Chapter 66

An introduction to probability

Questions answered in this chapter:

- What are the definitions of an experiment, a sample space, and an event?
- What are some axioms that event probabilities must satisfy?
- What is the Law of Complements?
- What are mutually exclusive events?
- What is the Additive Rule for Computing Probabilities?
- What are independent events?
- What is conditional probability?
- What is the Law of Total Probability?
- What is Bayes' theorem?

Just about the only thing we can be certain of is that we live in an uncertain world. To be an intelligent consumer of the barrage of statistics and probabilities that we see every day, you must have an understanding of basic probability.

Answers to this chapter's questions

What are the definitions of an experiment, a sample space, and an event?

An *experiment* is any procedure that can be repeated many times and has a well-defined set of outcomes. The set of all possible outcomes for an experiment is the *sample space*. Here are some examples of experiments and sample spaces:

You toss two fair dice. Each die is equally likely to show 1, 2, 3, 4, 5, or 6 dots. The sample space would consist of the following 36 points. The first number for each point is the number of dots showing on the first die, and the second number is the number of dots showing on the second die:

(1,1) (1,2) (1,3) (1,4) (1,5) (1,6) (2,1) (2,2) (2,3) (2,4) (2,5) (2,6) (3,1) (3,2) (3,3) (3,4) (3,5) (3,6)

(4,1) (4,2) (4,3) (4,4) (4,5) (4,6) (5,1) (5,2) (5,3) (5,4) (5,5) (5,6) (6,1) (6,2) (6,3) (6,4) (6,5) (6,6)

You toss two coins. The sample space consists of four equally likely points:

(H, H) (H, T) (T, H) (T, T)

An *event* is any subset of points in a sample space. The following are two examples of events:

- The total number of dots showing on two dice is eight.
- When two coins are tossed, a total of one head is observed.

What are some axioms that event probabilities must satisfy?

Event probabilities (we use *P* to denote probability) must satisfy the following axioms:

- **Axiom 1**: For any event *E* 0<=P(E)<=1.
- **Axiom 2**: If the event *E* consists of all the points in the sample space, then P(E) = 1.

If we assume that two fair dice are tossed, then each of the points in the sample space has the same probability *x*. Therefore, Axiom 2 implies 36x = 1 or x = 1/36.

Using this fact, you can determine the probability of throwing a total of 8 with two dice. The following five points in the sample space yield a total of 8: (2, 6), (3,5), (4,4), (5,3) and (6,2). Therefore, the chance of tossing a total of 8 with two dice is 5/36. This means that if you tossed two dice many times, on average you would expect 5/36 of the tosses to result in a total of 8.

What is the Law of Complements?

The Law of Complements is also known as the Complement Rule of Probability. For any event *A*, the event *Not A* consists of all points in the sample space not in *A*. The Law of Complements then states the following:

P(Not A) = 1 - P(A)

As an example of the Law of Complements, suppose you throw two dice. What is the chance that you do not throw a total of 2? Define *A* = Probability of throwing a total of 2. Since the only point in the sample space yielding a 2 is (1,1), then P(A) = 1/36. The event Not A = Total with two dice is not a 2. The Law of Complements then tells us the following:

P(Total not 2) = 1 – P(Total is 2) = 1 – 1/36 = 35/36

The importance of the Law of Complements is that once you compute the “easier” element of P(A) or P(Not A), then you know the other probability.

What are mutually exclusive events?

If two events cannot occur simultaneously, the events are *mutually exclusive*. Observe the following two examples:

- If event A = Total with two dice is 4 and event B = Total with two dice is 8, then the events A and B are mutually exclusive.

- If Event A = Dow Jones Index goes up by at least 10 percent in 2020 and Event B = Dow Jones Index goes up by at least 15 percent in 2020, then Events A and B are not mutually exclusive.

What is the Additive Rule for Computing Probabilities?

