# Numerical methods for conservation laws

Sheet 2

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**Exercise 1**

We consider the one dimensional Poisson equation on the unit interval $\Omega = [0, 1]$,

$$\begin{aligned} -u_{xx} &= S \quad &&\text{in} \quad \Omega, \\ u_x \cdot \mathbf{n} &= g_N \quad &&\text{on} \quad \Gamma_N, \\ u &= g_D \quad &&\text{on} \quad \Gamma_D, \end{aligned}$$

with source $S : \Omega \to \mathbb{R}$, boundary sets $\Gamma_N, \Gamma_D \subset \partial\Omega$ such that $\Gamma_N \cup \Gamma_D = \partial\Omega$, $\Gamma_N \cap \Gamma_D = \emptyset$ and $g_N : \Gamma_N \to \mathbb{R}$, $g_D : \Gamma_D \to \mathbb{R}$. In particular, we have Neumann boundary conditions on $\Gamma_N$ and Dirichlet boundary conditions on $\Gamma_D$.

We consider the cases

i). $S(x) = \pi^2 \sin(\pi x)$, $\Gamma_N = \emptyset$, $\Gamma_D = \partial\Omega$, $g_D = 0$ (homogeneous Dirichlet boundary conditions).

ii). $S(x) = \pi^2 \cos(\pi x)$, $\Gamma_N = \{1\}$, $g_N = 0$, $\Gamma_D = \{0\}$, $g_D = 1$ (Dirichlet conditions at the left and Neumann conditions at the right boundary).

(a). What is the solution of the Poisson equation in case i), respectively ii)?

(b). Derive the scheme of the finite volume method with $N \in \mathbb{N}$ cells of size $h = \frac{1}{N}$ for the Poisson equation with i) and ii). Use the midpoint rule for approximating the integrals of $S$ and finite differences for approximating the flux $u_x$. You should end up with a linear system of the form $AQ = \mathcal{S}$, where $Q$ is a vector consisting of the cell averages of the solution $u$.

(c). Implement the finite volume method (e.g. in MATLAB) with cell size $N = 50$ (or $h = 0.02$ respectively) in both cases i) and ii). Plot the approximation of the solution together with the exact solutions from (a).

(d). For $h = 2^{-i}$, $i = 5, \ldots, 15$ compute the approximation of the solution and calculate the error $e(h) = \max_{i=1,\ldots N} |u(x_i) - Q_i|$ where $u(x_i)$ is the exact solution at the cell midpoint $x_i$ and $Q_i$ is the approximation of the $i$-th cell average of the numerical solution. Plot the results in a logarithmic $(h, e(h))$-diagram for the both cases i) and ii).