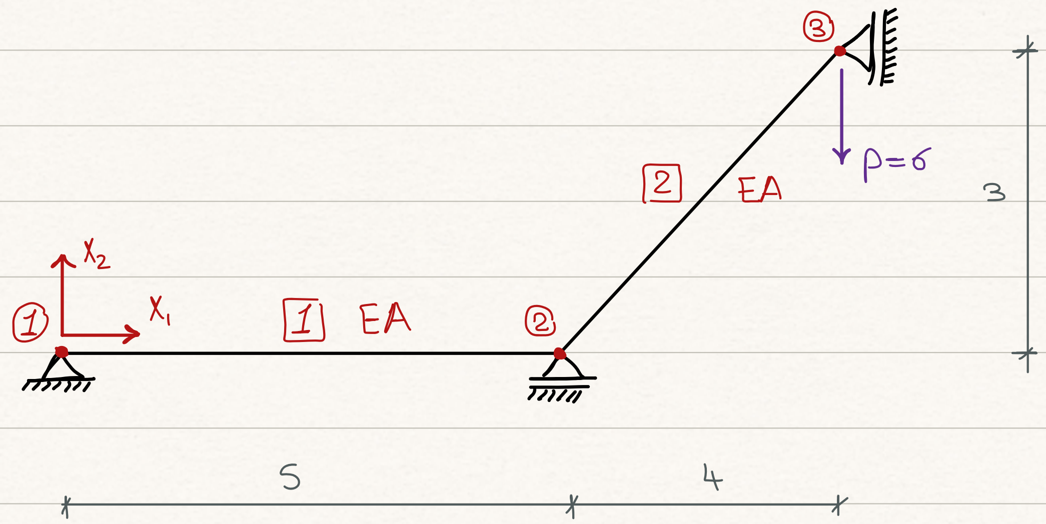
problem:

EA = 250.



Required:

1. Specify $\underline{f}_{int}$ of the non linear truss elements.
2. Specify the total internal force of the system. i.e. $\underline{f}_{total}$.
3. Specify the tangent stiffness matrix. $\underline{K}^e = \underline{K}^e_{geo} + \underline{K}^e_{mat}$
4. Find $\underline{K}_{total}$ of the system.

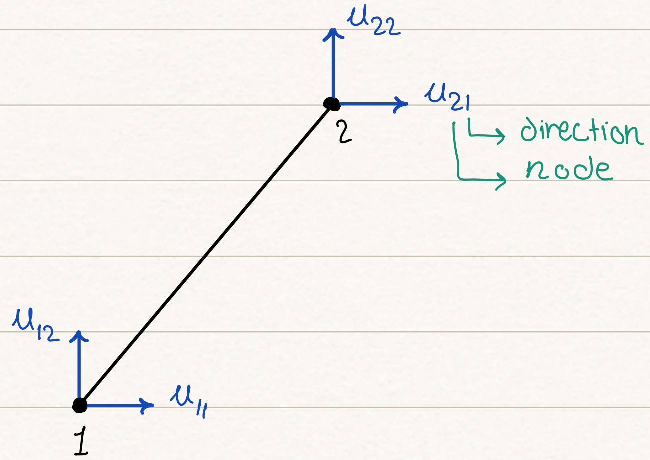
Solution

Recap:

In Local Coordinates:

$$\underline{f}^e_{int} = \begin{bmatrix} -1 \\ +1 \end{bmatrix} (1 - u_1 + u_2)\, EA\, (\ell^2 - L^2)$$

In Global Coordinates:



$$\underline{X}^e = \begin{bmatrix} X_{11} \\ X_{12} \\ X_{21} \\ X_{22} \end{bmatrix}, \quad \underline{u}^e = \begin{bmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{bmatrix}$$

Define a, b:

$$a = X_{11} + u_{11} - X_{21} - u_{21} \qquad (\underline{e}_1 \text{ direction})$$

$$b = X_{12} + u_{12} - X_{22} - u_{22} \qquad (\underline{e}_2 \text{ direction})$$

