Characterization and complexity of Thin Strip Graphs

Abdeselam El-Haman Abdeselam
Department of Computer Science
Universite Libre de Bruxelles

May 7, 2014

ABSTRACT

Abstract

1 Graphs and disks

1.1 Graphs

A graph G is defined as $G = (V, E)$, where V is the set of vertices and E the set of edges. A vertex $v \in V$ is the fundamental unit of a graph. An edge $e \in E$ links two vertices. The vertices $vw \in V$ that $e \in E$ links are called the *endpoints*.

Definition 1 *An embedding of a graph G is a representation of this graph on the plane.*

A graph G is planar if there is an embedding of this graph that doesn't have any crossing between the edges.

Theorem 2 (Kuratowski) *A graph G is planar iff it doesn't contain K_5 or $K_{3,3}$ as a minor.*

1.2 Intersection graphs

Given a geometric construction with multiple objects, an intersection graph is a graph that maps the objects into vertices and every intersection between objects is an edge between the corresponding vertices.

Definition 3 *A binary relation R on a set S is a*

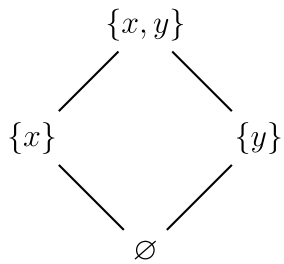

Figure 1: Hasse diagram of a poset of the power set of 2 elements ordered by inclusion. This graph is also a comparability graph, elements at each level are not comparable.

Definition 4 *A poset is a partially ordered set. A partially ordered set is a binary relation $\leq$ over a set A satisfying this axioms:*

- $a \leq a$ *(reflexivity).*
- *if* $a \leq b$ *and* $b \leq a$ *then* $a = b$ *(antisymmetry).*
- *if* $a \leq b$ *and* $b \leq c$ *then* $a \leq c$ *(transitivity).*

Definition 5 *A graph G is a comparibility graph if for each edge $\{u, v\} \in E$ there is a binary relation R such that $u \leq v$ or $v \leq u$. Equivalently, G is a comparability graph if it is the comparability graph of a poset. For example, the Hasse diagram (figure 1) is a comparability graph where the relation is inclusion.*

1.2.1 Interval graphs

Definition of interval Graphs
Properties
Definition of MIXED interval graphs

1.3 Realizations

Definition 6 *A realization of a graph G is a mapping of this graph in $\mathbb{R}^2$ respecting some properties, i.e. 2 points are linked if and only if their distance equals 1 (Unit Distance Graphs).*

The *graph realizability problem* is the problem that finds a realization of a given length $l(e)$ for a graph G (this means that the edge e has to be represented by a straight line of length $l(e)$ in $\mathbb{R}^2$).

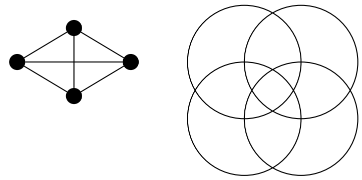

Figure 2: Realization of a UDG (Unit Disk Graph).

A unit distance graph G is a graph that has a realization where 2 points u, v have $\text{dist}(u, v) = 1$ if and only if their respective vertices are linked. This problem will be shown at chapter 2 to be $\exists\mathbb{R}$-complete. If this realization doesn't have any crossing then G is a *matchstick graph*.

A unit disk graph G is a graph that has a realization where 2 points have $\text{dist}(u, v) \leq 1$ if and only if their respective vertices are linked. Each point can be represented as the center of a disk of unit diameter and the edges can be represented as the intersection of 2 disks. This class of graphs is important for this thesis, as the Thin Strip Graphs are a sub-class of Unit Disk Graphs (section 4). Unit Disk Graph realizability is $\exists\mathbb{R}$-complete. We will refer to the Unit Disk Graph class as UDG and an example of a realization can be found in the figure 2.

2 Complexity

Complexity theory has the objective to establish lower bounds on how efficient an algorithm can be for a given problem [7]. This approach let us have a reference point to establish the difficulty of a problem.

Definition 7 *Let Σ be a finite alphabet, Σ^* every word derived from Σ, $L \subseteq \Sigma^*$ is a decision problem.*

Definition 8 *The algorithm A decides problem $L \subseteq \Sigma^*$ if for all word $w \in \Sigma^*$:*

- *A finishes and returns TRUE if $w \in L$.*
- *A finishes and returns FALSE if $w \notin L$.*

Definition 9 *A problem is verifiable if there's an algorithm that verifies it.*

Definition 10 *A problem is decidable if there's an algorithm that decides it.*

2.1 P vs NP

Definition 11 *A problem $L \in \mathcal{P}$ if L can be decided in polynomial time $\mathcal{O}(n^k)$.*

Definition 12 *A problem $L \in \mathcal{NP}$ if L can be verified in polynomial time $\mathcal{O}(n^k)$. Thus, $\mathcal{P} \subseteq \mathcal{NP}$.*

To prove a bound of complexity on an unknown problem L we have to find other problems with already known complexity and find equivalences between those 2. This can be achieved through *reductions*.

