge set is defined as: $\{vw : v, w \in V, vw \notin E\}$.

Definition 1.1.4. H is called a *minor* of G if H can be constructed by deleting edges and vertices, or contracting edges.

Theorem 1.1.5 (Kuratowski)**.** *A graph G is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor or a induced subgraph.*

Definition 1.1.6. A path P_n in a graph G is a sequence of vertices $v_1v_2v_3\dots v_n$ such that $v_iv_{i+1} \in E$.

Definition 1.1.7. A cycle C_n in a graph $G = (V, E)$ is a path $v_1 \dots v_n$ such that $v_1 = v_n$.

Definition 1.1.8. A chord of a cycle C_n with $n \geq 4$ is an edge that connects two non consecutive vertices of C_n.

Definition 1.1.9. A triangular chord of a cycle is a chord that will create a new triangle (C_3).

Definition 1.1.10. A graph $G = (V, E)$ is complete if every pair of distinct $v_1, v_2 \in V$ are adjacent. This is denoted K_n with n the size of the graph. If G is an induced graph of H then G is a clique of H.

Definition 1.1.11. A graph G is bipartite if there exist two disjoint subsets $A, B \subset V$ such that $A \cup B = V$ and each edge $e \in E$ has an endpoint on A and the another on B.

Definition 1.1.12. A bipartite graph G with bipartitions A and B is complete bipartite if every pair of vertices $v \in A, w \in B$ are adjacent. It is denoted as $K_{n,m}$, being n and m the size of each bipartition.

Some graphs can be characterized with properties. A property of a graph is a property that is preserved under all its isomophisms. These properties are called *hereditary* if they are also preserved under all its induced subgraphs; they are called *minor-hereditary* if they are also preserved under its minors (e.g. Kuratowski's planar graph characterization [1.1.5]).

Definition 1.1.13. An forbidden induced subgraph (minor) of a graph class X is a graph such that if it is the induced subgraph (minor) of a graph G, we know that $G \notin X$.

The coloration of a graph is a color assignment to each vertex such that the color of the two endpoints of every edge of the graph is different.

Definition 1.1.14. The chromatic number of a graph $\chi(G)$ is the smallest number of colors needed to have an acceptable coloration of G.

Definition 1.1.15. The clique number of a graph $\omega(G)$ is the size of the biggest clique of G. We can observe that for every graph: $\chi(G) \geq \omega(G)$.

Definition 1.1.16. A perfect graph is a graph that respects this condition for every induced subgraph:

$$\omega(G) = \chi(G)$$

Theorem 1.1.17 (Lovasz). *G is perfect if and only if $\overline{G}$ is perfect.*

1.1.2 Intersection graphs

Definition 1.1.18. The *intersection graph* of a collection ζ of objects is the graph (ζ, E) such that $c_1c_2 \in E \Leftrightarrow c_1 \cap c_2 \neq \varnothing$.

An intersection can be seen as a relationship between two objects. In this thesis, it will be important to define these relations more formally to characterize intersection graphs.

Definition 1.1.19. A partial order is a binary relation $\leq$ over a set A satisfying these axioms:

- if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry).
- if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).
- $a \leq a$ (reflexivity).

Definition 1.1.20. A total order is a partial order where the reflexivity order is replaced by the connex property:

$$a \leq b \text{ or } b \leq a$$

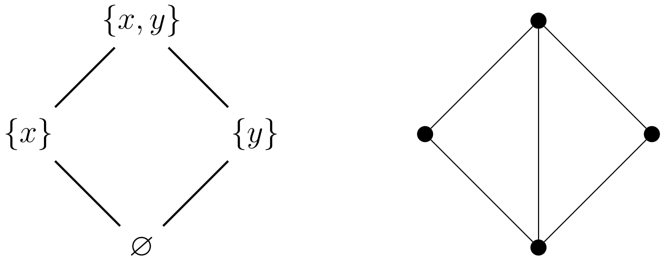


Figure 1.1: On the left, Hasse diagram of a poset of the power set of 2 elements ordered by inclusion. On the right, the comparability graph of this poset.

Definition 1.1.21. A partially ordered set or poset $(S, \leq)$ where S a set and $\leq$ a partial order on S.

Definition 1.1.22. A spanning order $(V, <)$ of a graph $G = (V, E)$ is a total order on V such that for any three vertices $u < v < w$:

$$uw \in E \rightarrow uv \in E \text{ or } vw \in E$$

Definition 1.1.23. A graph $G = (V, E)$ is a comparibility graph if there exists a partial order $\leq$ such that $vw \in E \Leftrightarrow v \leq w$ or $w \leq v$. Equivalently, G is a comparability graph if it is the comparability graph of a poset. For example, the Hasse diagram (figure 1.1) is a comparability graph where the relation is inclusion.

Definition 1.1.24. A graph $G = (V, E)$ is a co-comparability graph if its complement is a comparability graph.

There are multiple characterizations for the co-comparability graph class; we will see one that uses a poset to characterize it:

Theorem 1.1.25 (Damaschke [Dam92])**.** *A graph G is a co-comparability graph if and only if it has a spanning order.*

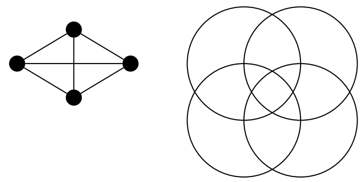


Figure 1.2: Realization of a UDG (Unit Disk Graph).

Disks

A disk graph G is a graph that is an intersection graph of disks on the plane, when the size of the disk is unitary, we talk about unit disk graphs. This class of graphs is important for this thesis, as thin strip graphs are a sub-class of unit disk graphs.

We will refer to the unit disk graph class as UDG and an example of a realization can be found in the figure 1.2.

Induced forbidden subgraphs The characterization of this class with respect to its induced forbidden subgraphs has been studied [AZ16].

Theorem 1.1.26 (Atminas-Zamaraev)**.** *For every integer $k > 1$, $\overline{K_2 + C_{2k+1}}$ is a minimal induced subgraph of UDG.*

Theorem 1.1.27 (Atminas-Zamaraev)**.** *For every integer $k > 4$, $\overline{C_{2k}}$ is a minimal induced subgraph of UDG.*

1.2 Complexity

Complexity theory has the objective to establish lower bounds on how efficient an algorithm can be for a given problem [Sip06]. This approach let us have a reference point to establish the difficulty of a problem.

Definition 1.2.1. Let Σ be a finite alphabet, Σ^* every word derived from Σ, $L \subseteq \Sigma^*$ is a decision problem.

Definition 1.2.2. A decider for a decision problem A is an deterministic algorithm V where

$$A = \{w | V \, accepts \; w\}$$

A is polynomially decidable if it has a polynomial time decider [Sip06].

Definition 1.2.3. A verifier for a decision problem A is an deterministic algorithm V where

$$A = \{w | V \, accepts \; \langle w, c \rangle \; for \; some \; string \; c\}$$

A is polynomially verifiable if it has a polynomial time verifier [Sip06].

1.2.1 P vs NP

Definition 1.2.4. A problem $L \in \mathcal{P}$ if L is polynomially decidable.

