Characterization and complexity of Thin Strip Graphs

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ABSTRACT

Abstract

1 Graphs and disks

1.1 Graphs

A graph G is defined as $G = (V, E)$, where V is the set of vertices and E the set of edges. A vertex $v \in V$ is the fundamental unit of a graph. An edge $e \in E$ is a structures that links two vertices. The vertices $vw \in V$ that $e \in E$ links are called the *endpoints*.

Definition 1 *An embedding of a graph G is a representation of this graph on the plane.*

A graph G is planar if there is an embedding of this graph that doesn't have any crossing between the edges.

Theorem 2 (Kuratowski) *A graph G is planar iff it doesn't contain K_5 or $K_{3,3}$ as a minor.*

1.2 Intersection graphs

Given a geometric construction with multiple objects, an intersection graph is a graph that maps the objects into vertices and every intersection between objects is an edge between the corresponding vertices.

Definition 3 *A graph G is a comparibility graph if for each edge $\{u, v\} \in E$ there is a partial order $\leq$ such that $u \leq v$ or $v \leq u$.*

1.2.1 Interval graphs

Definition of interval Graphs

Properties

Definition of MIXED interval graphs

1.2.2 Unit disk graphs

Definition of UDG.

Definition of a realization.

2 Complexity

Problem solving is based on the complexity of a problem and not only a particular algorithm that solves it [6].

Definition 4 *Let Σ be a finite alphabet, Σ^* every word derived from Σ, $L \subseteq \Sigma^*$ is a decision problem.*

Definition 5 *The algorithm A decides problem $L \subseteq \Sigma^*$ if for all word $w \in \Sigma^*$:*

- *A finishes and returns TRUE if $w \in L$.*
- *A finishes and returns FALSE if $w \notin L$.*

Definition 6 *A problem is verifiable if there's an algorithm that verifies it.*

Definition 7 *A problem is decidable if there's an algorithm that decides it.*

2.1 P vs NP

Definition 8 *A problem $L \in \mathcal{P}$ if L can be decided in polynomial time $\mathcal{O}(n^k)$.*

Definition 9 *A problem $L \in \mathcal{NP}$ if L can be verified in polynomial time $\mathcal{O}(n^k)$. Thus, $\mathcal{P} \subseteq \mathcal{NP}$.*

2.2 $\exists\mathbb{R}$ complexity class

$\exists\mathbb{R}$ is the class that describes the problems such that they can be reduced to *the existential theory of the reals*[1]. The decidability of the existential theory of the reals is the problem to decide if a sentence of this shape is true:

$$(\exists X_1 \dots \exists X_n) : F(\exists X_1, \dots, \exists X_n)$$

where F is a quantifier-free formula in the reals. In other words, it's a conjuntion of clauses where each clause is a real polynomial inequality where each variable X_k is a real number. We can see that ETR is NP-hard because SAT can be reduced to it.

Proof. Let's take an instance of SAT ϕ_{SAT} with clauses c_k and variables x_k, we can construct an instance of ETR ϕ_{ETR} where we can construct *booleans* variables with inequalities, so for each variable X_k:

$$0 \leq X_k \leq 1$$

$$X_k - X_k^2 = 0$$

Each litteral of each clause will be positive or negative depending if the litteral is cancelled in ϕ_{SAT}:

$$x_k \to l = X_k$$

$$\neg x_k \to l = -X_k$$

Then for each clause we can have a polynomial that will sum the value of every litteral in the clause must be greater that one, so at least one litteral is true:

$$\sum_{l \in c_k} l \geq 1$$

This done, it's easy to see that if ϕ_{ETR} is valid if and only if ϕ_{SAT} is also valid $\square$

This result can show us that $P \subseteq NP \subseteq \exists\mathbb{R}$

2.2.1 Problems in $\exists\mathbb{R}$

The art gallery problem is $\exists\mathbb{R}$-complete.[2]
Recognition of Unit Disk Graphs is $\exists\mathbb{R}$-complete. (corollary of graph realizability problem)[4]
Stretchability is $\exists\mathbb{R}$-complete.

