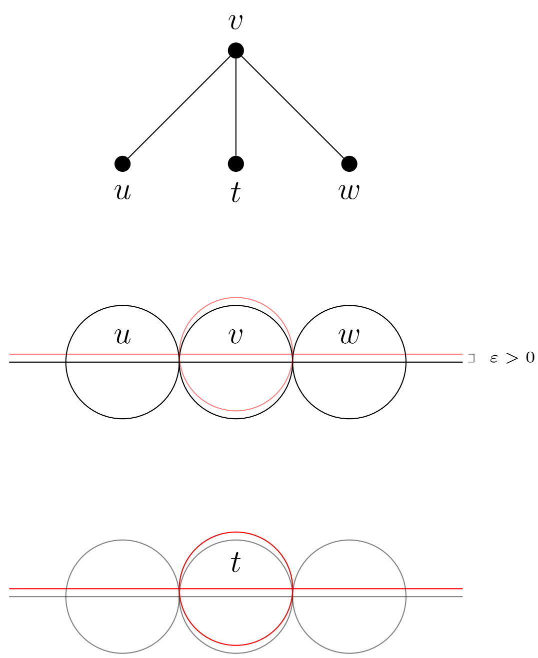

Figure 1: $K_{1,3}$ with its unique thin strip graph realization.

Lemma 0.1. *The only way to represent $K_{1,3}$ as a thin strip graph is for three points u, v, w such that $u_x = v_x - 1$ and $w_x = v_x + 1$ with the same y-coordinate and the fourth point t is placed such $t_x = v_x$ and $t_y \neq v_y$ (see Figure 1).*

Proof. We proceed to prove this lemma by contradiction. We begin constructing the realization of $K_{1,3}$ by taking its induced $P_3 \in K_{1,3}$. The middle point of P_3 has to be between the other two points horizontally, so we know then that $u_x < v_x < w_x$ with v the middle point.

Now we introduce t, the vertex that is adjacent to the middle point of P_3. We know by fact that $u_x < t_x < w_x$: if we take $t_x \leq u_x$, t has to be adjacent to u in order to intersect v which is not the case; viceversa for w.

Let $\alpha_{u,v} = \sqrt{1-(u_y-v_y)^2}$ be a real number that represents the *critical region* between two points. Note that if $|u_x - v_x| \le \alpha_{u,v}$ then u and v are intersecting and $0 < \alpha_{u,v} < 1$.

Now that we know that $u_x < t_x < w_x$, if we set $t_x < v_x$ and maximize the distance between u and v (so $u_x = v_x - 1$) we should have:

$$t_x > \alpha_{u,t} + u_x$$

for every t_y. We assume that $v_y = 0$ and $v_x = k$ with $k \in \mathbb{R}$ without loss of generality.

$$\begin{aligned}
&t_x > \alpha_{u,t} + v_x - 1 \\
&v_x > t_x > \alpha_{u,t} + v_x - 1 \\
&0 > \alpha_{u,t} - 1 \\
&1 > \alpha_{u,t}
\end{aligned}$$

Which is impossible because t and u are non-adjacent vertices which means that $\alpha_{u,t}$ has to be bigger than one by the definition of $\alpha_{u,t}$. The same occurs with w_x and $t_x > v_x$.

The only case left is when $t_x = v_x$. If u and t intersect then:

$$\begin{aligned}
&t_x \le \alpha_{u,t} + u_x \\
&t_x + 1 \le \alpha_{u,t} + v_x
\end{aligned}$$

we know that $t_x = v_x$ and $v_y = u_y = 0$ without loss of generality:

$$\begin{aligned}
v_x + 1 &\le \alpha_{u,t} + v_x \\
1 &\le \alpha_{u,t} \\
&\le \sqrt{1 - t_y^2}
\end{aligned}$$

which is impossible except when $t_y = 0$. The same happens when we suppose that t and w intersect.

Finally, we have to prove that the assumption of maximality $u_x = v_x - 1$ and $w_x = v_x + 1$ are the only solutions for this graph. By symmetry, we only have to prove this for u.

Let $u_x < v_x - 1$. In this case, u and v are not adjacent because $|u_x - v_x| > 1 > \alpha_{u,v}$. Otherwise, if $u_x > v_x - 1$ and we know that $t_x = v_x$, we can state that $u_x - k = v_x - 1$ where $0 < k < 1$ is a positive real number. We have to see if there always exists a value of $t_y > 0$ for every k so that u and t are adjacent to each other.

$$
\begin{aligned}
u_x - k &= t_x - 1 \\
t_x - u_x &= 1 - k \leq \alpha_{u,t} \\
1 - k &\leq \sqrt{1 - t_y^2} \\
(1-k)^2 &\leq 1 - t_y^2 \\
k^2 - 2k + 1 &\leq 1 - t_y^2 \\
2k - k^2 &\geq t_y^2 \\
\sqrt{2k - k^2} &\geq t_y
\end{aligned}
$$

We know that $\sqrt{2k - k^2} > 0$, so we always have a t_y such that $\sqrt{2k - k^2} > t_y > 0$. This is a contradiction to the main proposition because u and t do not have to be adjacent to each other despite of the last equation which shows us that for every deplacement to the right of u, there will always a t_y-strip graph for which u and t will be adjacent given if $u_x > t_x - 1$.

This means that the only solutions are when $t_y \neq v_y$, $t_x = v_x$, $u_x = v_x - 1$, $w_x = v_x + 1$ and $v_y = u_y = w_y$. $\square$