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Characterization and complexity of Thin Strip Graphs

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”” *Science isn’t about why – it’s about* **why not**.

— **Cave Johnson**

(Portal 2)

Acknowledgment

I want to thank ...

Contents

Chapter 1

Introduction

> *There's no such thing in the world as absolute reality. Most of what they call real is actually fiction; what you think you see is only as real as your brain tells you it is. It's not whether you were right or wrong, but how much faith you were willing to have, that decides the future.*
>
> — **Solid Snake**
> (Metal Gear Solid 2)

This work is mainly focused on the characterization and complexity of variants of unit disk graphs, where the domain of possible locations for the disks is limited. We are also going to see their close relation to a certain family of interval graphs. The main goal of this thesis is to compile the information of different branches of graph theory together, to build an understandable hierarchy of classes between disk graphs, interval graphs and co-comparability graphs as well as giving some personal observations and new theorems with respect to inclusion relations between classes. In this chapter, we will overview the open questions we will focus on and our main results. Further details about the results discussed in this chapter will be introduced later in the thesis as well as a background in Chapter 2.

Interval graphs

In Chapter 3 we introduce the concept of interval graphs and some of their use cases. An ***interval graph*** is a graph in which each one of its vertices is a closed interval on the real line and they are adjacent if they overlap; interval graphs where the length of its intervals is the same is called ***unit interval graphs (UIG)***.

Moreover, we introduce two new subclasses of graphs. ***Mixed unit interval graphs (MUIG)*** [17] can be seen as unit interval graphs but the endpoints of each interval can be open or closed. Another variant are ***unfettered unit interval graphs (UUIG)*** [15], where we can chose whether two touching intervals (so that one of their endpoints are in the same position) are adjacent or not.

Joos describes the class MUIG [17] with a list of graphs that cannot be MUIGs. Also, Hayashi et al. describe the class UUIG with the next theorem.

Theorem. *A graph is an UUIG if and only if it has a level structure such that each level is a clique.*

Finally, we take an algorithmic approach to study these classes of graphs. A graph recognition problem for a class of graphs is the problem to guess whether a given graph is of a certain class. The recognition of MUIG is of $\mathcal{O}(n^2)$ [25] and the recognition of UUIG is only overviewed. For the moment, we know that recognition of UUIG is in $\mathcal{NP}$.

Strip graphs

In Chapter 4 we introduce the main class of graphs of this thesis. ***Unit disk graphs (UDG)*** are intersection graphs of disks on a plane when the diameter of the disks are unitary. ***c-strip graphs (SG(c)*** [7] is a subclass of UDG, where the center of the disks can only be located between two horizontal lines with a separation of c. More formally, for each disk v in the graph G, $v_y \in [0, c]$. Breu [7] defined this class of graphs and studied early phases of its charaterization and recognition. However, this is not complete as there is still no answer to the complexity of TSG recognition.

Thin strip graphs

Thin strip graphs (TSG) is a subclass of UDG that that can be defined as the intersection of every SG(c) with $c > 0$. This is equivalent to say that $TSG = SG(\varepsilon)$ with ε and arbitrarily small number. Hayashi et al. [15] present this class of graphs in their work and found some interesting properties about them.

Theorem. *There is no constant* t *such that* $TSG = SG(t)$*.*

More importantly, TSG is well located in the hierarchy of the graphs seen until now. We know that $MUIG \subsetneq TSG \subsetneq UUIG$. This helps us to find a characterization for TSG because we know that the characterization of MUIG is complete. We also see as one of the results of this thesis that every forbidden graph for MUIG is realizable in TSG except for one of them which is also forbidden in TSG.

Chapter 2

Background

> *The right man in the wrong place can make all the difference in the world.*
>
> — **G-Man**
> (Half-Life 2)

In this chapter we review some definitions and notations used in this thesis. We limit ourselves to the basic notations used during the work. However, the bibliography of each subject will be referenced for further details about the topic.

2.1 Graph theory

A ***graph*** is defined as a tuple $G = (V, E)$ where V is the set of ***vertices*** and E is a set of ***edges*** where $E \subseteq \binom{V}{2}$. An ***orientation*** of a graph G is an assignment of a direction to each edge, we denote the orientation of the edges by $\overrightarrow{E}$. An orientation is ***transitive*** if $uv \in \overrightarrow{E}$ and $vwin\overrightarrow{E}$, then $uw \in \overrightarrow{E}$. If two vertices are share the same edge e they are called ***adjacent*** and also the ***endpoints*** of e. The ***neighbourhood*** of a vertex v is the subset of V of vertices that are adjacent to v and is denoted by $N(v)$. A ***subgraph*** $H = (V', E')$ of a graph G is a graph such that $V' \subseteq V$ and $E' \subseteq E$. An

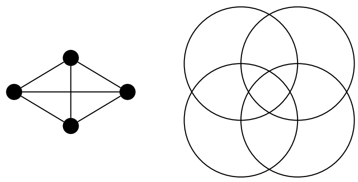

Figure 2.1: Realization of a UDG (unit disk graph).

induced subgraph of a graph is a subgraph H of a graph G such that for every edge of G is also in H if its two endpoints are in V'. A ***clique*** is a subgraph such that every vertex is adjacent to each other. A graph that is also a clique is called a ***complete graph*** and it is denoted as K_n. A graph is ***bipartite*** if there exist two disjoint subsets of the vertex set $A \cup B = V$ such that two vertices of the same subset are not adjacent. A ***complete bipartite graph*** $K_{n,m}$ is a bipartite graph such that $v \in A$ and $w \in B$ implies $vw \in E$ where n and m are the size of each bipartition.

A ***path*** $P_n = v_1 \ldots v_{n+1}$ of a graph is a sequence of pairwise distinct n vertices such that two consequent vertices are adjacent. A ***cycle*** is a path $C_n = v_1 \ldots v_n v_{n+1}$ such that $v_1 = v_{n+1}$. A graph is ***connected*** if there exists a path between every pair of vertices. A ***chord*** of a cycle C_n with $n \geqslant 4$ is an edge that connects two non adjacent vertices of the cycle. A graph is ***chordal*** if there is a chord in every cycle bigger than four.

Some graphs can be characterized with properties. An ***isomorphism*** between two graphs $G = (V, E)$ and $H = (V', E')$ is a bijection $f : V \to V'$ between the two vertex sets such that u, v are adjacent in G if and only if $f(u), f(v)$ are adjacent in H. A graph ***property*** is a property of the graph that is preserved in all its isomophisms; this will help us to set properties that are based on the abstraction of the graph and not only its drawings. A property is ***hereditary*** if it is also preserved under all taking subgraphs.

For notation in this thesis, sometimes the class of a certain type of graphs is denoted by its initials (*e.g.* the class of unit interval graphs is denoted by ***UIG***) to avoid extreme repetition.

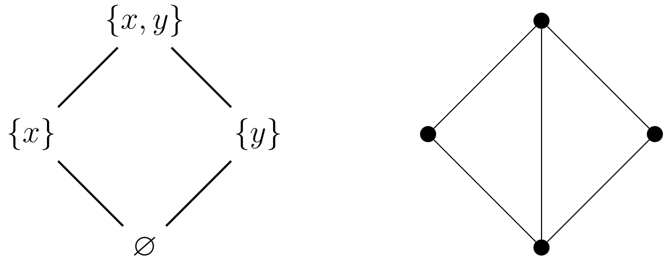

Figure 2.2: On the left, Hasse diagram of a poset of the power set of 2 elements ordered by inclusion. On the right, the comparability graph of this poset.

2.1.1 Intersection graphs

An ***intersection graph*** is a graph $G = (\zeta, E)$ of a collection of objects ζ is a graph such that $v, w \in \zeta$ and $v \cup w \neq \varnothing$ implies that $vw \in E$. An ***interval graph*** is an intersection graph of intervals on the plane; when the size of the intervals is equal they are called ***unit interval graphs***. A ***unit disk graph*** is an intersection graph of disks on a plane that have the same diameter - you can find an example in Figure 2.1.

For more details about graph theory we recommend the reading of *Graph Theory* by Diestel [2], *Graph Classes: A Survey* by Brändstadt *et al.* [6] and *Topics in Intersection Graph Theory* by McKee *et al.* [18].

2.2 Order and set theory

The ***powerset*** $\wp(S)$ of a set S is the set of subsets of S. A ***partial order*** is a binary relation $\leqslant$ over a set A satisfying three axioms:

- if $a \leqslant b$ and $b \leqslant a$ then $a = b$ (***antisymmetry***).
- if $a \leqslant b$ and $b \leqslant c$ then $a \leqslant c$ (***transitivity***).
- $a \leqslant a$ (***reflexivity***).

On the other side, a ***total order*** is a partial order where the reflexivity order is replaced by the ***connexity*** property – $a \leq b$ or $b \leq a$. A ***partially***

ordered set (or ***poset***) $(S, \leqslant)$ is a set such that the elements of S are partially ordered by the relation $\leqslant$. A good way to represent a poset is the ***Hasse diagram*** (Figure 2.2).

2.2.1 Comparability graphs

A ***spanning order*** $(V, <)$ on a graph $G = (V, E)$ is a total order on V such that for any three vertices $u < v < w$:

$$uw \in E \Rightarrow uv \in E \text{ or } vw \in E$$

The class of comparability graphs are built on the ideas of order theory. A graph G is a ***comparability graph*** if there exists a partial order $\leqslant$ such that $uv \in E \Leftrightarrow v \leqslant w$ or $w \leqslant v$. The complement of comparability graphs are called ***co-comparability graphs***.

2.3 Complexity

Complexity theory has the objective to establish lower bounds on how efficient an algorithm can be for a given problem. This approach let us have a reference point to establish the difficulty of a problem. A ***decision problem*** is a problem where we have to decide if a statement is true or false. A ***decider*** of a decision problem is defined as the deterministic machine that solves this problem. The problem is ***polynomially decidable*** if it has a polynomial time decider. A ***verifier*** of a decision problem is a deterministic machine that verifies whether an answer to the decision problem is true or false. Equally, a problem is ***polinomially verifiable*** if it has a polynomial time verifier. The problem of ***recognition*** is the problem to decide whether a graph G is in a class of graphs. We use $\mathcal{O}$ to describe the behaviour of a function compared to its input, we classify the time complexity of algorithms using this concept. We denote by $\mathcal{P}$ the class of polinomially decidable problems. On the other hand, $\mathcal{NP}$ denotes the class of polinomially verifiable problems. We can see that $\mathcal{P} \subseteq \mathcal{NP}$.

A ***reduction*** of a problem L to a problem M is a mapping of an instance of L (I_L) to an instance of M (I_M) such that I_L is true for the problem L if and only if I_M is true for the problem M. This is denoted by $L \leq M$ and $L \leq_P M$ if the reduction is done in polynomial time. We usually prove bounds of complexity for an unknown problem L by reducing it to another problem with an already known complexity. Thus, we can define the class ***$\mathcal{NP}$-hard*** as the set of problems such that we can reduce every $\mathcal{NP}$ problem to one of them. The set of problems that are both $\mathcal{NP}$ and $\mathcal{NP}$-hard are called ***$\mathcal{NP}$-complete***. For more details about complexity we recommend the reading of *Introduction to the Theory of Computation* by Sipser [24].

