Design and Control of a Double Inverted Pendulum for Precise and Accurate Motion

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Contents

1 Abstract

In the realm of modern control theory, the study and implementation of advanced control techniques hold paramount significance in tackling complex and dynamic systems. This project embarks upon the exploration of such techniques through the lens of a double inverted pendulum mounted upon a cart – a quintessential example of a nonlinear and inherently unstable system. Leveraging principles of control theory, this endeavor seeks to unravel the intricacies of stabilizing and orchestrating the motion of the pendulum-cart system.

2 Introduction

In the pursuit of comprehending and manipulating intricate physical systems, the discipline of control theory stands as an indispensable beacon. This project delves into the realm of modern control techniques, as they converge with the fascinating and challenging domain of a double inverted pendulum mounted upon a cart. The double inverted pendulum embodies an emblematic representation of a multivariable, nonlinear system - an archetype that epitomizes the complexities often encountered in real-world applications.

Our endeavor commences with an analysis of the system's dynamics, unraveling the differential equations governing its behavior. Through the lens of the Lagrangian method, we dissect the intricate dance of forces and torques that define the system's motion. To forge a bridge between theory and practice, we harness the power of Simulink to encapsulate the system's dynamics within a virtual realm.

The journey deepens as we venture into the heart of control design. We fashion a Linear Time-Invariant (LTI) state space model, bestowing us with a lens to peer into the very essence of the system's response. In parallel, we traverse the terrain of controllability, observability, and stability, painting a vivid portrait of the system's inherent characteristics.

Yet, our expedition truly unfurls with the design of control strategies. A symphony of full-state feedback controllers and Linear Quadratic Regulators (LQR) takes center stage, orchestrated to steer the system towards graceful stability. Through meticulous selection of eigenvalues, tuning of gains, and rigorous simulations, we usher forth a pantheon of approaches that vie for control supremacy.

Moreover, we illuminate the role of full-state observers, entrusted with the task of unraveling the system's internal state from mere observations. We intertwine these observers with the controllers, culminating in a symphony where the conductor's baton orchestrates harmony between physical reality and computational insight.

3 System Modeling and Assumptions

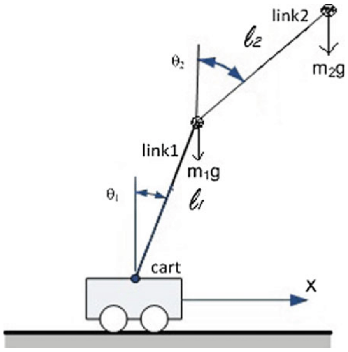

Figure 1: Double-inverted-pendulum on a cart

The system is graphically represented in as shown Figure 1. Firstly, we make some assumptions are massless rod, force applied on mass M, and the pendulum's mass as m_1 and m_2, respectively. The external force $u(t)$ is applied in x-direction. Here, $\theta_1(t)$ and $\theta_2(t)$ are the pendulum angles, $x(t)$ represents position.

3.1 Mathematical Modeling

To derive its equations of motion,we use Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_i}}\right) - \frac{\partial L}{\partial q_i} = Q$$

where $L = T - P$ is a Lagrangian, Q is a vector of generalized forces acting in the direction of generalized coordinates q_i (x,θ_1,θ_2) and not accounted for in formulation of kinetic energy T and potential energy P. Kinetic and potential energies of the system are given by the sum of energies of its individual components (a wheeled cart and two pendulums):

$$T = T_0 + T_1 + T_2$$
$$P = P_0 + P_1 + P_2$$

where

$$\begin{aligned}
T_0 &= \frac{1}{2} M \dot{x}^2 \\
T_1 &= \frac{1}{2} m_1 \left[\left(\dot{x} + l_1 \dot{\theta}_1 \cos\theta_1 \right)^2 + \left(l_1 \dot{\theta}_1 \sin\theta_1 \right)^2 \right] + \frac{1}{2} I_1 \dot{\theta}_1^2 \\
&= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} \left(m_1 l_1^2 + I_1 \right) \dot{\theta}_1^2 + m_1 l_1 \dot{x} \dot{\theta}_1 \cos\theta_1 \\
T_2 &= \frac{1}{2} m_2 \left[\left(\dot{x} + l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_2 \cos\theta_2 \right)^2 + \left(l_1 \dot{\theta}_1 \sin\theta_1 + l_2 \dot{\theta}_2 \sin\theta_2 \right)^2 \right] + \frac{1}{2} I_2 \dot{\theta}_2^2 \\
&= \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} \left(m_2 l_2^2 + I_2 \right) \dot{\theta}_2^2 + m_2 l_1 \dot{x} \dot{\theta}_1 \cos\theta_1 + m_2 l_2 \dot{x} \dot{\theta}_2 \cos\theta_2 \\
&+ m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\left(\theta_1 - \theta_2 \right) \\
P_0 &= 0 \\
P_1 &= m_1 g l_1 \cos\theta_1 \\
P_2 &= m_2 g \left(l_1 \cos\theta_1 + l_2 \cos\theta_2 \right)
\end{aligned}$$

thus the Lagrangian of the system is

$$\begin{aligned}
\mathcal{L} =& \frac{1}{2} \left[M + m_1 + m_2 \right] \dot{x}^2 + \frac{1}{2} \left[m_1 + m_2 \right] l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 \\
&+ \left[m_1 + m_2 \right] \dot{x} \dot{\theta}_1 l_1 \cos\theta_1 + m_2 \dot{x} \dot{\theta}_2 l_2 \cos\theta_2 \\
&+ l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\left[\theta_1 - \theta_2 \right] - \left[m_1 g l_1 \cos\theta_1 + m_2 g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2 \right]
\end{aligned}$$

Differentiating the Lagrangian by $\dot{q_i}$ ($\dot{x}$,$\dot{\theta}_1$,$\dot{\theta}_2$) and q_i (x,θ_1,θ_2) yield Lagrange equation :