Internal Force: $\underline{f}^e_{int}$

$$\underline{f}^e_{int} = \frac{EA}{2L^3}\left(l^2 - L^2\right) \left[\begin{array}{c} a \\ b \\ -a \\ -b \end{array}\right] \begin{array}{l} \delta u_{11} \\ \delta u_{12} \\ \delta u_{21} \\ \delta u_{22} \end{array}$$

Element tangent stiffness matrix:

tangent stiffness matrix

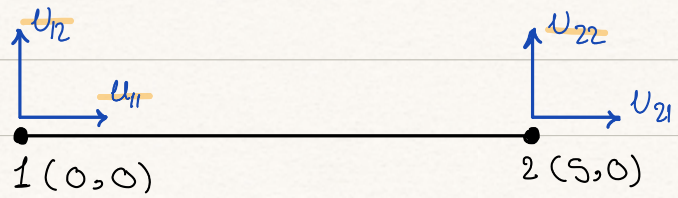
$$\underline{K}^e = \underline{K}^e_{geo} + \underline{K}^e_{mat}$$

$$\underline{K}^e_{geo} = \frac{EA}{L^3}\,\frac{\left(l^2 - L^2\right)}{2} \begin{array}{c} \begin{array}{cccc} u_{11} & u_{12} & u_{21} & u_{22} \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] \end{array} \begin{array}{l} \\ \delta u_{11} \\ \delta u_{12} \\ \delta u_{21} \\ \delta u_{22} \end{array}$$

$$\underline{K}^e_{mat} = \frac{EA}{L^3} \begin{array}{c} \begin{array}{cccc} u_{11} & u_{12} & u_{21} & u_{22} \end{array} \\ \left[\begin{array}{cccc} a^2 & ab & -a^2 & -ab \\ ab & b^2 & -ab & -b^2 \\ -a^2 & -ab & a^2 & ab \\ -ab & -b^2 & ab & b^2 \end{array}\right] \end{array} \begin{array}{l} \\ \delta u_{11} \\ \delta u_{12} \\ \delta u_{21} \\ \delta u_{22} \end{array}$$

Q1: f_{int}

Truss 1: 1 DoF: u_{21}



$$\underline{X}^1 = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \end{bmatrix}, \quad u^1 = \begin{bmatrix} 0 \\ 0 \\ u_{21} \\ 0 \end{bmatrix}$$

$$L^2 = 5^2, \quad \ell^2 = (5 + u_{21})^2$$

$$a = -5 - u_{21}$$

$$b = 0$$

$$f^1_{int} = \frac{EA}{2L^3}\left(\ell^2 - L^2\right) \begin{array}{c} \begin{bmatrix} a \\ b \\ -a \\ -b \end{bmatrix} \end{array} \begin{array}{l} \delta u_{11} \\ \delta u_{12} \\ \delta u_{21} \\ \delta u_{22} \end{array} = \frac{250}{2(5)^3}\left((5+u_{21})^2 - 25\right) \begin{array}{c} \begin{bmatrix} -5-u_{21} \\ 0 \\ 5+u_{21} \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \delta u_{11} \\ \delta u_{12} \\ \delta u_{21} \\ \delta u_{22} \end{array}$$

$$f^1_{int} = (\cancel{25} + u_{21}^2 + 10\,u_{21} \cancel{-25})(5 + u_{21})$$

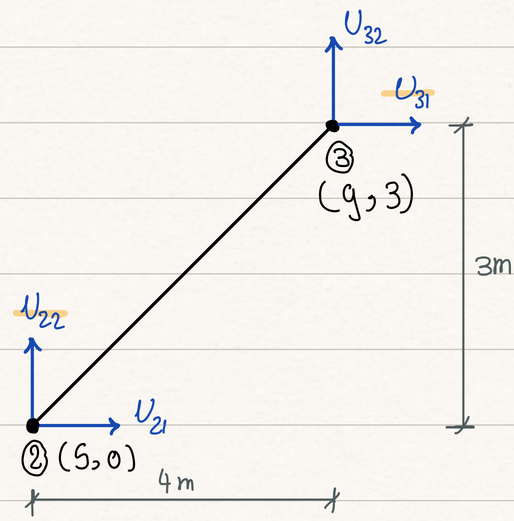
$$= 5\,u_{21}^2 + 50\,u_{21} + u_{21}^3 + 10\,u_{21}^2$$