2.4 Geometry

We must recall some really basic definitions of geometry. Every geometrical object of this thesis is located in $\mathbb{R}^2$ if it is not otherwise specified. The ***distance*** between two points as $\text{dist}(a, b)$. An object S is ***convex*** if for every point p, q the segment between the two points is also contained in S. More formally:

$$\forall \lambda \in [0, 1] : (1 - \lambda)p + \lambda q \in S$$

A ***stabbing*** is a point that traverses a set of intersecting objects. A lot of research has been done [21] on the minimal amount of stabbings to cover every object in a set. If instead of points we use more complex object, we denote it by a ***covering***. The ***Helly*** theorem says that:

Theorem (Helly ([16]). *Given a set S of objects in $\mathbb{R}^d$, if for each subset of S of size $d + 1$ their intersection is non empty, then $\bigcap_{s \in S} \neq \varnothing$.*

We say that a set S satisfies the ***Helly property*** if every subfamily of S composed of pairwise intersecting objects has also a non-empty intersection. For more details about algorithmic geometry, we recommend the reading of *Computational Geometry: algorithms and applications* by de Berg *et al.* [11].

Chapter 3

Interval graphs

> *If you like easy, my program isn't for you. Nothing great comes from easy.*
>
> — **Robert Callaghan**
>
> (Big Hero 6)

The goal of this chapter is to present the family of classes of interval graphs that are related to the class of thin strip graphs. We introduce the class of interval graphs, which is one of the most used classes of intersection graphs. There are multiple types of interval graphs and those that are the most relevant for the thesis are going to be defined below.

First, we recall the basic definition of an interval graphs and their multiple characterizations. Also, we present unit interval graphs, where we see their characterization and complexity such as Robert's characterization [19]. Then, we see some characterizations such as Joos's paper about mixed unit interval graphs [17] and the paper from Hayashi *et al.* [15] where the unfettered unit interval graphs are defined and also characterized as well as some equivalences with ***unit disk graphs*** are presented. Also, the complexity of the recognition for each one of the classes presented will be discussed.

3.1 Interval graphs

First we present the main characterizations of interval graphs. In the next sections we present some other subclasses of interval graphs that will help us characterize the thin strip graphs on Chapter 4. There are multiple characterizations of interval graphs that are equivalent, in this thesis we present Gilmore and Hoffman's characterization described in Theorem 3.1.1. From this theorem it is clear that *IG* class is a subclass of the *CO-CO* class.

Theorem 3.1.1 (Gilmore and Hoffman [14]). *G is an interval graph if and only if G does not contain C_4 as an induced subgraph and $\overline{G}$ can be ordered partially, in other words, $\overline{G}$ is a comparability graph.*

The first interesting subclass of IG is the class of ***unit interval graphs*** which is defined by the interval graphs that have intervals with the same length (or equal to one). This class of graphs is equivalent to the class of ***proper interval graph*** which is the class of intervals where no interval is a strict subset of another. This statement is powerful because the study of unit interval graphs can be more confortable because of the simplicity of its definition and characterization as seen in Theorem 3.1.2.

Theorem 3.1.2 (Roberts [19]). *An interval graph is a unit interval graph if and only if it has no induced subgraph $K_{1,3}$ [1].*

In terms of recognition, interval graphs as long as unit interval graphs can be recognized in ***linear time***. Interval graph linear time recognition was discovered by Booth *et al.* by doing so with a ***breadth-first search*** [5]. UIG recognition has also been proven to be linear [9].

3.2 Mixed unit interval graphs

We can define a new class of graphs that is related to UIG by its definition. This class is closely related to thin strip graphs as we will see in Chapter 4. ***Mixed unit interval graphs*** are graphs where the intervals have the same

[1] $K_{1,3}$ is also called ***claw***.

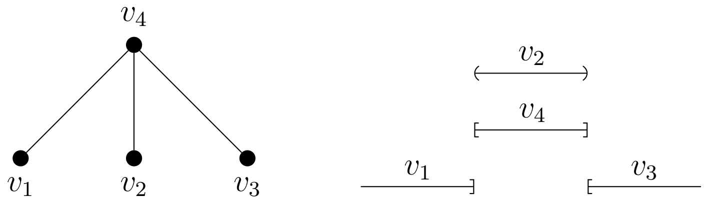

Figure 3.1: Representation of $K_{1,3}$ as a MUIG.

size as the unit interval graphs. However, in this class, the endpoints of the intervals can be open or closed - or one of each.

Formally, MUIG is defined by using the next classes of graphs:

$$\mathcal{U}^{++} = \{[x, y] : x, y \in \mathbb{R}, x \leq y\}$$

$$\mathcal{U}^{--} = \{(x, y) : x, y \in \mathbb{R}, x \leq y\}$$

$$\mathcal{U}^{+-} = \{[x, y) : x, y \in \mathbb{R}, x \leq y\}$$

$$\mathcal{U}^{-+} = \{(x, y] : x, y \in \mathbb{R}, x \leq y\}$$

where $\mathcal{U}^{xx}$ is the class of unit interval graphs where its intervals can be open or closed depending on their sign. For exemple, $\mathcal{U}^{++}$ = UIG.

Dourado, by defining these classes of unit interval graphs with open/closed intervals also found that, for unit interval graphs, it does not matter if the endpoints are open, closed, or closed open (Theorem 3.2.1).

Theorem 3.2.1 (Dourado et al. [12])**.** *The classes of the graphs* $\mathcal{U}^{--}$*,* $\mathcal{U}^{++}$*,* $\mathcal{U}^{-+}$*,* $\mathcal{U}^{+-}$*, and* $\mathcal{U}^{-+} \bigcup \mathcal{U}^{+-}$ *are the same.*

However, MUIG is defined as $\mathcal{U}^{++} \bigcup \mathcal{U}^{--} \bigcup \mathcal{U}^{+-} \bigcup \mathcal{U}^{-+}$ which is also denoted as $\mathcal{U}$. In this case it is clear that this class is not equivalent to UIG. As we have seen in Theorem 3.1.2, a UIG can be seen as a $K_{1,3}$-free IG. Nevertheless, MUIG can accept this graph as seen in Proposition 3.2.2.

Proposition 3.2.2 (Dourado *et al.* [13])**.** *MUIG has a* $K_{1,3}$ *representation. Also, for every MUIG representation* $\phi : V(K_{1,3}) \to \mathcal{U}$ *such that* $\phi(V(K_{1,3}))$ *contains:*

- $a = [x, x+1]$
- $b = (x, x+1)$
- $c = [x+1, x+2]$ *or* $[x+1, x+2)$
- $d = [x-1, x]$ *or* $[x-1, x)$

Proof. Let $\phi : V(K_{1,3}) \to \mathcal{U}$ be the representation of $K_{1,3}$ as a mixed unit interval intersection diagram as illustrated in Figure 3.1. Let $V(K_{1,3}) = \{v_1, v_2, v_3, v_4\}$ and $E(K_{1,3}) = \{v_1v_4, v_2v_4, v_3v_4\}$. Let $x(v_1) = I(v_1) \cap I(v_4)$, $x(v_2) = I(v_2) \cap I(v_4)$ and $x(v_3) = I(v_3) \cap I(v_4)$. Because v_1, v_2 and v_3 are not adjacent, we can assume that $x(v_1) < x(v_2) < x(v_3)$. Since $x(v_1) \in I(v_4)$ and $x(v_3) \in I(v_4)$, then $x(v_3) - x(v_1) \leqslant 1$. Since $I(v_1), I(v_2)$ and $I(v_3)$ are disjoint, $I(v_2)$ must be a proper subset of $(x(v_1), x(v_3))$. Since $I(v_2)$ is a mixed unit interval, then it implies that $x(v_3) = x(v_1) + 1$, $I(v_2) = (x(v_1), x(v_1 + 1))$, $I(v_4) = [x(v_1), x(v_1) + 1]$, $I(v_1) = \{(x(v_1) - 1, x(v_1)], [x(v_1) - 1, x(v_1)]\}$ and $I(v_3) = \{[x(v_3) - 1, x(v_3)), [x(v_3) - 1, x(v_3)]\}$. $\square$

Theorem 3.2.3 (Dourado *et al.* [13])**.** *UIG* $\subsetneq$ *MUIG.*

Proof. The strict inclusion is straightforward: we know that UIG $= \mathcal{U}^{++} \subset$ *MUIG* by definition. For the inequality, we prove it by Proposition 3.2.2, as $K_{1,3}$ is not realizable in UIG. $\square$

Nevertheless, MUIG still shares some properties with UIG. In the previous section we mentioned that the class of unit interval graphs is the same as the class of proper interval graphs. In our case, mixed unit interval graphs is also exactly the same as the mixed proper interval graphs – where no mixed interval can be a proper subset of another one.

Theorem 3.2.4. *For a graph G, the following two statements are equivalent.*

- *G is a mixed proper interval graph.*
- *G is a mixed unit interval graph.*

Shuchat *et al.* [22] describe an algorithm to recognize mixed unit interval graphs in polynomial time with a characterization. Proof and details about the algorithm will not be provided but we encourage the reading of their paper.

Theorem 3.2.5 (Schuchat et al. [22]). *The MUIG recognition problem is in $\mathcal{P}$. Moreover, there is an algorithm that solves it in $\mathcal{O}(|V|^2)$ for V the vertex set of a graph.*

3.2.1 Characterization

A complete characterization by induced forbidden subgraphs have been found independently by A. Schuchat et al. [23] and F. Joos [17]. In this section we will present briefly the characterization of MUIG given by Joos with forbidden subgraphs. We will also review each one of these forbidden subgraphs and discuss the properties compared of one of them that will be relevant in the next chapter. However, the proof of this characterization will not be given in this thesis because of its length. His work follows Dourado *et al.* [13] where they characterized ***diamond-free*** mixed unit interval graphs.

Theorem 3.2.6 (Joos [17]). *G is a MUIG if and only if it is a $\{F\} \cup \mathcal{R} \cup \mathcal{S} \cup \mathcal{S}'' \cup \mathcal{T}$-free interval graph.*

The forbidden families can be seen in Figures 3.3, 3.4, 3.5, 3.6, 3.7. You can notive that, without including F, every family of forbidden graphs of MUIG is infinite and is defined recursively by its predecessor. Our only goal in this section is to focus in the properties of $\mathcal{R}$ because, as we will see in Chapter 4, it is the only forbidden graph family for MUIG that is also forbidden for thin strip graphs.

We know that $\mathcal{R}$ is a family of forbidden subgraph for mixed unit interval graphs. If we look up in the graph classes hierarchy 3.2 we find that MUIG $\subsetneq$ CO-CO. The first step to see if $\mathcal{R}$ is a family of proper forbidden subgraph for MUIG is to prove if $\mathcal{R} \subsetneq$ CO-CO. In the first place, we present a characterization of cocomparability graphs.

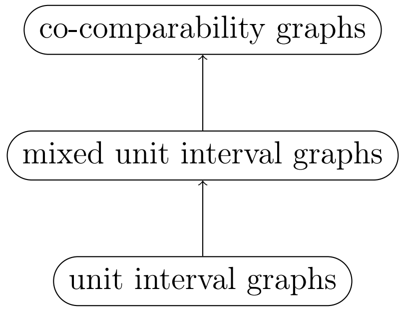

Figure 3.2: The hierarchy of the classes between ***UIG*** and ***CO-CO***. The arrows represent a relation of $\subsetneq$.