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = u$$

$$\begin{gathered}
\left(M + m_1 + m_2 \right) \ddot{x} + \left(m_1 + m_2 \right) l_1 \cos\left(\theta_1 \right) \ddot{\theta}_1 + m_2 l_2 \cos\left(\theta_2 \right) \ddot{\theta}_2 \\
- \left(m_1 + m_2 \right) l_1 \sin\left(\theta_1 \right) \dot{\theta}_1^2 - m_2 l_2 \sin\left(\theta_2 \right) \dot{\theta}_2^2 = u
\end{gathered}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$\begin{gathered}
\left(m_1 + m_2 \right) l_1 \cos\left(\theta_1 \right) \ddot{x} + \left(m_1 l_1^2 + m_2 l_1^2 + I_1 \right) \ddot{\theta}_1 + m_2 l_1 l_2 \cos\left(\theta_1 - \theta_2 \right) \ddot{\theta}_2 \\
+ m_2 l_1 l_2 \sin\left(\theta_1 - \theta_2 \right) \dot{\theta}_2^2 - \left(m_1 + m_2 \right) l_1 g \sin\theta_1 = 0
\end{gathered}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = 0$$

$$\begin{gathered}
m_2 l_2 \cos\left(\theta_2 \right) \ddot{x} + m_2 l_1 l_2 \cos\left(\theta_1 - \theta_2 \right) \ddot{\theta}_1 + \left(m_2 l_2^2 + I_2 \right) \ddot{\theta}_2 \\
- m_2 l_1 l_2 \sin\left(\theta_1 - \theta_2 \right) \dot{\theta}_1^2 - m_2 l_2 g \sin\theta_2 = 0
\end{gathered}$$

The equation of the system after linearizing the about an equilibrium point θ_1 and θ_2 equals zero, given that the mass is concentrated at the top of the rod resulting in the center of gravity coincides with the pendulum ball's center, the moment of inertia (I) is assumed to be negligible (I = 0) are :

$$(M + m_1 + m_2)\,\ddot{x} + (m_1 + m_2)\,l_1\ddot{\theta}_1 + m_2 l_2\ddot{\theta}_2 = u \tag{1}$$

$$(m_1 + m_2)\,l_1\ddot{x} + \left(m_1 l_1^2 + m_2 l_1^2\right)\ddot{\theta}_1 + m_2 l_1 l_2\ddot{\theta}_2 = (m_1 + m_2)\,l_1 g\theta_1 \tag{2}$$

$$m_2 l_2\ddot{x} + m_2 l_1 l_2\ddot{\theta}_1 + \left(m_2 l_2^2\right)\ddot{\theta}_2 = m_2 l_2 g\theta_2 \tag{3}$$

3.2 Simulink Modeling

The system parameters are:

$$\begin{array}{ll} M = 2\,Kg, & l_1 = 0.5\,m \\ m_1 = 0.1\,Kg, & l_2 = 0.5\,m \\ m_2 = 0.1\,Kg, & g = 9.81\,m/sec^2 \end{array}$$

After substituting in 1, 2, and 3:

$$2.2\,\ddot{x} + 0.1\,\ddot{\theta}_1 + 0.05\,\ddot{\theta}_2 = u$$

$$0.1019\,\ddot{x} + 0.051\,\ddot{\theta}_1 + 0.0255\,\ddot{\theta}_2 = \theta_1$$

$$0.1019\,\ddot{x} + 0.051\,\ddot{\theta}_1 + 0.051\,\ddot{\theta}_2 = \theta_2$$

$$\begin{bmatrix} 2.2 & 0.1 & 0.05 \\ 0.1019 & 0.051 & 0.0255 \\ 0.1019 & 0.051 & 0.051 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} u \\ \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.981 & 0 \\ -1 & 41.202 & -19.62 \\ 0 & -39.24 & 39.24 \end{bmatrix} \begin{bmatrix} u \\ \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\ddot{x} = 0.5\,u - 0.981\,\theta_1 \tag{4}$$

$$\ddot{\theta}_1 = -u + 41.202\,\theta_1 - 19.62\,\theta_2 \tag{5}$$

$$\ddot{\theta}_2 = -39.24\,\theta_1 + 39.24\,\theta_2 \tag{6}$$

Verifying using MatLab:

```
M=2; m1=0.1; m2=0.1; l1=0.5; l2=0.5; g=9.81;
A0=[(M+m1+m2) (m1+m2)*l1 m2*l2;
   ((m1+m2)*l1)/((m1+m2)*l1*g) ((m1+m2)*l2^2)/((m1+m2)*l1*g) (m2*l1*l2)/((m1+m2)*l1*g);
   (m2*l2)/(m2*l2*g) (m2*l1*l2)/(m2*l2*g) (m2*l2^2)/(m2*l2*g)];
eqmatrix = inv(A0);
```

SimuLink Model:

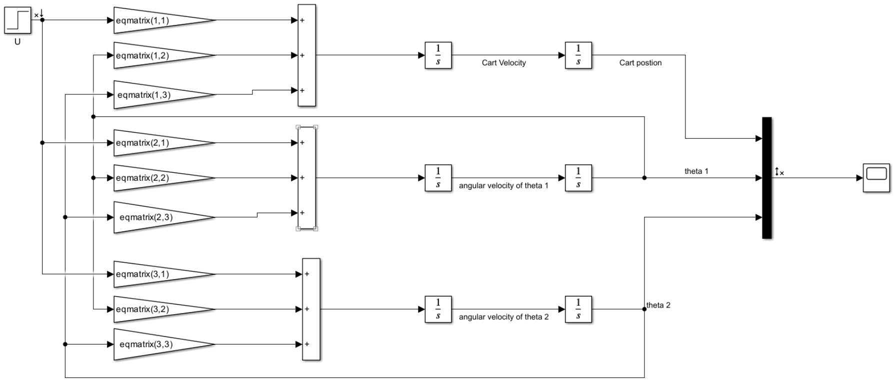

Figure 2: SimuLink Model

3.3 (LTI) state space model

Defining the state variables:

$$\begin{aligned} x_1 &= x \quad & x_3 &= \theta_1 \quad & x_5 &= \theta_2 \\ x_2 &= \dot{x} \quad & x_4 &= \dot{\theta}_1 \quad & x_6 &= \dot{\theta}_2 \end{aligned}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.981 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 41.202 & 0 & -19.62 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -39.24 & 0 & 39.24 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0.5 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 \end{bmatrix}$$

and we got:

$$G(s) = \frac{0.5(s+8.185)(s-8.185)(s+3.39)(s-3.39)}{s^2(s+8.2445)(s-8.245)(s+3.529)(s-3.529)}$$

4 System Properties

4.1 Controllability

First calculating the controllability matrix (Co)

$$\begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix}$$

Using Matlab to calculate it:

```matlab
A = [0,1,0,0,0,0;
     0,0,-0.981,0,0,0;
     0,0,0,1,0,0;
     0,0,41.202,0,-19.62,0;
     0,0,0,0,0,1;
     0,0,-39.24,0,39.24,0;];
B = [0;0.5;0;-1;0;0;];
Co = ctrb(A,B);
r = rank(Co);
```

$$Co = \begin{bmatrix} 0 & 0.5 & 0 & 1 & 0 & 40.4 \\ 0.5 & 0 & 1 & 0 & 40.4 & 0 \\ 0 & -1 & 0 & -41.2 & 0 & -2467.5 \\ -1 & 0 & -41.2 & 0 & -2467.5 & 0 \\ 0 & 0 & 0 & 39.2 & 0 & 3156.5 \\ 0 & 0 & 39.2 & 0 & 3156.5 & 0 \end{bmatrix}$$

The controllability matrix rank is 6. Hence, It is completely controllable system.

