Consider the 1D bar with the boundary conditions shown in Fig. 1. The bar is loaded by a

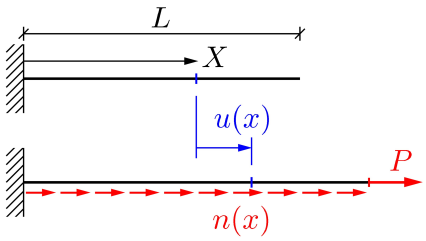


Figure 1: 1D bar under distributed load $n(x)$ and end load $P$.

distributed line load $n(x)$ that is constant per reference length, i.e. $n_0 := \lambda\, n = \text{const}$. The material is characterized by the constitutive law

$$N = EA\, \epsilon_0\,, \tag{1}$$

where $EA$ is constant and $\epsilon_0 := \ln \lambda$ is the so-called Hencky strain. Considering $P = 0$:

1. Show that the axial force in the bar is given by

$$N(X) = n_0\,(L - X)\,. \tag{2}$$

2. Determine $\lambda(X)$.

3. Determine $x = \varphi(X)$. Express your result in terms of the constant $c := EA/n_0$ and the maximum stretch $\lambda_{\max} := \lambda(0)$.

4. Determine the inverse function $X = \varphi^{-1}(x)$.

5. Determine $u(X)$ and $u(x)$. What is $u(L)$ for $c = L/2$?

6. Determine $N(x)$. Confirm that $N_{,x} + n = 0$.

7. Plot $\varphi(X)$, $\lambda(X)$, $u(X)$ and $u(x)$ for $c = L/2$.

# Question 01:

Consider the 1D bar with the boundary conditions shown in Fig. 1. The bar is loaded by a

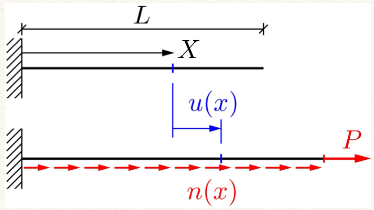


Figure 1: 1D bar under distributed load $n(x)$ and end load $P$.

distributed line load $n(x)$ that is constant per reference length, i.e. $n_0 := \lambda n =$ const. The material is characterized by the constitutive law

$$N = EA\,\epsilon_0\,, \tag{1}$$

where $EA$ is constant and $\epsilon_0 := \ln\lambda$ is the so-called Hencky strain. Considering $P = 0$:

1. Show that the axial force in the bar is given by

$$N(X) = n_0\,(L - X)\,. \tag{2}$$

## Solution

Equilibrium equation wrt initial Configuration:

$$N_{,X} + \underbrace{\lambda n}_{n_0} = 0 \iff N_{,X} + n_0 = 0$$

$$N_{,X} = -n_0 \iff \frac{\partial N(X)}{\partial X} = -n_0$$

$$\Rightarrow \quad N(X) = -n_0 X + C_1$$

B.C:

$$N(L) = 0 \Rightarrow 0 = -n_0 L + C_1 \iff C_1 = n_0 L$$

$$\therefore \quad N(X) = -n_0 X + n_0 L \iff \boxed{N(X) = n_0(L - X)} \longrightarrow \#$$

# Question 02:

2. Determine $\lambda(X)$.

Given: $N(X) = EA\,\varepsilon_0$ , $\varepsilon_0 = \ln\lambda$

$$n_0(L - X) = EA \ln\lambda \iff \ln\lambda = \frac{n_0(L-X)}{EA}$$

$$\Rightarrow e^{\ln\lambda} = e^{\frac{n_0(L-X)}{EA}}$$

$$\iff \boxed{\underbrace{\lambda}_{f(X)} = e^{\frac{n_0(L-X)}{EA}}} \rightarrow \#$$

# Question 03:

3. Determine $x = \varphi(X)$. Express your result in terms of the constant $c := EA/n_0$ and the maximum stretch $\lambda_{\max} := \lambda(0)$.