Definition 13 *A reduction of a problem L to a problem M is a mapping of an instance of L (I_L) to an isntance of M (I_M) such that I_L is true for the problem L if and only if I_M is true for the problem M. This is noted $L \leq M$ and $L \leq_P M$ if the reduction is done in polynomial time.*

With this concept we can define new complexity classes. $\mathcal{NP}$-hard is the set of problems so that we can reduce every $\mathcal{NP}$ problem to. The set of problems that are $\mathcal{NP}$-hard and $\mathcal{NP}$ are called $\mathcal{NP}$-complete. This is generalized to every complexity class ($\mathcal{P}$, $\exists\mathbb{R}$, RP, etc...)

Satisfiability problem The satisfiability problem (SAT) decides the satisfiability of a CNF formula ϕ. A CNF formula is a boolean formula that is a conjunction of multiple clauses c_k. A clause is a disjunction of multiple litterals. A litteral may be a variable or a negation of a variable.

Theorem 14 (Cook-Levin) *SAT is $\mathcal{NP}$-complete.*

2.2 $\exists\mathbb{R}$ complexity class

$\exists\mathbb{R}$ is the class that describes the problems that can be reduced to *the existential theory of the reals*[1]. The decidability of the existential theory of the reals is the problem that decides if a sentence of this form is true:

$$(\exists X_1 \ldots \exists X_n) : F(\exists X_1, \ldots, \exists X_n)$$

where F is a quantifier-free formula in the reals. In other words, it's a conjuntion of clauses where each clause is a real polynomial inequality where each variable X_k is a real number. We can see that ETR is NP-hard because SAT can be reduced to it.

Proof. Let's take an instance of SAT ϕ_{SAT} with clauses c_k and variables x_k, we can construct an instance of ETR ϕ_{ETR} where we can construct variables in the domain $\{0, 1\}$ with this equality, so for each variable X_k:

$$X_k - X_k^2 = 0$$

Each litteral of each clause will be positive or negative depending if the litteral is cancelled in ϕ_{SAT}:

$$x_k \to l = X_k$$
$$\neg x_k \to l = -X_k$$

Then for each clause we can have a polynomial that will sum the value of every litteral in the clause must be greater that one, so that at least one litteral is true:

$$\sum_{l \in c_k} l \geq 1$$

With this proof, it's easy to see that ϕ_{ETR} is valid if and only if ϕ_{SAT} is also valid. $\square$

This result can show us that $P \subseteq NP \subseteq \exists\mathbb{R}$.

2.2.1 Problems in $\exists\mathbb{R}$

In this section we will describe some problems that are $\exists\mathbb{R}$-complete and will give an overview about the proof since it is not the main goal of this paper (donner dtail de pourquoi je donne un overview).

The art gallery problem Given a simple polygon P (without crossings between every side), we introduce *guards*. A guard g is a point that every point of the polygon is watched by a guard. A point p is watched by a point q if the segment pq is contained in P. The subset G, being G the set of guards and $G \subseteq P$, is optimum if it has the minimal cardinality covering the whole polygon.

The art gallery problem decides given a polygon P and a number of guards k if there exists a configuration of k guards in G guarding the whole polygon. The art gallery problem is $\exists\mathbb{R}$-complete [2].

Proof idea First of all, we can see that the art gallery problem is in $\exists\mathbb{R}$ if we reduce this problem to ETR. If we have an instance (P, k) of the art gallery problem we can have a formula [3] like this:

$$\phi = \{\exists x_1 y_1, \dots x_k y_k \forall p_x p_y : \text{INSIDE-POLYGON}(p_x, p_y) \rightarrow \bigvee_{1 \leq i \leq k} \text{SEES}(x_i, y_i, p_x, p_y)\}$$

Where INSIDE-POLYGON returns 1 if $(p_x, p_y) \in P$ and SEES returns 1 if the segment $(x, y)(p_x, p_y) \in P$. ϕ is not a ETR formula, so we'd like to construct a quantifier-free formula with the idea of ϕ. To achieve this, the main idea is to have a small set of points $Q \subseteq P$ such that if these points are watched, the whole polygon is watched. This subset Q is called the *witness set*. The only thing is now to create a polynome for each point that ensures that the point is watched by a guard.

