Sheet (4)
Sequences and Mathematical Induction

Question 1:

Write the first four terms of the sequences defined by the following formulas:

1. $a_k = \frac{k}{10-k}$, for all integers k ≥ 1.
2. $c_i = \frac{(-1)^i}{3^i}$, for all integers i ≥ 0.

Question 2:

Find explicit formulas for the following sequences with the initial terms given:

1. 0, 1, -2, 3, -4, 5.
2. $\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \frac{6}{8}$.
3. $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}$.

Question 3:

Let $a_0 = 2$, $a_1 = 3$, $a_2 = -2$, $a_3 = 1$, $a_4 = 0$, $a_5 = -1$ and $a_6 = -2$. Compute each of the summations and products below.

1. $\sum_{i=0}^{0} a_i$.
2. $\sum_{j=1}^{3} a_{2j}$.
3. $\prod_{k=0}^{6} a_k$.
4. $\prod_{k=2}^{2} a_k$.

Question 4:

Write each of the following summation or product notations.

1. $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + 7^2$.
2. $n + \frac{n-1}{2!} + \frac{n-2}{3!} + \frac{n-3}{4!} + \ldots\ldots\ldots\ldots + \frac{1}{n!}$

Question 4:

Transform each of the following by making the change of variable j = i – 1.

1. $\sum_{i=1}^{n+1} \frac{(i-1)^2}{i}$.
2. $\sum_{i=3}^{n+1} \frac{i}{i+n-1}$.

Question 5:

a) Prove that n! + 2 is divisible by 2, for all integers n ≥ 2.
b) Prove that n! + k is divisible by k, for all integers n ≥ 2 and k = 2, 3, ………, n.

Question 6:

Use theorem 4.2.2 $\left(\sum_{i=1}^{n} i = \left(\frac{n}{2}\right) * (n+1)\right)$ to solve the following:

$$2 + 4 + 6 + \ldots\ldots\ldots\ldots + 2n = n^2 + n.$$

For all integers n ≥ 1.

Question 7:
Without using theorem 4.2.2, use mathematical induction to prove that:

$$1 + 5 + 9 + \ldots\ldots\ldots\ldots\ldots + (4n - 3) = n(2n - 1).$$

For all integers n ≥ 1.

Question 8:
Without using theorem 4.2.3: $1+r+r^2+\cdots+r^n = \dfrac{r^{n+1}-1}{r-1}$

use mathematical induction to prove that:

$$1 + 2 + 2^2 + \ldots\ldots\ldots\ldots\ldots + 2^n = 2^{n+1} - 1.$$

For all integers n ≥ 0.

Question 9:
Prove each of the following statements by mathematical induction:

1. $1^2 + 2^2 + \ldots\ldots\ldots\ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$, for all integers n ≥ 1.
2. $1^3 + 2^3 + \ldots\ldots\ldots\ldots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$, for all integers n ≥ 1.
3. $\dfrac{1}{1.2} + \dfrac{1}{2.3} + \ldots\ldots\ldots\ldots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$, for all integers n ≥ 1.
4. $\displaystyle\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$, for all integers n ≥ 2.