4.2 Observability

First We get the Observability matrix (Ob),

$$\begin{bmatrix} C^* & C^*A^* & C^*(A^*)^2 & C^*(A^*)^3 & C^*(A^*)^4 & C^*(A^*)^5 \end{bmatrix}$$

Using Matlab:

```matlab
A = [0,1,0,0,0,0;
    0,0,-0.981,0,0,0;
    0,0,0,1,0,0;
    0,0,41.202,0,-19.62,0;
    0,0,0,0,0,1;
    0,0,-39.24,0,39.24,0;
    ];
C= [1,0,0,0,0,0];
Ob = obsv(A,C);
r = rank(Ob);
```

$$Ob = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.981 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.981 & 0 & 0 \\ 0 & 0 & -40.4192 & 0 & 19.2472 & 0 \\ 0 & 0 & 0 & -40.4192 & 0 & 19.2472 \end{bmatrix}$$

The Observability matrix rank is 6. Hence, It is completely observable system.

4.3 Stability

First to get the characteristic equation we used the following MatLab code:

```
A = [0,1,0,0,0,0;
     0,0,-0.981,0,0,0;
     0,0,0,1,0,0;
     0,0,41.202,0,-19.62,0;
     0,0,0,0,0,1;
     0,0,-39.24,0,39.24,0;];
syms s;
sI = s*eye(6);
ch = det(inv(sI-A));
```

we got

$$det(sI - A) = s^6 - 80.442\, s^4 + 846.877\, s^2$$

then using isstable() command in matlab we got:

```
>> tf(1,[1 0 -80.442 0 846.877 0 0])

ans =

             1
  ---------------------------
  s^6 - 80.44 s^4 + 846.9 s^2

Continuous-time transfer function.

>>
>> G = tf(1,[1 0 -80.442 0 846.877 0 0])

G =

             1
  ---------------------------
  s^6 - 80.44 s^4 + 846.9 s^2

Continuous-time transfer function.

>> isstable(G)

ans =

  logical

   0

>> 
```

Figure 3: Stability Test for the system

the value 0 indicates that the system is not stable.

5 Full-State Feedback Controller

5.1 The Desired System Eigenvalues & The Controller Gains.

First Determining the overshoot percent (OS%) and the settling time (T_s):
Acceptable values will be: $OS\% = 20\%$ & $T_s = 2\,s$
By using the next MatLab Code:

```
Ts = 2;
OS = 0.2;
Re = 4/Ts;
zeta = -log(OS) / sqrt(pi^2 + log(OS)^2);
theta = acosd(zeta);
Im = Re*tand(theta);
P = [-Re+1i*Im  -Re-1i*Im  -5*Re  -5.1*Re  -5.2*Re  -5.3*Re  -5.4*Re];
```

We got the desired closed-loop system eigenvalues to be:

$$\begin{array}{ll} \mu_1 = -2 + 3.9040j, & \mu_5 = -10.4 \\ \mu_2 = -2 + 3.9040j, & \mu_6 = -10.6 \\ \mu_3 = -10, & \mu_7 = -10.8 \\ \mu_4 = -10.2 & \end{array}$$

Then we got to get $\hat{A}$ & $\hat{B}$:

$$\hat{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.981 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 41.202 & 0 & -19.62 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -39.24 & 0 & 39.24 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By Using the following MatLab Code:

```
Ah = [A; -C];  Ah = [Ah, zeros(7,1)];
Bh = [B; 0];
K = place(Ah,Bh,P);
Ki = -K(end);
```

$$\hat{K} = \begin{bmatrix} K & -k_I \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & -k_I \end{bmatrix}$$
$$= \begin{bmatrix} 4182.3 & 2103.3 & 702.1 & 995.7 & 6623.9 & 1452.5 & -6070.0 \end{bmatrix}$$