In general, for two events *A* and *B,* the Additive Rule for Computing Probabilities states the following:

P(A or B) = P(A) + P(B) – P(A and B)

You need to subtract P(A and B) or else you double-count the probability of points in the sample space that are common to events A and B. As a special case, if *A* and *B* are mutually exclusive, then P(A and B) = 0 and the Additive Rule reduces to the following:

P(A or B) = P(A) + P(B)

The following are two examples of the Additive Rule in action:

- If there is a 50 percent chance it rains on Saturday and there is a 50 percent chance it rains on Sunday, is there a 100 percent chance of rain during the weekend?

  If we let *A* = event it rains on Saturday and *B* = event it rains on Sunday, then we see the following:

  P(Rain during weekend) = P(Rain on Saturday) + P(Rain on Sunday) – P(Rain on both Saturday and Sunday = 0.5 + 0.5 – P(Rain on both Saturday and Sunday) = 1 – P(Rain on both Saturday and Sunday)

  Since *P*(Rain on both Saturday and Sunday) > 0, we find that the *P*(Rain during weekend) < 1, so it is not sure whether it rains on the weekend (except in Seattle).

As a second example of the Additive Rule, suppose you toss two dice. What is the chance that you get at least one 4? If we let *A* = event first die is 4 and *B* = event second die is 4, we have P(A) = 1/6, P(B) = 1/6, and P(A and B) = 1/36. Then you see the following:

P(1st die or 2nd die shows 4) = (1/6) + (1/6) – (1/36) = 11/36

What are independent events?

Two events, *A* and *B*, are *independent* if knowing that one event has occurred does not change your estimate of the probability of the other event occurring. Therefore, the two events *A* and *B* are independent if and only if P(A and B) = P(A)*P(B). For more than two events, the rigorous definition of independent events is beyond our scope, but if *n* events *A1*, *A2*, … *An* are independent, then the following is true:

P(A1 and A2, and ,… An) = P(A1)*P(A2),… *P(An-1)*P(An)

The following are two “intuitive” examples that should clarify the definition of independent events:

- Let *A* = Event that the Dow Jones Index increases in 2018 and *B* = Event that the Houston Texans win the 2018 Super Bowl. Knowing that one of these events happened should not change your estimate of the probability of the other event, so these events are independent.

- Let *A* = Event the Chicago Cubs (please!) win the 2017 World Series and *B* = Event the Houston Astros win the 2017 World Series. Clearly if *A* happens, then the chance of event *B* drops to 0, so events *A* and *B* are not independent. Clearly, events *A* and *B* cannot be both mutually exclusive and independent (see Problem 6).

- The following examples should cement your understanding of the concept of independent events:

- Suppose you toss a fair coin and throw a fair die. What is the probability that the coin comes up heads and you throw a 6 on the die? If we define *A* = event coin comes up heads and *B* = Event a 6 is thrown, then it is clear that *A* and *B* are independent. Since P(A) = 1/2 and P(B) = 1/6, then P(Coin comes up Heads and a 6 is thrown) = (1/2)*(1/6) = 1/12.

  Suppose a randomly chosen card is drawn from a deck of cards. You draw a single card. Are the events you draw a spade *and* draw an ace independent events? Let *A* = Event card is a spade and *B* = Event card is an ace. Since there are 13 spades and 4 aces in a deck of 52 cards, P(A) = 13/52 and P(B) = 4/52. Also, the only card in the deck that is an ace and a spade is the ace of spades, so P(Spade and Ace) = 1/52. Since P(Spade)*P(Ace) = (13/52)*(4/52) = 1/52, we find that events *A* and *B* are independent.

  Now suppose that before drawing the card, we remove the 2 of spades from the deck. Are the events *A* and *B* still independent? Now P(A) = 12/51, P(B) = 4/51, and P(A and B) = 1/51. Since (12/51)*(4/51) is not (1/51), events *A* and *B* are dependent.