Definition 1.2.5. A problem $L \in \mathcal{NP}$ if L is polynomially verifiable. Thus, $\mathcal{P} \subseteq \mathcal{NP}$.

To prove a bound of complexity on an unknown problem L we have to find another problem with already known complexity and find equivalences between those two. This can be achieved through *reductions*.

Definition 1.2.6. A reduction of a problem L to a problem M is a mapping of an instance of L (I_L) to an isntance of M (I_M) such that I_L is true for the problem L if and only if I_M is true for the problem M. This is noted $L \leq M$ and $L \leq_P M$ if the reduction is done in polynomial time.

With this concept we can define new complexity classes. $\mathcal{NP}$-hard is the set of problems so that we can reduce every $\mathcal{NP}$ problem to. The set of problems that are both $\mathcal{NP}$-hard and $\mathcal{NP}$ are called $\mathcal{NP}$-complete. This is generalized to every complexity class ($\mathcal{P}$, $\exists\mathbb{R}$, RP, etc...)

Satisfiability problem The satisfiability problem (SAT) is to decide the satisfiability of a CNF formula ϕ. A CNF formula is a boolean formula that is a conjunction of multiple clauses c_k. A clause is a disjunction of multiple literals. A literal may be a variable or a negation of a variable.

Theorem 1.2.7 (Cook-Levin)**.** *SAT is $\mathcal{NP}$-complete.*

Clique problem The clique problem is to find a maximum clique of a graph G.

Theorem 1.2.8. *CLIQUE is $\mathcal{NP}$-complete. [Kar72]*

Theorem 1.2.9. *CLIQUE is QPTAS when applied to disk graphs. [BGK+17]*

Theorem 1.2.10 (Clark-Colbourn)**.** *CLIQUE is $\mathcal{P}$ when applied to unit disk graphs. [CCJ90]*

1.2.2 $\exists\mathbb{R}$ complexity class

$\exists\mathbb{R}$ is the class that describes the problems that can be reduced to *the existential theory of the reals*[Exi06a]. The existential theory of the reals is the problem of deciding if a sentence of this form is true:

$$(\exists X_1 \ldots \exists X_n) : F(\exists X_1, \ldots, \exists X_n)$$

where F is a quantifier-free formula in the reals. In other words, it is a conjuntion of clauses where each clause is a real polynomial inequality where each variable X_k is a real number. We can see that ETR is NP-hard because SAT can be reduced to it.

Proof. Let's take an instance of SAT ϕ_{SAT} with clauses c_k and variables x_k, we can construct an instance of ETR ϕ_{ETR} where we can construct variables in the domain $\{0, 1\}$ with this equality, so for each variable X_k:

$$X_k - X_k^2 = 0$$

Each literal of each clause will be positive or negative depending if the literal is cancelled in ϕ_{SAT}:

$$x_k \to l = X_k$$

$$\neg x_k \to l = (1 - X_k)$$

Then for each clause we can have a polynomial for which the sum of the values of every literal in the clause must be greater than one, so that at least one literal is true:

$$\sum_{l \in c_k} l \geq 1$$

With this proof, it is easy to see that ϕ_{ETR} is valid if and only if ϕ_{SAT} is also valid. □

□

This result can show us that $P \subseteq NP \subseteq \exists\mathbb{R}$.

Problems in $\exists\mathbb{R}$

In this section we will describe some problems that are $\exists\mathbb{R}$-complete and will give an overview of the proof.

The art gallery problem Given a simple polygon P (without crossings between every side), we introduce *guards*. A guard g is a point such that every point of the polygon is watched by a guard. A point p is watched by a point q if the segment pq is contained in P. The subset G, being G the set of guards and $G \subseteq P$, is optimum if it has the minimal cardinality covering the whole polygon.

The art gallery problem is to decide, given a polygon P and a number of guards k, whether there exists a configuration of k guards in G guarding the whole polygon. The art gallery problem is $\exists\mathbb{R}$-complete [AAM17].

Proof idea First of all, we can see that the art gallery problem is in $\exists\mathbb{R}$ if we reduce this problem to ETR. If we have an instance (P, k) of the art gallery problem we can have a formula [EH06] like this:

$$\phi = \{\exists x_1 y_1, \ldots x_k y_k \forall p_x p_y : \text{INSIDE-POLYGON}(p_x, p_y) \rightarrow \bigvee_{1 \leq i \leq k} \text{SEES}(x_i, y_i, p_x, p_y)\}$$

Where INSIDE-POLYGON returns 1 if $(p_x, p_y) \in P$ and SEES returns 1 if the segment $(x, y)(p_x, p_y) \in P$. ϕ is not a ETR formula, so we would like to construct a quantifier-free formula with the idea of ϕ. To achieve this, the main idea is to have a small set of points $Q \subseteq P$ such that if these points are watched, the whole polygon is watched. This subset Q is called the *witness set*. The only thing is now to create a polynomial for each point that ensures that the point is watched by a guard.

To finish the proof we have to prove that the art gallery problem is $\exists\mathbb{R}$-hard. For this part an $\exists\mathbb{R}$-complete problem has been deducted from ETR. For the problem ETR-INV we have a set of variables $\{x_1, \ldots, x_n\}$ and a set of equations of this form:

$$x = 1, \quad x + y = z, \quad x \cdot y = 1$$

and the problem decides if it exists a solution to this set of equations such that the value of each variable is real in $[\frac{1}{2}, 2]$.

A reduction of ETR-INV is found to the art gallery problem by constructing a polygon P and finding a number g for that polygon such that the instance of ETR-INT is true if and only if P is covered by at most g guards.

Stretchability A pseudoline is a simple closed curve in the plane. The stretchability problem is to decide if given a pseudoline arrangement, it is equivalent to an arrangement of straight lines.

Proof idea ETR can be reduced to STRETCHABILITY due to Mnev's universality theorem. [Sch10]

Unit disk graph recognition The unit disk graph recognition is the problem that decides if a graph G is a unit disk graph. Unit disk graph recognition is $\exists\mathbb{R}$-complete. [Sch13a]

Proof idea UDG recognition is a corollary of deciding whether a graph with a given length is realizable. This problem is $\exists\mathbb{R}$-complete.

The reduction is done from STRETCHABILITY [Sch13a]. The reduction is done by adding a vertex to V for each pseudoline intersection. For each three consecutive points u_1, u_2, u_3 along a pseudoline a widget will be added that will be only realizable if and only if the pseudoline can be stretched with the same arrangement.

1.3 Geometry

Definition 1.3.1. $\operatorname{dist}(a, b)$ denotes the distance between the points a and b and is calculated with:

$$\operatorname{dist}(a, b) = \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2}$$

The intersection of convex objects is a matter well studied for multiple subjects. In our case, it is interesting to know some properties about the intersection of disks, those ones being convex objects.

A set S is convex if:

$$\forall p, q \in S \ \forall \lambda \in [0, 1] : (1 - \lambda)p + \lambda q \in S$$

1.3.1 Stabbing

A *stabbing* is a point that traverses a set of intersecting objects. A lot of research has been done [Sch13b] on the minimal amount of stabbings to cover every object in a set. Stabbings can also be done with more complex structures than points, in that case we are talking about *coverings*.

Theorem 1.3.2 (Helly). *Given a set Q of objects in $\mathbb{R}^d$, if for each subset of Q of size $d + 1$ their intersection is non empty, then $\bigcap_{q \in Q} \neq \varnothing$. [Hel23]*

Theorem 1.3.3. *The problem that for a set of n disks whether there exists a regular n-gon whose vertices stab every disk of the set can be decided in $O(n^{10.5}/\sqrt{\log(n)})$ [Sch13b]*

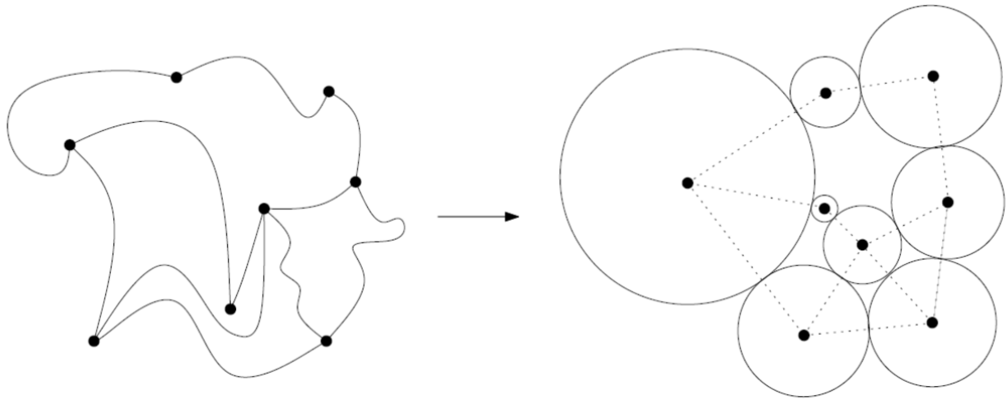


Figure 1.3: Circle packing of a planar graph. [Nac16]

1.3.2 Coin graphs

Penny graphs can be defined as disk graphs where the disks can just touch each other without overlapping. A famous theorem is derived from this class of graphs: the circle packing theorem.

Theorem 1.3.4 (Circle packing theorem)**.** *The circle packing theorem states that every simple connected planar graph G is a penny graph. [BS93]*

Corollary 1.3.5. *Planar graphs $\subseteq$ disk graphs [Spi12].*

Chapter 2

Interval graphs

In this chapter we are going to introduce the class of interval graphs, which is one of the most used classes of intersection graphs at all. There are multiple types of interval graphs and the most relevant for the thesis are going to be defined below.

The mixed unit interval graphs have been introduced by Dourado et al. [DLP$^+$12a] and its characterization is given by Joos [Joo13]. The proof of the characterization will not be given because of the length of it, but each family of forbidden subgraphs will be presented.

Last, we present unfettered unit interval graphs, which have been defined by Hayashi et al. [HKO$^+$17] while defining thin strip graphs in their paper.

2.1 Interval graphs

An interval graph is a graph G that is the intersection graph of a collection of closed intervals in $\mathbb{R}$.

First we present the main characterizations of interval graphs. In the next sections we present some other subclasses of interval graphs that will help us characterize the thin strip graphs on chapter 3. There are multiple characterizations of interval graphs that are equivalent, in this thesis we are going to present only one, which is the most relevant for our research:

Theorem 2.1.1. *G is an interval graph if and only if G does not contain C_4 as an induced subgraph and $\overline{G}$ has a transitive orientation (it is a co-comparability graph). (Gilmore and Hoffman [GH64])*

Proof. We want to prove that G is an interval graph if and only if does not contain C_4 and it is

2.1.1 Unit interval graphs

When every interval has the same length (or *unitary*), the intersection graph of this interval set is referred to as a unit interval graph (or UIG). Roberts [Rob68] shows in his paper about indifference relations a characterization of UIG and an interesting equivalence with another class:

Theorem 2.1.2 (Roberts [Rob68])**.** *A graph is a unit interval graph if and only if it is a proper interval graph (an interval graph where no interval is a strict subset of another).*

Theorem 2.1.3 (Roberts [Rob68])**.** *An interval graph is a unit interval graph if and only if it has no induced subgraph $K_{1,3}$ (or* claw*).*

2.2 Mixed unit interval graphs

The next class of interval graphs that we present are mixed unit interval graphs, where each interval is unitary and can be closed, open, open-closed or closed-open.

In this paper we define four classes of unitary interval graphs:

$$\mathcal{U}^{++} = \{[x, y] : x, y \in \mathbb{R}, x \leq y\}$$

$$\mathcal{U}^{--} = \{(x, y) : x, y \in \mathbb{R}, x \leq y\}$$

$$\mathcal{U}^{+-} = \{[x, y) : x, y \in \mathbb{R}, x \leq y\}$$

$$\mathcal{U}^{-+} = \{(x, y] : x, y \in \mathbb{R}, x \leq y\}$$

where $\mathcal{U}^{++} = \text{UIG}$.

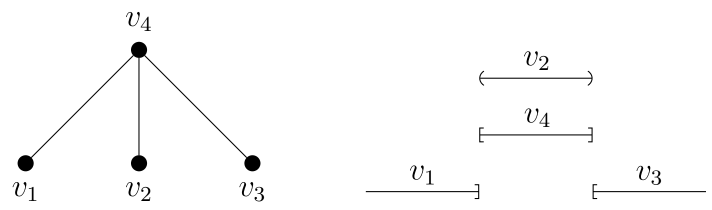


Figure 2.1: Representation of $K_{1,3}$ as a MUIG.

Theorem 2.2.1 (Dourado et al. [DLP+12b]). *The classes of the graphs $\mathcal{U}^{--}$, $\mathcal{U}^{++}$, $\mathcal{U}^{-+}$, $\mathcal{U}^{+-}$, and $\mathcal{U}^{-+} \cup \mathcal{U}^{+-}$ are the same.*

We can already show an equivalence between UIG and MUIG:

Theorem 2.2.2 (Dourado et al. [DLP+12a]). *UIG $\subsetneq$ MUIG.*

Proof. The strict inclusion is straightforward: we know that UIG $= \mathcal{U}^{++} \subset$ $MUIG$ by definition.

And we only have to find a forbidden subgraph in UIG that is a MUIG. By theorem 2.1.3 we have $K_{1,3}$ forbidden in UIG, but it is in fact in MUIG if we take an interval set a, b, c, d such that:

- $a = [x, x+1]$
- $b = (x, x+1)$
- $c = [x+1, x+2]$ (or $[x+1, x+2)$)
- $d = [x-1, x]$ (or $[x-1, x)$)

for all $x \in \mathbb{R}$ (see Figure 2.1). □

A complete characterization by induced forbidden subgraphs have been found independently by A. Schuchat et al. [SSTW14a] and F. Joos [Joo13]. However, the Schuchat paper gives a polynomial algorithm to recognize MUIGs:

Theorem 2.2.3 (Schuchat et al. [SSTW14b]). *The MUIG recognition problem is in $\mathcal{P}$. Moreover, there is an algorithm that solves it in $O(|V|^2)$ for V the vertex set of a graph.*

2.2.1 Characterization

In this section we will go over the characterization of MUIG given by Joos with forbidden subgraphs. We will also review each one of these forbidden subgraphs and discuss them:

Theorem 2.2.4 (Joos [Joo13]). *G is a MUIG if and only if it is a $\{F\} \cup \mathcal{R} \cup \mathcal{S} \cup \mathcal{S}'' \cup \mathcal{T}$-free interval graph.*

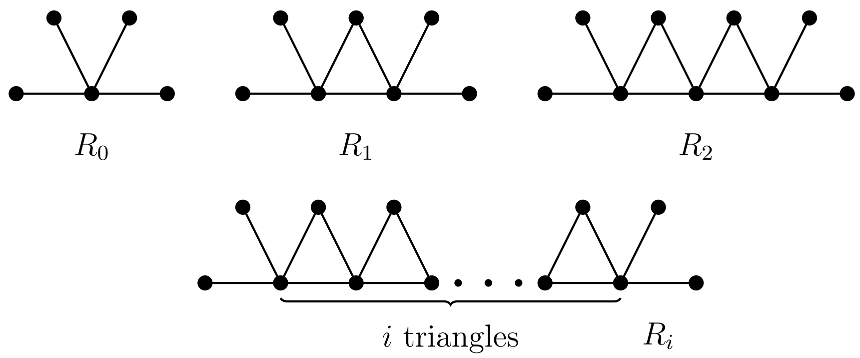


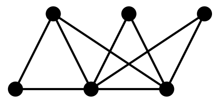
Figure 2.2: The class $\mathcal{R}$. [Joo13]

Without including F, every family of forbidden graphs of MUIG is infinite, and is defined recursively by its precedent: then every property of these graphs has to be proved recursively. We begin first with $\mathcal{R}$.

Lemma 2.2.5. *$\mathcal{R}$ is a family of co-comparability graphs.*

Proof. If we recall Theorem 1.1.25, in order to prove that $\mathcal{R}$ is a family of co-comparability graphs we will have to find a spanning order for every R_i with $i \geq 0$. We will proceed to label our vertices with a mapping function $f : V \to \mathbb{N}$ such that $f(v) \in [1, |V|]$. This mapping will give us a spanning order by induction:

- $i = 0$: We assign the number 1 to the vertex with maximum degree v_1. We assign then the rest of the numbers to the other vertices. We see then that $\forall u < v < w : uw \in E \to uv \in E$ because every vertex is adjacent to v_1.



F

Figure 2.4: The graph F. [Joo13]

- $i = i+1$: We define $\lambda_i = 5 + 2i$ where $\lambda_i = |V(R_i)|$. We add two vertices on each graph, where their labels are $\lambda_i + 1$ and $\lambda_i + 2$ and we also add three new edges: $v_{\lambda_i}v_{\lambda_i - 1}, v_{\lambda_i}v_{\lambda_i+1}, v_{\lambda_i}v_{\lambda_i+2} \in E$.

  By induction we only have to see if it holds with the new edges. We can say that it still holds with $v_{\lambda_i}v_{\lambda_i-1}$ and $v_{\lambda_i}v_{\lambda_i+1}$ because:

$$\nexists k \in \mathbb{N} : i < k < i+1$$

  Finally, we see that $v_{\lambda_i}v_{\lambda_i+2}$ is a valid edge because $v_{\lambda_i}v_{\lambda_i+1} \in E$. $\square$

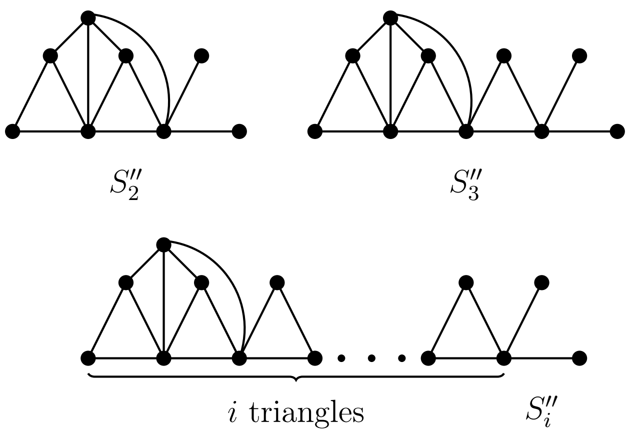


Figure 2.3: The class $\mathcal{S}''$. [Joo13]

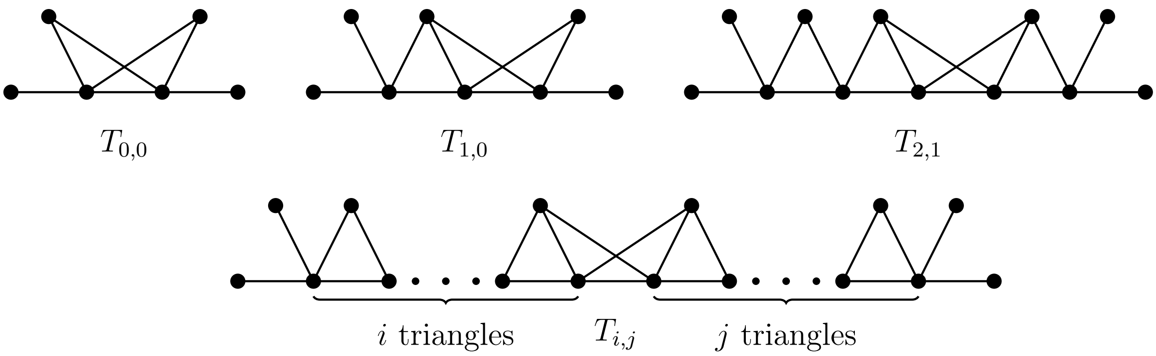


Figure 2.5: The class $\mathcal{T}$. [Joo13]

2.3 Unfettered unit interval graphs

An unfettered unit interval graph can be defined as an unit interval graph such that for every touching endpoints we can chose either if they are adjacent or not.

Hayashi has characterized this class of graphs by levels. A *level structure* of a graph $G = (V, E)$ is a partition $L = \{L_i : i \in [1, t]\}$ of V such that

$$v \in L_k \to N(v) \subseteq L_{k-1} \cup L_k \cup L_{k+1}$$

where $L_0 = L_t + 1 = \varnothing$.

Theorem 2.3.1 (Hayashi et al. [HKO$^+$17])**.** *A graph G is an unfettered unit interval graph if and only if it has a level structure where each level is a clique.*

We can clearly see that MUIG $\in$ UUIG. However, we still have to see what is the location of UUIG in the higher graph classes hierarchy:

Proposition 2.3.2. *UUIG $\subset$ co-comparability.*

Proof. This proposition is equivalent to say that a graph G is a UUIG if and only if it has a spanning order.

For each vertex of a partition L_k of UUIG (Theorem 2.3.1) we assign arbitrarily a number $i \in [\max(V(L_{k-1})) + 1, \max(V(L_{k-1})) + |V(L_k)| + 1]$;

intuitively, we assign every available number from the beginning in increasing order ($|V(L_1)|$ first numbers on the first partition and consecutively).

Because we know that each partition L_k is a clique, we can say that for each three vertices $u < v < w$, if $vw \in E \to uv \in E$ or $vw \in E$. We know this because given $u \in L_i$ and $w \in L_j$: if $uw \in E$ it means that levels L_i and L_j are adjacent, which means that $v \in L_i$ or $v \in L_j$ so v will be adjacent either to u or w. This is a spanning order. □

If we recall the characterization of MUIG in section 2.2.1, we can see that every forbidden graph of MUIG is an UUIG (except for $\mathcal{R}$); which means that they are also co-comparability graphs.

In the other hand, we can find a graph in UUIG that is not an UDG. This theorem will be used in chapter 3.

Theorem 2.3.3 (Hayashi et al. [HKO+17]). *UUIG $\neq$ UDG.*

Proof. We can define $G = (L_1 \cup L_2, E)$ a UUIG with two levels $L_1 = \{v_1, v_2, v_3, v_4\}$ and $L_2 = \wp(L_1)$ and $E = \binom{L_1}{2} \cup \binom{L_2}{2} \cup \{vw : w \in L_2, v \in w\}$.

We can see L_1 as a Venn diagram with four sets. We know by instance that a Venn diagram cannot be constructed if the number of sets is bigger than four [Ven80]. Thus, $G \notin$ UDG. □

2.3.1 Recognition

As we mentionned in the previous section, UUIG is a class of graphs very relevant to define TSG and that is why we are interested in knowing how this class of graphs is recognized.

Lemma 2.3.4. *Let G be a connected UUIG with a level structure with levels $L_1, \ldots, L_n$. $G \setminus L_i$ is a graph where each connected component is also an UUIG and the number of connected components is not bigger than two.*

Proof. By definition for a graph with a level structure, if $v \in L_i$, $N(v) = L_{i-1} \cup L_i \cup L_{i+1}$. This said, if we delete a level L_i, L_{i-1} and L_{i+1} are disconnected, but they are still connected to the other consecutive levels (L_{i-1} is connected to L_{i-2}, which is connected to L_{i-3}... and viceversa with L_{i+1}).

And because a level is only adjacent to two other levels, we only have two connected components, only one if $L_i = L_1$ or $L_i = L_n$. □

By this lemma we can suppose that the input graph G is a connected graph. Given an input graph G, we can compute a level structure where each level is a clique in exponential time.

Theorem 2.3.5. *UUIG recognition is in $\mathcal{NP}$.*

Proof. We can design a deterministic algorithm that solves UUIG recognition in exponential time when G is a connected graph. We begin by taking an arbitrary vertex $v \in G$. By instance, this vertex is included in the maximal clique $K \subseteq G$.

We have $P(K \setminus \{v\})$ the powerset of the clique K excluding v. For each subset $s \in P(K \setminus \{v\})$, we have a subgraph $H = G \setminus (s \cup \{v\})$. We recall that $q(G)$ denotes the set of connected components of G. We can have three cases:

1. $|q(H)| > 2$: If the number of components of H is bigger than 2, the current chosen clique is connected to more than two different cliques (or different levels); this is an invalid level, and we choose another clique from $s \in P(K \setminus \{v\})$.

2. $0 < |q(H)| \leq 2$: The current chosen clique is connected to one or two levels, which still respects our definition of level structure for the current chosen level. We check recursively if those connected components are also UUIG.

3. $|q(H)| = 0$: The current clique is an UUIG with only one level. This is a valid valid.

To prove that UUIG recognition is in UUIG, we have to prove that it is in $\mathcal{NP}$. We have an upper bound on the complexity of this algorithm that is given by:

$$T(n) \leq 2^{\omega(G)-1} T(n-1)$$

Which gives us:

$$T(n) \leq (2^{\omega(G)-1})^n = 2^{O(n\omega(G))}$$

with $\omega(G)$ the size of the maximum clique of G. $\square$

Future work on the recognition of unit unfettered interval graphs would be to adapt this algorithm to avoid combinatorial complexity. In our case we are interested in seeing the recognition of UUIG for unit disk graphs. We know that the CLIQUE problem is in $\mathcal{P}$ for unit disk graphs and the first hypothesis was that given an UUIG G, at least one level of G is a maximal clique of the graph. Nevertheless, we have a counterexample in T_0 (Fig. 2.5) where the levels of the graph are $\{K_1, K_2, K_2, K_1\}$ while $\omega(T_0) = 3$.

Observation 2.3.6. *Given an* UUIG G, *no level of* G *has to be a maximal clique.*

Chapter 3

Thin strip graphs

The goal of this chapter is to introduce you to the main subject of this thesis. Thin strip graphs is a class of graphs that lay between unit disk graphs and mixed interval graphs. We can define formally a c-strip graph as a unit disk graph such that the centers of the disks belong to $\{(x, y) : -\infty < x < \infty, 0 \leq y \leq c\}$, more intuitively we can see this as a unit disk graph where the centers of the disks lay between two parallel horizontal lines with a distance of c between them. We denote this class by SG(c). We have then that SG(0) = UIG and SG(∞) = UDG.

The definition and main work for this class comes from Breu in his thesis [Bre96]. However, Hayashi et al. [HKO$^+$17] expand his work by defining the class of *thin strip graphs*.

3.1 Thin strip graphs

A thin strip graph can be intuitively defined as a c-strip graph where c is an arbitrarily little ε. Also, we can see that SG(k) $\subseteq$ SG(l) with $k < l$. A more strict definition emerges from this observation:

Definition 3.1.1. Thin strip graphs are defined as TSG $= \bigcap_{c>0}$ SG(c).

Remark 3.1.2. SG(0) $\neq$ TSG. We can construct a $K_{1,3}$ such that we have 3 vertices with the coordinates $(1, 0)$, $(0, 0)$, $(1, 0)$ and a last one $(0, \varepsilon)$ with $\varepsilon > 0$ and arbitrarily small as seen in Figure 3.1.

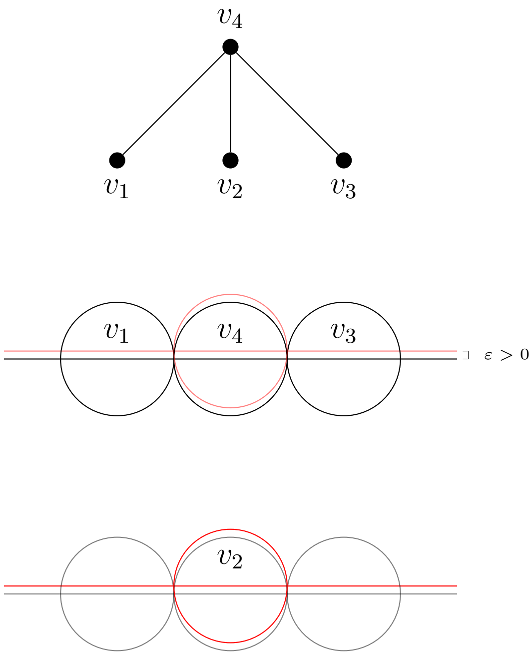


Figure 3.1: A construction of $K_{1,3}$ with a disk realization, being this graph a TSG.