5.2 Simulation of Closed-Loop System using Simulink

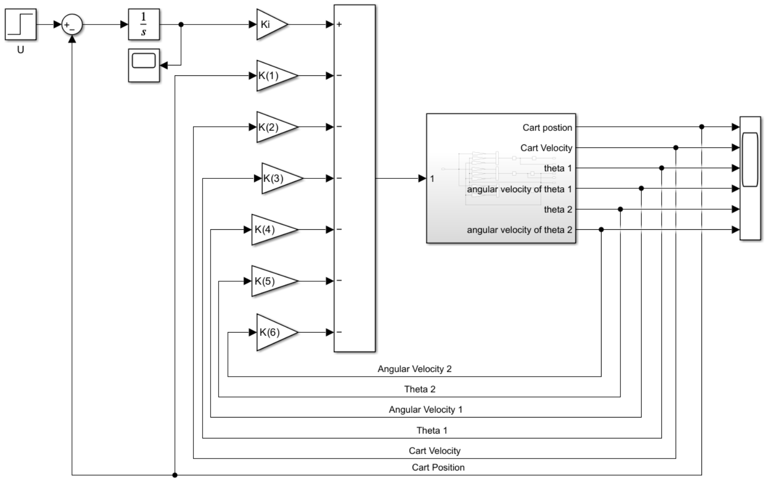

Figure 4: Simulation of Closed-Loop System using Simulink

5.3 Results

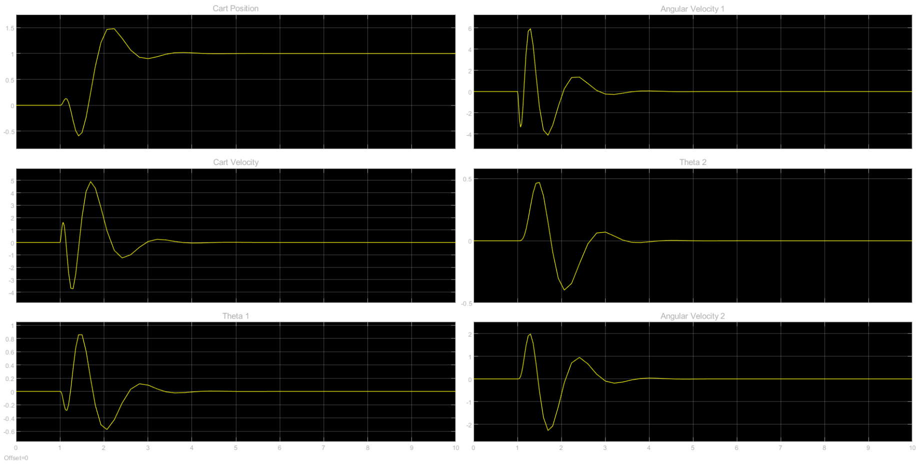

Figure 5: All Stats versus Time

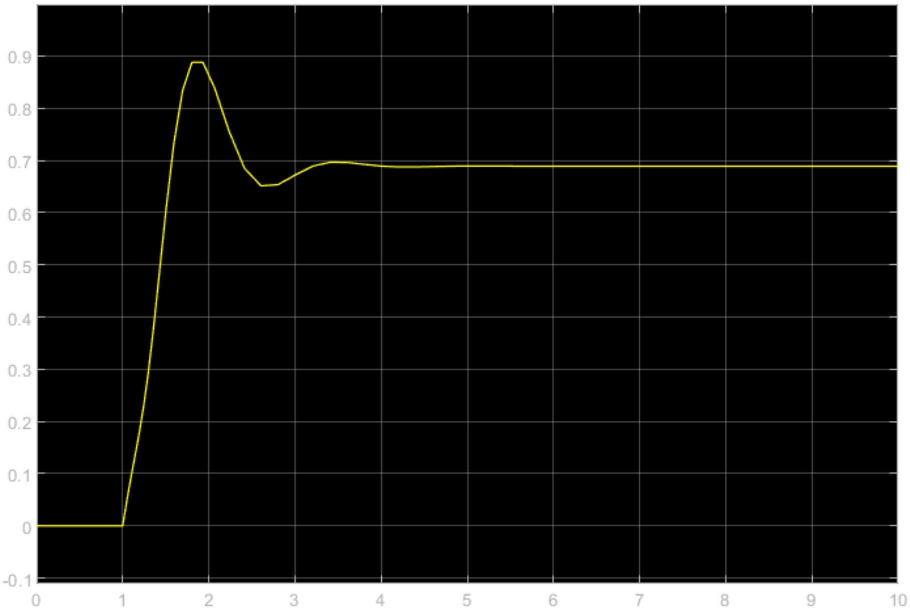

Figure 6: Input versus Time

6 Linear Quadratic Regulator (LQR) Controller

6.1 The Values of Q & R and Their Controller Gains

Two different values for Q and R were selected:

First values are :

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad R_1 = 1$$

Second values are :

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad R_2 = 0.01$$

By using the following MatLab Code:

```
% LQR
Ah = [A; -C];  Ah = [Ah, zeros(7,1)];
Bh = [B; 0];

Q1 = [1 0 0 0 0 0 0;
      0 1 0 0 0 0 0;
      0 0 1 0 0 0 0;
      0 0 0 1 0 0 0;
      0 0 0 0 1 0 0;
      0 0 0 0 0 1 0;
      0 0 0 0 0 0 1];
R1 = 1;
Q2 = [1 0 0 0 0 0 0;
      0 1 0 0 0 0 0;
      0 0 100 0 0 0 0;
      0 0 0 1 0 0 0;
      0 0 0 0 100 0 0;
      0 0 0 0 0 1 0;
      0 0 0 0 0 0 1];
R2 = 0.01;
K1 = lqr(Ah,Bh,Q1,R1); K1i = -K1(end);
K2 = lqr(Ah,Bh,Q2,R2); K2i = -K2(end);
```

$$K_1 = \begin{bmatrix} K & -k_I \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & -k_I \end{bmatrix}$$

$$= \begin{bmatrix} 3.7190 & 6.4153 & -317.4937 & -22.0897 & 357.8236 & 55.3703 & -1.0000 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} K & -k_I \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & -k_I \end{bmatrix}$$

$$= \begin{bmatrix} 31.3512 & 44.1449 & -573.8894 & -14.0403 & 884.0479 & 147.2630 & -10.0000 \end{bmatrix}$$

6.2 Results

6.2.1 First values of Q & R results

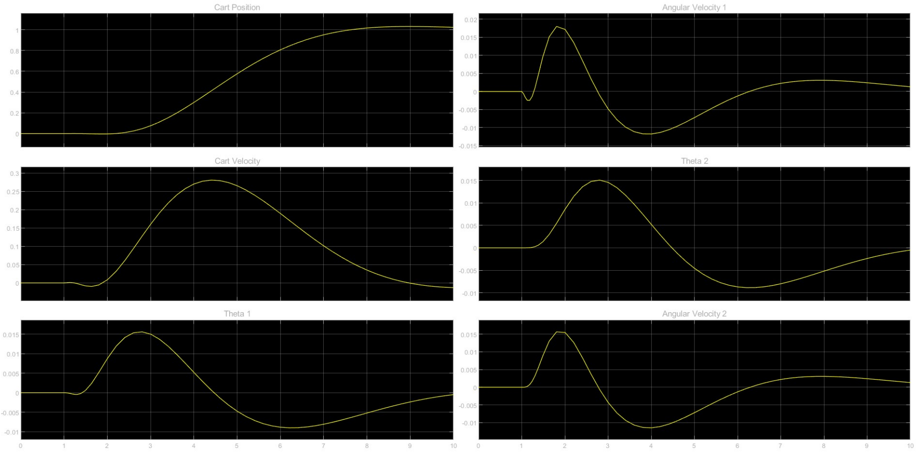

Figure 7: All Stats versus Time

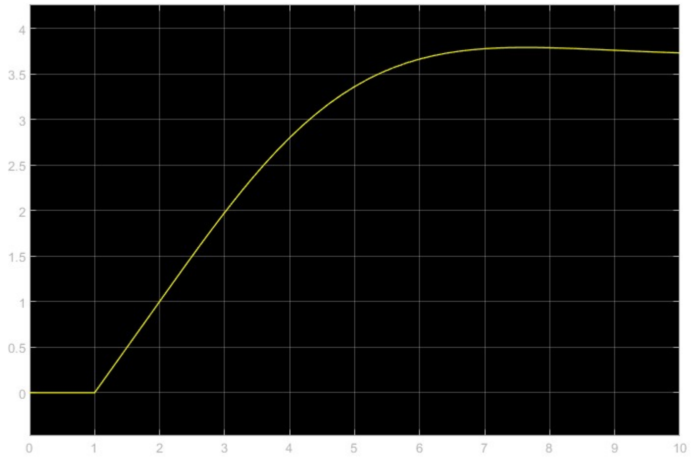

Figure 8: Input versus Time

6.2.2 Second values of Q & R results

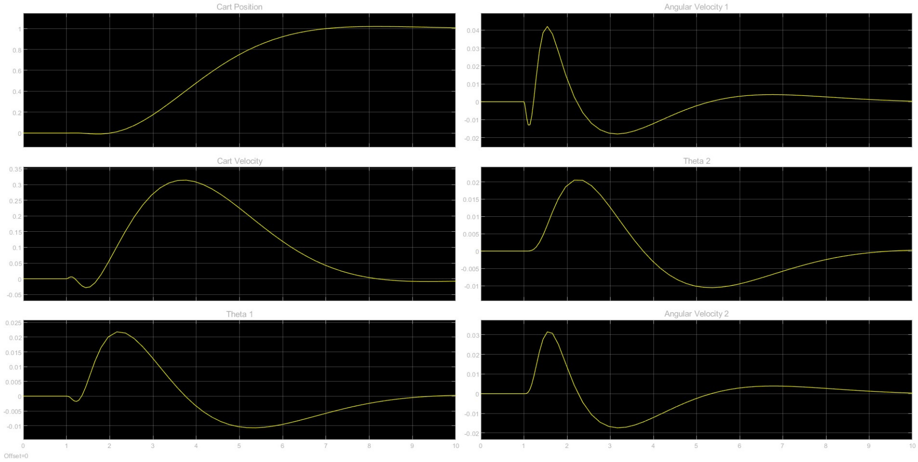

Figure 9: All Stats versus Time

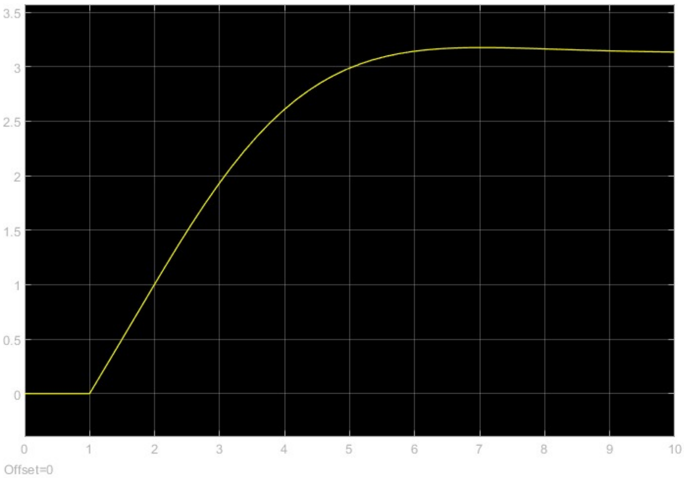

Figure 10: Input versus Time

6.3 Discussion

6.3.1 First and Second values of Q & R

First value of Q & R : The Q matrix had chosen with an equal weight, where the diagonal elements are assigned a value of 1. This choice of Q implies that all state variables have equal importance in the control process. Additionally, the R-value had been set to be equal to 1, we assign equal weight to the control effort. This choice can lead to a balanced performance across all state variables and control efforts.
Second value of Q & R : we customize the weight matrix Q to prioritize the state deviations of the pendulum angles θ_1 and θ_2 by assigning a higher weight (100) to their respective positions in the state vector. The remaining state variables, such as angular velocities, are assigned relatively lower weights (1). Moreover, we minimize the control effort by selecting a smaller R-value of 0.01. This configuration underscores the significance of stabilizing the pendulum angles while keeping the control effort at a reasonable level

The Graphical Representation Analysis: For the first chosen set of Q and R values, we should apply a unit step input of 3.8 units to the system which corresponding input control effort required to achieve stabilization is visually represented in Figure 8, and The pendulum angles, θ_1, and θ_2, exhibit deviations of 0.016 and 0.015, respectively, from their equilibrium positions as shown in figure 7. Meanwhile, for the Second chosen values for Q and R, the control effort is reduced to 3.2, as illustrated in Figure 10 which will reflect the controller's improved efficiency in minimizing control effort. Simultaneously, the values of θ_1 and θ_2 exhibit changed behaviors, with deviations of 0.022 and 0.021, respectively, from their equilibrium positions as shown in Figure 9.

By Comparing the two graphical representations, we observe that the second set of Q and R values yields a more effective control effort reduction while maintaining the stability of the pendulum angles. The modified weight matrix Q prioritizes the stabilization of theta1 and theta2, leading to improved control performance in terms of both state deviations and control effort.

6.3.2 Linear Quadratic Regulator and Full-State Feedback

For **Full-State Feedback** Control approach, a unit step input of 0.9 units was applied to the system. This input resulted in the pendulum angles, theta1 and theta2, exhibiting deviations of 0.9 and 0.45, respectively, from their equilibrium positions. This behavior is illustrated in Figures 5 & 6.

On the other hand, in **the LQR controller**, utilizing the second set of chosen values for Q and R, The control effort represented in unite step input to the system increased to 3.2 units. However, the corresponding values of θ_1 and θ_2 displayed significantly reduced deviations from their equilibrium positions, amounting to 0.022 and 0.021, respectively. This reduction represents a substantial difference in the state deviation values between the two methods. as shown in Figures 9 & 10.

Comparative Analysis and Trade-offs : Comparing the outcomes of the Full-State Feedback and LQR strategies, it is evident that the LQR approach excels in minimizing the state deviations of the pendulum angles. The significant reduction in θ_1 and θ_2 deviations underscores the effectiveness of the LQR controller in achieving precise state regulation. However, this enhanced control performance of the LQR strategy comes at the expense of higher control effort. The input effort to the system increases when implementing the LQR controller, as indicated by the higher control effort observed in Figure 10. This trade-off between state deviation minimization and control effort is a fundamental consideration when selecting the appropriate control strategy.