$$\boxed{f^1_{int} = u_{21}^3 + 15\,u_{21}^2 + 50\,u_{21}} \longrightarrow \#1$$

Note:

we only need the internal force component that corresponds with δu_{21}, because other components (even if they have non zero) won't contribute to the internal energy $\rightarrow W^e_{int} = \delta \underline{u}^e \cdot f^e_{int}$, δu^e has to satisfy Direchilet B.Cs and otherwise arbitrary.

Truss 2: 2 DOFs: v_{21}, v_{32}



$$\underline{X}^2 = \begin{bmatrix} 5 \\ 0 \\ 9 \\ 3 \end{bmatrix}, \quad \underline{u}^2 = \begin{bmatrix} v_{21} \\ 0 \\ 0 \\ v_{32} \end{bmatrix}$$

$$L^2 = 5^2 \quad , \quad \ell^2 = (4 - v_{21})^2 + (3 + v_{32})^2$$

$$a = 5 + v_{21} - 9 - 0 = -4 + v_{21}$$

$$b = -3 - v_{32}$$

$$f^2_{int} = \frac{EA}{2L^3}\left(\ell^2 - L^2\right) \begin{array}{c} \begin{bmatrix} a \\ b \\ -a \\ -b \end{bmatrix} \end{array} \begin{array}{l} \delta v_{21} \\ \delta v_{22} \\ \delta v_{31} \\ \delta v_{32} \end{array} = \frac{250}{2(5)^3}\left(\left((4 - v_{21})^2 + (3 + v_{32})^2 - 25\right)\right) \begin{bmatrix} -4 + v_{21} \\ 3 + v_{32} \end{bmatrix} \begin{array}{l} \delta v_{21} \\ \delta v_{32} \end{array}$$

$$= \left(\cancel{16} - 8 v_{21} + v_{21}^2 + \cancel{9} + 6 v_{32} + v_{32}^2 - \cancel{25}\right) \begin{bmatrix} -4 + v_{21} \\ 3 + v_{32} \end{bmatrix} \begin{array}{l} \delta v_{21} \\ \delta v_{32} \end{array}$$

$$\boxed{f^2_{int} = \left(v_{32}^2 + 6 v_{32} - 8 v_{21} + v_{21}^2\right) \begin{bmatrix} -4 + v_{21} \\ 3 + v_{32} \end{bmatrix} \begin{array}{l} \delta v_{21} \\ \delta v_{32} \end{array}} \longrightarrow \#1$$

Q2: Total internal force (f^{total}_{int}):

$$\underline{f}^{total}_{int} = \begin{bmatrix} f^1_{21} + f^2_{21} \\ f^2_{32} \end{bmatrix} \begin{array}{l} \delta v_{21} \\ \delta v_{32} \end{array}$$

Notation:

f^1_{21}: 1 → element; 2 → direction; 1 → node.

$$\underline{f}_{int}^{total} = \begin{array}{c} \left[\begin{array}{c} U_{21}^3 + 15\,U_{21}^2 + 50\,U_{21} + (U_{32}^2 + 6\,U_{32} - 8\,U_{21} + U_{21}^2)(-4+U_{21}) \\ (U_{32}^2 + 6\,U_{32} - 8\,U_{21} + U_{21}^2)(3+U_{32}) \end{array}\right] \end{array} \begin{array}{l} \delta U_{21} \\ \delta U_{32} \end{array} \longrightarrow \#2$$

Q3: Tangent Stiffness Matrix:

Truss 1:

$$\underline{K}^1 = \underbrace{\frac{EA}{L^3}\,\frac{(\ell^2 - L^2)}{2} \begin{array}{c} U_{21} \\ \left[\,1\,\right] \end{array} \begin{array}{l} \\ \delta U_{21} \end{array}}_{\underline{K}^{geo}} + \underbrace{\frac{EA}{L^3} \begin{array}{c} U_{21} \\ \left[\,a^2\,\right] \end{array} \begin{array}{l} \\ \delta U_{21} \end{array}}_{\underline{K}^{mat}}$$

$$= \frac{250}{5^3}\,\frac{(5+U_{21})^2 - 5^2}{2} + \frac{250}{5^3}\,(-5-U_{21})^2$$

$$\underline{K}^1 = \left[\cancel{25} + U_{21}^2 + 10\,U_{21} - \cancel{25}\right] + \left[25 + U_{21}^2 + 10\,U_{21}\right] \times 2$$

$$\underline{K}^1 = 3\,U_{21}^2 + 30\,U_{21} + 50 \longrightarrow \#3$$

Truss 2:

$$a = U_{21} - 4$$

$$b = -U_{32} - 3$$

$$\underline{K}^2 = \frac{EA}{L^3}\,\frac{(\ell^2 - L^2)}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{EA}{L^3} \begin{bmatrix} a^2 & -ab \\ -ab & b^2 \end{bmatrix}$$

$$\frac{EA}{L^3}\,\frac{\ell^2 - L^2}{2} = \frac{250}{5^3 \times 2}\left[(4-U_{21})^2 + (3+U_{32})^2 - 25\right]$$

$$= \cancel{16} + U_{21}^2 - 8\,U_{21} + \cancel{9} + U_{32}^2 + 6\,U_{32} - \cancel{25} = U_{32}^2 + 6\,U_{32} - 8\,U_{21} + U_{21}^2$$

$$a^2 = (v_{21}-4)^2 = v_{21}^2 - 8v_{21} + 16$$

$$b^2 = (-v_{32}-3)^2 = v_{32}^2 + 6v_{32} + 9$$

$$ab = (v_{21}-4)(v_{32}+3) = v_{21}v_{32} - 4v_{32} + 3v_{21} - 12$$

$$\underline{K}^2 = \begin{bmatrix} v_{32}^2 + 6v_{32} - 8v_{21} + v_{21}^2 + 2v_{21}^2 - 16v_{21} + 32 & 2v_{21}v_{32} - 8v_{32} + 6v_{21} - 24 \\ 2v_{21}v_{32} - 8v_{32} + 6v_{21} - 24 & v_{32}^2 + 6v_{32} - 8v_{21} + v_{21}^2 + 2v_{32}^2 + 12v_{32} + 18 \end{bmatrix}$$

$$K_{11}^{total} = v_{32}^2 + 6v_{32} - 8v_{21} + v_{21}^2 + 2v_{21}^2 - 16v_{21} + 32 + 3v_{21}^2 + 30v_{21} + 50$$
$$= 6v_{21}^2 + 6v_{21} + 6v_{32} + v_{32}^2 + 82$$

Simpler Method:

$$\underline{K} = \frac{\partial \underline{f}_{int}}{\partial \underline{v}} = \begin{bmatrix} \partial f_1/\partial v_1 & \partial f_1/\partial v_2 \\ \partial f_2/\partial v_1 & \partial f_2/\partial v_2 \end{bmatrix}$$

$$\underline{f}_{int}^{total} = \begin{bmatrix} v_{21}^3 + 15v_{21}^2 + 50v_{21} + (v_{32}^2 + 6v_{32} - 8v_{21} + v_{21}^2)(-4+v_{21}) \\ (v_{32}^2 + 6v_{32} - 8v_{21} + v_{21}^2)(3+v_{32}) \end{bmatrix} \begin{matrix} \leftarrow f_1 \\ \leftarrow f_2 \end{matrix}$$

$$\underline{v} = \begin{bmatrix} v_{21} \\ v_{32} \end{bmatrix} \begin{matrix} \leftarrow v_1 \\ \leftarrow v_2 \end{matrix}$$

$$K_{11} = 3v_{21}^2 + 30v_{21} + 50 + \underbrace{(-8+2v_{21})(-4+v_{21})}_{32 \; -16v_{21} \; +2v_{21}^2} + (v_{32}^2 + 6v_{32} - 8v_{21} + v_{21}^2)$$

$$= 6v_{21}^2 + 6v_{21} + 6v_{32} + v_{32}^2 + 82$$