Real POVMs on the plane : integral quantization, Naimark theorem and linear polarisation of the light

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Based on
R. Beneduci, E. Frion, J.-P. G., A. Perri,
Real POVMS on the plane : integral quantization, Naimark theorem and linear polarization of the light, (submitted, 2021)
arXiv :2108.04086v1 [quant-ph]

- Many works in quantum information, quantum measurement, quantum foundations, ..., are illustrated with manipulations of the two real Pauli matrices and their tensor products

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

- No complex numbers, i.e. no $\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, just the Euclidean plane and its Cartesian or tensor products
- Let us explore the elementary model of 2d real Hilbert space with its mathematical and interpretative resources
- Let us demonstrate that it is not only a toy model...[1]

1. See for instance Orientations in the Plane as Quantum States, H. Bergeron, E. Curado, J.-P. G., L. Rodrigues Brazilian Journal of Physics **49** 391-401 (2019)

Outline1

- PV, POVM, and related theorems
- Integral Quantization
- Euclidean plane as Hilbert space of quantum states
- Integral quantization with 2×2 real density matrices
- Interaction polarizer-partially linear polarized light as a quantum measurement
- Summary of our other results
- Entanglement and isomorphisms

- A normalized Positive-Operator Valued measure (POVM) is a map $F : \mathcal{B}(\Omega) \to \mathcal{L}_s^+(\mathcal{H})$ from the Borel σ-algebra of a topological space Ω to the space of linear positive self-adjoint operators such that :

$$F\left(\bigcup_{n=1}^{\infty} \Delta_n\right) = \sum_{n=1}^{\infty} F(\Delta_n) \,, \tag{2}$$

$$F(\Omega) = \mathbb{1} \,, \tag{3}$$

  where $\{\Delta_n\}$ is a countable family of disjoint sets in $\mathcal{B}(\Omega)$ and the series converges in the weak operator topology.
- The POVM is said to be real if $\Omega = \mathbb{R}$.
- A projection-valued measure (PVM) is a POVM such that $F(\Delta)$ is a projection operator for every $\Delta \in \mathcal{B}(\Omega)$.

- Quantum framework : a complex and separable Hilbert space $\mathcal{H}$ is associated with each system, and the states are represented by density operators, *i.e.* non-negative, bounded self-adjoint operators with trace 1.
- (Holevo) There is a one-to-one correspondence between POVMs $F : \mathcal{B}(\Omega) \to \mathcal{L}_s^+(\mathcal{H})$ and affine maps $S(\mathcal{H}) \mapsto \mathcal{M}_+(\Omega)$ from states to probability measures which is given by

$$\mu(\Delta) = Tr(\rho F(\Delta))\,,$$

$$\Delta \in \mathcal{B}(\Omega)\,, \quad \rho \in S(\mathcal{H})\,, \quad \mu \in \mathcal{M}_+(\Omega)$$

- More in P. Busch et al. Quantum measurement. Vol. 23. Springer, 2016

- (Naimark, 1943) Let $F : \mathcal{B}(\Omega) \to \mathcal{L}_s^+(\mathcal{H})$ be a POVM in a Hilbert space $\mathcal{H}$. Then, there is an extended Hilbert space $\mathcal{H}^+$ and a projector-valued (PV) measure $E^+ : \mathcal{B}(\Omega) \to \mathcal{L}_s(\mathcal{H}^+)$ such that

$$PE^+(\Delta)\psi = F(\Delta)\psi, \quad \psi \in \mathcal{H}, \quad \Delta \in \mathcal{B}(\Omega)\,,$$

  where P is the projection operator onto $\mathcal{H}$.

- Naimark's theorem provides a necessary and sufficient condition for the compatibility of two POVMs

- **Definition** : two POVMs $F_1 : \mathcal{B}(\Omega_1) \to \mathcal{L}_s^+(\mathcal{H})$ and $F_2 : \mathcal{B}(\Omega_2) \to \mathcal{L}_s^+(\mathcal{H})$ are compatible if there is a third POVM $F : \mathcal{B}(\Omega_1 \times \Omega_2) \to \mathcal{L}_s^+(\mathcal{H})$ of which F_1 and F_2 are the marginals, *i.e.*,

$$F_1(\Delta_1) = F(\Delta_1 \times \Omega_2), \quad F_2(\Delta_2) = F(\Omega_1 \times \Delta_2).$$

- **Theorem** Two POVMs $F_1 : \mathcal{B}(\Omega_1) \to \mathcal{L}_s^+(\mathcal{H})$ and $F_2 : \mathcal{B}(\Omega_2) \to \mathcal{L}_s^+(\mathcal{H})$ are *compatible* if and only if there are two Naimark extensions $E_1^+ : \mathcal{B}(\Omega_1) \to \mathcal{L}_s(\mathcal{H}^+)$ and $E_2^+ : \mathcal{B}(\Omega_2) \to \mathcal{L}_s(\mathcal{H}^+)$ such that $[E_1^+, E_2^+] = 0$.

- More in R. Beneduci. "Joint measurability through Naimark's dilation theorem", Reports on Mathematical Physics 79.2 (2017)

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- Integral quantization [2] of $f(x)$ on a measure space (X, ν) is the linear map :

$$f \mapsto A_f = \int_X \mathsf{M}(x)\, f(x)\, \mathrm{d}\nu(x)\,, \tag{4}$$

where the family of operators $\mathsf{M}(x)$ solves the identity as

$$X \ni x \mapsto \mathsf{M}(x)\,, \quad \int_X \mathsf{M}(x)\, \mathrm{d}\nu(x) = \mathbb{1}\,. \tag{5}$$

- If the $\mathsf{M}(x)$ are non-negative, the quantum operator related to the characteristic function on Δ, $A(\chi_\Delta)$, defines a POVM through the quantization map

$$F(\Delta) := A(\chi_\Delta) = \int_X \mathsf{M}(x)\, \chi_\Delta(x)\, \mathrm{d}\nu(x) = \int_\Delta \mathsf{M}(x)\, \mathrm{d}\nu(x)\,, \tag{6}$$

- Therefore, two key roles of POVMs : they are the mathematical representatives of observables and they provide a quantization procedure.

2. J.-P. G. & H. Bergeron, Integral quantizations with two basic examples, Annals of Physics 344 (2014)

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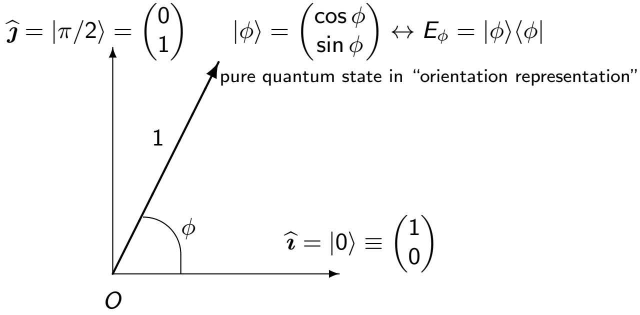
Euclidean plane as Hilbert space of quantum states

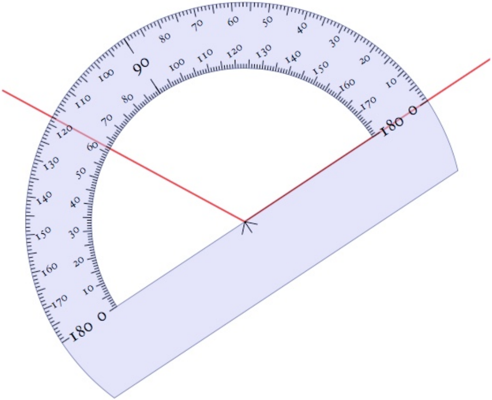


A pure state in the horizontal-vertical representation can be decomposed as

$$|\phi\rangle = \cos\phi\,|0\rangle + \sin\phi\,\left|\frac{\pi}{2}\right\rangle\,, \quad \langle 0|\phi\rangle = \cos\phi\,, \qquad \left\langle \frac{\pi}{2}\right|\phi\rangle = \sin\phi\,, \qquad (7)$$

To $|\phi\rangle$ corresponds the orthogonal projector $E_\phi = \begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{pmatrix}$

Pure states in the Euclidean plane as a protractor



The identification with $\mathbb{S}^1/\{-1,1\}$ is clearly apparent on the picture, with $0^\circ \equiv -180^\circ$, $10^\circ \equiv -170^\circ$, etc, sign minus being attributed to the internal graduation.

- Density matrices are non-negative unit trace matrices with spectral decomposition

$$\rho = \left(\frac{1+r}{2}\right) E_{\phi} + \left(\frac{1-r}{2}\right) E_{\phi+\pi/2}\,, \quad 0 \leq r \leq 1 \tag{8}$$

- With polar coordinates (r, ϕ) for the upper half unit disk,

$$\begin{aligned}\rho \equiv \rho_{r,\phi} &= \frac{1}{2}\mathbb{1} + \frac{r}{2}\mathcal{R}(\phi)\sigma_3\mathcal{R}(-\phi)\\ &= \begin{pmatrix} \frac{1}{2} + \frac{r}{2}\cos 2\phi & \frac{r}{2}\sin 2\phi \\ \frac{r}{2}\sin 2\phi & \frac{1}{2} - \frac{r}{2}\cos 2\phi \end{pmatrix} = \frac{1}{2}\left(\mathbb{1} + r\sigma_{2\phi}\right)\,.\end{aligned} \tag{9}$$

where $\mathcal{R}(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$

- and where

$$\sigma_{\phi} := \cos\phi\,\sigma_3 + \sin\phi\,\sigma_1 \equiv \overrightarrow{\boldsymbol{\sigma}} \cdot \widehat{\mathbf{u}}_{\phi} = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} = \mathcal{R}(\phi)\,\sigma_3\,, \tag{10}$$

is a typical observable (e.g. Bell, Peres, Mermin...)