- Finally, suppose you throw three dice. What is the chance you throw at least one 6? Let *Ai* = event die *i* does not show a 6. Then P(Ai) = 1 – (1/6) = 5/6. By the Law of Complements, we see the following:

P(At least 1 six) = 1 – P(0 sixes)

Since successive die rolls are independent, P(0 sixes) = P(A1)*P(A2)*P(A3) = $(5/6)^3$ = 125/216. Therefore, we see the following:

P(At least 1 six) = 1 – (125/216) = 91/216 = 0.42

What is conditional probability?

Often, we want to know how knowledge that one event has occurred changes the probability of another event. More formally, we define the *conditional probability* of event *B* occurring, given that event *A* has occurred as P(B|A). You read this as Probability Event *B* occurs, given that event *A* has occurred. P(B|A) may be computed by the following equation:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$ G66xx01

We often rewrite this equation as P(A and B) = P(A)*P(B|A) or P(A and B) = P(B)*P(A|B).

Events *A* and *B* are independent if and only if P(B|A) = P(B).

The following are two examples to help you understand conditional probability:

- Suppose you toss two dice. Define *A* = Event you get at least one 6 and *B* = Event total of the two dice is 10. What is P(B|A)? P(A and B) = 2/36 and P(A) = 1-P(No sixes) = 1-(5/6)^2 = 11/36. Therefore, you see the following:

P(B|A) = (2/36)/(11/36) = 2/11

- Suppose we draw an ace from a deck of cards. If we then draw a second card, what is the chance the card is an ace? If we let event *A* = Event first card is an ace and *B* = Event second card is an ace, after we take the first card (an ace) out of the deck, the deck contains 51 cards and 3 aces, so P(B|A) = 3/51 = 1/17.

What is the law of total probability?

The *law of total probability* refers to computing the probability of an event by adding together the probabilities of several mutually exclusive events. Sometimes we can compute the probability of an event by conditioning it on other events. For example, suppose 5 percent of used cars have flood damage and 80 percent of those cars later develop engine problems. Also suppose that 10 percent of cars that are not flood damaged later develop engine problems. What is the chance that a randomly chosen car will have engine problems?

To solve this problem, let *EP* = Event car has engine problems later, *FL* = Event car has flood damage. Then the event *EP* can be decomposed into the following two mutually exclusive events: A car with flood damage has problems and a car without flood damage has problems, as you see in this formulation:

P(EP) = P(EP and FL) + P(EP and no FL) = P(EP|FL)*P(FL) + P(EP|No FL)*P(No FL) = (0.80)*(.05) +(0.10)*(0.95) = 0.135

You can also create a *contingency table*, which shows all the possibilities. The following is an example of a contingency table:

	Engine problem	**No engine problem**
FL	0.04	0.01
No FL	0.095	0.855

The numbers in the No Engine Problem column are included to show how the first row should add to 0.05 and the second row should add to 0.95.

What is Bayes' theorem?

Once you understand conditional probability and the law of total probability**,** *Bayes theorem* is easy to understand. In many situations, we are trying to estimate the probabilities of various states of the world. Then we receive information that we use to change our probability estimates. For example, consider a 40-year-old woman with no risk factors for breast cancer. The states of the world can be defined by the events *C* = Event woman has cancer and *NC* = Event woman does not have cancer. Given no other information, the probabilities of these events (known as *prior* or *a priori* probabilities) are given by P(C) = 0.004 and P(NC) = 0.996.

Now we receive more information (results of a mammogram) that change our estimates of prior probabilities. Suppose a mammogram yields a positive (**+**) test result. To update our probability estimates, we need to know the likelihood of a positive test result for each state of the world. The likelihoods for a positive test result are known to be P(+|C) = 0.80 and P((+|NC) = 0.10. Now we want to update our prior probability (0.004) of cancer after receiving the positive test result. This new probability (P(C|+) is called a *posterior* or *a posteriori* probability. Applying the definitions of conditional probability and the law of total probability, we find the following:

$$P(C|+) = \frac{P(C\cap+)}{P(+)} = \frac{P(+|C)P(C)}{P(+|C)*P(C)+P(+|NC)*P(NC)} = \frac{.80*.004}{.80*.004+.996(.10)} = 0.031$$