Theorem 3.2.7 (Damaschke [10])**.** *A graph G is a co-comparability graph if and only if it has a spanning order.*

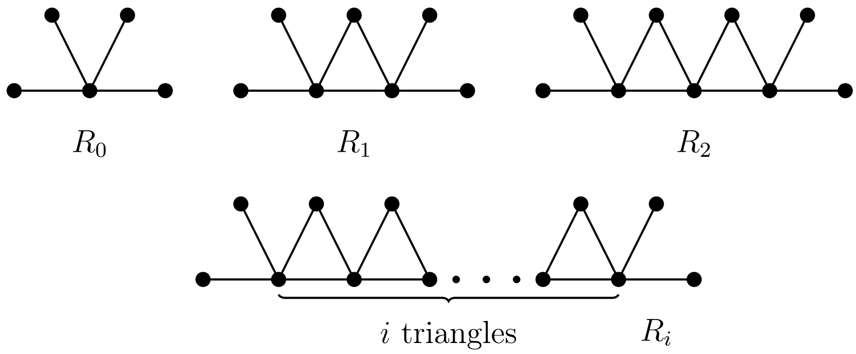

Figure 3.3: The class $\mathcal{R}$. [17]

Lemma 3.2.8. *$\mathcal{R}$ is a family of co-comparability graphs.*

Proof. If we recall Theorem 3.2.7, in order to prove that $\mathcal{R}$ is a family of co-comparability graphs we will have to find a spanning order for every R_i with $i \geq 0$. We will proceed to label our vertices with a mapping function $f : V \to \mathbb{N}$ such that $f(v) \in \{1, \ldots, |V|\}$. This mapping will give us a spanning order by induction:

- $i = 0$: We assign the number 1 to the vertex with maximum degree v_1. We assign then the rest of the numbers to the other vertices. We see

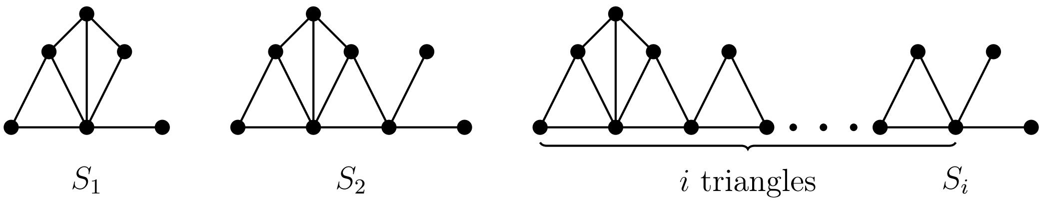

Figure 3.4: The class $\mathcal{S}$ [17].

then that $\forall u < v < w : uw \in E \to uv \in E$ because every vertex is adjacent to v_1.

- $i = i + 1$: We define $\lambda_i = 5 + 2i$ where $\lambda_i = |V(R_i)|$. We add two vertices on each graph, where their labels are $\lambda_i + 1$ and $\lambda_i + 2$ and we also add three new edges: $v_{\lambda_i}v_{\lambda_i-1}, v_{\lambda_i}v_{\lambda_i+1}, v_{\lambda_i}v_{\lambda_i+2} \in E$.

  By induction we only have to see if it holds with the new edges. We can say that it still holds with $v_{\lambda_i}v_{\lambda_i-1}$ and $v_{\lambda_i}v_{\lambda_i+1}$ because:

$$\nexists k \in \mathbb{N} : i < k < i + 1$$

  Finally, we see that $v_{\lambda_i}v_{\lambda_i+2}$ is a valid edge because $v_{\lambda_i}v_{\lambda_i+1} \in E$. $\square$

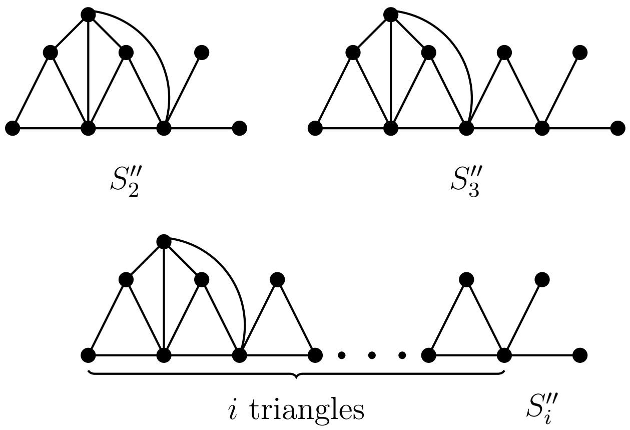

Figure 3.5: The class $\mathcal{S}''$. [17]

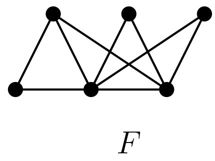

Figure 3.6: The graph F. [17]

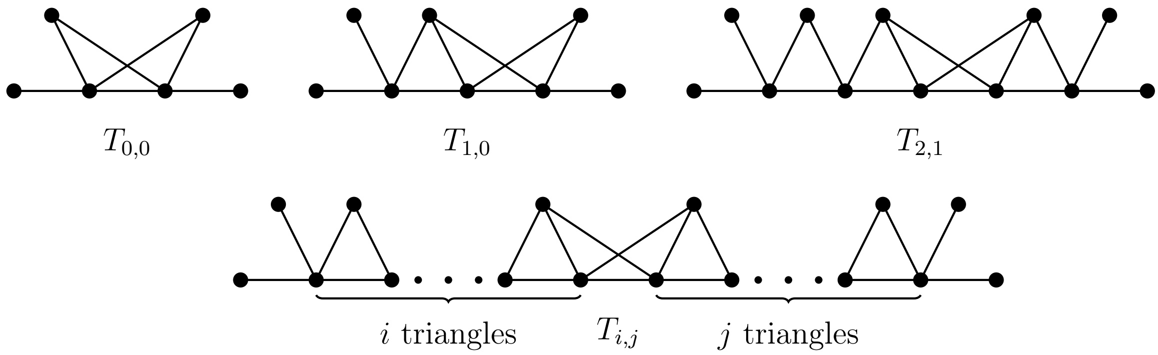

Figure 3.7: The class $\mathcal{T}$. [17]

3.3 Unfettered unit interval graphs

In this section we detail the properties of unfettered . An unfettered unit interval graph can be defined as an unit interval graph such that for every touching endpoints we can chose either if they are adjacent or not. We remark that by definition, every unit interval graph is feasible in UUIG. This class is a minimal superclass of TSG, *i.e.* TSG $\subsetneq$ UUIG.

This class has a characterization by levels done by Hayashi *et al.* where levels are used. A ***level structure*** of a graph $G = (V, E)$ is a partition $L = \{L_i : i \in [1, t]\}$ of V such that

$$v \in L_k \Rightarrow N(v) \subseteq L_{k-1} \cup L_k \cup L_{k+1}$$

where $L_0 = L_{t+1} = \varnothing$.

Theorem 3.3.1 (Hayashi et al. [15])**.** *A graph G is an unfettered unit interval graph if and only if it has a level structure where each level is a clique.*

Proof. We begin by proving the if-part. Let G be a graph with levels $L_1, \ldots, L_t$ where every level is a clique. For every vertex $v \in L_i$, we assign an interval $[i-1, i]$. We see that every interval within a level is in the same position, so they are all adjacent. Then, for L_i we have its adjacent levels L_{i-1} and L_{i+1}. The right endpoints of the intervals L_{i-1} match the left endpoints of L_i. On the other side, the left endpoints of L_{i+1} match the right endpoints of L_i. As we know, we can chose whether the endpoints touch or not between levels. This will construct its respective UUIG.

Now we prove the only-if. Let G be a UUIG and $I(v)$ the interval representation of $v \in V(G)$ and $\ell(I(v))$ the left side of an interval. Let $I'(v) = [\lfloor \ell(I(v)) \rfloor, \lfloor \ell(I(v)) \rfloor + 1]$. This gives us exactly the same graph because the following holds:

$$\begin{aligned} \ell(I(v)) - \ell(I(v)) \leqslant 1 \Rightarrow & \lfloor \ell(I(v)) \rfloor - \lfloor \ell(I(v)) \rfloor \leqslant 1 \\ \ell(I(v)) - \ell(I(v)) \geqslant 1 \Rightarrow & \lfloor \ell(I(v)) \rfloor - \lfloor \ell(I(v)) \rfloor \geqslant 1 \end{aligned} \tag{3.1}$$

We can have a partition $L_i = \{v : \ell(I'(v)) = i\}$ where every L_i is a clique. Also, this partition is a level structure because the endpoints of L_i meet the endpoints of L_{i-1} and L_{i+1}. □

We can clearly see that MUIG $\in$ UUIG. However, we still have to see what is the location of UUIG in the higher graph classes hierarchy:

Proposition 3.3.2. *UUIG $\subset$ co-comparability.*

Proof. This proposition is equivalent to say that if a graph G is a UUIG, then it also has a spanning order.

For each vertex of a partition L_k of UUIG (Theorem 3.3.1) we assign arbitrarily a number $i \in \{\max(V(L_{k-1})) + 1, \ldots, \max(V(L_{k-1})) + |V(L_k)| + 1\}$; intuitively, we assign every available number from the beginning in increasing order ($|V(L_1)|$ first numbers on the first partition and consecutively).

Because we know that each partition L_k is a clique, we can say that for each three vertices $u < v < w$, if $vw \in E \Rightarrow uv \in E$ or $vw \in E$. We know this because given $u \in L_i$ and $w \in L_j$: if $uw \in E$ it means that levels L_i and L_j are adjacent, which means that $v \in L_i$ or $v \in L_j$ so v will be adjacent

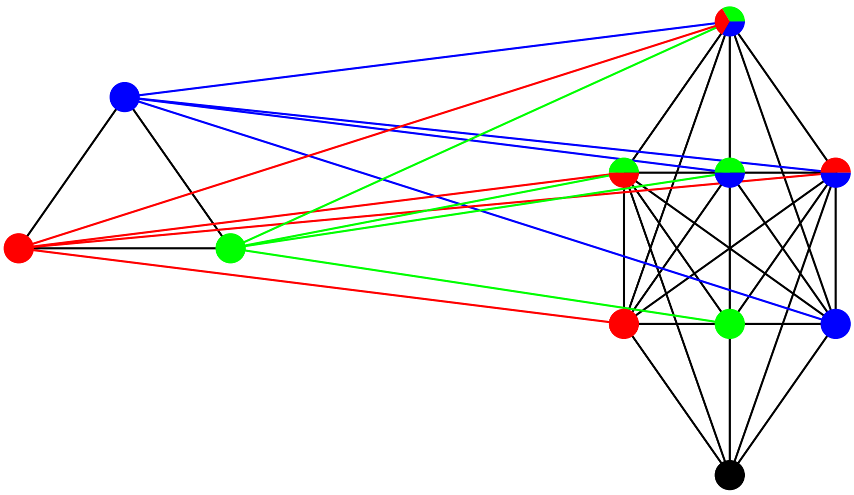

Figure 3.8: Representation of T with three vertices in the first level instead of four where every level is a clique. The colors represent the three vertices of the first level. The multicolored vertices represent sets of the vertices in the first level.

either to u or w. This is a spanning order. $\square$

If we recall the characterization of MUIG in section 3.2.1, we can see that every forbidden graph of MUIG is an UUIG (except for $\mathcal{R}$); which means that they are also co-comparability graphs.

