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تكامل المحدود و تطبيقاته

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Introduction

A great achievement of classical geometry was obtaining formulas for the areas and volumes of triangles, spheres, and cones. In this chapter we develop a method to calculate the areas and volumes of very general shapes. This method, called integration, is a way to calculate much more than areas and volumes. The definite integral is the key tool in calculus for defining and calculating many important quantities, such as areas, volumes, lengths of curved paths, probabilities, averages, energy consumption, the weights of various objects, and the forces against a dam's floodgates, just to mention a few. Many of these applications are studied in subsequent chapters. As with the derivative, the definite integral also arises as a limit, this time of increasingly fine approximations to the quantity of interest. The idea behind the integral is that we can effectively compute such quantities by breaking them into small pieces, and then summing the contributions from each piece. We then consider what happens when more and more, smaller and smaller pieces are taken in the summation process. As the number of terms contributing to the sum approaches infinity and we take the limit of these sums in a way described in Section 5.3, the result is a definite integral. By considering the rate of change of the area under a graph, we prove that definite integrals are connected to antiderivatives, a connection that gives one of the most important relationships in calculus .

Elements

1- Definite Integral
2- Applications of Definite Integral
3- Properties of Definite Integrals
4- Area between curves

1- Definite Integral

A definite integral is an integral

$$\int_a^b f(x)dx$$

Example 1: Using the definition of the definite integral compute the following.

$$\int_0^2 x^2 + 1dx$$

First, we can't actually use the definition unless we determine which points in each interval that well use for x*i

.In order to make our life easier we'll use the right endpoints of each interval.

From the previous section we know that for a general n

the width of each subinterval is,

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

The subintervals are then,

$$[0, \frac{2}{n}], [\frac{2}{n}, \frac{4}{n}], [\frac{4}{n}, \frac{6}{n}], \dots, [\frac{2(i-1)}{n}, \frac{2i}{n}], \dots, [\frac{2(n-1)}{n}, 2]$$

As we can see the right endpoint of the i^{th} subinterval is

$$x_i^* = \frac{2i}{n}$$

The summation in the definition of the definite integral is then,

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i^*)\Delta x &= \sum_{i=1}^{n} f(\frac{2i}{n})(\frac{2}{n}) \\
&= \sum_{i=1}^{n} ((\frac{2i}{n})^2 + 1)\,(\frac{2}{n}) \\
&= \sum_{i=1}^{n} (\frac{8i^2}{n^3} + \frac{2}{n})
\end{aligned}$$

Now, we are going to have to take a limit of this. That means that we are going to need to “evaluate” this summation. In other words, we are going to have to use the formulas given in the summation notation review to eliminate the actual summation and get a formula for this for a general

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n} \frac{8i^2}{n^3} + \sum_{i=1}^{n} \frac{2}{n}$$

$$= \frac{8}{n^3}\sum_{i=1}^{n} i^2 + \frac{1}{n}\sum_{i=1}^{n} 2$$

$$= \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{n}(2n)$$

$$= \frac{4(n+1)(2n+1)}{3n^2} + 2$$

$$= \frac{14n^2 + 12n + 4}{3n^2}$$

We can now compute the definite integral.

$$\int_0^2 x^2 + 1dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$= \lim_{n\to\infty} \frac{14n^2 + 12n + 4}{3n^2}$$

$$= \frac{14}{3}$$

We’ve seen several methods for dealing with the limit in this problem so we’ll leave it to you to verify the results.

