Chapter 69

The Poisson and exponential random variable

Questions answered in this chapter:

- What is the Poisson random variable?
- How do I compute probabilities for the Poisson random variable?
- If the number of customers arriving at a bank is governed by a Poisson random variable, what random variable governs the time between arrivals?

Answers to this chapter's questions

What is the Poisson random variable?

The *Poisson random variable* is a discrete random variable that is useful for describing probabilities for situations in which events (such as customer arrivals at a bank or orders placed for a product) have a small probability of occurring during a short time interval. More specifically, during a short time interval, denoted as t, either zero or one event will occur, and the probability of one event occurring during a short interval of length t is (for some λ) given by λt. Here, λ (*lambda*) is the mean number of occurrences per time unit.

Situations in which the Poisson random variable can be applied include the following:

- Number of units of a product demanded during a month.
- Number of deaths per year by horse kick in the Prussian army.
- Number of car accidents you have during a year.
- Number of copies of *The Seat of the Soul* ordered today at Amazon.com.
- Number of workers' compensation claims filed at a company this month.
- Number of defects in 100 yards of string. (Here, 1 yard of string plays the role of time.)

How do I compute probabilities for the Poisson random variable?

You can use the Microsoft Excel 2016 POISSON.DIST function to compute probabilities involving the

Poisson random variable. In versions of Excel prior to Excel 2013, the POISSON function yielded POISSON probabilities. Just remember that in a length of time *t*, the mean of a Poisson random variable is *t*. The syntax of the POISSON function is as follows:

- POISSON.DIST(x,*λ*,True or 1) calculates the probability that a Poisson random variable with a mean equal to *λ* is less than or equal to *x*.
- POISSON.DIST(x,*λ*,False or 0) calculates the probability that a Poisson random variable with a mean equal to *λ* is equal to *x*.

Here are some examples of how to compute probabilities for Poisson random variables. You can find these examples in the file Poisson.xlsx, shown in Figure 69-1.

	B	C	D
1	Calls per hour	30	
2	Mean	60	
3			
4	Prob 60 calls in two hours	0.05143	=POISSON.DIST(60,C2,FALSE)
5	Prob<= 60 calls in two hours	0.53426	=POISSON.DIST(60,C2,1)
6	Prob between 50 and 100 calls(inclusive) in two hour	0.91559	=POISSON.DIST(100,C2,TRUE)-POISSON.DIST(49,C2,TRUE)

F69xx01

FIGURE 69-1 Using the Poisson random variable.

F69xx01: This figure shows examples of computations involving the Poisson random variable.

Suppose that my consulting business receives an average of 30 phone calls per hour. During a two-hour period, I want to determine the following:

- The probability that exactly 60 calls will be received in the next two hours
- The probability that the number of calls received in the next two hours will be fewer than or equal to 60
- The probability that from 50 through 100 calls will be received in the next two hours

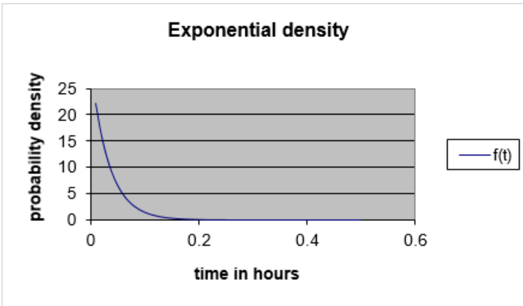
During a two-hour period, the mean number of calls is 60. In cell C4, I find the probability (0.05) that exactly 60 calls will be received in the next two hours, by using the formula =POISSON.DIST(60,C2,False). In cell C5, I find the probability (0.534) that at most 60 calls will be received in two hours by using the formula =POISSON.DIST(60,C2,1). (I use 1 as the alternate value for True.) In cell C6, I find the probability (0.916) that from 50 through 100 calls will be received in two hours by using the formula =POISSON.DIST(100,C2,True)–POISSON.DIST(49,C2,True).

Note You can always use 1 instead of True as an argument in any Excel function.

If the number of customers arriving at a bank is governed by a Poisson random variable, what random variable governs the time between arrivals?

The time between arrivals can be any value, which means that the time between arrivals is a continuous random variable. If an average of *λ* arrivals occur per time unit, the time between arrivals follows an exponential random variable having for $t>=0$ the probability density function (PDF) of $f(t)=\lambda e^{-\lambda t}$.

This random variable has a mean, or average, value equal to $1/\lambda$. For $\lambda=30$, a graph of the exponential PDF is shown in Figure 69-2. You can find this chart and the data for this example in the Density worksheet in the file Exponentialdist.xlsx.



F69xx02

FIGURE 69-2 An exponential probability density function.

F69xx02: This figure shows an example of the exponential density function.

Recall from Chapter 67, “An Introduction to random variables,” that for a continuous random variable, the height of the PDF for a number *x* reflects the likelihood that the random variable assumes a value near *x*. You can see in Figure 69-2 that extremely short times between customer arrivals at a bank (for example, less than 0.05 hours) are very likely, but that for longer times, the PDF drops off sharply.

Even though the average time between arrivals is 1/30=0.033 hours, there’s a reasonable chance that the time between arrivals will be as much as 0.20 hours. The formula EXPON.DIST(x,1/mean,True or 1) will give the probability that an exponential random variable with a given mean will assume a value less than or equal to *x*. Thus, the second argument to the EXPON.DIST function is the rate per time unit at which events occur. For example, in the Computation worksheet, to compute the probability that the time between arrivals is at least 5, 10, or 15 minutes, I copied from cell D5 to D7 the formula =1–EXPON.DIST(C5,D2,True). In earlier versions of Excel, the EXPONDIST function yields the same results as EXPON.DIST.

Note that I first converted minutes to hours (5 minutes equals 1/12 hour, and so on). Also, the mean time between arrivals is 0.033 hours, so I entered the formula **1/Mean=1/0.033=30**. In short, I entered the arrival rate per time unit, as you can see in Figure 69-3 and in the Computation worksheet.

	B	C	D	E
1		mean	0.033333333	
2		1/mean	30	
3				
4		x = Time between arrivals	Prob time >=x	
5	5 minutes	0.08333333	0.082084999	=1-EXPON.DIST(C5,D2,TRUE)
6	10 minutes	0.16666667	0.006737947	=1-EXPON.DIST(C6,D2,TRUE)
7	15 minutes	0.25	0.000553084	=1-EXPON.DIST(C7,D2,TRUE)

F69xx03

FIGURE 69-3 Computations of exponential probabilities.

F69xx03: This figure shows examples of computing exponential probabilities.

Problems

1. Beer drinkers order an average of 40 pitchers of beer per hour at Nick's Pub in Bloomington, Indiana. Answer the following questions:

   - What is the probability that at least 100 pitchers are ordered in a two-hour period?
   - What is the chance that the time between ordered pitchers will be 30 seconds or less?

2. Suppose that teenage drivers have an average of 0.3 accidents per year. Answer the following questions:

   - What is the probability that a teenager will have no more than one accident during a year?
   - What is the probability that the time between accidents will be six months or less?

3. I am next in line at a fast-food restaurant in which customers wait in a single line, and the time to serve a customer follows an exponential distribution with a mean of three minutes. What is the chance that I will have to wait at least five minutes to be served?

4. Since 1900, a fraction (0.00124) of all major league baseball games have resulted in no-hitters. A team plays 162 games in a season. What is the chance that a team pitches two or more no-hitters during a season?

5. An average of 80 customers arrive each hour at the Central Forest Coffee Shop. What is the probability that at least 150 customers arrive in a two-hour period?

6. What hourly arrival rate would ensure that the chance that at least 150 customers arrive at Central Forest in three hours would equal 0.5?