7 Observer-Based Controller for Full-State Feedback Controller

We can use the same method used in section 5 to determine the poles of the controller first;

```
Ts = 4;
OS = 0.2;
Re = 4/Ts;
zeta = -log(OS) / sqrt(pi^2 + log(OS)^2);
theta = acosd(zeta);
Im = Re*tand(theta);
P = [-Re+1i*Im  -Re-1i*Im  -5*Re  -5.1*Re -5.2*Re -5.3*Re -5.4*Re];
Ah = [A; -C];  Ah = [Ah, zeros(7,1)];
Bh = [B; 0];
K = place(Ah,Bh,P);
Ki = -K(end);
K0 = K(1:6);
```

then we choose the poles of the observer to be 15 times away from the dominant poles;

```
Po = [-15*Re  -15.1*Re -15.2*Re -15.3*Re -15.4*Re -15.5*Re];
Ke = place(A',C',Po)';
```

7.1 Simulation of Closed-Loop System using Simulink

the following simulink model is the same as the full-state controller after adding the Observer;

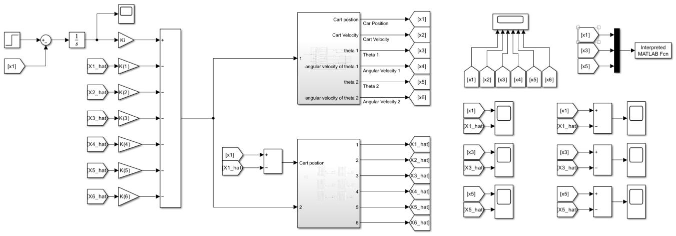

Figure 11: Simulink Observer-Based Controller

7.2 The states initial conditions

for the initial conditions, we don't have the flexibility to change the values as the model is linearized about exact point. So, we made the intial conditions:

$$x_1 = x = -0.001\, m$$
$$x_3 = \theta_1 = 0.079\, rad$$
$$x_5 = \theta_2 = 0.079\, rad$$

7.3 Results

The results of the observer yield a maximum distance of 3 meters.

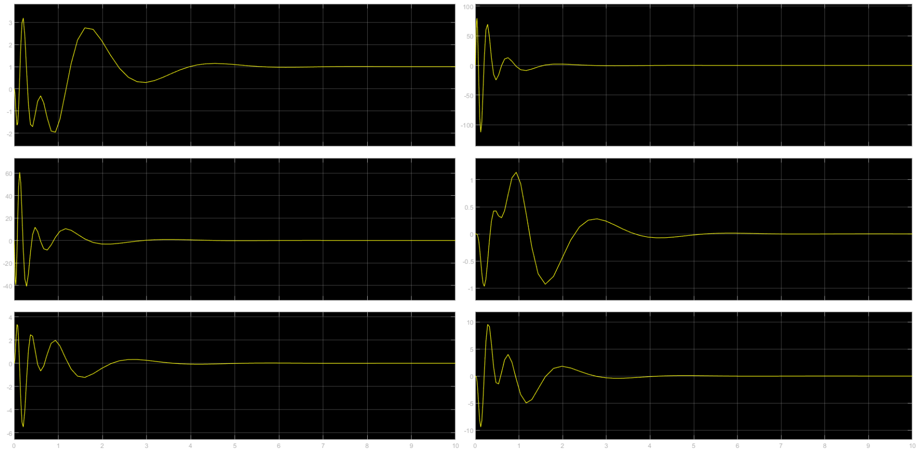

Figure 12: Results of Simulink Observer-Based Controller

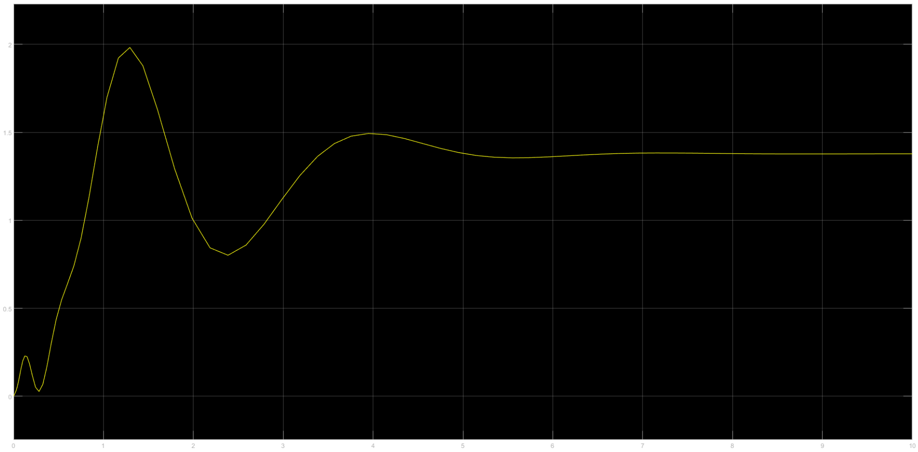

Figure 13: Results of the input

7.4 Discussion

The results of the observer vary from the actual full-state controller as the change of the initial conditions leads to a change of making the pendulum oscillate more with higher amplitude. this shows the difference, in the beginning, leads to more chaotic result until it maintains the same value as the actual.