In the other hand, we can find a graph in UUIG that is not an UDG. This theorem will be used in Chapter 4.

Theorem 3.3.3 (Hayashi et al. [15]). *UUIG $\neq$ UDG.*

Proof. We can define $T = (L_1 \cup L_2, E)$ a UUIG with two levels $L_1 = \{v_1, v_2, v_3, v_4\}$ and $L_2 = \wp(L_1)$ and $E = \binom{L_1}{2} \cup \binom{L_2}{2} \cup \{vw : w \in L_2, v \in w\}$. For a better visualisation, you can find in Figure 3.8 the representation as an UDG in the case where $L_1 = \{v_1, v_2, v_3\}$.

We can see the UDG representation of G as a Venn diagram of four disks (L_1) where there is a disk of L_2 that intersects only with its subset associated. We know by instance that a Venn diagram cannot be constructed with disks if

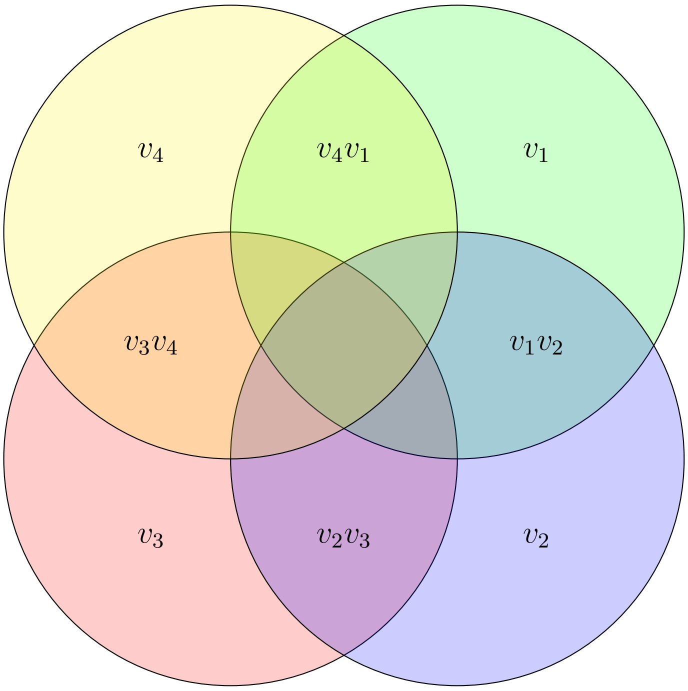

Figure 3.9: A disk Venn diagram of four sets. Each circle of color represent a set. You may notice that some subsets are not represented here (*e.g.* v_2v_4 or v_1v_3). So a disk that touches v_4 and v_2 in this representation is not possible without intersecting also v_3 or v_1.

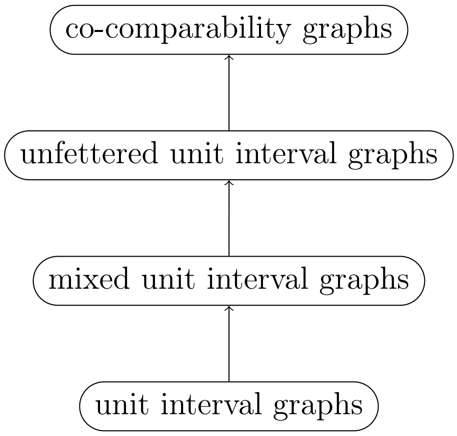

Figure 3.10: Extension of the graph classes diagram from Figure 3.2 with ***UUIG***.

the number of sets is bigger than four [26] as you can see in Figure 3.9. Thus, there will be at least one disk that is not able to intersect with its associated subset because there is at least one subset that is not representable by a disk Venn Diagram. Which means that $G \notin$ UDG. □

3.3.1 Recognition

As we mentioned in the previous section, UUIG is a class of graphs very relevant to define TSG and that is why we are interested in knowing how this class of graphs is recognized.

Lemma 3.3.4. *Let G be a connected UUIG with a level structure with levels $L_1, \ldots, L_n$. $G \setminus L_i$ is a graph where each connected component is also an UUIG and the number of connected components is not bigger than two.*

Proof. By definition for a graph with a level structure, if $v \in L_i$, $N(v) = L_{i-1} \cup L_i \cup L_{i+1}$. This said, if we delete a level L_i, L_{i-1} and L_{i+1} are disconnected, but they are still connected to the other consecutive levels (L_{i-1} is connected to L_{i-2}, which is connected to L_{i-3}... and viceversa with L_{i+1}).

And because a level is only adjacent to two other levels, we only have two connected components, only one if $L_i = L_1$ or $L_i = L_n$. $\square$

By this lemma we can suppose that the input graph G is a connected graph. This observation reduces the complexity of the problem for a graph G from $\mathcal{O}(f(|V(G)|)$ to $\mathcal{O}(f(|V(H)|))$ where $H \subseteq G$ the biggest component of G.

Theorem 3.3.5. *UUIG recognition is in $\mathcal{NP}$.*

Proof. The UUIG recognition of a graph G is in $\mathcal{NP}$ because we can build a ***polynomial time verifier*** that takes a level structure of G and check whether each level is a clique or not. Viceversa, we can build another one that takes a partition and check whether each clique is a level of a level structure. $\square$

Future work on the recognition of unfettered unit interval graphs would be to adapt this algorithm to avoid combinatorial complexity. In our case we are interested in seeing the recognition of UUIG for unit disk graphs. We know that the CLIQUE problem is in $\mathcal{P}$ for unit disk graphs [8] and the first hypothesis was that given an UUIG G, at least one level of G is a maximal clique of the graph. Nevertheless, we have a counterexample in $T_{0,0}$ (Fig. 3.7) where the levels of the graph are $\{K_1, K_2, K_2, K_1\}$ while $\omega(T_{0,0}) = 3$.

Observation 3.3.6. *Given an UUIG G, a level of G does not have to be necessarily a maximal clique.*

Chapter 4

Thin strip graphs

> *Sometimes it is the people no one imagines anything of who do the things that no one can imagine.*
>
> — **Alan Turing**
> (The Imitation Game)

The goal of this chapter is to introduce the main subject of this thesis which is a class of graphs that lie between unit disk graphs and mixed interval graphs called . We can define formally a as a unit disk graph such that the centers of the disks belong to $\{(x, y) : -\infty < x < \infty, 0 \leq y \leq c\}$, more intuitively we can see this as a unit disk graph where the centers of the disks lie between two parallel horizontal lines with a distance of c between them. We denote this class by SG(c). We then have that SG(0) = UIG and SG(∞) = UDG.

The definition and main work for this class comes from Breu in his thesis [7]. However, Hayashi *et al.* [15] expanded his work by defining the class of thin strip graphs. A first review of unit disk graphs will be done in the first section of this chapter, based on the original paper of Clark *et al.* [8] where unit disks graphs were defined with some interesting results.

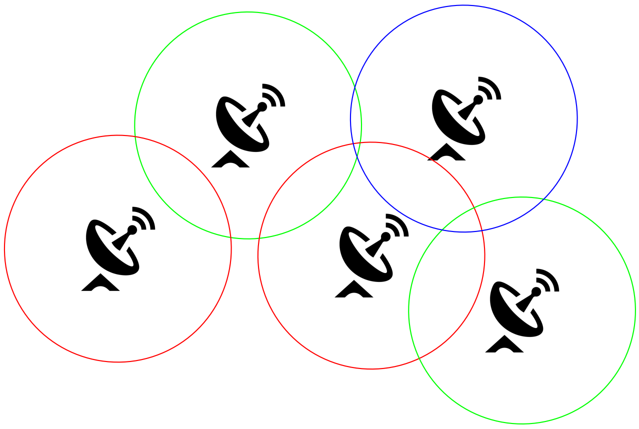

Figure 4.1: A broadcast network with its respective unit disk graph model. The color of each disk represents the frequency of the signals sent by its broadcast node. It has to be different for adjacent antennas to avoid signal interference. The broadcast problem is equivalent to the coloring problem of the graph.

4.1 Unit disk graphs

An is an intersection graph of equal-sized disks on a plane - also called ***unitary***. The main interest of this class of graphs is its application. They can be used to create a graph-theoretic model for any kind broadcast networks. This can be useful in the case where a broadcast node needs to have a different frequency from another broadcast node that is close enough. With the unit disk graph model, we can solve this problem using algorithms for well known graph-theoretic problems such that the coloration problem.

The most studied problem for this class of graphs is its recognition and characterization. It has been proven that its recognition problem is [20]. On the other hand, its characterization is still not complete and is an open question. Atminas *et al.* tackled this problem by finding forbidden subgraphs to some of its subclasses [3]. However, from its definition, we know that some graph-theoretic problems have different complexity when applied to unit disk graphs like the CLIQUE problem which has been proven to be polynomial when applied to UDGs [8]. The approximation complexity of these problems has also been studied [4].

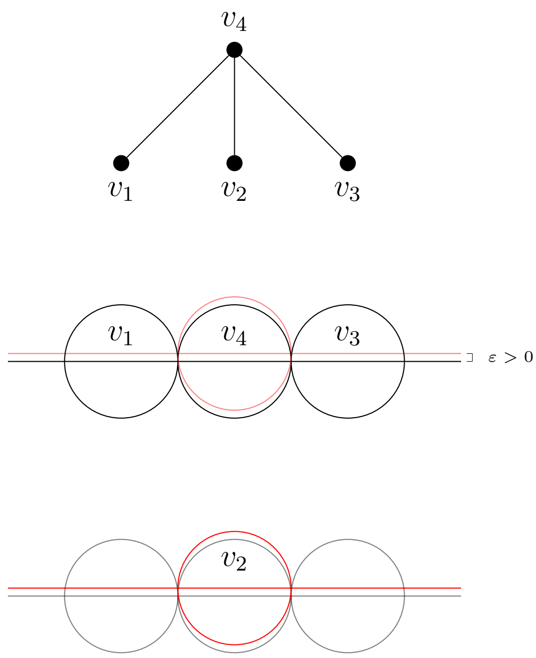

Figure 4.2: A construction of $K_{1,3}$ with a disk realization, being this graph a TSG.

4.2 c-strip graphs

A c-strip graph can be defined as a unit disk graph such that the centers of the disks are located between two horizontal parallel lines with a distance of c between both of them.

4.3 Thin strip graphs

A thin strip graph can be intuitively defined as a c-strip graph where c is an arbitrarily little ε. Also, we can see that $\mathrm{SG}(k) \subseteq \mathrm{SG}(l)$ with $k < l$. A more strict definition emerges from this observation:

Definition 4.3.1. Thin strip graphs are defined as $\mathrm{TSG} = \bigcap_{c>0} \mathrm{SG}(c)$.

Remark 4.3.2. $\mathrm{SG}(0) \neq \mathrm{TSG}$. We can construct a $K_{1,3}$ such that we have 3 vertices with the coordinates $(1,0)$, $(0,0)$, $(1,0)$ and a last one $(0,\varepsilon)$ with $\varepsilon > 0$ and arbitrarily small as seen in Figure 4.2.

Theorem 4.3.3 (Hayashi et al. [15]). *There is no constant t such that SG(t) = TSG.*

Theorem 4.3.4 (Hayashi et al. [15]). *There is no constant t such that SG(t) = UDG.*

Hayashi et al. left some open problems. We try to elaborate the knowledge around some of these problems to understand them better, mainly for the recognition of this class of graphs. Before that, we see where this class lays in the hierarchy of classes. We know by definition that TSG $\subsetneq$ UDG.

4.3.1 Interval graphs

Thin strip graphs shares their geometrical structure with interval graphs (remember SG(0) = UIG). In this subsection, we overview the results of Hayashi et al. [15] where they find maximal and minimal superclasses for TSG in the interval graphs presented in chapter 3. The following theorem will be proven by taking the proof written by Hayashi et al. in order to use their mapping in other theorems (e.g. **??**).

Theorem 4.3.5 (Hayashi et al. [15]). *MUIG $\subsetneq$ TSG.*

Proof. First, we prove that MUIG $\neq$ TSG. This can be proven because $C_4 \in$ TSG if we take as points $(0,0),(0,\varepsilon),(1,0),(1,\varepsilon)$ with $1 > \varepsilon > 0$ and $C_4 \notin$ MUIG because it is a chordal graph.

Then, we have to prove that MUIG $\subseteq$ TSG. Let $G = (V,E) \in$ MUIG where each vertex is a unit mixed interval denoted as I_v. We define $t = \min\{|I_u \cap I_v| : |I_u \cap I_v| > 0, \{I_u, I_v\} \subseteq V\}$ and $s = \min\{\ell(I_v) - r(I_u) : \ell(I_v) > r(I_u), \{I_u, I_v\} \subseteq V\}$. We have then t being the minimum length of an intersection bigger than zero (that is, not endpoint-adjacent) and s is the minimum distance between non-adjacent vertices (also not endpoint-adjacent). We also define $c(I_v) = \frac{\ell(I_v)+r(I_v)}{2}$ as the center of the interval and $p(I_v) = (-1)^{\lfloor c(I_v) \rfloor}$.

Let d be a real such that $0 < d < \frac{2}{3}$, $d \leq \frac{t}{4}$, $d < \frac{s}{2}$ and $\varepsilon \geq 2\sqrt{d - d^2}$. If we let $h = \sqrt{d - d^2}$, then we can create a $2h$-realization of G with the following mapping:

$$\phi(v) = \begin{cases} (c(I_v), 0) & \text{if } I_v \text{ is a closed interval} \\ (c(I_v), hp(I_v)) & \text{if } I_v \text{ is an open interval} \\ (c(I_v) - d, hp(I_v)) & \text{if } I_v \text{ is a closed-open interval} \\ (c(I_v) + d, hp(I_v)) & \text{if } I_v \text{ is an open-closed interval} \end{cases}$$

For two vertices u and v of G such that $u \leq v$, we have the three following cases:

1. $r(I_u) < \ell(I_v)$:

   I_u and I_v are not adjacents, which means that $\text{dist}(\phi(u), \phi(v)) > 1$. If we minimize the distance between them we have $\phi(u) = (c(I_u) + d, hp(I_u))$ and $\phi(v) = (c(I_v) - d, hp(I_v))$ with $p(I_u) = p(I_v)$. Therefore, we only have to compare their x-coordinates:

   $$\text{dist}(\phi(u), \phi(v)) \geq (c(I_v) - d) - (c(I_u) + d) = c(I_v) - c(I_u) - 2d$$

   By definition, $s \leq l(I_v) - r(I_u)$. If we take the centers, then $s \leq c(I_v) - c(I_u) - 1$, which means finally that $s + 1 \leq c(I_v) - c(I_u)$

   $$\text{dist}(\phi(u), \phi(v)) \geq s + 1 - 2d > 1$$

2. $r(I_u) > \ell(I_v)$: In this case u and v are adjacent. We maximize $\text{dist}(\phi(u), \phi(v))$ when $\phi(u) = (c(I_u) - d, hp(I_u))$ and $\phi(v) = (c(I_v) + d, hp(I_v))$ with $p(I_u) \neq p(I_v)$. Therefore,

   $$\begin{aligned} \text{dist}(\phi(u), \phi(v)) &\leq \sqrt{((c(I_v) + d) - (c(I_u) - d))^2 + (h + h)^2} \\ &\quad \text{with the same reasoning as before } c(I_v) - c(I_u) \leq 1 - t \\ &\leq \sqrt{(1 - t + 2d)^2 + 4h^2} \\ &\leq \sqrt{(1 - 4d + 2d)^2 + 4(d - d^2)} \\ &= \sqrt{1 - 4d + 4d^2 + 4d - 4d^2} = 1 \end{aligned}$$

3. $r(I_u) = \ell(I_v)$:

   In this case, u and v are adjacent only if $r(I_u)$ and I_v are closed. We know that $c(I_v) = c(I_u) + 1$ and $p(I_u) \neq p(I_v)$. Without loss of generality, we suppose that $p(I_u) = 1$ and $p(I_v) = -1$. We have two cases:

   (a) Both ends are closed. So we have this set of possible assignments for each one of the vertices:

   $$\phi(u) \in \{(c(I_u), 0), (c(I_u) + d, h)\}$$
   $$\phi(v) \in \{(c(I_u) + 1, 0), (c(I_u) + 1 - d, -h)\}$$

   This gives us a rectangle with its diagonal smaller than one.

   (b) One of the ends is closed, we suppose $r(I_u)$ is open. In this case, we have these solutions:

   $$\phi(u) \in \{(c(I_u) - d, h), (c(I_u), h)\}$$
   $$\phi(v) \in \{(c(I_u) + 1, 0), (c(I_u) + 1, -h), (c(I_u) + 1 \pm d, -h)\}$$

   Every distance between every points is greater than 1 if we take into consideration the domain of d. □

From this theorem, UIG $\subsetneq$ TSG. Actually, there exists a stronger connection between these two classes:

Theorem 4.3.6 (Breu [7]). *Let G a c-strip graph with $c \in \mathbb{R}_0^+$. G has an induced $K_{1,3}$ or C_4 if and only if G is not an unit interval graph.*

Thin strip graphs can also be seen as unfettered unit interval graphs, which means that if a graph is a thin strip graph, then we can partition this graph with a level structure where each level is a clique. This information will be relevant in the next section.

Theorem 4.3.7 (Hayashi et al. [15]). *TSG $\subsetneq$ UUIG.*

Proof. The inequality is proven by Theorem 3.3.3 because TSG $\subset$ UDG by definition. We only have to show that TSG $\subseteq$ UUIG. Let $G = (V, E)$ a TSG with $V = \{v_1, v_2, \ldots, v_n\}$ and $n \geqslant 2$. Let $\varepsilon \leqslant \frac{\sqrt{8n+1}}{4n+1}$ and ϕ the ε-realization of G. We sort the vertices from left to right, thus $0 \leqslant \phi_x(v_1) \leqslant \phi_x(v_2) \leqslant \phi_x(v_3) \leqslant \cdots \leqslant \phi_x(v_n)$. Let $\delta = \min(\{\sqrt{1+\varepsilon^2} - 1\} \cup \{\text{dist}(\phi(v_i), \phi(v_j)) - 1 : \{v_i, v_j\} \notin E\})$.

Now we construct another ε-realization ϕ' that is built by moving every disk to the left as much as possible, by keeping the same y-coordinates and the condition of adjacency $0 \leqslant \phi'_x(v_1) \leqslant \phi'_x(v_2) \leqslant \phi'_x(v_3) \leqslant \cdots \leqslant \phi'_x(v_n)$. This ϕ' realization can be built as an optimal solution of this program:

$$\begin{aligned}
\max \quad & \sum_{1 \leqslant i \leqslant n} x_i \\
\text{subject to} \quad & (x_i - x_j)^2 + (\phi_y(v_i) - \phi_y(v_j))^2 \leqslant 1 \quad \text{s.t. } \{v_i, v_j\} \in E \\
& (x_i - x_j)^2 + (\phi_y(v_i) - \phi_y(v_j))^2 \leqslant 1 + \delta \quad \text{s.t. } \{v_i, v_j\} \notin E \\
& 0 \leqslant x_1 \leqslant x_2 \leqslant x_3 \leqslant \cdots \leqslant x_n
\end{aligned}$$

Recall that $\phi_y(v_i)$ stays the same so they are constants in our program. If we develop the conditions of our program we find that this is a linear program and it has an optimal solution:

$$\begin{aligned}
\max \quad & \sum_{1 \leqslant i \leqslant n} x_i \\
\text{subject to} \quad & x_i - x_j \leqslant \sqrt{1 - (\phi_y(v_i) - \phi_y(v_j))^2} \quad \text{s.t. } \{v_i, v_j\} \in E \\
& x_i - x_j \leqslant \sqrt{(1 + \delta) - (\phi_y(v_i) - \phi_y(v_j))^2} \quad \text{s.t. } \{v_i, v_j\} \notin E \\
& 0 \leqslant x_1 \leqslant x_2 \leqslant x_3 \leqslant \cdots \leqslant x_n
\end{aligned}$$

We construct a digraph $D = (V, A)$ based on the positions of the vertices of G in ϕ'. An arc $(v_i, v_j) \in A$ if and only if:

1. $i > j$, $\{v_i, v_j\} \in E$ and $\text{dist}(\phi'(v_i), \phi'(v_j)) = 1$ called ***left arcs***.
2. $i < j$, $\{v_i, v_j\} \notin E$ and $\text{dist}(\phi'(v_i), \phi'(v_j)) = 1 + \delta$ called ***right arcs***.
3. $i < j$ and $\phi'_x(v_i) = \phi'_x(v_j)$ called ***vertical arcs***.

Based on the optimality of the linear program, we should have a path between v_1 and v_i with $1 \leq i \leqslant n$. We can prove this in the next claim.

Claim 4.3.8. *In D, there is a directed path from v_1 to v_i for each $v_i \in V$.*

Proof of Claim 4.3.8. Let S be a non-empty subset of V and $v_1 \in V \setminus S$. Assume that there is no arc from $V \setminus S$ to S. That means that every vertex of S can be shifted to the left to meet the condition of optimality. This contradicts the optimality of x_i. Therefore, there is an arc from $V \setminus S$ to S for all $S \subset V$ such that $v_i \notin S$. This implies the claim. □

For each k, P_k is the shortest directed path $v_1 \ldots v_k$ in D. Let ℓ_k and r_k be the number of left arcs and right arcs in P_k, respectively. Clearly, $r_k + \ell_k < n$.