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a) $\int_0^2 x^2 + 1dx$

Solution :

$$\int_2^0 x^2 + 1dx = -\int_0^2 x^2 + 1dx$$

$$= -\frac{14}{3}$$

b) $\int_0^2 10x^2 + 10dx$

Solution :

$$\int_0^2 10x^2 + 10dx = \int_0^2 10(x^2+1)dx$$
$$= 10\int_0^2 x^2 + 1dx$$
$$= 10(\frac{14}{3})$$
$$= \frac{140}{3}$$

c) $\int_0^2 t^2 + 1dt$

solution :

$$\int_0^2 t^2 + 1dt = \int_0^2 x^2 + 1dx = \frac{14}{3}$$

<u>Example 3:</u> Evaluate the following definite integral.

$$\int_{130}^{130} \frac{x^3 - x\sin(x) + \cos(x)}{x^2+1} dx$$

Solution :

$$\int_{130}^{130} \frac{x^3 - x\sin(x) + \cos(x)}{x^2+1} dx = 0$$

Example 4 Given that : $\int_6^{-10} f(x)dx = 23$and $\int_{-10}^{6} g(x)dx = -9$ determine the value of $\int_{-10}^{6} 2f(x) - 10g(x)dx$

Solution :

$$\int_{-10}^{6} 2f(x) - 10g(x)dx = \int_{-10}^{6} 2f(x)dx - \int_{-10}^{6} 10g(x)dx$$
$$= 2\int_{-10}^{6} f(x)dx - 10\int_{-10}^{6} g(x)dx$$

$$\int_{-10}^{6} 2f(x) - 10g(x)dx \quad = -2\int_{6}^{-10} f(x)dx - 10\int_{-10}^{6} g(x)dx$$
$$= -2(23) - 10(-9)$$
$$= 44$$

Example 5 Given that $\int_{12}^{-10} f(x)dx = 6, \int_{100}^{-10} f(x)dx = -2$,and $\int_{100}^{-5} f(x)dx = 4$ determine the value of $\int_{-5}^{12} f(x)dx$

Solution :

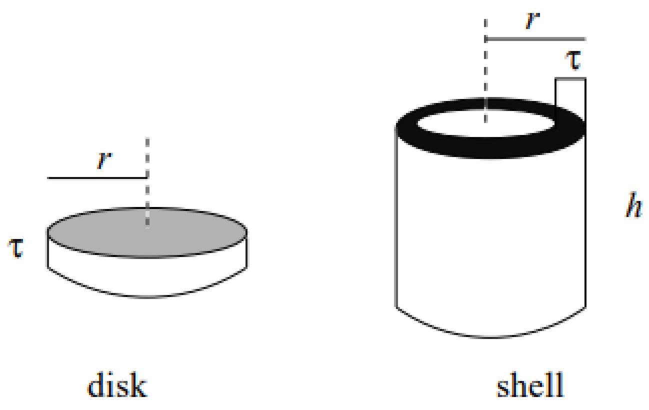
$$\int_{-5}^{12} f(x)dx = \int_{-5}^{100} f(x)dx + \int_{100}^{12} f(x)dx$$

$$\int_{-5}^{12} f(x)dx = \int_{-5}^{100} f(x)dx + \int_{100}^{-10} f(x)dx + \int_{-10}^{12} f(x)dx$$

$$\int_{-5}^{12} f(x)dx \quad = -\int_{100}^{-5} f(x)dx + \int_{100}^{-10} f(x)dx - \int_{12}^{-10} f(x)dx$$
$$= -4 - 2 - 6$$
$$= -12$$

2- Application of Definite Integral

we consider applications of the definite integral to calculating geometric quantities
such as volumes. The idea will be to dissect the three dimensional objects into pieces that resemble
disks or shells, whose volumes we can approximate with simple formulae. The volume of the entire
object is obtained by summing up volumes of a stack of disks or a set of embedded shells, and
considering the limit as the thickness of the dissection cuts gets thinner.
In Figure 5.1 we first remind the reader of the volumes of some of the geometric shapes that
will be used as elementary pieces into which our shapes will be carved. Recall that, from earlier
discussion, we have



1- The volume of a cylinder of height h having circular base of radius r, is

$$V_{cylinder} = \pi r^2 h$$

2. The volume of a circular disk of thickness τ , and radius r as a special case of the above, is

$$V_{shell} = 2\pi r h \tau$$

(This approximation holds for τ << r.)