To finish the proof we have to prove that the art gallery problem is $\exists\mathbb{R}$-hard. For this part an $\exists\mathbb{R}$-complete problem has been deducted from ETR. For the problem ETR-INV we have a set of variables $\{x_1, \dots, x_n\}$ and a set of equations of this form:

$$x = 1, \quad x + y = z, \quad x \cdot y = 1$$

and the problem decides if it exists a solution to this set of equations such that the value of each variable is real in $[\frac{1}{2}, 2]$.

A reduction of ETR-INV is found to the art gallery problem by constructing a polygon P and finding a number g for that polygon such that the instance of ETR-INT is true if and only if P is covered by at most g guards.

Unit Disk Graph recognition The Unit Disk Graph recognition is the problem that decides if a graph G has a realization ϕ as a Unit Disk Graph. Unit Disk Graph recognition is $\exists\mathbb{R}$-complete.

Recognition of Unit Disk Graphs is $\exists\mathbb{R}$-complete. (corollary of graph realizability problem)[5]

Stretchability is $\exists\mathbb{R}$-complete.

3 Geometry

The intersection of convex objects is a matter well studied for multiple subjects. In our case, it is interesting to know some properties about the intersection of disks, those being convex objects.

A set S is convex if:

$$\forall p, q \in S \ \forall \lambda \in [0, 1] : (1 - \lambda)p + \lambda q \in S$$

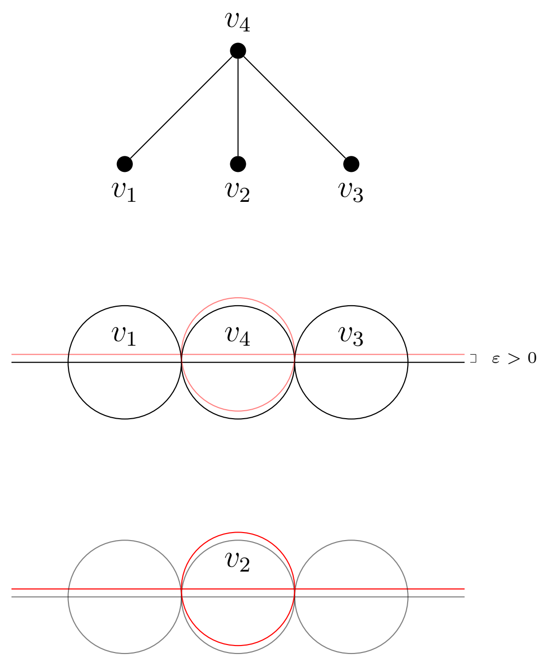

Figure 3: A construction of $K_{1,3}$ with a disk realization, being this graph a TSG.

3.1 Stabbing

Definition of stabbing.

Stabbing geometric structures.[6]

Koebe's planar $\subseteq$ disk = Planar graph duality

Helly's theorem

4 Thin Strip Graphs

c-strip graphs are unit disk graphs such that the centers of the disks are delimited on the area $\{(x, y) : -\infty < x < \infty, 0 < y \leq c\}$ and its class noted SG(c). We can say that SG(0) = UIG and SG(∞) = UDG. [4]

Definition 15 *Thin strip graphs are defined as TSG* $= \bigcap_{c>0}$ *SG(c).*

Remark 16 *SG(0)* $\neq$ *TSG. We can construct a* $K_{1,3}$ *such that we have 3 vertices with the coordinates* $(1, 0)$, $(0, 0)$, $(1, 0)$ *and a last one* $(0, \varepsilon)$ *with* $\varepsilon > 0$ *as seen in Figure 3.*

It has been proven that MUIG $\subsetneq$ TSG.

Denote that there's not constant t such that $\mathrm{SG}(t) = \mathrm{TSG}$.

Unfettered unit interval graphs = UUIG

MUIG $\subsetneq$ TSG $\subsetneq$ UUIG

UUIG $\subseteq$ co-comparability graphs (to prove).
In the following sections we state the problems that are being studied for the thesis.

4.1 Forbidden subgraphs of Thin Strip Graphs

We've proven that MUIG $\subsetneq$ TSG $\subsetneq$ UUIG. Knowing the
(Why F_k is a co-comparability unit disk graph?)

4.2 Complexity class of TSG recognition

We've shown in section 2 that some intersection geometric problems are in $\exists\mathbb{R}$ (unit disk graph recognition problem or the stretchability problem) and we'd like to know if TSG recognition or even SG(c) recognition is in NP knowing that TSG $\subseteq$ UDG.

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