Theorem 3.1.3 (Hayashi et al. [HKO+17])**.** *There is no constant t such that SG(t) = TSG.*

Theorem 3.1.4 (Hayashi et al. [HKO+17])**.** *There is no constant t such that SG(t) = UDG.*

Hayashi et al. left some open problems. I try to expand the knowledge around some of these problems to help the understanding of them, largely the recognition of this class of graphs. Before, we see where this class lays in the hierarchy of classes. We know by definition that TSG $\subsetneq$ UDG.

3.1.1 Interval graphs

Thin strip graphs shares their geometrical structure with interval graphs (remember SG(0) = UIG). In this subsection, we overview the results of Hayashi et al. [HKO+17] where they find maximal and minimal superclasses for TSG in the interval graphs presented in chapter 2. The following theorem will be proven by taking the proof written by Hayashi et al. in order to use their mapping in other theorems (e.g. 4).

Theorem 3.1.5 (Hayashi et al. [HKO+17]). *MUIG* $\subsetneq$ *TSG.*

Proof. First, we prove that MUIG $\neq$ TSG. This can be proven because $C_4 \in$ TSG if we take as points $(0,0),(0,\varepsilon),(1,0),(1,\varepsilon)$ with $1 > \varepsilon > 0$ and $C_4 \notin$ MUIG because it is a chordal graph.

Then, we have to prove that MUIG $\subseteq$ TSG. Let $G = (V, E) \in$ MUIG where each vertex is a unit mixed interval denoted as I_v. We define $t = \min\{|I_u \cap I_v| : |I_u \cap I_v| > 0, \{I_u, I_v\} \subseteq V\}$ and $s = \min\{\ell(I_v) - r(I_u) : \ell(I_v) > r(I_u), \{I_u, I_v\} \subseteq V\}$. We have then t being the minimum length of an intersection bigger than zero (that is, not endpoint-adjacent) and s is the minimum distance between non-adjacent vertices (also not endpoint-adjacent). We also define $c(I_v) = \frac{\ell(I_v)+r(I_v)}{2}$ as the center of the interval and $p(I_v) = (-1)^{\lfloor c(I_v) \rfloor}$.

Let d be a real such that $0 < d < \frac{2}{3}$, $d \leq \frac{t}{4}$, $d < \frac{s}{2}$ and $\varepsilon \geq 2\sqrt{d - d^2}$. If we let $h = \sqrt{d - d^2}$, then we can create a $2h$-realization of G with the following mapping:

$$\phi(v) = \begin{cases} (c(I_v), 0) & \text{if } I_v \text{ is a closed interval} \\ (c(I_v), hp(I_v)) & \text{if } I_v \text{ is an open interval} \\ (c(I_v) - d, hp(I_v)) & \text{if } I_v \text{ is a closed-open interval} \\ (c(I_v) + d, hp(I_v)) & \text{if } I_v \text{ is an open-closed interval} \end{cases}$$

For two vertices u and v of G such that $u < v$, we have the three following cases:

1. $r(I_u) < \ell(I_v)$:

   I_u and I_v are not adjacents, which means that $\text{dist}(\phi(u), \phi(v)) > 1$. If we minimize the distance between them we have $\phi(u) = (c(I_v) + d, hp(I_v))$ and $\phi(v) = (c(I_v) - d, hp(I_v))$. Therefore,

2. $r(I_u) > \ell(I_v)$:

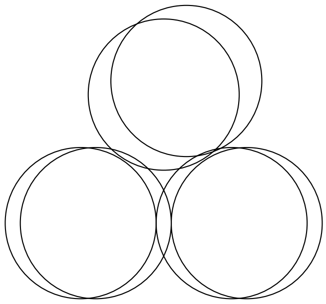


Figure 3.2: A construction of F with a unit disk realization.

3. $r(I_u) = \ell(I_v)$:

We can define a new class of graphs: unfettered unit interval graphs. These graphs are unit interval graphs where if two intersections touch, we can decide whether they intersect or not. We denote this class UUIG.

Theorem 3.1.6 (Hayashi et al. [HKO+17])**.** *TSG* $\subsetneq$ *UUIG.*

Proof. See [HKO+17].

longer demonstration, maybe is going to go to the appendix.

3.2 Characterization of thin strip graphs

One of the main goals of this thesis is to characterize TSG. by forbidden induced subgraphs. To approach this, we will see how many induced forbidden subgraphs are also forbidden for TSG. We have described the familes of forbidden induced subgraphs for MUIG in section 2.2 and one of these familes has been proven to be a forbidden induced subgraph for TSG:

Theorem 3.2.1 (Hayashi et al. [HKO+17])**.** $\mathcal{R}$ *is a forbidden induced subgraph family of TSG.*

With the properties presented in this chapter, we can begin to state our first hypothesis:

Hypothesis 3.2.2. $F \in (UDG \cap UUIG) \setminus TSG.$

Claim 3.2.3. $F \in (UDG \cap UUIG)$

Proof. We can see that $F \in$ UDG because we can have a unit disk realization (Figure 3.2) and also has a level structure $L = \{K_2, K_3, K_1\}$. □

Theorem 3.2.4. *$F \notin TSG$.*

this is bullshit, F is in fact in TSG...

Proof. The distance between a and d has to be at most 2, because there is at least one vertex that is adjacent to both. And clearly, the other points have to be between them because they are adjacent to both, so if they were not.

3.3 Recognition

The recognition of this class of graphs is stated by Breu in his thesis .

explain everything about dangerous cycles and complement oriented graphs in Breu's paper

Chapter 4

Thin two-level graphs

Breu [Bre96] has presented in his thesis a similar class of constrained unit disk graphs where the disks are placed on k horizontal parallel lines. More formally: a disk (x, y) can be placed in $x \in (-\infty, \infty)$ and $y \in L$ with $|L| = k$.

In this chapter I define thin two-level graphs as a two-level graph where $L = \{0, \epsilon\}$ and ϵ is an arbitrarily small real number.

4.1 Characterization of TTL

Proposition 4.1.1. *An σ-SG(c) graph G (with $c < 1$) can be characterized by computing $\delta : A \times B \to E$ where $A, B \subseteq G$, A and B are UIG, and $A \cup B = \varnothing$:*

$$\delta(x, y) = \begin{cases} xy & \textit{if } dist(x, y) \leq 1 \\ \varnothing, & \textit{otherwise} \end{cases}$$

this proposition is ugly, re-do this with Breu's notations.

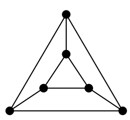


Figure 4.1: Forbidden graph in TTL

This class of graphs is close to our main class TSG. But we have to know at what point we can rely in this class of graphs to study TSG:

Lemma 4.1.2 (Breu [Bre96])**.** *Let abcda be a chordless 4-cycle in a two-level graph $G = (V, E)$. Then ad and bc are level edges (they are adjacent in the same level), and the others are cross edges for every realization ϕ of G for which $\phi(a)_x < \phi(c)_x$ and $\phi(b)_x < \phi(d)_x$.*

With this preliminary result, we proceed to one of our main results:

Theorem 4.1.3. *TTL $\subsetneq$ TSG*

Proof. By definition, we know that TTL $\subset$ TSG because the area where the disks can be placed in TTL is included in the area in TSG.

We can prove that TTL $\neq$ TSG because we can construct a graph G such that $G \in$ TSG and $G \notin$ TTL. This graph D is a net* graph as described in Figure 4.1.

