

Characterization and complexity of Thin Strip Graphs

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ABSTRACT

Abstract

1 Graphs and disks

1.1 Graphs

A graph G is defined as $G = (V, E)$, where V is the set of vertices and E the set of edges. A vertex $v \in V$ is the fundamental unit of a graph. An edge $e \in E$ links two vertices. The vertices $vw \in V$ that $e \in E$ links are called the *endpoints*.

Definition 1 *An embedding of a graph G is a representation of this graph on the plane.*

A graph G is planar if there is an embedding of this graph that doesn't have any crossing between the edges.

Theorem 2 (Kuratowski) *A graph G is planar iff it doesn't contain K_5 or $K_{3,3}$ as a minor.*

1.2 Intersection graphs

Given a geometric construction with multiple objects, an intersection graph is a graph that maps the objects into vertices and every intersection between objects is an edge between the corresponding vertices.

Definition 3 *A graph G is a comparability graph if for each edge $\{u, v\} \in E$ there is a partial order \leq such that $u \leq v$ or $v \leq u$.*

1.2.1 Interval graphs

Definition of interval Graphs

Properties

Definition of MIXED interval graphs

1.2.2 Unit disk graphs

Definition of UDG.

Definition of a realization.

2 Complexity

Problem solving is based on the complexity of a problem and not only a particular algorithm that solves it [7].

Definition 4 Let Σ be a finite alphabet, Σ^* every word derived from Σ , $L \subseteq \Sigma^*$ is a decision problem.

Definition 5 The algorithm A decides problem $L \subseteq \Sigma^*$ if for all word $w \in \Sigma^*$:

- A finishes and returns *TRUE* if $w \in L$.
- A finishes and returns *FALSE* if $w \notin L$.

Definition 6 A problem is verifiable if there's an algorithm that verifies it.

Definition 7 A problem is decidable if there's an algorithm that decides it.

2.1 P vs NP

Definition 8 A problem $L \in \mathcal{P}$ if L can be decided in polynomial time $\mathcal{O}(n^k)$.

Definition 9 A problem $L \in \mathcal{NP}$ if L can be verified in polynomial time $\mathcal{O}(n^k)$. Thus, $\mathcal{P} \subseteq \mathcal{NP}$.

2.2 $\exists\mathbb{R}$ complexity class

$\exists\mathbb{R}$ is the class that describes the problems that can be reduced to *the existential theory of the reals*[1]. The decidability of the existential theory of the reals is the problem that decides if a sentence of this form is true:

$$(\exists X_1 \dots \exists X_n) : F(\exists X_1, \dots, \exists X_n)$$

where F is a quantifier-free formula in the reals. In other words, it's a conjunction of clauses where each clause is a real polynomial inequality where each variable X_k is a real number. We can see that ETR is NP-hard because SAT can be reduced to it.

Proof. Let's take an instance of SAT ϕ_{SAT} with clauses c_k and variables x_k , we can construct an instance of ETR ϕ_{ETR} where we can construct variables in the domain $\{0, 1\}$ with this equality, so for each variable X_k :

$$X_k - X_k^2 = 0$$

Each literal of each clause will be positive or negative depending if the literal is cancelled in ϕ_{SAT} :

$$\begin{aligned} x_k \rightarrow l &= X_k \\ \neg x_k \rightarrow l &= -X_k \end{aligned}$$

Then for each clause we can have a polynomial that will sum the value of every literal in the clause must be greater than one, so that at least one literal is true:

$$\sum_{l \in c_k} l \geq 1$$

With this proof, it's easy to see that ϕ_{ETR} is valid if and only if ϕ_{SAT} is also valid. \square

This result can show us that $P \subseteq NP \subseteq \exists\mathbb{R}$.

2.2.1 Problems in $\exists\mathbb{R}$

In this section we will describe some problems that are $\exists\mathbb{R}$ -complete and will give an overview about the proof since it is not the main goal of this paper (donner d'tail de pourquoi je donne un overview).

The art gallery problem Given a simple polygon P (without crossings between every side), we introduce *guards*. A guard g is a point that every point of the polygon is watched by a guard. A point p is watched by a point q if the segment pq is contained in P . The subset G , being G the set of guards and $G \subseteq P$, is optimum if it has the minimal cardinality covering the whole polygon.

The art gallery problem decides given a polygon P and a number of guards k if there exists a configuration of k guards in G guarding the whole polygon. The art gallery problem is $\exists\mathbb{R}$ -complete [2].