G66xx02

Perhaps surprisingly, even after a positive test result, there is a small chance (thankfully!) that the woman has cancer. This is because of the fact that most women do not have breast cancer, so many of their mammograms will result in false positives. Another way to see this is to look at a typical sample of 10,000 women. Then a contingency table would show that the 10,000 women would be classified as follows:

	E	F	G
7		+ Test	- Test
8	Cancer	10,000*(.004)*(.8) = 32	10000*(.004)*(1-.8)= 8
9	No Cancer	10,000*(.996)*(.1) = 996	10,000*(.996)*(.8)=8964

G66xx03

Given a positive test result, we are working with the 1,028 women in column F. Therefore, after a positive test result, the chance the woman has cancer is 32/1028 = 0.031.

As a final example of Bayes' theorem, consider the classic Let's Make a Deal problem popularized by Marilyn Vos Savant in her "Ask Marilyn" column in *Parade Magazine.*

A car is behind one of three doors, and there is a goat behind the other two doors. I choose a door (let's say door 1). Now Monty Hall (the host of *Let's Make a Deal*) chooses to open a door (door 2 or door 3) and reveals a goat. You are now allowed to switch doors. Should you?

Let's assume Monty opens door 2. The following are the relevant events:

- The states of the world are *D1*, *D2*, *D3*, which are the events that the car is behind door *i*.

- Define S1, S2, S3 to be events that Monty says the car is behind door *i*.
- We know P(D1) = P(D2) = P(D3) = 1/3 are the prior probabilities.
- The likelihoods are P(S3|D1) = 1/2, P(S2|D1) = 1/2, P(S3|D2) = 1, and P(S2|D3)=1.

Now we know the car is behind either door 3 or door 1. By means of Bayes' theorem, we calculate the following:

$$P(D3|S2) = \frac{P(D3 \text{ and } S2)}{P(D3 \text{ and } S2) + P(D1 \text{ and } S2)} = \frac{\left(\frac{1}{3}\right)*1}{\left(\frac{1}{3}\right)*1+\left(\frac{1}{3}\right)*\frac{1}{2}} = \frac{2}{3}$$ G66xx04

Therefore, we find that P(D1|S2) = 1 - (2/3) = 1/3. Thus, we should switch our guess to door 3!

Note: The answers to this chapter's problems are in a Word file, not an Excel file.

Problems

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1. If you toss two dice, find the probability of each possible total.
2. If you throw three dice, what is the chance the total is 9?
3. Of Carver High School students, 20 percent play baseball and 15 percent play basketball. Five percent play both. What fraction of students play baseball or basketball?
4. Let *A* = Event a die comes up even and *B* = Event die shows one or two dots. Are events *A* and *B* independent?
5. Are the events *A* and *Not A* mutually exclusive?
6. Can mutually exclusive events be independent?
7. In the game of craps, you roll two dice. A total of 7 or 11 on the first roll wins. What is the chance you win a game on the first roll?
8. If you draw two cards (without replacement) from a deck of cards, what is the chance that both are clubs?
9. A roulette wheel contains the numbers 0, 00, 1, 2,…, 36. Suppose you bet on each of 25 spins that 00 will show. After 25 spins, what is chance you have won at least once?
10. Suppose that 10 percent of all adults watch *The Bachelor*. Assume that 80 percent of *The Bachelor*'s viewers are women and half of all adults are men. Compute the probability that a given woman or man is a viewer of *The Bachelor*.
11. What fraction of men and women watch *The Bachelor*?

12. In our flood-engine example, you are told that an engine had problems. What is the probability the engine had prior flood damage?

13. A chest has two drawers. Each drawer contains two coins; you know one drawer has two gold coins, and the other drawer has one gold and one silver coin. You randomly pick a drawer and a coin; the selected coin is gold. What is the chance that the other coin in the drawer is also gold?

14. Of all the cabs, 85 percent are blue and the rest are green. A cab identified in a hit-and-run accident is identified as green. People can correctly identify the color of a cab 80 percent of the time. When surveyed, most Stanford students thought there was an 80 percent chance the cab is actually green. Do you agree?

15. An urn contains nine balls. Each ball has one of the numbers 1, 2, …, 9 painted on the ball. Draw two balls (with replacement). What is the chance the two numbers are the same?

16. Toss a die and let *A* = Event you roll an odd number and *B* = Event you roll a number >=4. Find P(A|B).

17. Two engines each have a 0.9 chance of working. The success or failure of the engines is independent. If the engines are in a series, both engines need to work for the system to work. If the engines are in parallel, the system works if at least one engine works. What is the probability that a series or parallel system works?

18. The following table shows admission statistics for men and women at a large US university. The total number applying for admittance is in parentheses, and the number admitted is not in parentheses. You can see the percentage of men admitted is more than the percentage of women admitted. Using statistics, do you think the university discriminated against women?

	Men	**Women**
Easy major	864 (1386)	106 (133)
Difficult major	334 (1306)	451 (1702)

19. A company has a plant in Houston and in Dallas. Seventy percent of the employees work in Houston, and 30 percent work in Dallas. Each year, 3 percent of the Houston employees are involved in an accident; 5 percent of the Dallas employees are involved in an accident. If you randomly choose an employee who had an accident last year, what is the chance he or she works in Houston?

20. An MBA student is studying finance and marketing. Assume the student has a 90 percent chance of getting an A in finance and an 80 percent chance of getting an A in marketing. If her

performance in each of the two courses is independent of the other, what is the probability that she gets at least one A?

21. A bowl contains four red balls and six blue balls. Two balls are drawn (without replacement) from the bowl. Given that the second ball is blue, what is the chance the first ball is blue?

22. Suppose you toss two dice. Let *A* = Event first die shows a 3 and *B* = Event total of the two dice is 8. Are events *A* and *B* independent?

23. Of an insurance company's policyholders, 80 percent are high risk and 20 percent are low risk. Assume nobody has more than one accident in a year. Also assume 10 percent of high-risk people have an accident during a year and 3 percent of low-risk people have an accident during a year. If you randomly choose a policyholder who had an accident last year, what is the chance he or she is a high-risk policyholder?

24. In each year's NCAA basketball tournament, there are four teams with a 3 percent chance of beating a #1 seed. Before the tournament started, what is the chance that at least one #16seed team beats a #1-seed team?

25. Assume one in every 1,000 people is a liar (incapable of telling the truth). Also suppose a liedetector test is 98 percent accurate. That is, if a person is lying, there is a 98 percent chance the test will indicate that person is lying. Also, if the person is not lying, there is a 98 percent chance the test will indicate the person is not lying. If the lie-detector test indicates the person is lying, what is the chance the person is lying?

26. After throwing two dice, define event *A* = Total of the dice is even and *B* = First die shows a five. Are events *A* and *B* independent?

27. During 40 percent of all weeks, a supermarket cuts the price on macaroni and cheese. During 20 percent of all weeks, the supermarket puts macaroni and cheese on display, and during 15 percent of all weeks, the supermarket cuts the price of macaroni and cheese and puts macaroni and cheese on display. During what fraction of weeks is macaroni and cheese sold at a discount or on display?

28. You are told two cards have been drawn from a deck of cards, and both are hearts. What is the probability that the first card was the two of hearts?

29. In a drawer, there are two normal quarters and one quarter with two heads. With your eyes closed, you choose one of the coins; your friend flips the coin and reports the coin came up heads. What is the probability you selected the two-headed coin?

30. You toss two dice. Let event *A* = Total is 10 and *B* = Event first die shows an odd number. Are the events *A* and *B* independent?

31. Of an insurance company's policyholders, 20 percent are high risk, 40 percent are low risk, and 40 percent are intermediate risk. Of the low-risk policyholders, 2 percent have an accident

during a year, 4 percent of intermediate-risk policyholders have an accident during a year, and 20 percent of high-risk policyholders have an accident during a year. If a policyholder has an accident, what is the chance he or she was a high-risk policyholder?