3 Geometry

4 Thin Strip Graphs

4.1 Stabbing disks

Definition of stabbing.
Stabbing geometric structures.[5]

4.2 Thin Strip Graphs

c-strip graphs are unit disk graphs such that the centers of the disks are delimited on the area $\{(x, y) : -\infty < x < \infty, 0 < y \leq c\}$ and its class noted SG(c). We can say that SG(0) = UIG and SG(∞) = UDG. [3]

Definition 10 *Thin strip graphs are defined as TSG* $= \bigcap_{c>0}$ *SG(c).*

Remark 11 *SG(0)* $\neq$ *TSG. We can construct a* $K_{1,3}$ *such that we have 3 vertices with the coordinates* $(1, 0)$, $(0, 0)$, $(1, 0)$ *and a last one* $(0, \varepsilon)$ *with* $\varepsilon > 0$ *as seen in Figure 1.*

It has been proven that MUIG $\subsetneq$ TSG.

Denote that there's not constant t such that SG(t) = TSG.

Unfettered unit interval graphs = UUIG

MUIG $\subsetneq$ TSG $\subsetneq$ UUIG

UUIG $\subseteq$ co-comparability graphs (to prove).
In the following sections we state the problems that are being studied for the thesis.

4.2.1 Forbidden subgraphs of Thin Strip Graphs

We've proven that MUIG $\subsetneq$ TSG $\subsetneq$ UUIG. Knowing the
(Why F_k is a co-comparability unit disk graph?)

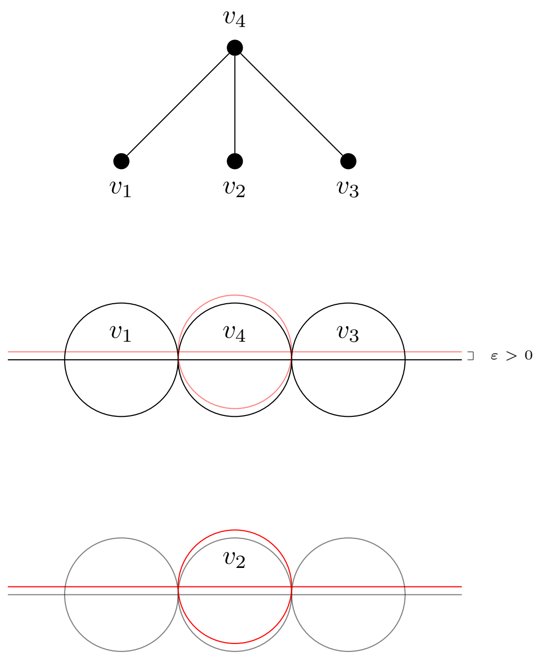

Figure 1: A construction of $K_{1,3}$ with a disk realization, being this graph a TSG.

4.2.2 Complexity class of TSG recognition

We've shown in section 2 that some intersection geometric problems are in $\exists\mathbb{R}$ (unit disk graph recognition problem or the art gallery problem) and we'd like to know if TSG recognition or even SG(c) recognition is in NP knowing that TSG $\subseteq$ UDG.

References

[1] Existential Theory of the Reals. In *Algorithms in Real Algebraic Geometry*, volume 10, pages 505–532. Springer Berlin Heidelberg.

[2] Mikkel Abrahamsen, Anna Adamaszek, and Tillmann Miltzow. The Art Gallery Problem is $\exists \mathbb{R}$-complete.

[3] Takashi Hayashi, Akitoshi Kawamura, Yota Otachi, Hidehiro Shinohara, and Koichi Yamazaki. Thin Strip Graphs. 216:203–210.

[4] Marcus Schaefer. Realizability of Graphs and Linkages. In Jnos Pach, editor, *Thirty Essays on Geometric Graph Theory*, pages 461–482. Springer New York.

[5] L.M. Schlipf. *Stabbing and Covering Geometric Objects in the Plane.*

[6] Michael Sipser. *Introduction to the Theory of Computation.* Course Technology, second edition.