Part 1. *D is a TSG because we can realize it as a TSG if we take as center of disks $(0,0)$, $(0,z)$, $(0,\epsilon)$, $(1,0)$, $(1,z)$, $(1,\epsilon)$ such that $0 < z < \epsilon$.*

Part 2. *Now we have to prove that D is a forbidden induced subgraph of TTL. We will try to construct it by taking a induced subgraph that is realizable: we take $D_{-1} = D - x$ with $x \in V(D)$. We notice that $V(D_{-1})$ is a chordless C_4 (abcd) with a vertex e adjacent to any two consecutive vertices $x, y \in V(C_4)$ creating the triangle xye.*

By Lemma 4.1.2 we know that abcda is a cycle if ab and cd are level edges. We can classify these vertices in two sets: $\ell(V) = a, d$ and $r(V) = b, c$ where $\forall u \in \ell(V) v \in r(V) : \phi(u)_x < \phi(v)_x$.

To realize D_{-1} we have to add a vertex e to C_4. In our case, e cannot be added in two

Finish proof here

4.1.1 Relation with MUIG

A big question that was asked during this research is: What is the relationship between TTL and MUIG? We know that the net*-graph is forbidden for every two-level graph, this graph is also forbidden for MUIG because it includes

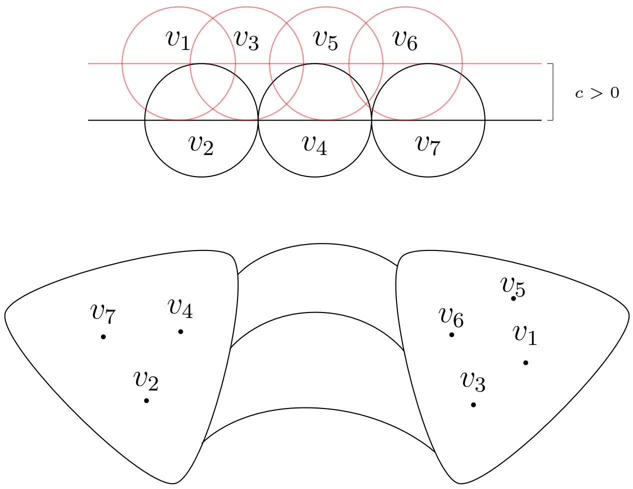


Figure 4.2: A representation of a TL(c)

an induced C_4; we can also see that every forbidden induced subgraph for MUIG also is for TTL.

However, C_4 is realizable for two-level graphs with $c = 1$ in general, which means TTL $\neq$ interval $=$ MUIG. We then know that TTL $\cup$ MUIG $\subseteq$ TSG with TTL $\neq$ MUIG.

Rewrite this better.

4.2 Induced forbidden subgraphs

Chapter 5

Conclusion

This is a conclusion

Conclusions

The conclusions are to be written with care, because it will be sometimes the part that could convince a potential reader to read the whole document.

Appendices

Appendix A

Graph classes hierarchy

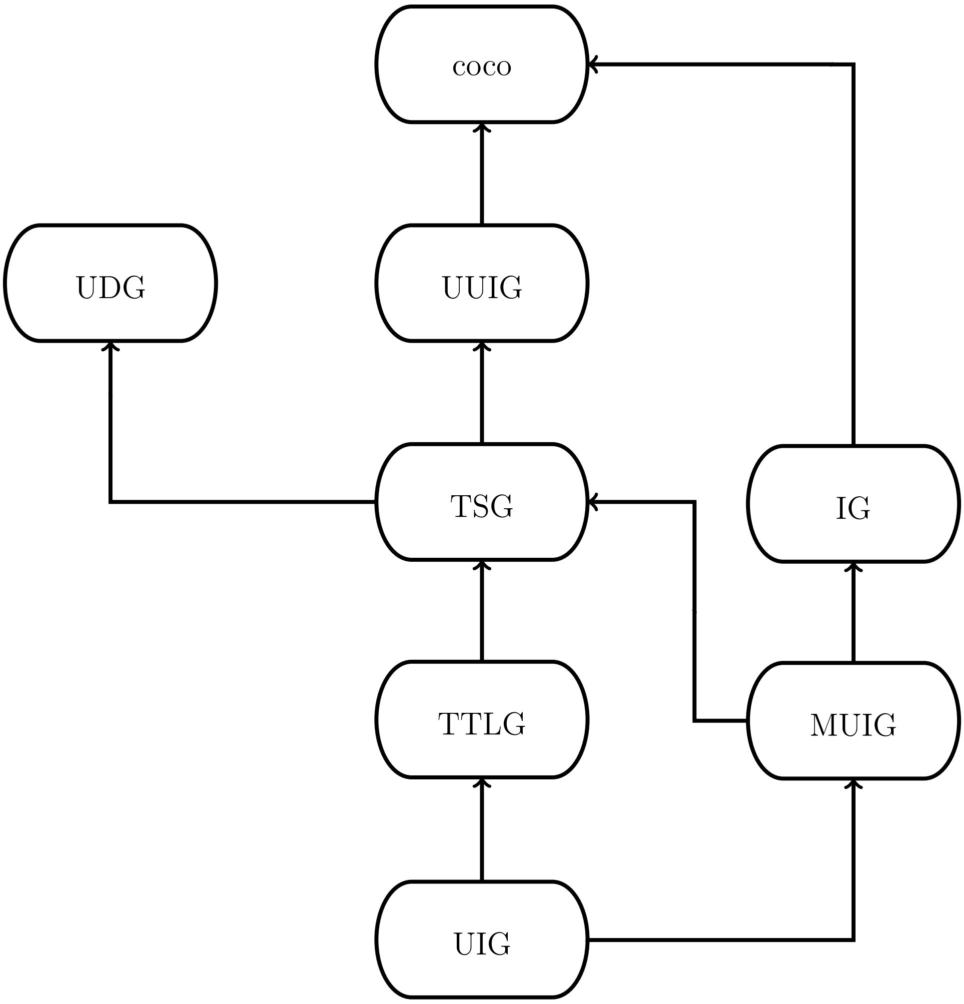


Figure A.1: A hierarchy of every relevant graph of this document. The relation $\text{class}_1 \to \text{class}_2$ means that $\text{class}_1 \subset \text{class}_2$.