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- The measure space $(X, \mathrm{d}\nu(x))$ for the Euclidean plane is the unit circle with its uniform (Lebesgue) measure :

$$X = \mathbb{S}^1\,, \quad \mathrm{d}\nu(x) = \frac{\mathrm{d}\phi}{\pi}\,, \quad \phi \in [0, 2\pi)\,. \tag{11}$$

- Resolution of the identity by the density matrices $\rho_{r,\phi+\phi_0}$, ϕ_0 arbitrary,

$$\int_0^{2\pi} \rho_{r,\phi+\phi_0}\, \frac{\mathrm{d}\phi}{\pi} = \mathbb{1}\,. \tag{12}$$

- Quantization of a function (or distribution) $f(\phi)$ on the circle.

$$\begin{aligned} f \mapsto A_f &= \int_0^{2\pi} f(\phi)\rho_{r,\phi+\phi_0}\, \frac{\mathrm{d}\phi}{\pi} = \begin{pmatrix} \langle f \rangle + \frac{r}{2} C_c\left(R_{\phi_0} f\right) & \frac{r}{2} C_s\left(R_{\phi_0} f\right) \\ \frac{r}{2} C_s\left(R_{\phi_0} f\right) & \langle f \rangle - \frac{r}{2} C_c\left(R_{\phi_0} f\right) \end{pmatrix} \\ &= \langle f \rangle\, \mathbb{1} + \frac{r}{2}\left[C_c\left(R_{\phi_0} f\right) \sigma_3 + C_s\left(R_{\phi_0} f\right) \sigma_1\right]\,, \end{aligned} \tag{13}$$

where $\langle f \rangle := \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\, \mathrm{d}\phi$ is the average of f on the unit circle and $R_{\phi_0}(f)(\phi) := f(\phi - \phi_0)$.

- C_c and C_s : cosine and sine doubled angle Fourier coefficients of f,

$$C_c(f) = \int_0^{2\pi} f(\phi) \cos 2\phi\, \frac{\mathrm{d}\phi}{\pi}\,, \quad C_s(f) = \int_0^{2\pi} f(\phi) \sin 2\phi\, \frac{\mathrm{d}\phi}{\pi}\,. \tag{14}$$

- As a consequence, representation of a given mixed state as a continuous superposition of mixed states :

$$\rho_{s,\theta} = \int_0^{2\pi} \underbrace{\left[\frac{1}{2} + \frac{s}{r}\, \cos 2\phi\right]}_{f(\phi)} \rho_{r,\phi+\theta}\, \frac{\mathrm{d}\phi}{\pi}\,, \tag{15}$$

- It is convex for $r \geq 2s$ and $\phi \mapsto \left[\frac{1}{2} + \frac{s}{r}\, \cos 2\phi\right]$ is a probability distribution.
- It provides one more illustration of a *typical property of quantum-mechanical ensembles in comparison with their classical counterparts.*

- Let us identify $\mathbb{R}^3$ with the subspace in $L^2(\mathbb{S}^1, \mathrm{d}\phi/\pi)$,

$$\mathbb{R}^3 \sim V_3 = \mathrm{Span}\left\{ e_0(\phi) := \frac{1}{\sqrt{2}}, e_1(\phi) := \cos 2\phi, e_2(\phi) := \sin 2\phi \right\} \quad (16)$$

- Then, the integral quantization map with $\rho_{r,\phi+\phi_0}$ yields a non-commutative version of $\mathbb{R}^3$

$$A_{e_0} = \frac{\mathbb{1}}{\sqrt{2}}\,, \quad (17)$$

$$A_{e_1} = \frac{r}{2}[\cos 2\phi_0\, \sigma_3 + \sin 2\phi_0\, \sigma_1] \equiv \frac{r}{2}\sigma_{2\phi_0}\,, \quad (18)$$

$$A_{e_2} = \frac{r}{2}[-\sin 2\phi_0\, \sigma_3 + \cos 2\phi_0\, \sigma_1] \equiv \frac{r}{2}\sigma_{2\phi_0+\pi/2}\,, \quad (19)$$

- Non-zero commutation rule

$$[A_{e_1}, A_{e_2}] = -\frac{r^2}{2}\tau_2\,, \quad \tau_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\mathrm{i}\sigma_2\,. \quad (20)$$

- Remind (Naimark, 1943) Let $F : \mathcal{B}(\Omega) \to \mathcal{L}_s^+(\mathcal{H})$ be a POVM in a Hilbert space $\mathcal{H}$. Then, there is an extended Hilbert space $\mathcal{H}^+$ and a projector-valued (PV) measure $E^+ : \mathcal{B}(\Omega) \to \mathcal{L}_s(\mathcal{H}^+)$ such that

$$PE^+(\Delta)\psi = F(\Delta)\psi, \quad \psi \in \mathcal{H}, \quad \Delta \in \mathcal{B}(\Omega)\,,$$

  where P is the projection operator onto $\mathcal{H}$.

- The quantization of the circle offers a nice illustration of this theorem in combination with Toeplitz quantization.

- One can show that there exist orthogonal projectors from $L^2(\mathbb{S}^1, \mathrm{d}\phi/\pi)$ to $\mathbb{R}^2$ such that for a function $f(\phi)$ the multiplication operator on $L^2(\mathbb{S}^1, \mathrm{d}\phi/\pi)$ defined by

$$v \mapsto M_f v = fv\,. \tag{21}$$

  map M_f to A_f.

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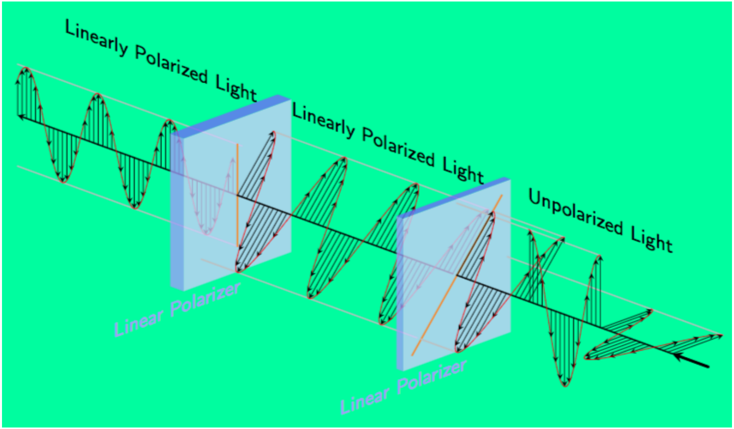
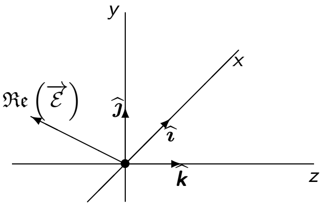


If we send polarized linearly light ($|\phi\rangle$) into 2d polaroid, at an angle $\phi - \eta$ to the passing direction (here $\eta = \pi/2$), the amplitude which comes out of the polaroid is only the $\cos(\phi - \eta)$ part, the $\sin(\phi - \eta)$ being absorbed. The energy or intensity of light which passes through the polaroid is proportional to $\cos^2(\phi - \eta)$ (Malus's law)

- Electric field for a propagating quasi-monochromatic e.m. wave along the z-axis : $\overrightarrow{\mathcal{E}}(t) = \overrightarrow{\mathcal{E}_0}(t)\, e^{i\omega t} = \mathcal{E}_x \widehat{\imath} + \mathcal{E}_y \widehat{\jmath} = (\mathcal{E}_\alpha)$,
- $\overrightarrow{\mathcal{E}_0}(t)$ determines the polarization, has slow temporal variation and is measured through Nicol prisms, or other devices, by measuring the intensity of the light yielded by mean values $\propto \mathcal{E}_\alpha \mathcal{E}_\beta$, $\mathcal{E}_\alpha \mathcal{E}^*_\beta$ and conjugates
- Due to rapidly oscillating factors and so null temporal average $\langle\cdot\rangle_t$, partially polarized light is described by the 2×2 Hermitian matrix (Stokes parameters)

$$\frac{1}{J}\begin{pmatrix} \left\langle \mathcal{E}_{0x}\mathcal{E}^*_{0x}\right\rangle_t & \left\langle \mathcal{E}_{0x}\mathcal{E}^*_{0y}\right\rangle_t \\ \left\langle \mathcal{E}_{0y}\mathcal{E}^*_{0x}\right\rangle_t & \left\langle \mathcal{E}_{0y}\mathcal{E}^*_{0y}\right\rangle_t \end{pmatrix} \equiv \rho_{r,\phi} + \frac{A}{2}\sigma_2 = \frac{1+r}{2}E_\phi + \frac{1-r}{2}E_{\phi+\pi/2} + i\frac{A}{2}\tau_2$$

J : wave intensity ; $0\leq r\leq 1$: linear polarization ; $-1\leq A\leq 1$: circular polarization (put $=0$ here)



- Two planes and their tensor product : the first one is the Hilbert space on which act the states $\rho^M_{s,\theta}$ of the polarizer viewed as an orientation *pointer*.
- Note that the action of the generator of rotations $\tau_2 = -\mathrm{i}\sigma_2$ on these states corresponds to a $\pi/2$ rotation :

$$\tau_2 \rho^M_{s,\theta} \tau_2^{-1} = -\tau_2 \rho^M_{s,\theta} \tau_2 = \rho^M_{s,\theta+\pi/2} \,. \tag{22}$$

- The second plane is the Hilbert space on which act the partially linearized polarization states $\rho^L_{r,\phi}$ of the plane wave crossing the polarizer.
- Its spectral decomposition corresponds to the incoherent superposition of two completely linearly polarized waves

$$\rho^L_{r,\phi} = \frac{1+r}{2}\, E_\phi + \frac{1-r}{2}\, E_{\phi+\pi/2} \,. \tag{23}$$

- The pointer is designed to detect the orientation in the plane determined by the angle ϕ.
- The interaction pointer-system generating a measurement whose time duration is the interval $I_M = (t_M - \eta, t_M + \eta)$ centred at t_M is described by the (pseudo-) Hamiltonian operator

$$\widetilde{H}_{\mathrm{int}}(t) = g^\eta_M(t) \tau_2 \otimes \rho^L_{r,\phi} \,, \tag{24}$$

  where g^η_M is a Dirac sequence with support in I_M, *i.e.*,

$$\lim_{\eta\to 0} \int_{-\infty}^{+\infty} \mathrm{d}t\, f(t)\, g^\eta_M(t) = f(t_M) \,.$$

- $\widetilde{H}_{\mathrm{int}}(t) = g^{\eta}_{M}(t)\tau_2 \otimes \rho^{L}_{r,\phi}$ is the tensor product of an antisymmetric operator for the pointer with an operator for the system which is symmetric (*i.e.* Hamiltonian)
- The operator defined for $t_0 < t_M - \eta$ as

$$U(t,t_0) = \exp\left[\int_{t_0}^{t} \mathrm{d}t'\, g^{\eta}_{M}(t')\,\tau_2 \otimes \rho^{L}_{r,\phi}\right] = \exp\left[G^{\eta}_{M}(t)\,\tau_2 \otimes \rho^{L}_{r,\phi}\right]\,, \qquad (25)$$

  with $G^{\eta}_{M}(t) = \int_{t_0}^{t} \mathrm{d}t'\; g^{\eta}_{M}(t')$, is a **unitary evolution operator**.
- From the formula involving an orthogonal projector P,

$$\exp(\theta\tau_2 \otimes P) = \mathcal{R}(\theta) \otimes P + \mathbb{1} \otimes (\mathbb{1} - P)\,, \qquad (26)$$

  we obtain

$$U(t,t_0) = \mathcal{R}\left(G^{\eta}_{M}(t)\,\frac{1+r}{2}\right) \otimes E_{\phi} + \mathcal{R}\left(G^{\eta}_{M}(t)\,\frac{1-r}{2}\right) \otimes E_{\phi+\pi/2}\,. \qquad (27)$$

  For $t_0 < t_M - \eta$ and $t > t_M + \eta$, we finally obtain

$$U(t,t_0) = \mathcal{R}\left(\frac{1+r}{2}\right) \otimes E_{\phi} + \mathcal{R}\left(\frac{1-r}{2}\right) \otimes E_{\phi+\pi/2}\,. \qquad (28)$$

- After having prepared the polarizer in the state $\rho^M_{s_0,\theta_0}$, the evolution $U(t,t_0)\,\rho^M_{s_0,\theta_0}\otimes\rho^L_{r_0,\phi_0}\,U(t,t_0)^\dagger$ of the initial state reads for $t > t_M + \eta$

$$\rho^M_{s_0,\theta_0+\frac{1+r}{2}}\otimes\frac{1+r_0\cos 2(\phi-\phi_0)}{2}\,E_\phi+\rho^M_{s_0,\theta_0+\frac{1-r}{2}}\otimes\frac{1-r_0\cos 2(\phi-\phi_0)}{2}\,E_{\phi+\pi/2}$$
$$+\frac{1}{4}\left(\mathcal{R}(r)+s_0\sigma_{2\theta_0+1}\right)\otimes r_0\sin 2(\phi-\phi_0)\,E_\phi\tau_2$$
$$-\frac{1}{4}\left(\mathcal{R}(-r)+s_0\sigma_{2\theta_0+1}\right)\otimes r_0\sin 2(\phi-\phi_0)\,\tau_2E_\phi\,.$$

- As expected from the standard theory of quantum measurement, this formula indicates that the probability for the pointer to rotate by $\frac{1+r}{2}$, corresponding to the polarization along the orientation ϕ, is

$$\mathrm{Tr}\left[\left(U(t,t_0)\,\rho^M_{s_0,\theta_0}\otimes\rho^L_{r_0,\phi_0}\,U(t,t_0)^\dagger\right)\left(\mathbb{1}\otimes E_\phi\right)\right]=\frac{1+r_0\cos 2(\phi-\phi_0)}{2}\,,\quad(29)$$

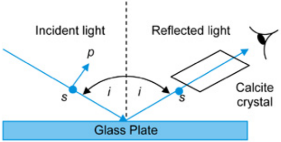
whereas it is

$$\mathrm{Tr}\left[\left(U(t,t_0)\,\rho^M_{s_0,\theta_0}\otimes\rho^L_{r_0,\phi_0}\,U(t,t_0)^\dagger\right)\left(\mathbb{1}\otimes E_{\phi+\pi/2}\right)\right]=\frac{1-r_0\cos 2(\phi-\phi_0)}{2}\,,\quad(30)$$

for the perpendicular orientation $\phi+\pi/2$ and the pointer rotation by $\frac{1-r}{2}$. For the completely linear polarization of the light, i.e. $r_0=1$, we recover the familiar *Malus laws*, $\cos^2(\phi-\phi_0)$ and $\sin^2(\phi-\phi_0)$ respectively.

