CHAPTER 1

SPEAKING MATHEMATICALLY

1.2 The Language of Sets

The Language of Sets

Use of the word *set* as a formal mathematical term was introduced in 1879 by Georg Cantor (1845–1918). For most mathematical purposes we can think of a set intuitively, as Cantor did, simply as a collection of elements.

Set-Roster Notation

If S is a set, the notation $\boldsymbol{x \in S}$ means that x is an element of S. The notation $\boldsymbol{x \notin S}$ means that x is not an element of S. A set may be specified using the **set-roster notation** by writing all of its elements between braces. For example, **{1, 2, 3}** denotes the set whose elements are 1, 2, and 3. A variation of the notation is sometimes used to describe a very large set, as when we write **{1, 2, 3, . . . , 100}** to refer to the set of all integers from 1 to 100. A similar notation can also describe an infinite set, as when we write **{1, 2, 3, . . . }** to refer to the set of all positive integers. (The symbol . . . is called an **ellipsis** and is read "and so forth.")

The Language of Sets

The **axiom of extension** says that a set is completely determined by what its elements are—not the order in which they might be listed or the fact that some elements might be listed more than once.

Example 1.2.1 – *Using the Set-Roster Notation*

a. Let $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$, and $C = \{1, 1, 2, 3, 3, 3\}$. What are the elements of A, B, and C? How are A, B, and C related?

b. Is $\{0\} = 0$?

c. How many elements are in the set $\{1, \{1\}\}$?

d. For each nonnegative integer n, let $U_n = \{n, -n\}$. Find U_1, U_2, and U_0.

Example 1.2.1 – *Solution*

a. A, B, and C have exactly the same three elements: 1, 2, and 3. Therefore, A, B, and C are simply different ways to represent the same set.

b. $\{0\} \neq 0$ because $\{0\}$ is a set with one element, namely 0, whereas 0 is just the symbol that represents the number zero.

Example 1.2.1 – *Solution*

continued

c. The set {1, {1}} has two elements: 1 and the set whose only element is 1.

d. $U_1 = \{1, -1\}$, $U_2 = \{2, -2\}$, $U_0 = \{0, -0\} = \{0, 0\} = \{0\}$.

The Language of Sets

Certain sets of numbers are so frequently referred to that they are given special symbolic names. These are summarized in the following table.

Symbol	Set
R	the set of all real numbers
Z	the set of all integers
Q	the set of all rational numbers, or quotients of integers

The Language of Sets

Addition of a superscript + or − or the letters *nonneg* indicates that only the positive or negative or nonnegative elements of the set, respectively, are to be included. Thus $\mathbf{R}^{+}$ denotes the set of positive real numbers, and $\mathbf{Z}^{nonneg}$ refers to the set of nonnegative integers: 0, 1, 2, 3, 4, and so forth.

Some authors refer to the set of nonnegative integers as the set of **natural numbers** and denote it as **N**.

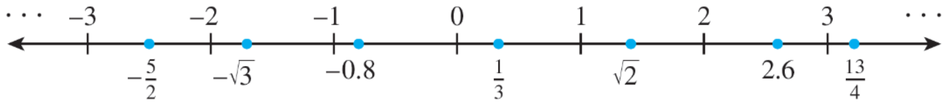
The Language of Sets

The set of real numbers is usually pictured as the set of all points on a line. The number 0 corresponds to a middle point, called the *origin*.

A unit of distance is marked off, and each point to the right of the origin corresponds to a positive real number found by computing its distance from the origin.

The Language of Sets

Each point to the left of the origin corresponds to a negative real number, which is denoted by computing its distance from the origin and putting a minus sign in front of the resulting number. The set of real numbers is therefore divided into three parts: the set of positive real numbers, the set of negative real numbers, and the number 0. *Note that 0 is neither positive nor negative*. Labels are given for a few real numbers corresponding to points on the line shown below.



The Language of Sets

Set-Builder Notation

Let S denote a set and let $P(x)$ be a property that elements of S may or may not satisfy. We may define a new set to be **the set of all elements x in S such that $P(x)$ is true**. We denote this set as follows:

$$\underbrace{\{}_{\text{the set of all}} x \in S \underbrace{\mid}_{\text{such that}} P(x)\}$$

Example 1.2.2 – *Using the Set-Builder Notation*

Given that **R** denotes the set of all real numbers, **Z** the set of all integers, and $\mathbf{Z}^+$ the set of all positive integers, describe each of the following sets.