Solids of revolution

In our first approach to volumes, we will restrict attention to solids of revolution, i.e. volumes

enclosed by some curve (described by a function such as y = f(x)), when it is rotated about one of

the axes. In Figure 5.2(a) we show one such curve, and the surface it forms when it is revolved about

the x axis. We note that if this surface is cut into slices along the x axis, the cross-sections look like

circles. (The circle will have a radius that depends on the position of the cut.) In Figure 5.2(b) we

show how a set of disks of various radii can approximately represent the shape of interest. The total

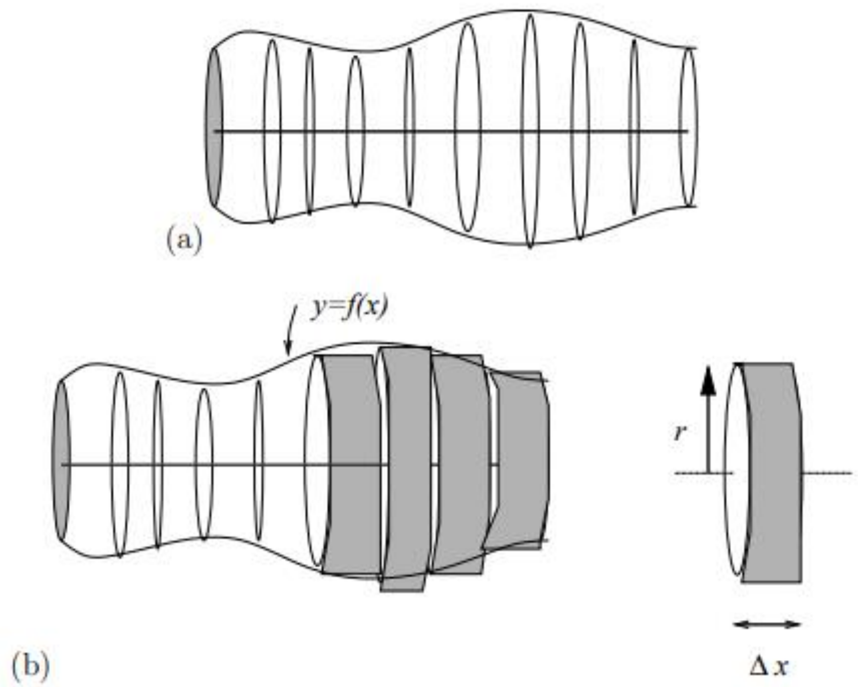
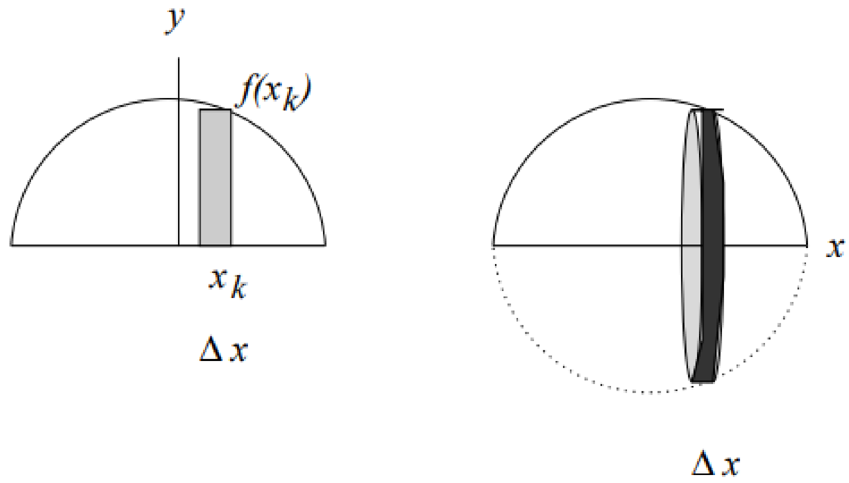


Figure 5.2: (a) A solid of revolution, showing dissection into slices along its axis of rotation. (b)The same volume is approximated by a set of disks. Each disk has some radius r (that varies along the length of the object) and thickness Δx. Note that the thickness is in the direction of the x axis: this will remind us that we integrate with respect to x.

volume of these disks is not the same, clearly, as the volume of the object, since some of these stick out beyond the surface. However, if we make the thickness of these disks very small, we will get a good approximation of the desired volume. In the limit, as the thickness becomes infinitesimal, we arrive at the true volume. In most of the examples discussed in this chapter, the key step is to make careful observation of the way that the radius of a given disk depends on the function that generates the surface. (By this we mean the function that specifies the curve that forms the surface of revolution.) We also pay attention to the dimension that forms the disk thickness. Some of our examples will involve surfaces revolved about the x axis, and others will be revolved about the y axis. In setting up these examples, a diagram is usually quite helpful

Example 1: Volume of a sphere



When the semicircle (on the left) is rotated about the x axis, it generates a sphere. On the right, we show one disk generated by the revolution of the shaded rectangle.We can think of a sphere of radius R as a solid formed by rotating a semi-circle about its long axis.

$$y = f(x) = \sqrt{R^2 - x^2}$$

We show the sphere dissected into a set of disks, each of width Δx. The disks are lined up along the x axis with coordinates x_k. These are just integer multiples of the slice thickness Δx, so for example,

$$x_k = k\Delta x$$

The radius of the disk depends on its position. Indeed, the radius of a disk through the x axis at a point x_k is specified by the function

$$r_k = y_k = f(x_k)$$

The volume of the k'th disk is

$$V_k = \pi r_k^2 \Delta x$$

By the above remarks, using the fact that the function actually determines the radius, we have

$$V_k = \pi (f(x_k))^2 \Delta x$$

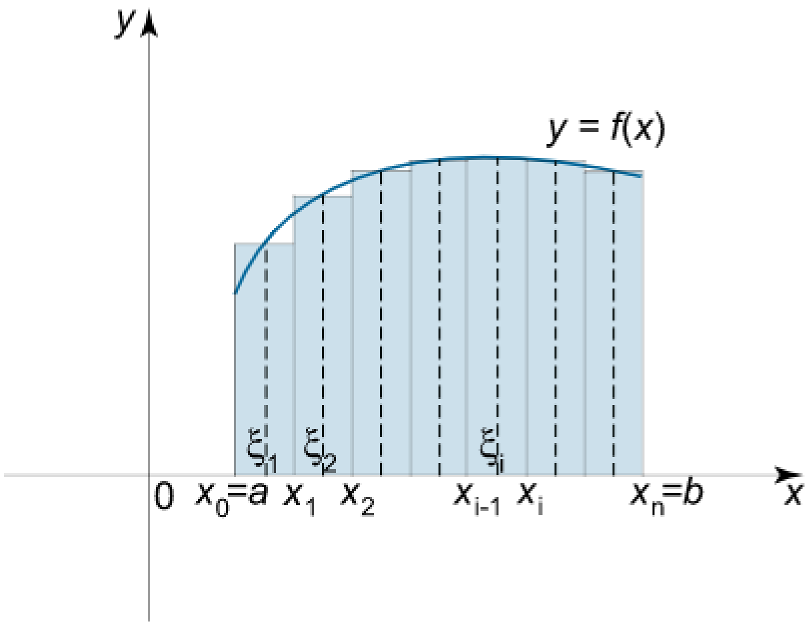
$$V_k = \pi (\sqrt{R^2 - x_k^2})^2 \Delta x = \pi (R^2 - x_k^2) \Delta x$$

