Chapter 4

Latin Square Design

Latin Square Designs are probably not used as much as they should be - they are very efficient designs. Latin square designs allow for two blocking factors. In other words, these designs are used to simultaneously control (or eliminate) two sources of nuisance variability. For instance, if you had a plot of land the fertility of this land might change in both directions, North -- South and East -- West due to soil or moisture gradients. So, both rows and columns can be used as blocking factors. However, you can use Latin squares in lots of other settings. As we shall see, Latin squares can be used as much as the RCBD in industrial experimentation as well as other experiments. Whenever, you have more than one blocking factor a Latin square design will allow you to remove the variation for these two sources from the error variation. So, consider we had a plot of land, we might have blocked it in columns and rows, i.e. each row is a level of the row factor, and each column is a level of the column factor. We can remove the variation from our measured response in both directions if we consider both rows and columns as factors in our design. The Latin Square Design gets its name from the fact that we can write it as a square with Latin letters to correspond to the treatments. The treatment factor levels are the Latin letters in the Latin square design. The number of rows and columns has to correspond to the number of treatment levels. So, if we have four treatments then we would need to have four rows and four columns in order to create a Latin square. This gives us a design where we have each of the treatments and in each row and in each column. This is just one of many 4×4 squares that you could create. In fact, you can make any size square you want, for any number of treatments - it just needs to have the following property associated with it - that each treatment occurs only once in each row and once in each column. Consider another example in an industrial setting: the rows are the batch of raw material, the columns are the operator of the equipment, and the treatments (A, B, C and D) are an industrial process or protocol for producing a particular product.

Latin Square Design

- Commonly called as LSD.
- LSD is a design where the experimental material is divided into ‘r’ rows, ‘r’ columns and ‘r’ treatments assigned by randomization method to rows and columns.
- The randomization is in such a way that each treatment occurs only once in each row and each column.

Advantages of LSD

- Statistical analysis is relative simple(complicated than CRD and RBD)
- LSD is more efficient than RBD or CRD.
- This is because of double grouping that will result in small experimental error.
- Statistical analysis is simple if one value is missing.

Disadvantages of LSD

- This design is not as flexible as RBD or CRD as the number of treatments is limited to the number of rows and columns.
- LSD is seldom used when the number of treatments is more than 12.

- LSD is not suitable for treatments less than five. Because of the limitations on the number of treatments, LSD is not widely used in agricultural experiments.
- Statistical analysis is complicated when two or more values are missing

Latin square design is a form of complete block design that can be used when there are two blocking criteria. This is layout is used in field experiments when environmental gradients, such as irrigation and soil type, are expected to differ by rows and by columns. Every treatment is randomly assigned once in every row and every column. Each row and each column are complete blocks. The major disadvantage to this layout is that the number of rows and columns must be equal.

The Latin square design is used to eliminate two nuisance sources of variability; that is, it systematically allows blocking in two directions. Thus, the rows and columns actually represent two restrictions on randomization. In general, a Latin square for **r** factors, or a **r×r** Latin square, is a square containing **r** rows and **r** columns. Each of the resulting $\mathbf{r^2}$ cells contains one of the **r** letters that corresponds to the treatments, and each letter occurs once and only once in each row and column. Some examples of Latin squares are

4 × 4	5 × 5	6 × 6
A B D C	*A D B E C*	*A D C E B F*
B C A D	*D A C B E*	*B A E C F D*
C D B A	*C B E D A*	*C E D F A B*
D A C B	*B E A C D*	*D C F B E A*
	E C D A B	*F B A D C E*
		E F B A D C

8			4		6			7
						4		
	1					6	5	
5		9		3		7	8	
				7				
	4	8		2		1		3
	5	2					9	
		1						
3			9		2			5

(image of sudoku game)

Latin squares are closely related to a popular puzzle called a sudoku puzzle that originated in Japan (sudoku means “single number” in Japanese). The puzzle typically consists of a 9×9 grid, with nine additional 3×3 blocks contained within. A few of the spaces contain numbers and the others are blank. The goal is to fill the blanks with the integers from 1 to 9 so that each row, each column, and each of the nine 3×3 blocks making up the grid contains just one of each of the nine integers. The additional constraint that a standard 9×9 sudoku puzzle have 3×3 blocks that also contain each of the nine integers reduces the large number of possible 9×9 Latin squares to a smaller but still quite large number, approximately 6×10^{21}.

Linear Model for the LSD

The statistical model for a Latin square is

$$y_{ijk} = \mu + \alpha_i + \beta_j + \tau_k + \varepsilon_{ijk}, \qquad i = 1,2,\dots,r, \quad j = 1,2,\dots,r, \quad k = 1,2,\dots,r$$

where y_{ijk} is the observation in the ith row and in jth column for the kth treatment

μ is an overall mean

α_i is the ith row effect,

β_j is the jth column effect,

τ_k is the kth treatment effect

ε_{ijk} is the random error

Note that this is an effects model. The model is completely additive; that is, there is no interaction between rows, columns, and treatments. Because there is only one observation in each cell, only two of the three subscripts i, j, and k are needed to denote a particular observation.

Such that

$$\sum_{i=1}^{r}\alpha_i = 0 \ , \quad \sum_{j=1}^{r}\beta_j = 0 \text{ and } \sum_{k=1}^{r}\tau_k = 0$$

and ε_{ijk} is the usual NID (0, σ^2) random error term. We will initially consider treatments and blocks to be fixed factors.