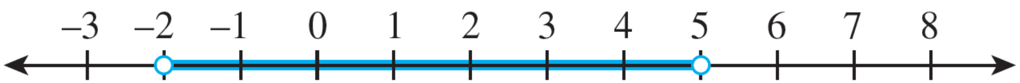
a. $\{x \in \mathbf{R} \mid -2 < x < 5\}$

b. $\{x \in \mathbf{Z} \mid -2 < x < 5\}$

c. $\{x \in \mathbf{Z}^+ \mid -2 < x < 5\}$

Example 1.2.2 – *Solution*

a. $\{x \in \mathbf{R} \mid -2 < x < 5\}$ is the open interval of real numbers (strictly) between −2 and 5. It is pictured as follows:



b. $\{x \in \mathbf{Z} \mid -2 < x < 5\}$ is the set of all integers (strictly) between −2 and 5. It is equal to the set {−1, 0, 1, 2, 3, 4}.

c. Since all the integers in $\mathbf{Z}^+$ are positive,

$\{x \in \mathbf{Z}^+ \mid -2 < x < 5\} = \{1, 2, 3, 4\}.$

Subsets

Subsets

A basic relation between sets is that of subset.

Definition

If A and B are sets, then A is called a **subset** of B, written $\boldsymbol{A \subseteq B}$, if, and only if, every element of A is also an element of B.

Symbolically:

$$A \subseteq B \quad \text{means that} \quad \text{for every element } x, \text{ if } x \in A \text{ then } x \in B.$$

The phrases A *is contained in* B and B *contains* A are alternative ways of saying that A is a subset of B.

Subsets

It follows from the definition of subset that for a set *A* not to be a subset of a set *B* means that there is at least one element of *A* that is not an element of *B*. Symbolically:

$A \not\subseteq B$ means that there is at least one element x such that $x \in A$ and $x \notin B$.

Definition

Let A and B be sets. A is a **proper subset** of B if, and only if, every element of A is in B but there is at least one element of B that is not in A.

Example 1.2.3 – *Subsets*

Let $A = \mathbf{Z}^+, B = \{n \in \mathbf{Z} \mid 0 \le n \le 100\}$, and C = {100, 200, 300, 400, 500}. Evaluate the truth and falsity of each of the following statements.

a. $B \subseteq A$

b. C is a proper subset of A

c. C and B have at least one element in common

d. $C \subseteq B$

e. $C \subseteq C$

Example 1.2.3 – *Solution*

a. False. Zero is not a positive integer. Thus zero is in B but zero is not in A, and so $B \not\subseteq A$.

b. True. Each element in C is a positive integer and, hence, is in A, but there are elements in A that are not in C. For instance, 1 is in A and not in C.

c. True. For example, 100 is in both C and B.

Example 1.2.3 – *Solution*

continued

d. False. For example, 200 is in C but not in B.

e. True. Every element in C is in C. In general, the definition of subset implies that all sets are subsets of themselves.

Example 1.2.4 – *Distinction between* $\in$ *and* $\subseteq$

Which of the following are true statements?

a. $2 \in \{1, 2, 3\}$

b. $\{2\} \in \{1, 2, 3\}$

c. $2 \subseteq \{1, 2, 3\}$

d. $\{2\} \subseteq \{1, 2, 3\}$

e. $\{2\} \subseteq \{\{1\}, \{2\}\}$

f. $\{2\} \in \{\{1\}, \{2\}\}$

Example 1.2.4 – *Solution*

Only (a), (d), and (f) are true.

For (b) to be true, the set {1, 2, 3} would have to contain the element {2}. But the only elements of {1, 2, 3} are 1, 2, and 3, and 2 is not equal to {2}. Hence (b) is false.

For (c) to be true, the number 2 would have to be a set and every element in the set 2 would have to be an element of {1, 2, 3}. This is not the case, so (c) is false.

Example 1.2.4 – *Solution*

continued

For (e) to be true, every element in the set containing only the number 2 would have to be an element of the set whose elements are {1} and {2}. But 2 is not equal to either {1} or {2}, and so (e) is false.

Cartesian Products

Cartesian Products

Notation

Given elements a and b, the symbol (a, b) denotes the **ordered pair** consisting of a and b together with the specification that a is the first element of the pair and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$. Symbolically:

$$(a, b) = (c, d) \quad \text{means that} \quad a = c \text{ and } b = d.$$

Example 1.2.5 – *Ordered Pairs*

a. Is (1, 2) = (2, 1)?

b. Is $\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)$?

c. What is the first element of (1, 1)?

Example 1.2.5 – *Solution*

a. No. By definition of equality of ordered pairs,

$$(1, 2) = (2, 1) \text{ if, and only if, } 1 = 2 \text{ and } 2 = 1.$$

But 1 ≠ 2, and so the ordered pairs are not equal.

b. Yes. By definition of equality of ordered pairs,

$$\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right) \text{ if, and only if, } 3 = \sqrt{9} \text{ and } \frac{5}{10} = \frac{1}{2}.$$

Because these equations are both true, the ordered pairs are equal.

Example 1.2.5 – *Solution*

continued

c. In the ordered pair (1, 1), the first and the second elements are both 1.

Cartesian Products

The notation for an *ordered n-tuple* generalizes the notation for an ordered pair to a set with any finite number of elements. It also takes both order and multiplicity into account.

Definition

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered n-tuple, $(x_1, x_2, \ldots, x_n)$**, consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair**, and an ordered 3-tuple is called an **ordered triple**.

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1, x_2 = y_2, \ldots,$ and $x_n = y_n$.

Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

Example 1.2.6 – *Ordered n-tuples*

a. Is (1, 2, 3, 4) = (1, 2, 4, 3)?

b. Is $\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right)$?

Example 1.2.6 – *Solution*

a. No. By definition of equality of ordered 4-tuples,

$$(1, 2, 3, 4) = (1, 2, 4, 3) \Leftrightarrow 1 = 1, 2 = 2, 3 = 4, \text{ and } 4 = 3$$

But $3 \neq 4$, and so the ordered 4-tuples are not equal.

Example 1.2.6 – *Solution*

continued

b. Yes. By definition of equality of ordered triples,

$$\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right) \quad \Leftrightarrow \quad 3 = \sqrt{9} \text{ and } (-2)^2 = 4 \text{ and } \frac{1}{2} = \frac{3}{6}.$$

Because these equations are all true, the two ordered triples are equal.

Cartesian Products

Definition

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted $\boldsymbol{A_1 \times A_2 \times \cdots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1$, $a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2.

Example 1.2.7 – *Cartesian Products*

Let $A = \{x, y\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$.

a. Find $A \times B$.

b. Find $B \times A$.

c. Find $A \times A$.

d. How many elements are in $A \times B$, $B \times A$, and $A \times A$?

e. Find $(A \times B) \times C$

f. Find $A \times B \times C$

g. Let $\mathbf{R}$ denote the set of all real numbers. Describe $\mathbf{R} \times \mathbf{R}$.

Example 1.2.7 – *Solution*

a. $A \times B = \{(x, 1), (y, 1), (x, 2), (y, 2), (x, 3), (y, 3)\}$

b. $B \times A = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$

c. $A \times A = \{(x, x), (x, y), (y, x), (y, y)\}$

d. $A \times B$ has 6 elements. Note that this is the number of elements in *A* times the number of elements in *B*. $B \times A$ has 6 elements, the number of elements in *B* times the number of elements in *A*. $A \times A$ has 4 elements, the number of elements in *A* times the number of elements in *A*.

Example 1.2.7 – *Solution*

continued

e. The Cartesian product of *A* and *B* is a set, so it may be used as one of the sets making up another Cartesian product. This is the case for $(A \times B) \times C$.

$$\begin{aligned}
(A \times B) \times C &= \{(u, v) \mid u \in A \times B \text{ and } v \in C\} && \text{by definition of Cartesian product} \\
&= \{((x, 1), a), ((x, 2), a), ((x, 3), a), ((y, 1), a), \\
&\quad ((y, 2), a), ((y, 3), a), ((x, 1), b), ((x, 2), b), ((x, 3), b), \\
&\quad ((y, 1), b), ((y, 2), b), ((y, 3), b)\}
\end{aligned}$$