Solution

$$\lambda(X) = \frac{\partial \overbrace{x}^{\phi(X)}}{\partial X} \iff \partial x = \lambda\, \partial X$$

By integrating both sides.

$$x = \int \lambda\, \partial X = \int e^{\frac{n_0(L-X)}{EA}}\, \partial X = e^{\frac{n_0(L-X)}{EA}} \cdot \underbrace{\frac{-EA}{n_0}}_{=:c} + C_1$$

$$x = -c\, e^{\frac{n_0(L-X)}{EA}} + C_1$$

B.C: $x(0) = 0$ i.e. No deformation

$$0 = -c\, e^{\frac{n_0 L}{EA}} + C_1 \iff C_1 = c\, \underbrace{e^{\frac{n_0 L}{EA}}}_{=:\lambda_{max}}$$

$$\therefore \boxed{x = -c\, e^{\frac{n_0 (L - X)}{EA}} + c\lambda_{max}} \longrightarrow \#$$

# Question 4:

4. Determine the inverse function $X = \varphi^{-1}(x)$.

Solution

$$x = -c\, e^{\frac{n_0 (L - X)}{EA}} + c\lambda_{max}$$

$$\Leftrightarrow e^{\frac{n_0 (L - X)}{EA}} = \frac{c\lambda_{max} - x}{c}$$

$$\frac{n_0 (L - X)}{EA} = \ln\left[\frac{c\lambda_{max} - x}{c}\right]$$

$$X = L - \underbrace{\frac{EA}{n_0}}_{c} \ln\left[\frac{c\lambda_{max} - x}{c}\right]$$

$$\therefore \boxed{X = L - c\ln\left[\frac{c\lambda_{max} - x}{c}\right]} \longrightarrow \#$$

# Question 05:

5. Determine $u(X)$ and $u(x)$. What is $u(L)$ for $c = L/2$?

Solution

$x = u + X \iff u = x - X$ ← Initial Config.

↑ Current Config.

# wrt Initial Configuration (i.e. $X$):

$$u(X) = -c\, e^{\frac{n_0 (L - X)}{EA}} + c\lambda_{max} - X$$

# wrt Current Configuration (i.e. $x$):

$$u(x) = x - X(x) = x - L + c \ln\left[\frac{c\lambda_{max} - x}{c}\right]$$

# Question 6:

6. Determine $N(x)$. Confirm that $N_{,x} + n = 0$.

## Solution

Already established: $N(\bar{X}) = n_o(L - \bar{X})$

$$\bar{X} = L - c\ln\left[\frac{c\lambda_{max} - x}{c}\right]$$

$$N(x) = n_o\left[\underbrace{c}_{\frac{EA}{n_o}} \ln \frac{c\lambda_{max} - x}{c}\right]$$

$$N(x) = EA \ln\left[\frac{c\lambda_{max} - x}{c}\right]$$

$$N(x) = EA\left[\ln(c\lambda_{max} - x) - \ln(c)\right]$$

$$N_{,x} = EA \frac{-1}{c\lambda_{max} - x} = \frac{-EA}{c\lambda_{max} - x} \rightarrow (1)$$

$$n = \frac{n_o}{\lambda} = \frac{n_o}{e^{\frac{n_o(L-\bar{X})}{EA}}} = \frac{n_o}{e^{\frac{-n_o(\bar{X}-L)}{EA}}}$$

$$n = n_o e^{\frac{n_o(\bar{X}-L)}{EA}}$$

$$= n_o e^{\frac{n_o\bar{X}}{EA}} \cdot e^{-\frac{n_o L}{EA}}$$

$$= n_o e^{\frac{n_o\bar{X}}{EA}} \cdot \frac{1}{\lambda_{max}}$$

$$n = \frac{n_o}{\lambda_{max}} e^{\frac{n_o}{EA}\left[L - c\ln\left(\frac{c\lambda_{max} - x}{c}\right)\right]}$$

$$n = \frac{n_o}{\cancel{\lambda_{max}}} \cdot \cancel{\lambda_{max}} \cdot e^{-\frac{\cancel{n_o}}{\cancel{EA}} \cdot \cancel{c} \ln(\quad)} \quad , c = \frac{EA}{n_o}$$

$$n = n_0 \cdot \frac{c\,\overset{\frac{EA}{n_0}}{S}}{c\lambda_{max} - x} = \frac{EA}{c\lambda_{max} - x} \longrightarrow ②$$

From ①, ②: $N_{,x} + n = 0$

# Question 07: