# Cubic Spline Method

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August 21, 2024

## Recostruction of a 3D bottle using Cubic Spline Method (CSM)

The aim of this project is the numerical reconstruction of a rotationally symmetric bottle profile given the undefined function smooth enough, that represents the shape of the bottle in the 2D plane. To do this, we use the Cubic Spline interpolation Method.

Using the properties of the CSM, and using the Lagrange basis, we designed the numerical method to model the bottle.

### 0.1 Problem setup

Let $[a, b]$ be an interval for $a, b \in \mathbb{R}$ and $a \leq x_0 < x_1 < \ldots < x_n \leq b$ be a subdivision of the interval $[a, b]$. Let's $f$ be an undefined function measured at regular interval, that is we know the value of $f(x_i) = f_i$ at each $x_i$ for $i = 0, 1, \ldots, n$.

#### 0.1.1 Proposition 1

There exists a unique polynomial $P \in \mathbb{R}_n[X]$ such that $P(x_i) = f_i$ for each $x_i$, for $i = 0, 1, \ldots, n$. $\mathbb{R}_n[X]$ is the set of polynomials of degree less than or equal to $n$ with coefficients in $\mathbb{R}$.

#### 0.1.2 Proof

Let assume there is $P_1, P_2 \in \mathbb{R}_n$ such that $P_1(x_i) = P_2(x_i) = f_i$ for each $x_i$, for $i = 0, 1, \ldots, n$. Let $Q(x) = P_1(x) - P_2(x)$; then $Q \in \mathbb{R}_n[X]$. $Q$ has at least $n + 1$ well known solutions $x_0, x_1, \ldots, x_n$. Therefore $Q = 0_{\mathbb{R}_n}$ where $0_{\mathbb{R}_n}(x) = 0$ for every $x \in \mathbb{R}$.

#### 0.1.3 Proposition 2

There exists a basis $B = \{\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_n\}$ in $\mathbb{R}_n[X]$ such that

$$P(x) = \sum_{k=0}^{n} P(x_k)\mathcal{L}_k(x),$$

with $P(x_i) = f(x_i) = f_i$ for $i = 0, 1, \ldots, n$ and $\mathcal{L}_k(x_i) = \delta_{ki}$ where $\delta_{ki}$ is the Kronecker symbol which is 1 if $k = i$ and 0 else. $\mathcal{L}_k$ is defined as

$$\mathcal{L}_k(x) = \prod_{j=0, j\neq k}^{n} \left(\frac{x - x_j}{x_k - x_j}\right).$$

### 0.1.4 Proof

1. Let $\lambda_k \in \mathbb{R}$ for $k = 0, 1, \ldots, n$ such that $\sum_{k=0}^{n} \lambda_k \mathcal{L}_k = 0_{\mathbb{R}_n}$. Therefore for all $x \in \mathbb{R}$, $\sum_{k=0}^{n} \lambda_k \mathcal{L}_k(x) = 0_{\mathbb{R}_n}(x) = 0$. In particular, for $x = x_i$ for an arbitrary $i = 0, 1, \ldots, n$, $\sum_{k=0}^{n} \lambda_k \mathcal{L}_k(x_i) = \sum_{k=0}^{n} \lambda_k \delta_{ki} = \lambda_i = 0$.
2. Let $Q \in \mathbb{R}_n[X]$. We are looking for $\psi_k \in \mathbb{R}$ such that $Q(x) = \sum_{k=0}^{n} \psi_k \mathcal{L}_k(x)$. Let $x_j \in \mathbb{R}$ for an arbitrary $j = 0, 1, \ldots, n$. $Q(x_j) = \sum_{k=0}^{n} \psi_k \mathcal{L}_k(x_j) = \sum_{k=0}^{n} \psi_k \delta_{kj} = \psi_j$.

Therefore $B = \{\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_n\}$ forms a basis in $\mathbb{R}_n[X]$.

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I am busy at the moment, but the rest is coming soon.
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