Claim 4.3.9. *For every k, $(r_k - \ell_k) - \frac{\sqrt{1-\varepsilon^2}}{2} \leqslant x_k \leqslant (r_k - \ell_k) + \frac{\sqrt{1-\varepsilon^2}}{2}$.*

Proof of Claim 4.3.9. With a left arc, the x-coordinate increases by at least $\sqrt{(1+\delta)^2 - \varepsilon^2}$ and at most by $\sqrt{1+\delta}$. With a left arc, it decreases by at least $\sqrt{1-\varepsilon^2}$ and at most 1. Clearly, vertical arcs do not change the x-coordinate. It holds that:

$$r_k\sqrt{(1+\delta)^2 - \varepsilon^2} - \ell_k \leqslant x_k \leqslant r_k(1+\delta) - \ell_k\sqrt{1-\varepsilon^2}$$

which implies:

$$(r_k - \ell_k) - r_k\Big(1 - \sqrt{(1+\delta)^2 - \varepsilon^2}\Big) \leqslant x_k \leqslant (r_k - \ell_k) + r_k\delta + \ell_k\Big(1 - \sqrt{1-\varepsilon^2}\Big)$$

Which is similar to the inequality to prove. Actually, to prove the inequality of the claim we have to prove the following equations:

$$r_k\Big(1 - \sqrt{(1+\delta)^2 - \varepsilon^2}\Big) \leqslant \frac{\sqrt{1-\varepsilon^2}}{2} \tag{4.1}$$

$$r_k\delta + \ell_k\Big(1 - \sqrt{1-\varepsilon^2}\Big) \leqslant \frac{\sqrt{1-\varepsilon^2}}{2} \tag{4.2}$$

By the value of δ, the left side of both equations are nonnegative. We can then show only the next equation:

$$r_k\Big(1 - \sqrt{(1+\delta)^2 - \varepsilon^2}\Big) + r_k\delta + \ell_k\Big(1 - \sqrt{1-\varepsilon^2}\Big) \leqslant \frac{\sqrt{1-\varepsilon^2}}{2} \tag{4.3}$$

If we suppose that Equation 4.3 does not hold and since $n > r_k + \ell_k$:

$$n\Big(\Big(1 - \sqrt{(1+\delta)^2 - \varepsilon^2}\Big) + \delta + \Big(1 - \sqrt{1-\varepsilon^2}\Big)\Big) > \frac{\sqrt{1-\varepsilon^2}}{2} \tag{4.4}$$

By developing, we show that $1 - \sqrt{(1+\delta)^2 - \varepsilon^2} + \delta \leqslant 1 - \sqrt{1-\varepsilon^2}$ for $0 \leqslant \varepsilon \leqslant 1$ and $\delta > 0$. Then it holds that $2n\Big(1 - \sqrt{1-\varepsilon^2}\Big) > \frac{\sqrt{1-\varepsilon^2}}{2}$. This implies that $\varepsilon > \frac{\sqrt{8n+1}}{4n+1}$, which is a contradiction. Thus, Equation 4.3 is correct which proves the claim. $\square$

By Claim 4.3.9, we can classify the vertices in levels $L_k = \{v_i : r_k - \ell_k = k\}$. By the claim we imply that if $u, w \in L_k$, then $\phi'(u)$ and $\phi'(w)$ are placed on a rectangle $\varepsilon \times \sqrt{1-\varepsilon^2}$. Thus, L_k is a clique. Now if we take another level k' such that $|k - k'| \geqslant 2$ with $v_k \in L_k$ and $v_{k'} \in L_{k'}$, then $\phi'_x(v_k) - \phi'_x(v_{k'}) \geqslant k - k' - \sqrt{1-\varepsilon^2} \geqslant 2 - \sqrt{1-\varepsilon^2} > 1$, which means that for $v_k \in L_k$: $N(v_k) \subseteq L(v_{k-1}) \cup L(v_k) \cup L(v_{k+1})$. Thus, G is an unfettered unit interval graph. $\square$

4.4 Characterization of thin strip graphs

One of the main goals of this thesis is to characterize thin strip graphs with forbidden induced subgraphs. We know that TSG is an hereditary class, then a way to characterize this class of graphs is by looking for its forbidden subgraphs the same way as MUIG has been characterized by Joos. Furthermore, MUIG $\subsetneq$ TSG by Theorem 4.3.5, so the first we can do is to check if the forbidden subgraphs of MUIG are also for TSG.

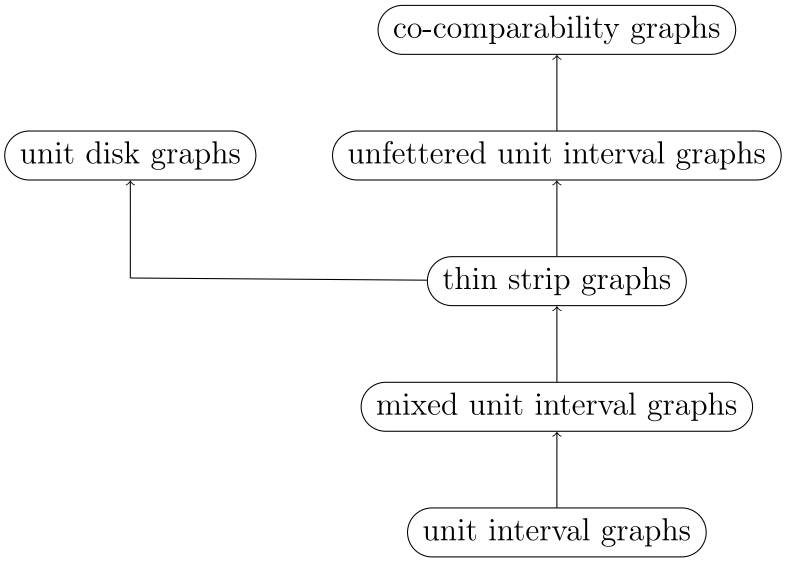

Figure 4.3: Extension of the graph classes diagram from Figure 4.3 with ***TSG*** and ***UDG***.

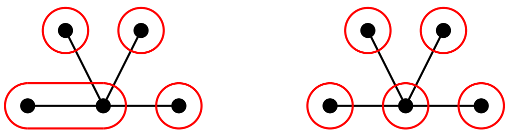

Figure 4.4: Every possible clique partition of R_0. You may notice that none of the partition is a level structure.

4.4.1 Mixed unit interval graph forbidden subgraphs

In the previous section we have shown that MUIG $\subsetneq$ TSG 4.3.5. Here we are going to overview these forbidden induced subgraphs and we their inclusion in TSG. Moreover, we will verify if one of these families is at least in UUIG. This will help us to add thin strip graphs to the hierarchy of classes we are building.

In this subsection, we are going to see the relationship between thin strip graphs and mixed unit interval graphs.

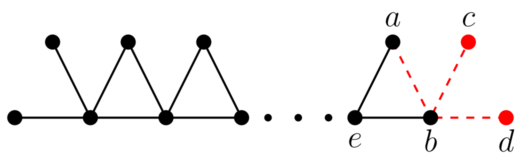

Figure 4.5: The graph R_{i+1}. You can see that the red edges and vertices are what differ from R_i.

Theorem 4.4.1 (Hayashi et al. [15]). *$\mathcal{R}$ is a forbidden induced subgraph family of UUIG.*

Proof. We can prove this by induction on i.

- ***Case*** $i = 0$: $R_0 \notin$ UUIG because there is no clique partition of R_0 that is also a level structure as seen in Figure 4.4.
- ***Case*** $i = i + 1$: We suppose that every valid clique partition of R_i is not a level structure. See in Figure 4.5 the edges and vertices that we add to generate R_{i+1}. We call a, b the vertices that were disjoint in R_i and c, d the new vertices. These two vertices are adjacent to b.

  Let $\{b, c\}$ or $\{b, d\}$ be a level of our clique partition. By the hypothesis of induction we know that this is partition is not a level structure because this partition is a valid partition of R_i because a and b are in different levels. The only way to create a new partition that is not a valid clique partition of R_i is if $\{a, b\}$ is a level. In this case, however, the clique level $\{a, b\}$ will be adjacent to three cliques $\{c\}, \{d\}$ and $\{e, \dots\}$ so this clique partition is not a level structure either.

This proves that R_i for every $i \in \mathbb{N}_0$ has not a clique level structure; thus, it is not an UUIG. $\square$

We see that $\mathcal{R}$ is a family of forbidden subgraphs of TSG. Nevertheless, the rest of the forbidden subgraphs for MUIG are thin strip graphs. The main reason is because they are unfettered unit interval graphs. We see our first example with the forbidden graph for MUIG F.

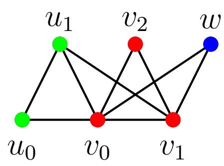

Figure 4.6: The graph F where each level is represented by a different color.

Theorem 4.4.2. *$F \in$ TSG.*

Proof. To prove this we have to find an ε-realization for our graph $F = (V, E)$ with ε arbitrarily small. Let $\phi(v)$ be the mapping of our vertices on the plane. F has a level structure $L = \{\{u_0, u_1\}, \{v_0, v_1, v_2\}, \{w\}\}$ as shown in Figure 4.6.

We begin to construct the representation as a thin strip graph by placing the vertices v_0, v_1, v_2 on the plane. Simply, we place them on the same x coordinate with an equal distance from v_0 to v_1 and v_1 to v_2 such that they are all adjacent, being those on a y coordinate smaller than ε:

$$\phi(v_k) = \left(0, \varepsilon\frac{k}{2}\right)$$

for $k \in \{0, 1, 2\}$.

We continue with w; w has to be adjacent to v_0 and v_1, but not v_2. We pursue to place it on a y coordinate that is located in the middle between 0 and $\frac{\varepsilon}{2}$ (the y coordinates of v_0 and v_1), which is $\frac{\varepsilon}{4}$. Now we have to find a x coordinate for w such that it touches both v_0 and v_1 but does not intersect

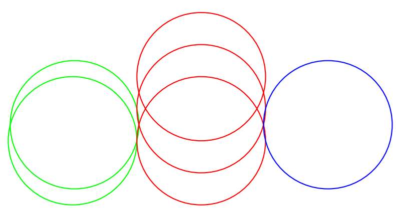

Figure 4.7: Realization of F as a thin strip graph.

with v_2. By symmetry, we only have to check the adjacency of v_0 and v_2. We can calculate a point such that the distance between w and v_0 equals one:

$$\begin{aligned}
\sqrt{\phi(w)_y^2 + \left(\frac{\varepsilon}{4}\right)^2} &= 1 \\
\phi(w)_x^2 + \left(\frac{\varepsilon}{4}\right)^2 &= 1 \\
\phi(w)_x^2 &= 1 - \left(\frac{\varepsilon}{4}\right)^2 \\
\phi(w)_x &= \sqrt{1 - \left(\frac{\varepsilon}{4}\right)^2} \\
\phi(w)_x &= \sqrt{\frac{16-\varepsilon^2}{16}} \\
\phi(w)_x &= \frac{\sqrt{16-\varepsilon^2}}{4}
\end{aligned}$$

and by symmetry, $-\frac{\sqrt{16-\varepsilon^2}}{4}$ is also a candidate. We only have to see if it touches v_2 for every ε:

$$\begin{aligned}
\sqrt{\left(\frac{\sqrt{16-\varepsilon^2}}{4}\right)^2 + \varepsilon^2} &> 1 \\
\sqrt{\frac{16-\varepsilon^2}{16} + \varepsilon^2} &> 1 \\
\sqrt{\frac{16-\varepsilon^2+16\varepsilon}{16}} &> 1 \\
\sqrt{\frac{16+15\varepsilon^2}{16}} &> 1 \\
\frac{1}{4}\sqrt{16+15\varepsilon^2} &> 1
\end{aligned}$$

the expression on the left will always be bigger than one if $\varepsilon \neq 0$, which means that w will never be adjacent to v_2.

$$\phi(w) = \left(\frac{\sqrt{16-\varepsilon^2}}{4}, \frac{\varepsilon}{4}\right)$$

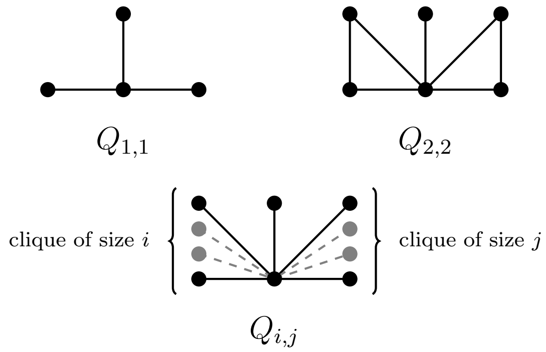

Figure 4.8: The family of graphs $\mathcal{Q}$.

Finally, we have to place u_0, u_1. We can remark that the neighbours of u_1 in the second level correspond to the neighbours of w , so it will be placed symmetrically with respect to 0 with the same y coordinate, as we have proven before. Finally, u_0 has to be adjacent to v_0. We can place it at $(-1, 0)$ with the same argument as the construction of $K_{1,3}$ (see Figure 4.2). The other vertices of the second level v_1 and v_2 will not be adjacent to u_0 unless their y coordinate is 0, which is not the case.

$$\phi(u_1) = \left(-\frac{\sqrt{16-\varepsilon^2}}{4}, \frac{\varepsilon}{4}\right)$$

$$\phi(u_2) = (-1, 0)$$

You can find a visual representation of this graph in Figure 4.7. □

To prove the realization of $\mathcal{T}, \mathcal{S}$ and $\mathcal{S}''$ as thin strip graphs, we first must define a new family of graphs. The family of graphs $\mathcal{Q}$ can be defined as a $K_{1,3}$ where two of its vertices of degree one are cliques. You can find an example in Figure 4.8.

Claim 4.4.3. *The family of graphs $\mathcal{Q}$ is a subset of the class of thin strip graphs and can be realized such that the position of every vertex is different.*

Proof. We proceed to realize $Q_{1,1}$ and $Q_{2,2}$ as thin strip graphs. With their realizations we can also deduce the realization of $Q_{i,j}$ for every $i, j \in \mathbb{N}_0$. The clique of size i is called A and the clique of size j is called B based on Figure 4.8.

$Q_{1,1}$ is $K_{1,3}$ and it has been shown that it is realizable as a thin strip graph with coordinates $(0,0), (-1,0), (1,0), (0,\varepsilon)$ for $\varepsilon > 0$. We can realize $Q_{i,j}$ if the position of every vertex in A equals $(-1,0)$ and $(1,0)$ for B. However, we want that every vertex of A and B has a different position, so we proceed to construct a realization that holds for $Q_{2,2}$.

Let a, b, c, d be the vertices of a graph $K_{1,3}$ where c is the vertex of highest degree. a and b are the leftmost and rightmost vertices when realized as a thin strip graph, so their positions are $(-1,0)$ and $(1,0)$ respectively and c is the top disk with coordinates $(0,\varepsilon)$ as seen in Figure 4.9. If we add now the vertices e and f with the same neighbourhood as a and b, we have to find out how much they can move with respect to a and b to still be in contact with d and not with c. We can build this new position by taking the same y-coordinate as a and b, so for the moment $\phi_y(e) = 0$ and $\phi_y(f) = 0$. We can see that the procedure is the same for e and f by symmetry, so we are going to prove this for e.

If $\phi_y(e) = 0$, then we have to find $\phi_x(e)$ such that e is not in contact with c with a position $\phi(c) = (0,\varepsilon)$.

$$\begin{aligned}
\sqrt{\phi_x(e)^2 + \varepsilon^2} &> 1 \\
\phi_x(e)^2 + \varepsilon^2 &> 1 \\
\phi_x(e)^2 &> 1 - \varepsilon^2 \\
\phi_x(e) &> \sqrt{1-\varepsilon^2} \quad \text{or} \quad \phi_x(e) < -\sqrt{1-\varepsilon^2}
\end{aligned}$$

with $\varepsilon > 0$.

With this, we can see that $\phi_x(e) \in (-1, -\sqrt{1-\varepsilon^2})$ and $\phi_x(f) \in (\sqrt{1-\varepsilon^2}, 1)$. To finalize this proof, we can see that every vertex of $v \in A$ such that $\phi_x(v) \in (-1, -\sqrt{1-\varepsilon^2})$, we have infinitely different positions to choose. The same holds for B. $\square$

With this result, we proceed to show the realization of $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{S}''$ as

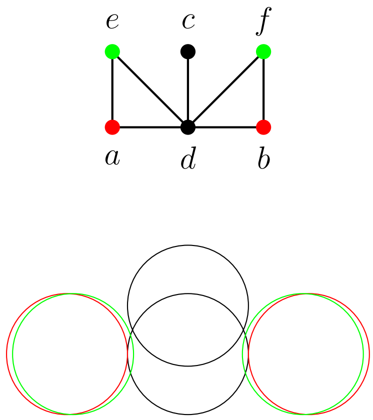

Figure 4.9: The graph $Q_{2,2}$ and its realization. You can see that the green disk is shifted towards the center with respect to the red one and still does not touch the top disk.

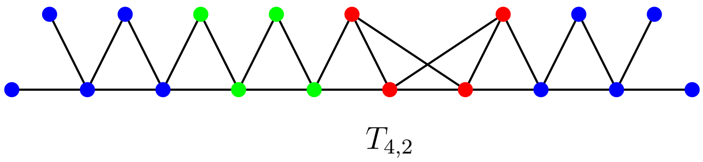

Figure 4.10: The graph $T_{4,2}$ with the diamond in red and the arms in green. The $Q_{2,1}$ from previous lemma are in blue.

thin strip graphs.

Theorem 4.4.4. *$\mathcal{T}$ is a family of thin strip graphs.*

Proof. We take as a reference $T_{4,2}$ in Figure 4.10. We can partition this graph in three parts: the ***hands*** (blue), the ***arms*** (green) and the ***diamond*** in the center (red). You may notice that the "hands" are actually $Q_{2,1}$ from Claim 4.4.3. We begin placing the left "hand" by minimizing the x-coordinate of the fifth node as stated in the claim, we give it a x-coordinate with value:

$$\tau_0 = \sqrt{1 - \varepsilon^2} + \delta$$

with $0 < \delta < 1 - \sqrt{1 - \varepsilon^2}$. And its y-coordinate equals 0, following the same procedure as in the claim.

Next, we place the disks that represent the "arm" of our graph. We place them by clique levels of two indexed by $i \geqslant 1$ from left to right with respect to Figure 4.10. The y-coordinate of the entire arm equals 0, it is actually an unit interval graph. We only have to set the x-coordinate of each one of the disks u_i, v_i of our level indexed by i. The disk u_i will be placed at the x-coordinate $i+1$, it will be adjacent to u_{i-1} or the rightmost vertex of the "hand" if $i = 1$. On the other hand, our second disk v_i will be placed in the x-coordinate:

$$\tau_i = \tau_{i-1} + 1 + \delta$$

with $0 < \delta < i - \tau_{i-1}$ so that v_i will be on the left of u_i. We do this consecutively for each level until we arrive to the diamond. The two leftmost vertices of the diamond are built as a level of the "arm". However, the right two vertices have the same coordinates of the left ones but shifted by one on the x-coordinate. By symmetry, the right arm is built equally, by taking care that the last level corresponds with the coordinates of the diamond. You can find the realization of $T_{4,2}$ as a thin strip graph in Figure 4.11.

This proof will be used as a basis that we are going to use to construct the realization of $\mathcal{S}$ and $\mathcal{S}''$ because as you can see, their structure is very similar. Actually, the "hand" and "arm" that have been used during the proof are

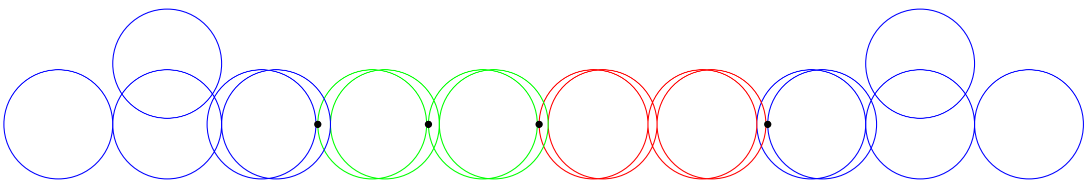

Figure 4.11: A realization as a thin strip graph of $T_{4,2}$ from Figure 4.10. The distance between the disks disminishes with the value of ε, but the disks designated by the black points will never touch.

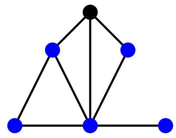

Figure 4.12: The graph S_1. We have colored the induced $Q_{1,2}$ in blue.

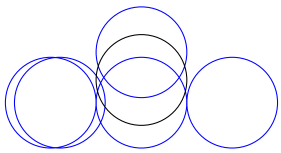

Figure 4.13: A realization of S_1 as a thin strip graph.

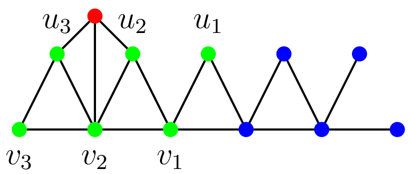

Figure 4.14: The graph S_4 with the arm in green and the induced $Q_{2,1}$ in blue. We also noted the vertices of each level of the arm.

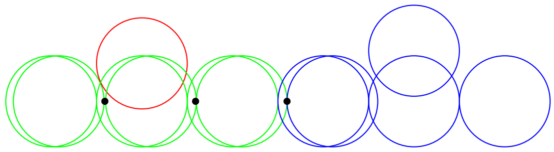

Figure 4.15: A realization of S_4 as a thin strip graph based on Figure 4.14. The black points indicate that the disks do not touch at that spot.

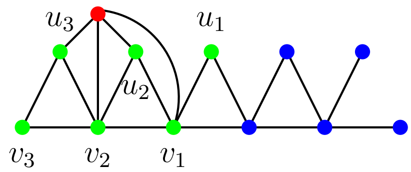

Figure 4.16: The graph S_4'' with the arm in green and the induced $Q_{2,1}$ in blue. We also noted the vertices of each level of the arm.

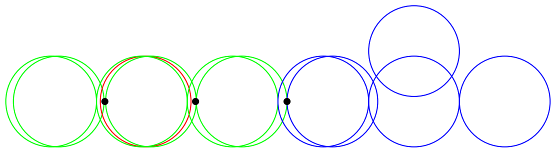

Figure 4.17: A realization of S_4'' as a thin strip graph based on Figure 4.16. The black points indicate that the disks do not touch at that spot.

directly applicable for the other graphs.

Theorem 4.4.5. *$\mathcal{S}$ is a family of thin strip graphs.*

Proof. In this case, we are going to construct S_1 and S_i with $i > 1$ differently. For S_1, we build the induced $Q_{2,1}$ as described on 4.12. Then, we add another disk on the center such that its y-coordinate is bigger than 0 and smaller than ε with a certain value that it touches the left disk as shown in Figure 4.13.

In the other hand, for S_i with $i > 1$ the construction is quite different. Indeed, we can see by Figure 4.14 that the "arm" and "hand" subgraphs are exactly the same as for $\mathcal{T}$ in Figure 4.10. The same construction works in this case. Now, we only have to add the red vertex w that differs, as seen in 4.14. If we take the same notation as in Theorem 4.4.4, we construct the arms by clique levels of two vertices of u_i and v_i. With this information, we see that we have an arm of $i-1$ levels in a graph S_i with $i > 1$. The last vertex w can be placed at the x-coordinate of v_{i-2} at a y-coordinate smaller than ε so that w is adjacent to u_3 and does not touch a vertex from the hand when $i = 2$. You can find the final realization of our example S_4 in Figure 4.14. $\square$

Theorem 4.4.6. *$\mathcal{S}''$ is a family of thin strip graphs.*

Proof. This family of graphs is a variant of $\mathcal{S}$ and as proven on Theorem 4.4.5 it is a subset of TSG. The only difference here is that now the red vertex w is also adjacent to v_1. In this case, we can simply place w at the y-coordinate 0 like the other vertices of the arm. The x-coordinate can be any position between the x-coordinates of v_2 and v_1. A realization can be found in 4.17.

Now we have a slightly better understanding of the structure of TSG. We have proven in this section that $\mathcal{R}$ is the only family of forbidden subgraphs of MUIG that is also forbidden for TSG. Moreover, $\mathcal{R}$ is also forbidden for UUIG. A good starting point, as stated by Hayashi *et al.* [15], is to find a graph that is in (UDG $\cap$ UUIG) $\setminus$ TSG. This graph will be the key for understanding what are the graph structures that are not likely to be a TSG.

4.5 Recognition

The recognition of this class of graphs is approached by Breu in his thesis [7]. He gives a polynomial-time algorithm to recognise strip graphs for a given input with an assignment of y-coordinates for each vertex of the graph and an orientation of the edges of its complement.

Theorem 4.5.1 (Breu [7])**.** *Let $G = (V, E, \gamma, \overrightarrow{E})$ a graph where $\gamma : V \to [0, c]$ is a function associating a y-coordinate (or a level) to each vertex and $\overrightarrow{E}$ an orientation of the complement of the graph. The recognition of c-strip graphs with this input is in $\mathcal{P}$.*

Observation 4.5.2. *Recognition of c-strip graphs without a given $\overrightarrow{E}$ is in $\mathcal{NP}$.*

Proof. Given a polynomial-time algorithm with a complexity of $\mathcal{O}(f(n))$ to solve recognition of $G = (V, E, \gamma, \overrightarrow{E})$, we can run again this algorithm by testing every possible orientation of its complement. This algorithm would take $\mathcal{O}(f(n))2^{|E|-1} = \mathcal{O}(f(n)2^{|E|})$ time to execute. $\square$

We would like to have an algorithm that solves this problem without knowing the y-coordinates of the vertices. Nevertheless, further research would concentrate on recognition of UUIGs. We know that TSG $\subsetneq$ UUIG, and recognition of UUIGs is $\mathcal{NP}$. If we the problem of recognising TSG given a UUIG and is solved in polynomial time, then TSG recognition would be $\mathcal{NP}$. However, given the observations in the end of chapter 3, there may be a polynomial-time algorithm for UUIG.

Conclusions

This thesis has achieved several observations and results for some problems. We have reviewed related works from several researchers and regrouped them in a hierarchy of classes. In this hierarchy of classes we added the co-comparability graphs class, which is the supergraph of almost every class of graphs we have seen in this work, except for unit disk graphs. For this, we have proven that the unfettered unit interval graph class is a subset of the co-comparability graphs class.

The most important results in the thesis have been the work on forbidden subgraphs. We have shown that a family of subgraphs of mixed unit interval graphs is also forbidden for unfettered unit interval graphs, which means that this family of forbidden subgraphs is not characteristic of thin strip graphs. For future work, we should research forbidden subgraphs for thin strip graphs that are unit disk graphs and unit interval graphs. This will help us understand the structure of thin strip graphs and give us better results. On the other hand, we have proven that the other forbidden subgraphs for mixed unit interval graphs are in fact thin strip graphs by giving a realization for them.

Finally, the recognition of the graphs that have been looked here have been only overlooked and some observations to tackle unfettered unit interval graphs have been given. The complexity of UUIG can be explored further as this question has not been addressed yet since the definition of Hayashi *et al* [15].

A subject that has not been addressed in this thesis is the two-level graphs defined by Breu in his doctoral thesis [7] about constrained disk graphs. An even understanding of thin strip graphs or c-strip graphs can be achieved.

This class of graphs is close to c-strip graphs and is an union of interval graphs [7]. There has not been any comparison between c-strip graphs and two-level graphs for the moment in terms of complexity or characterization and some open problems can be questioned for two-level graphs.

An exhaustive list of open questions that appeared during the work of this thesis have been compiled in the appendices.

Appendices

Appendix A

Problems in forbidden induced subgraph characterization

- **MUIG**: Joos [17] gives us a complete characterization of forbidden graphs.
- **TSG (Open)**: Hayashi [15] says that MUIG's forbidden induced subgraphs also are in TSG. He claims that finding a graph $F \in (\mathrm{UDG} \cap \mathrm{UUIG}) \setminus \mathrm{TSG}$ could be a good starting point. In my thesis I show that a forbidden induced subgraph for MUIG is in UDG $\cap$ UUIG.
- **TTLG (Open)**: There are many properties about these graphs in Breu's thesis [7].
- **UDG (Open)**: There is no complete characterization of UDG. Can the results of this thesis help find new ones?U

Appendix B

Problems in complexity

- **UIG/IG recognition**: Both of these problems are polynomial.
- **MUIG recognition**: Schuchat et al. give a linear algorithm ($\mathcal{O}(|V|^2)$) to recognise MUIGs [22].
- **UDG recognition**: $\exists\mathbb{R}$-complete [1].
- **SG(c) recognition (Open)**: Breu [7] states that SG(c) recognition is polynomial if a complement edge orientation and a mapping $\phi : V \rightarrow [0, c]$ is polynomial as an input of the decision problem.
- **TSG recognition (Open)**: Can we get rid of the mapping as input to recognise TSGs? In that case the problem would be at least NP.
- **UUIG recognition (Open)**: Informally the recognition of this class of graphs **cannot** be polynomial because we have to find all the cliques of the graph; the CLIQUE problem is NP-complete.

Index

Bibliography

[1] Existential Theory of the Reals. In *Algorithms in Real Algebraic Geometry*, volume 10, pages 505–532. Springer Berlin Heidelberg, 2006.

[2] *Graph Theory*. Springer Berlin Heidelberg, New York, NY, 2017.

[3] Aistis Atminas and Viktor Zamaraev. On forbidden induced subgraphs for unit disk graphs. *arXiv:1602.08148 [math]*, February 2016.

[4] Édouard Bonnet, Panos Giannopoulos, Eun Jung Kim, Paweł Rzążewski, and Florian Sikora. QPTAS and Subexponential Algorithm for Maximum Clique on Disk Graphs. *CoRR*, abs/1712.05010, 2017.

[5] Kellogg S. Booth and George S. Lueker. Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms. *Journal of Computer and System Sciences*, 13(3):335–379, December 1976.

[6] Andreas Brandstädt, Van Bang Le, and Jeremy P. Spinrad. *Graph Classes: A Survey*. Society for Industrial and Applied Mathematics, January 1999.

[7] Heinz Breu. *Algorithmic Aspects of Constrained Unit Disk Graphs*. PhD thesis, National Library of Canada = Bibliothèque nationale du Canada, Ottawa, 1996. OCLC: 46501382.

[8] Brent N. Clark, Charles J. Colbourn, and David S. Johnson. Unit Disk Graphs. *Discrete Mathematics*, 86(1):165–177, 1990.

[9] Derek G Corneil. Extensions of permutation and interval graphs. In *Proc. 18th Southeastern Conference on Combinatorics, Graph Theory and Computing*, pages 267–276, 1987.

[10] Peter Damaschke. Distances in cocomparability graphs and their powers. *Discrete Applied Mathematics*, 35(1):67–72, January 1992.

[11] Mark de Berg, editor. *Computational Geometry: Algorithms and Applications*. Springer, Berlin, 3rd ed edition, 2008.

[12] Mitre C. Dourado, Van Bang Le, Fábio Protti, Dieter Rautenbach, and Jayme L. Szwarcfiter. Mixed unit interval graphs. *Discrete Mathematics*, 312(22):3357–3363, 2012.

[13] Mitre C. Dourado, Van Bang Le, Fábio Protti, Dieter Rautenbach, and Jayme L. Szwarcfiter. Mixed unit interval graphs. *Discrete Mathematics*, 312(22):3357–3363, November 2012.

[14] P. C. Gilmore and A. J. Hoffman. A characterization of comparability graphs and of interval graphs. *Canadian Journal of Mathematics*, 16(0):539–548, January 1964.

[15] Takashi Hayashi, Akitoshi Kawamura, Yota Otachi, Hidehiro Shinohara, and Koichi Yamazaki. Thin Strip Graphs. *Special Graph Classes and Algorithms — in Honor of Professor Andreas Brandstädt on the Occasion of His 65th Birthday*, 216:203–210, January 2017.

[16] E. Helly. Über Mengen konvexer Körper mit gemeinschaftlichen Punkten. *Jahresber. Deutsch. Math.-Verein.*, 32:175–176, 1923. cited By 187.

[17] Felix Joos. A Characterization of Mixed Unit Interval Graphs. *arXiv:1312.0729 [math]*, December 2013.

[18] Terry A McKee, F. R McMorris, and Society for Industrial and Applied Mathematics. *Topics in Intersection Graph Theory*. Society for Industrial and Applied Mathematics (SIAM, 3600 Market Street, Floor 6, Philadelphia, PA 19104), Philadelphia, Pa., 1999. OCLC: 697895279.

[19] Fred S Roberts. *Representations of Indifference Relations*. Department of Mathematics, Stanford University., 1968.

[20] Marcus Schaefer. Realizability of Graphs and Linkages. In János Pach, editor, *Thirty Essays on Geometric Graph Theory*, pages 461–482. Springer New York, New York, NY, 2013.

[21] L.M. Schlipf. *Stabbing and Covering Geometric Objects in the Plane*. 2013.

[22] Alan Shuchat, Randy Shull, Ann N. Trenk, and Lee C. West. Unit Mixed Interval Graphs. *arXiv:1405.4247 [math]*, May 2014.

[23] Alan Shuchat, Randy Shull, Ann N. Trenk, and Lee C. West. Unit Mixed Interval Graphs. *arXiv:1405.4247 [math]*, May 2014.

[24] Michael Sipser. *Introduction to the Theory of Computation*. Course Technology, second edition, 2006.

[25] Alexandre Talon and Jan Kratochvíl. Completion of the mixed unit interval graphs hierarchy. *arXiv:1412.0540 [cs, math]*, December 2014.

[26] J. Venn. I. ***On the diagrammatic and mechanical representation of propositions and reasonings***. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 10(59):1–18, July 1880.