Malus's law from https ://spie.org/publications/fg05_p03_maluss_law ?SSO=1

At the beginning of the nineteenth century the only known way to generate polarized light was with a calcite crystal. In 1808, using a calcite crystal, Malus discovered that natural incident light became polarized when it was reflected by a glass surface, and that the light reflected close to an angle of incidence of 57° could be extinguished when viewed through the crystal. He then proposed that natural light consisted of the s- and p-polarizations, which were perpendicular to each other.



Since the intensity of the reflected light varied from a maximum to a minimum as the crystal was rotated, Malus proposed that the amplitude of the reflected beam must be $A = A_0 \cos\theta$. However, in order to obtain the intensity, Malus squared the amplitude relation so that the intensity equation $I(\theta)$ of the reflected polarized light was

$$I(\theta) = I_0^2 \cos^2\theta \quad \text{(Malus's Law)} \quad I_0 = A_0^2$$

1808 : Quantum formalism was already at work !

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Summary of our other results in arXiv :2108.04086 quant-ph :

- A dichotomic POVM F is a pair $F = \{A, \mathbb{1} - A\}$ with A an effect, *i.e.* a symmetric operator A such that $0 \leq A \leq \mathbb{1}$, but not a projection operator (it is a multiple of a projection operator).
- We show that two measurements in which a light ray goes first through an oblique polarizer before passing through a vertical polarizer is described by a dichotomic POVM, while a measurement in the reverse order is described by another dichotomic POVM, showing the incompatibility of the measurement procedures.
- We also find the necessary condition for the compatibility of two dichotomic POVMs in a real bidimensional Hilbert space.
- Finally we relate the density matrix's parameters to the Stokes parameters defining a polarization tensor for linearly polarized light. It turns out that we can identify the degree of mixing of a density matrix with the fuzziness of a quantum observable.
- In conclusion compatibility conditions of two POVMs can be expressed in terms of Stokes parameters of the polarization matrix.

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- Quantum entanglement of states is a logical consequence of the construction of tensor products of Hilbert spaces for describing quantum states of composite systems.
- In the present case, we are in presence of a remarquable sequence of vector space isomorphisms due to the fact that $2 \times 2 = 2 + 2$ [3] :
$$\mathbb{R}^2 \otimes \mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{R}^2 \oplus \mathbb{R}^2 \cong \mathbb{C}^2 \cong \mathbb{H}\,,$$
where $\mathbb{H}$ is the field of quaternions.
- It is straightforward to transpose into the present setting the 1964 analysis and result presented by Bell in his discussion about the EPR paper and about the subsequent Bohm's approaches based on the assumption of hidden variables.
- Just replace the Bell spin one-half particles with the horizontal (i.e., $+1$) and vertical (i.e., -1) quantum orientations in the plane as the only possible issues of the observable $\sigma_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$ supposing that there exists a pointer device designed for measuring such orientations with outcomes ± 1 only.

3. Remind that $\dim(V \otimes W) = \dim V \dim W$ while $\dim(V \times W) = \dim V + \dim W$ for 2 finite-dimensional vector spaces V and W

- Let us first write the (canonical) orthonormal basis of the tensor product $\mathbb{R}^2_A \otimes \mathbb{R}^2_B$, the first factor being for system "A" and the other for system "B", as

$$|0\rangle_A \otimes |0\rangle_B \,, \quad \left|\frac{\pi}{2}\right\rangle_A \otimes \left|\frac{\pi}{2}\right\rangle_B \,, \quad |0\rangle_A \otimes \left|\frac{\pi}{2}\right\rangle_B \,, \quad \left|\frac{\pi}{2}\right\rangle_A \otimes |0\rangle_B \,.$$

- Qubits $|0\rangle$, $\left|\frac{\pi}{2}\right\rangle$ pertaining to A or to B, can be associated to a pointer measuring the horizontal (resp. vertical) direction or polarisation described by the state $|0\rangle$ (resp. $\left|\frac{\pi}{2}\right\rangle$).
- Celebrated Bell pure states in $\mathbb{R}^2_A \otimes \mathbb{R}^2_B$: orthonormal basis of $\mathbb{R}^2_A \otimes \mathbb{R}^2_B$:

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A \otimes |0\rangle_B \pm \left|\frac{\pi}{2}\right\rangle_A \otimes \left|\frac{\pi}{2}\right\rangle_B \right) \,,$$
$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A \otimes \left|\frac{\pi}{2}\right\rangle_B \pm \left|\frac{\pi}{2}\right\rangle_A \otimes |0\rangle_B \right) \,,$$

- They are four specific *maximally entangled* quantum states of two qubits.