The total volume of all the disks is

$$V = \sum_k V_k = \pi(f(x_k))^2 \Delta x$$

3-Properties of Definite Integrals

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ max\Delta x_i \to 0}} \sum_{i=1}^{n} f(\xi_i)\Delta x_i \text{ ,where} \Delta x_i = x_i - x_{i-1}, x_{i-1} \le \xi_i \le x_i$$



1- The definite integral of is equal to the length of the interval of integration:

$$\int_a^b 1dx = b - a$$

2- A constant factor can be moved across the integral sign:

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

3- The definite integral of the sum of two functions is equal to the sum of the integrals of these functions:

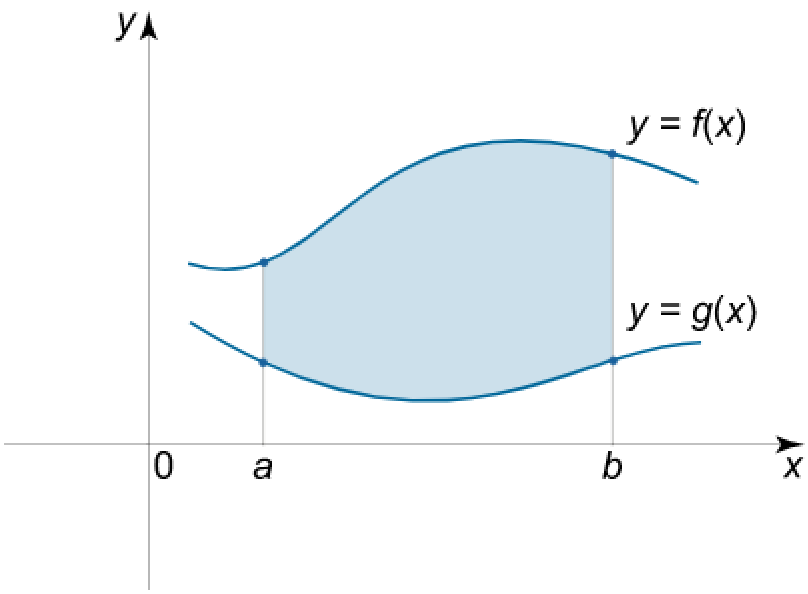
$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

4- The definite integral of the difference of two functions is equal to the difference of the integrals of these functions:

$$\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

4-area between curves

$$S = \int_a^b [f(x) - g(x)]dx = F(b) - G(b) - F(a) + G(a), \text{where} F'(x) = f(x)$$
$$G'(x) = g(x)$$



Problems and solutions

a) $[x^3 + 2x^2 + x]_1^5$

Solution: $[(5)^3 + 2(5)^2 + 5] = 125 + 50 + 5 = 180$

$[(1)^3 + 2(1)^2 + 1] = 1 + 2 + 1 = 4$

180 – 4 = 176

$$\int_1^5 (3x^2 + 4x + 1)\, dx \qquad = [x^3 + 2x^2 + x]_1^5$$
$$= [(5)^3 + 2(5)^2 + 5] - [(1)^3 + 2(1)^2 + 1]$$
$$= 180 - 4$$
$$= 176$$

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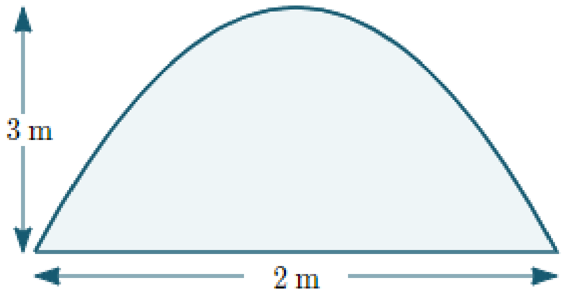
b) If the arch is 2 m wide at the bottom and is 3 m high,
   1- find the area under each arch using integration.