Appendix B

Problems in inclusion

- **MUIG $\subsetneq$ TSG $\subsetneq$ UUIG** : Hayashi [HKO+17]
- **MUIG $\neq$ TTLG (Open)**: To prove that MUIG $\subsetneq$ TSG, Hayashi [HKO+17] could simulate MUIGs with 4 different levels. Having only two levels, I conjecture that this is not possible. However, MUIG can have C_4, so an inclusion between these two classes is impossible (it has to be rewritten).
- **TTLG $\subsetneq$ TSG (Open)**: This problem has been solved in my thesis by finding a forbidden graph for TTLG, theorem 4.1.3.
- **TLG $\subset$ TSG (Open)**: This is a plausible stronger statement than the one before. However, this result could make the study of TTLG less relevant. Thus, this result would imply:

$$G \in \text{TLG}(j) \rightarrow G \in \text{SG}(k) : j, k \in \mathbb{R}$$

Appendix C

Problems in forbidden induced subgraph characterization

- **MUIG**: Joos [Joo13] gives us a complete characterization of forbidden graphs.
- **TSG (Open)**: Hayashi [HKO+17] says that MUIG's forbidden induced subgraphs also are in TSG. He claims that finding a graph $F \in (\text{UDG} \cap \text{UUIG}) \setminus \text{TSG}$ could be a good starting point. In my thesis I show that a forbidden induced subgraph for MUIG is in UDG$\cap$UUIG.
- **TTLG (Open)**: There are many properties about these graphs in Breu's thesis [Bre96].
- **UDG (Open)**: There is no complete characterization of UDG. Can the results of this thesis help find new ones?U

Appendix D

Problems in complexity

- **UIG/IG recognition**: Both of these problems are polynomial.
- **MUIG recognition**: Schuchat et al. give a linear algorithm ($O(|V|^2)$) to recognise MUIGs [SSTW14b].
- **UDG recognition**: $\exists\mathbb{R}$-complete [Exi06b].
- **SG(c) recognition (Open)**: Breu [Bre96] states that SG(c) recognition is polynomial if a complement edge orientation and a mapping $\phi : V \to [0, c]$ is polynomial as an input of the decision problem.
- **TSG recognition (Open)**: Can we get rid of the mapping as input to recognise TSGs? In that case the problem would be at least NP.
- **UUIG recognition (Open)**: Informally the recognition of this class of graphs **cannot** be polynomial because we have to find all the cliques of the graph; the CLIQUE problem is NP-complete.

Index

Bibliography

[AAM17] Mikkel Abrahamsen, Anna Adamaszek, and Tillmann Miltzow. The Art Gallery Problem is $\exists \mathbb{R}$-complete. *arXiv:1704.06969 [cs]*, April 2017.

[AZ16] Aistis Atminas and Viktor Zamaraev. On forbidden induced subgraphs for unit disk graphs. *arXiv:1602.08148 [math]*, February 2016.

[BGK$^+$17] Édouard Bonnet, Panos Giannopoulos, Eun Jung Kim, Paweł Rzążewski, and Florian Sikora. QPTAS and Subexponential Algorithm for Maximum Clique on Disk Graphs. *CoRR*, abs/1712.05010, 2017.

[Bre96] Heinz Breu. *Algorithmic Aspects of Constrained Unit Disk Graphs.* PhD thesis, National Library of Canada = Bibliothèque nationale du Canada, Ottawa, 1996. OCLC: 46501382.

[BS93] Graham R. Brightwell and Edward R. Scheinerman. Representations of Planar Graphs. *SIAM Journal on Discrete Mathematics*, 6(2):214–229, 1993.

[CCJ90] Brent N. Clark, Charles J. Colbourn, and David S. Johnson. Unit Disk Graphs. *Discrete Mathematics*, 86(1):165–177, 1990.

[Dam92] Peter Damaschke. Distances in cocomparability graphs and their powers. *Discrete Applied Mathematics*, 35(1):67–72, January 1992.

[DLP+12a] Mitre C. Dourado, Van Bang Le, Fábio Protti, Dieter Rautenbach, and Jayme L. Szwarcfiter. Mixed unit interval graphs. *Discrete Mathematics*, 312(22):3357–3363, November 2012.

[DLP+12b] Mitre C. Dourado, Van Bang Le, Fábio Protti, Dieter Rautenbach, and Jayme L. Szwarcfiter. Mixed unit interval graphs. *Discrete Mathematics*, 312(22):3357–3363, 2012.

[EH06] Alon Efrat and Sariel Har-Peled. Guarding galleries and terrains. *Information Processing Letters*, 100(6):238–245, 2006.

[Exi06a] Existential Theory of the Reals. In *Algorithms in Real Algebraic Geometry*, volume 10, pages 505–532. Springer Berlin Heidelberg, 2006.

[Exi06b] Existential Theory of the Reals. In *Algorithms in Real Algebraic Geometry*, volume 10, pages 505–532. Springer Berlin Heidelberg, 2006.

[GF17] Palash Goyal and Emilio Ferrara. Graph Embedding Techniques, Applications, and Performance: A Survey. *arXiv:1705.02801 [physics]*, May 2017.

[GH64] P. C. Gilmore and A. J. Hoffman. A characterization of comparability graphs and of interval graphs. *Canadian Journal of Mathematics*, 16(0):539–548, January 1964.

[Hel23] E. Helly. über Mengen konvexer Körper mit gemeinschaftlichen Punkten. *Jahresber. Deutsch. Math.-Verein.*, 32:175–176, 1923. cited By 187.

[HKO+17] Takashi Hayashi, Akitoshi Kawamura, Yota Otachi, Hidehiro Shinohara, and Koichi Yamazaki. Thin Strip Graphs. *Special Graph Classes and Algorithms — in Honor of Professor Andreas Brandstädt on the Occasion of His 65th Birthday*, 216:203–210, January 2017.

[Joo13] Felix Joos. A Characterization of Mixed Unit Interval Graphs. *arXiv:1312.0729 [math]*, December 2013.

[Kar72] Richard M. Karp. Reducibility among Combinatorial Problems. In Raymond E. Miller, James W. Thatcher, and Jean D. Bohlinger, editors, *Complexity of Computer Computations: Proceedings of a Symposium on the Complexity of Computer Computations, Held March 20–22, 1972, at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York, and Sponsored by the Office of Naval Research, Mathematics Program, IBM World Trade Corporation, and the IBM Research Mathematical Sciences Department*, pages 85–103. Springer US, Boston, MA, 1972.

[Nac16] Asaf Nachmias. Planar maps, random walks and the circle packing theorem, September 2016.

[Rob68] Fred S Roberts. *Representations of Indifference Relations*. Department of Mathematics, Stanford University., 1968.

[Sch10] Marcus Schaefer. Complexity of Some Geometric and Topological Problems. In David Eppstein and Emden R. Gansner, editors, *Graph Drawing*, pages 334–344, Berlin, Heidelberg, 2010. Springer Berlin Heidelberg.

[Sch13a] Marcus Schaefer. Realizability of Graphs and Linkages. In János Pach, editor, *Thirty Essays on Geometric Graph Theory*, pages 461–482. Springer New York, New York, NY, 2013.

[Sch13b] L.M. Schlipf. *Stabbing and Covering Geometric Objects in the Plane*. 2013.

[Sip06] Michael Sipser. *Introduction to the Theory of Computation*. Course Technology, second edition, 2006.

[Spi12] Jeremy P Spinrad. *Efficient Graph Representations.* American Mathematical Society, Providence, R.I, 2012. OCLC: 1030370765.

[SSTW14a] Alan Shuchat, Randy Shull, Ann N. Trenk, and Lee C. West. Unit Mixed Interval Graphs. *arXiv:1405.4247 [math]*, May 2014.

[SSTW14b] Alan Shuchat, Randy Shull, Ann N. Trenk, and Lee C. West. Unit Mixed Interval Graphs. *arXiv:1405.4247 [math]*, May 2014.

[Ven80] J. Venn. I. *On the diagrammatic and mechanical representation of propositions and reasonings. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 10(59):1–18, July 1880.

Todo list