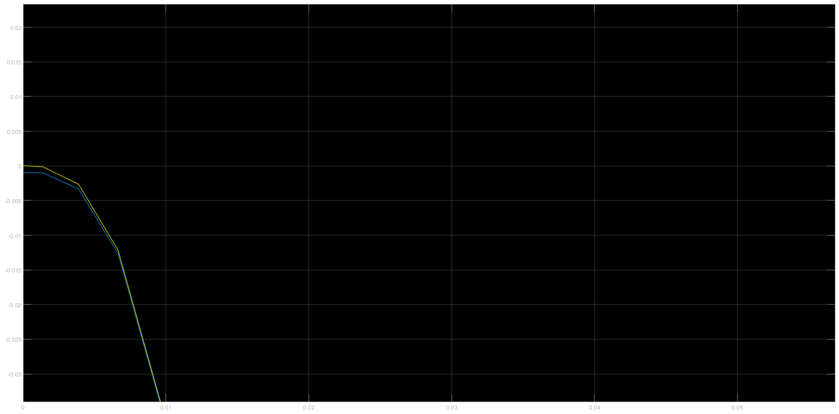

Figure 14: Position actual value vs. estimated value

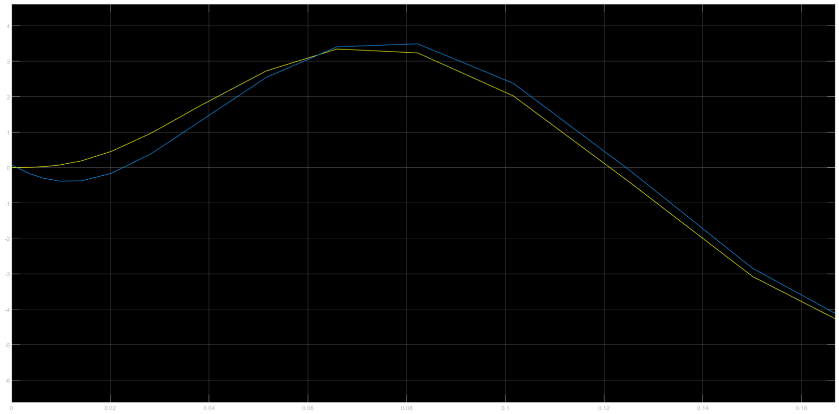

Figure 15: theta 1 actual value vs. estimated value

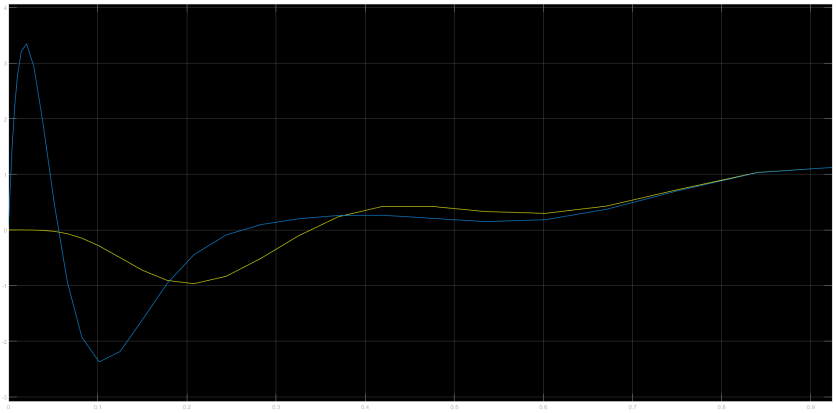

Figure 16: theta 2 actual value vs. estimated value

8 Observer-Based Linear Quadratic Regulator

we chose the second Values of the Q and R matrix to calculate the desired controller gains values as in section 6

```
% % LQR
 Ah = [A; -C];  Ah = [Ah, zeros(7,1)];
 Bh = [B; 0];

 Q2 = [1 0 0 0 0 0 0;
      0 1 0 0 0 0 0;
      0 0 1 0 0 0 0;
      0 0 0 1 0 0 0;
      0 0 0 0 1 0 0;
      0 0 0 0 0 1 0;
      0 0 0 0 0 0 1];
 R2 = 0.01;
 K = lqr(Ah,Bh,Q2,R2);
 Ki = -K(end);
 KO = K(1:6);
```

then we choose the poles of the observer to be 10 times away from the dominant poles;

```
Po = [-10*Re  -10.1*Re -10.2*Re -10.3*Re -10.4*Re -10.5*Re];
Ke = place(A',C',Po)';
```

We used the same Simulink model in section 7.1 by changing the values of gain to the new K and K$_e$.

8.1 The states initial conditions

for the initial conditions, we don't have the flexibility to change the values as the model is linearized about an exact point. So, we made the initial conditions:

$$x_1 = x = -0.001\, m$$
$$x_3 = \theta_1 = 0.069\, rad$$
$$x_5 = \theta_2 = 0.069\, rad$$

8.2 Results

The results of the observer yield a maximum distance of 2.1 meters.

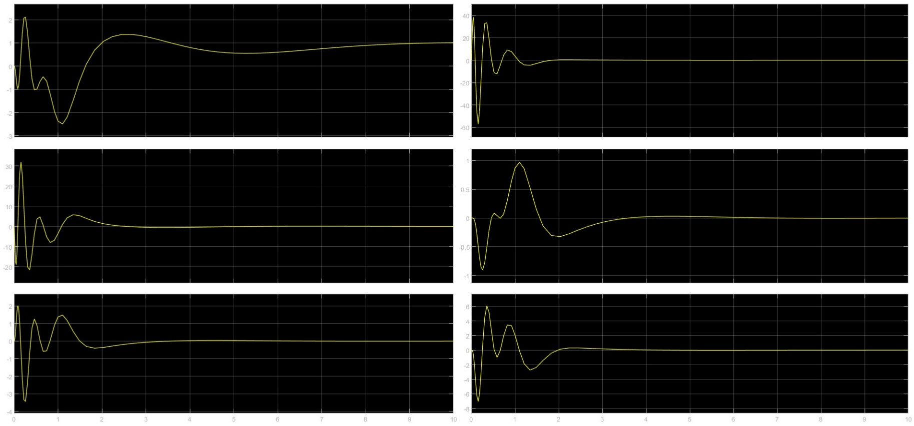

Figure 17: All States results of Simulink Observer-Based LQR

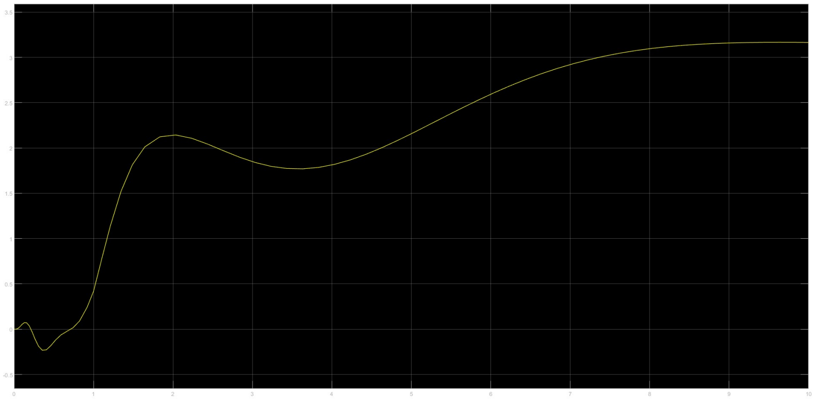

Figure 18: The state input results of Simulink Observer-Based LQR

8.3 Discussion

The results of the observer vary from the actual Linear Quadratic Regulator as the change of the initial conditions leads to a change of making the pendulum oscillate more with higher amplitude.
this shows the difference, in the beginning, leads to more chaotic result until it maintains the same value as the actual.