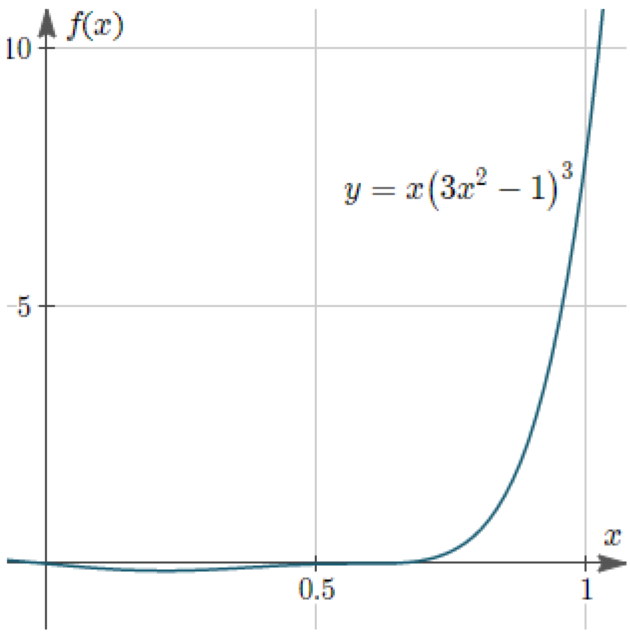
Parabolic arch.

Solution :

$$\int_0^2 (-3x^2 + 6x)\,dx \qquad = [-x^3 + 3x^2]_0^2$$

$$= [-(2^3) + 3(2)^2] - [0 + 0]$$
$$= [-8 + 12]$$
$$= 4\text{m}^2$$

C) Find the average value of $y = x(3x^2 - 1)^3$ from 0 to 1



The curve $y = x(3x^2 - 1)^3$.

$$\int_0^1 x(3x^2-1)^3dx$$

ut$u = 3x^2 - 1$,then$du = 6xdx$

$$= \frac{du}{6} = xdx$$

$$= \frac{1}{6}[\frac{u^4}{4}]_{x=0}^{x=1}$$

$$= \frac{1}{24}[(3x^2-1)^4]_0^1$$

$$= \frac{1}{24}[(3(1)^2-1)^4 - (3(0)^2-1)^4]$$

$$= \frac{1}{24}[16-1]$$

$$= \frac{15}{24}$$

$$= \frac{1}{6}$$

Conclusion

We have seen that in situations where it is impossible to know the function governing some phenomenon exactly, it is still possible to derive a reasonable estimate for the integral of the function based on data points. The idea is to choose a model function going through the data points and integrate the model function. The definition of an integral as a limit of Reimann sums shows that if you chose enough data points, the integral of the model function converges to the integral of the unkown function, so theoretically, numerical integration is on solid ground.

We have also seen that there are many practical factors that influence how well numerical integration works. Simple model functions may not emulate the behavior of the unkown function well. Complicated model functions are hard to work with. Problems with the number of data points, or the way in which the data was collected can have a major impact, and while we have explored some simple ways of estimating how accurate a particular numerical integral will be, this can be quite complicated in general.

Nonetheless, by using common sense, together with a solid grasp of what the integral means and how it is related to the geometry of the function being integrated, a creative scientist, mathematician or engineer can accomplish a great deal with numerical integration.

Sources

1- [12]"Properties of Definite Integrals", Math24, 2020. [Online]. Available: https://www.math24.net/properties-definite-integrals/. [Accessed: 06- Jun- 2020].
2- [13]Ugrad.math.ubc.ca, 2020. [Online]. Available: http://ugrad.math.ubc.ca/coursedoc/math103/keshet.notes/chapter5Notes.pdf. [Accessed: 06- Jun- 2020].
3- [14]"Calculus I - Applications of Integrals", Tutorial.math.lamar.edu, 2020. [Online]. Available: https://tutorial.math.lamar.edu/Classes/CalcI/IntAppsIntro.aspx. [Accessed: 06- Jun- 2020].
4- Book of Mathematics 2 (Integral Calculus)