Proof idea First of all, we can see that the art gallery problem is in $\exists\mathbb{R}$ if we reduce this problem to ETR. If we have an instance (P, k) of the art gallery problem we can have a formula [3] like this:

$$\phi = \{\exists x_1 y_1, \dots, x_k y_k \forall p_x p_y : \text{INSIDE-POLYGON}(p_x, p_y) \rightarrow \bigvee_{1 \leq i \leq k} \text{SEES}(x_i, y_i, p_x, p_y)\}$$

Where INSIDE-POLYGON returns 1 if $(p_x, p_y) \in P$ and SEES returns 1 if the segment $(x, y)(p_x, p_y) \in P$. ϕ is not a ETR formula, so we'd like to construct a quantifier-free formula with the idea of ϕ . To achieve this, the main idea is to have a small set of points $Q \subseteq P$ such that if these points are watched, the whole polygon is watched. This subset Q is called the *witness set*. The only thing is now to create a polynome for each point that ensures the point is watched by a guard.

To finish the proof we have to prove that the art gallery problem is $\exists\mathbb{R}$ -hard. For this part an $\exists\mathbb{R}$ -complete problem has been deducted from ETR. For the problem ETR-INV we have a set of variables $\{x_1, \dots, x_n\}$ and a set of equations of this form:

$$x = 1, \quad x + y = z, \quad x \cdot y = 1$$

and the problem decides if it exists a solution to this set of equations such that the value of each variable is real in $[\frac{1}{2}, 2]$.

A reduction of ETR-INV is found to the art gallery problem by constructing a polygon P and finding a number g for that polygon such that the instance of ETR-INT is true if and only if P is covered by at most g guards.

Unit Disk Graph recognition The Unit Disk Graph recognition is the problem that decides if a graph G has a realization ϕ as a Unit Disk Graph. Unit Disk Graph recognition is $\exists\mathbb{R}$ -complete.

Recognition of Unit Disk Graphs is $\exists\mathbb{R}$ -complete. (corollary of graph realizability problem)[5]

Stretchability is $\exists\mathbb{R}$ -complete.

3 Geometry

4 Thin Strip Graphs

4.1 Stabbing disks

Definition of stabbing.

Stabbing geometric structures.[6]

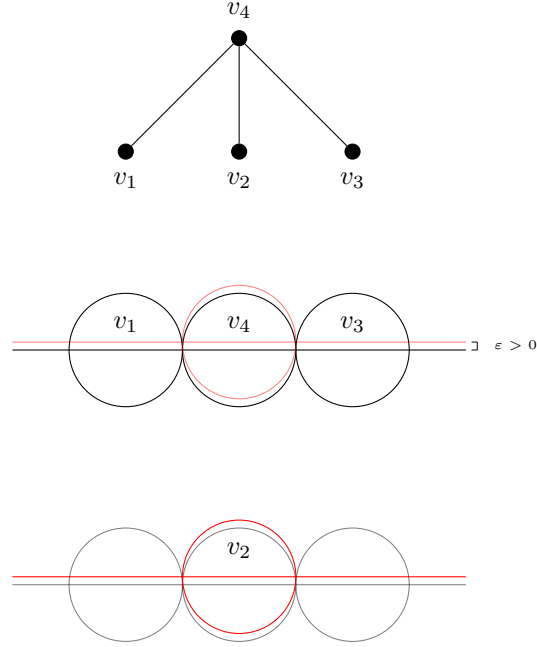


Figure 1: A construction of $K_{1,3}$ with a disk realization, being this graph a TSG.

4.2 Thin Strip Graphs

c -strip graphs are unit disk graphs such that the centers of the disks are delimited on the area $\{(x, y) : -\infty < x < \infty, 0 < y \leq c\}$ and its class noted $SG(c)$. We can say that $SG(0) = \text{UIG}$ and $SG(\infty) = \text{UDG}$. [4]

Definition 10 *Thin strip graphs are defined as $TSG = \bigcap_{c>0} SG(c)$.*

Remark 11 $SG(0) \neq TSG$. We can construct a $K_{1,3}$ such that we have 3 vertices with the coordinates $(1, 0)$, $(0, 0)$, $(-1, 0)$ and a last one $(0, \varepsilon)$ with $\varepsilon > 0$ as seen in Figure 1.

It has been proven that $\text{MUIG} \subsetneq \text{TSG}$.

Denote that there's not constant t such that $SG(t) = \text{TSG}$.

Unfettered unit interval graphs = UIIG

$\text{MUIG} \subsetneq \text{TSG} \subsetneq \text{UIIG}$

$\text{UIIG} \subseteq \text{co-comparability graphs}$ (to prove).

In the following sections we state the problems that are being studied for the thesis.

4.2.1 Forbidden subgraphs of Thin Strip Graphs

We've proven that $\text{MUIG} \subsetneq \text{TSG} \subsetneq \text{UIIG}$. Knowing the (Why F_k is a co-comparability unit disk graph?)

4.2.2 Complexity class of TSG recognition

We've shown in section 2 that some intersection geometric problems are in $\exists\mathbb{R}$ (unit disk graph recognition problem or the art gallery problem) and we'd like to know if TSG recognition or even $\text{SG}(c)$ recognition is in NP knowing that $\text{TSG} \subseteq \text{UDG}$.

References

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