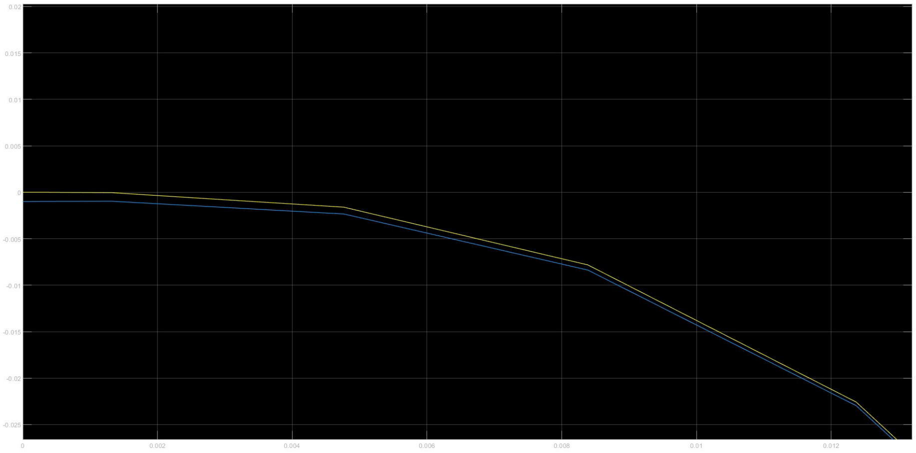

Figure 19: Position actual value vs. estimated value

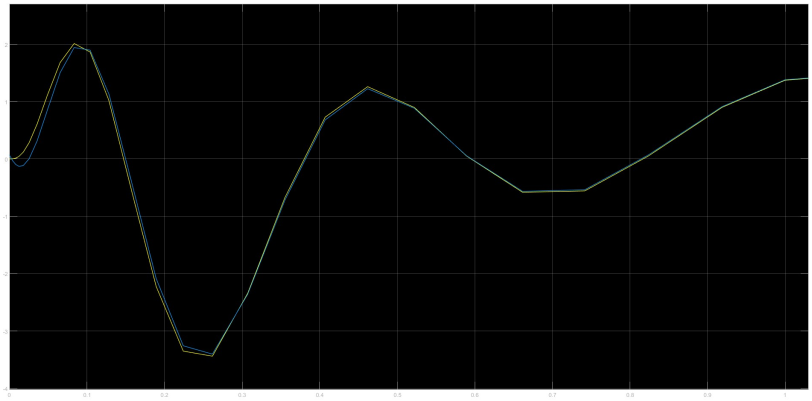

Figure 20: theta 1 actual value vs. estimated value

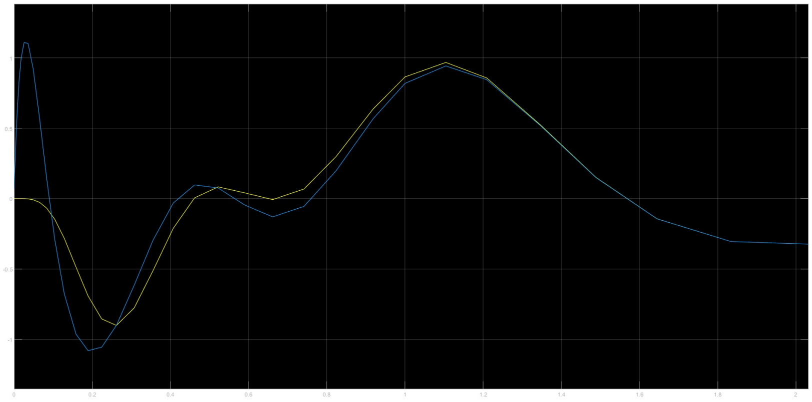

Figure 21: theta 2 actual value vs. estimated value

9 Animation of the Pendulum

we decided to make a MatLab function to animate the pendulum;

```
function call(u)
coder.extrinsic(’pause’)
if ~ishandle(1)
     figure(1);
end
clf
PlotS2A(u);
pause(0.02)
end

function h = circle2(x,y,r)
d = r*2;
px = x-r;
py = y-r;
h = rectangle(’Position’,[px py d d],’Curvature’,[1,1],’FaceColor’,[0.4 0.5 0.7]);
daspect([1,1,1])
end

function PlotS2A(u)
x = u(1);
y = u(2);
z=u(3);
rectangle(’Position’,[x-0.25 0 0.5 0.5],...
 ’FaceColor’,[0.4 0.5 0.7]);
a = 3*sin(y);
b= 3*cos(y);
line([x;a+x],[0.5;b+0.5],  ’LineWidth’,2,   ’Color’,[1 0.4 0.1]);
a2 = 3*sin(z);
b2 = 3*cos(z);
line([a+x;a2+x+a],[b+0.5;b2+0.5+b],  ’LineWidth’,2,   ’Color’,[1 0.4 0.1]);
circle2(a+x,b+0.5,0.2)
```

```
circle2(a2+x+a,b2+0.5+b,0.2)
axis([-5 5 -8 8]);
end
```

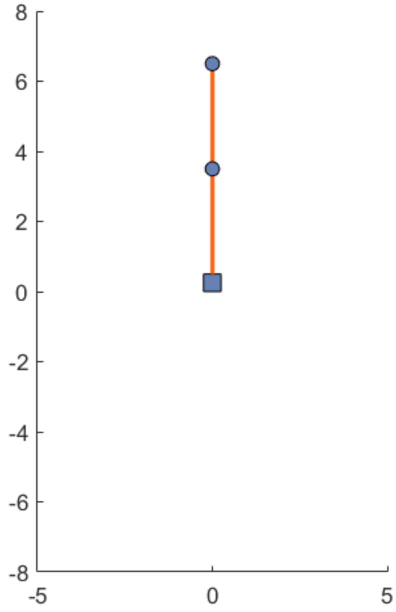

Figure 22: figure of the pendulum

10 Observations and conclusions

In this study, we navigated the realm of modern control theory to tackle the complexities of stabilizing a double inverted pendulum-cart system. Through rigorous mathematical modeling and simulation, we explored control strategies to manage this inherently unstable system.

Beginning with the derivation of the system's mathematical model using Lagrangian formulation, we obtained a set of coupled differential equations. MATLAB and Simulink facilitated the translation of these equations into a workable simulation model.

Our analysis encompassed the controllability and observability of the system, confirming the ability to fully control and estimate all states. Leveraging Full-State Feedback (FSF) and Linear Quadratic Regulator (LQR) control strategies, we achieved stabilization and desired performance.

The FSF approach allowed us to fine-tune eigenvalues for specific performance goals, while the LQR technique optimized control efforts and state deviations through careful weighting. Simulation results underscored the efficacy of both methods.

This study underscores the pertinence of control theory in comprehending intricate systems. By bridging theory with practical application, we harnessed the potential of FSF and LQR to overcome challenges posed by the double inverted pendulum-cart system's complexity. This journey highlights control theory's evolving role in crafting solutions for real-world instability and complexity, ushering in stability and efficiency.

11 References

1- Bogdanov, A. (2004). Optimal control of a double inverted pendulum on a cart. Oregon Health and Science University, Tech. Rep. CSE-04-006, OGI School of Science and Engineering, Beaverton, OR.

2- Ogata, K. (2010). Modern control engineering fifth edition.