Example 1.2.7 – *Solution*

continued

f. The Cartesian product $A \times B \times C$ is superficially similar to but is not quite the same mathematical object as $(A \times B) \times C$. $(A \times B) \times C$ is a set of ordered pairs of which one element is itself an ordered pair, whereas $A \times B \times C$ is a set of ordered triples. By definition of Cartesian product,

$$\begin{aligned} A \times B \times C &= \{(u, v, w) \mid u \in A, v \in B, \text{ and } w \in C\} \\ &= \{(x, 1, a), (x, 2, a), (x, 3, a), (y, 1, a), (y, 2, a), (y, 3, a), (x, 1, b), \\ &\quad (x, 2, b), (x, 3, b), (y, 1, b), (y, 2, b), (y, 3, b)\}. \end{aligned}$$

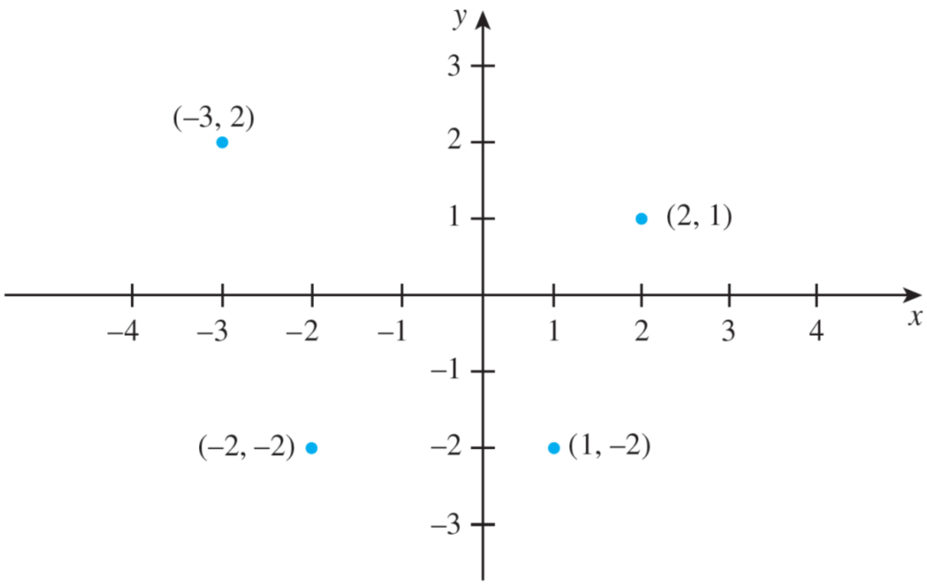
Example 1.2.7 – *Solution*

continued

g. $\mathbf{R} \times \mathbf{R}$ is the set of all ordered pairs (x, y) where both x and y are real numbers. If horizontal and vertical axes are drawn on a plane and a unit length is marked off, then each ordered pair in $\mathbf{R} \times \mathbf{R}$ corresponds to a unique point in the plane, with the first and second elements of the pair indicating, respectively, the horizontal and vertical positions of the point.

Example 1.2.7 – *Solution* continued

The term **Cartesian plane** is often used to refer to a plane with this coordinate system, as illustrated in Figure 1.2.1.



A Cartesian Plane

Figure 1.2.1

Cartesian Products

Another notation, which is important in both mathematics and computer science, denotes objects called *strings*.

Definition

Let n be a positive integer. Given a finite set A, a **string of length *n* over *A*** is an ordered n-tuple of elements of A written without parentheses or commas. The elements of A are called the **characters** of the string. The **null string** over A is defined to be the "string" with no characters. It is often denoted λ and is said to have length 0. If $A = \{0, 1\}$, then a string over A is called a **bit string**.

Example 1.2.8 – *Strings*

Let $A = \{a, b\}$. List all the strings of length 3 over A with at least two characters that are the same.

Example 1.2.8 – *Solution*

aab, *aba*, *baa*, *aaa*, *bba*, *bab*, *abb*, *bbb*

In computer programming it is important to distinguish among different kinds of data structures and to respect the notations that are used for them. Similarly in mathematics, it is important to distinguish among, say, $\{a, b, c\}$, $\{\{a, b\}, c\}$, (a, b, c), $(a, (b, c))$, *abc*, and so forth, because these are all significantly different objects.