- As a complex vector space, $\mathbb{C}^2$, with canonical basis $\mathbf{e}_1$, $\mathbf{e}_2$, has a real structure, i.e. is isomorphic to a real vector space which makes it isomorphic to $\mathbb{R}^4$, itself isomorphic to $\mathbb{R}^2 \otimes \mathbb{R}^2$,
- A **real** structure is obtained by considering the vector expansion

$$\mathbb{C}^2 \in \mathbf{v} = z_1\mathbf{e}_1 + z_2\mathbf{e}_2 = x_1\mathbf{e}_1 + y_1\left(\mathrm{i}\mathbf{e}_1\right) + x_2\mathbf{e}_2 + y_2\left(\mathrm{i}\mathbf{e}_2\right) \, ,$$

- i.e., by writing $z_1 = x_1 + \mathrm{i}y_1$, $z_2 = x_2 + \mathrm{i}y_2$, and considering the set of vectors

$$\{\mathbf{e}_1, \mathbf{e}_2, \left(\mathrm{i}\mathbf{e}_1\right), \left(\mathrm{i}\mathbf{e}_2\right)\}$$

as forming a basis of $\mathbb{R}^4$.

- Forgetting about the superfluous subscripts A and B, the map Euclidean plane $\mathbb{R}^2 \mapsto$ complex "plane" $\mathbb{C}$ is determined by

$$|0\rangle \mapsto 1\,, \qquad \left|\frac{\pi}{2}\right\rangle \mapsto \mathrm{i}\,.$$

- "Consistently", we write the correspondence between bases as

$$|0\rangle \otimes |0\rangle = \mathbf{e}_1\,, \quad \left|\frac{\pi}{2}\right\rangle \otimes \left|\frac{\pi}{2}\right\rangle = -\mathbf{e}_2\,, \quad |0\rangle \otimes \left|\frac{\pi}{2}\right\rangle = (\mathrm{i}\mathbf{e}_1)\,, \quad \left|\frac{\pi}{2}\right\rangle \otimes |0\rangle = (\mathrm{i}\mathbf{e}_2)$$

- Real composite nature to "up" and "down" complex objects :
  $\mathbf{e}_1 \equiv |\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\qquad \mathbf{e}_2 \equiv |\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

- Unitary map from the Bell basis to the basis of real structure of $\mathbb{C}^2$

$$\left(|\Phi^+\rangle \quad |\Phi^-\rangle \quad |\Psi^+\rangle \quad |\Psi^-\rangle\right) = \left(\mathbf{e}_1 \quad \mathbf{e}_2 \quad (\mathrm{i}\mathbf{e}_1) \quad (\mathrm{i}\mathbf{e}_2)\right) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} .$$

- In terms of respective components of vectors in their respective spaces,

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x^+ \\ x^- \\ y^+ \\ y^- \end{pmatrix} .$$

- Equivalently, in complex notations, with $z^\pm = x^\pm + \mathrm{i}y^\pm$,

$$\begin{pmatrix} z^+ \\ z^- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathsf{C} \\ \mathsf{C} & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \equiv \mathcal{C}_{@} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} ,$$

- Here we have introduced the conjugaison operator $\mathsf{C}z = \bar{z}$, i.e. the mirror symmetry with respect to the real axis, $-\mathsf{C}$ being the mirror symmetry with respect to the imaginary axis.

- Operator $\mathcal{C}_{@}$ can be expressed as

$$\mathcal{C}_{@} = \frac{1}{\sqrt{2}}\left(I + \mathsf{F}\right)\,, \quad \mathsf{F} := C\hat{\jmath} = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}\,, \quad \hat{\jmath} \equiv -\mathrm{i}\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \tau_2\,.$$

- Therefore, with the above choice of isomorphisms, Bell entanglement in $\mathbb{R}^2 \otimes \mathbb{R}^2$ is <u>not</u> represented by a linear superposition in $\mathbb{C}^2$. It involves also the two mirror symmetries $\pm$C.

- Operator F is a kind of "flip" whereas "cat" operator $\mathcal{C}_{@}$ builds from the *up* and *down* basic states the two elementary Schrödinger cats

$$\mathsf{F}\,|\uparrow\rangle = |\downarrow\rangle\,, \quad \mathsf{F}\,|\downarrow\rangle = -|\uparrow\rangle\,,$$
$$\mathcal{C}_{@}\,|\uparrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)\,, \quad \mathcal{C}_{@}\,|\downarrow\rangle = \frac{1}{\sqrt{2}}(-|\uparrow\rangle + |\downarrow\rangle)\,.$$

- Interesting too is the appearance of the flip F in the construction of the spin one-half coherent state defined in terms of spherical coordinates (θ, ϕ) by

$$\mathbb{S}^2 \ni \hat{\mathbf{n}}(\theta,\varphi) \mapsto |\theta,\varphi\rangle = \left(\cos\frac{\theta}{2}\,|\uparrow\rangle + e^{\mathrm{i}\phi}\sin\frac{\theta}{2}\,|\downarrow\rangle\right) \equiv \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{\mathrm{i}\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$= D^{\frac{1}{2}}\left(\xi_{\hat{\mathbf{n}}}^{-1}\right)|\uparrow\rangle \equiv \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2}e^{-\mathrm{i}\phi} \\ \sin\frac{\theta}{2}e^{\mathrm{i}\phi} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- Here, $\xi_{\hat{\mathbf{n}}}$ corresponds, through homomorphism $\mathrm{SO}(3) \mapsto \mathrm{SU}(2)$, to the specific rotation $\mathcal{R}_{\hat{\mathbf{n}}}$ mapping the unit vector pointing to the north pole, $\hat{\mathbf{k}} = (0,0,1)$, to $\hat{\mathbf{n}}$
- UIR operator $D^{\frac{1}{2}}\left(\xi_{\hat{\mathbf{n}}}^{-1}\right)$ represents the element $\xi_{\hat{\mathbf{n}}}^{-1}$ in SU(2)
- Second column of $D^{\frac{1}{2}}\left(\xi_{\hat{\mathbf{n}}}^{-1}\right)$ is precisely the flip of the first one,

$$D^{\frac{1}{2}}\left(\xi_{\hat{\mathbf{n}}}^{-1}\right) = \left(|\theta,\phi\rangle \quad \mathsf{F}|\theta,\varphi\rangle\right)\,.$$

- This is the key for grasping the isomorphisms $\mathbb{C}^2 \cong \mathbb{H} \cong \mathbb{R}_+ \times \mathrm{SU}(2)$. Using quaternionic algebra, e.g., $\hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}}\hat{\mathbf{k}}$ + even permutations,

$$\mathbb{H} \ni q = q_0 + q_1\hat{\boldsymbol{\imath}} + q_2\hat{\boldsymbol{\jmath}} + q_3\hat{\mathbf{k}} = q_0 + q_3\hat{\mathbf{k}} + \hat{\boldsymbol{\jmath}}\left(q_1\hat{\mathbf{k}} + q_2\right) \equiv \begin{pmatrix} q_0 + \mathrm{i}q_3 \\ q_2 + \mathrm{i}q_1 \end{pmatrix} \equiv Z_q \in \mathbb{C}^2\,,$$

after identifying $\hat{\mathbf{k}} \equiv \mathrm{i}$ as both are roots of -1.

- Then the flip appears naturally in the final identification $\mathbb{H} \cong \mathbb{R}_+ \times \mathrm{SU}(2)$ as

$$q \equiv \begin{pmatrix} q_0 + \mathrm{i}q_3 & -q_2 + \mathrm{i}q_1 \\ q_2 + \mathrm{i}q_1 & q_0 - \mathrm{i}q_3 \end{pmatrix} = \begin{pmatrix} Z_q & \mathsf{F}Z_q \end{pmatrix}\,. \tag{31}$$

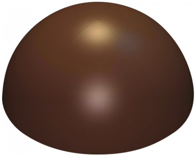
- As "cat states" in $\mathbb{C}^2$

$$\mathbb{C}^2 \ni |\theta, \varphi\rangle = \left(\cos \frac{\theta}{2} \,|\uparrow\rangle + e^{i\phi} \sin \frac{\theta}{2} \,|\downarrow\rangle \right)$$

- As entangled states in $\mathbb{R}^2_A \otimes \mathbb{R}^2_B$, $|\theta, \varphi\rangle \mapsto \begin{pmatrix} \cos \frac{\theta}{2} \\ -\sin \frac{\theta}{2} \cos \phi \\ \sin \frac{\theta}{2} \sin \phi \\ 0 \end{pmatrix}$

$$|\theta, \varphi\rangle = \cos \frac{\theta}{2} |0\rangle_A \otimes |0\rangle_B - \sin \frac{\theta}{2} \cos \phi \left| \frac{\pi}{2} \right\rangle_A \otimes \left| \frac{\pi}{2} \right\rangle_B + \sin \frac{\theta}{2} \sin \phi |0\rangle_A \otimes \left| \frac{\pi}{2} \right\rangle_B + 0 \left| \frac{\pi}{2} \right\rangle_A \otimes |0\rangle_B$$

- i.e., set of orientations in space $\mathbb{R}^3$ as spin 1/2 coherent states $\sim$ 2 entangled orientations in plane



$|\uparrow\rangle \sim |0\rangle_A \otimes |0\rangle_B$ and $|\downarrow\rangle \sim \left|\frac{\pi}{2}\right\rangle_A \otimes \left|\frac{\pi}{2}\right\rangle_B$

THANK YOU FOR YOUR ATTENTION
AND TAKE CARE !