We shall employ the usual notation for row, column and treatment totals and means

$$y_{i..} = \sum_{j=1}^{r} y_{ijk}\,, \qquad \bar{y}_{i..} = \frac{y_{i..}}{r}$$

$$y_{.j.} = \sum_{i=1}^{r} y_{ijk}\,, \qquad \bar{y}_{.j.} = \frac{y_{.j.}}{r}$$

$$y_{..k} = \sum_{i,j} y_{ijk}\,, \qquad \bar{y}_{..k} = \frac{y_{..k}}{r}$$

The overall total and mean are denoted as usual by:

$$y_{...} = \sum_{i=1}^{r}\sum_{j=1}^{r} y_{ijk}\,, \qquad \bar{y}_{...} = \frac{y_{...}}{r^2}$$

Decomposition of the Total Sum of Squares

The analysis of variance consists of partitioning the total sum of squares of the N = r^2 observations into components for rows, columns, treatments, and error, for example,

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\left(y_{ijk}-\bar{y}_{...}\right)^2 = r\sum_{i=1}^{r}(\bar{y}_{i..}-\bar{y}_{...})^2 + r\sum_{j=1}^{r}\left(\bar{y}_{.j.}-\bar{y}_{...}\right)^2 + r\sum_{k=1}^{r}(\bar{y}_{..k}-\bar{y}_{...})^2$$
$$+\sum_{i=1}^{r}\sum_{j=1}^{r}\left(y_{ijk}-\bar{y}_{i..}-\bar{y}_{.j.}-\bar{y}_{..k}+2\bar{y}_{...}\right)^2$$

$$SST = SSTr + SSR + SSC + SSE$$

with respective degrees of freedom

$$r^2\text{-}1 = (r\text{-}1) + (r\text{-}1) + (r\text{-}1) + (r\text{-}1)(r\text{-}2)$$

Computing formulas can be expressed in terms of treatment and block totals. These formulas are

$$SST = \sum_{i=1}^{r}\sum_{j=1}^{r} y_{ijk}^2 - \frac{y_{...}^2}{N}$$

$$SSTr = \frac{1}{r}\sum_{i=1}^{r} y_{..k}^2 - \frac{y_{...}^2}{N}$$

$$SSR = \frac{1}{r}\sum_{j=1}^{r} y_{i..}^2 - \frac{y_{...}^2}{N}$$

$$SSC = \frac{1}{r}\sum_{j=1}^{r} y_{.j.}^2 - \frac{y_{...}^2}{N}$$

and the error sum of squares is obtained by subtraction as

$$SSE = SST - SSTr - SSR - SSC$$

ANOVA F –Test

To test for treatment effects:

1-$H_0: \tau_1 = \tau_2 = \cdots = \tau_k = 0$ $H_a: \tau_k \neq 0$ for at least one k , $k = 1,2,3,4$

2-Test statistic

$$F_{tr} = \frac{MSTr}{MSE}$$

3-Rejection region:

We reject H_0 if $F_{tr} > F_{\alpha,(r-1),(r-1)(r-2)}$ or, p-value < α.

4-Calculations.

5-conclusion.

To test for rows effect:

$H_0: \alpha_1 = \alpha_2 = \cdots = \alpha_i = 0$ $\quad H_a: \alpha_i \neq 0$ for at least one i $\quad , i = 1,2, \dots , r$

2-Test statistic

$$F_R = \frac{MSR}{MSE}$$

3-Rejection region:

We reject H_0 if $F_R > F_{\alpha,(r-1),(r-1)(r-2)}$

To test for columns effect:

$H_0: \beta_1 = \beta_2 = \cdots = \beta_j = 0$ $\quad H_a: \beta_j \neq 0$ for at least one j $\quad , j = 1,2, . . , r$

2-Test statistic

$$F_C = \frac{MSC}{MSE}$$

3-Rejection region:

We reject H_0 if $F_C > F_{\alpha,(r-1),(r-1)(r-2)}$

The computational procedure for the ANOVA in terms of treatment, row, and column totals is shown in Table 4.1. From the computational formulas for the sums of squares, we see that the analysis is a simple extension of the RCBD, with the sum of squares resulting from rows obtained from the row totals.

Table 4.1.Analysis of variance for a Latin Square Design

Source of variation	Sum of squares	Degrees of freedom	Mean square	F
Treatments	$SSTr$	$r-1$	$MSTr=\frac{SSTr}{r-1}$	$F_{tr}=\frac{MSTr}{MSE}$
Rows	SSR	$r-1$	$MSR=\frac{SSR}{r-1}$	$F_R=\frac{MSR}{MSE}$
Columns	SSC	$r-1$	$MSC=\frac{SSC}{r-1}$	$F_C=\frac{MSC}{MSE}$
Error	$SSE=SST-SSTr-SSB-SSC$	$(r-1)(r-2)$	$MSE=\frac{SSE}{(r-1)(r-2)}$	
Total	SST	$N-1$=r^2-1		

Estimating Parameters

The least squares and maximum likelihood estimators of the parameters in latin square model

Parameter	Estimator
μ	$\hat{\mu}=\bar{y}_{...}$
α_i	$\hat{\alpha}_i=\bar{y}_{i..}-\bar{y}_{...}$
β_j	$\hat{\beta}_j=\bar{y}_{.j.}-\bar{y}_{...}$
τ_k	$\hat{\tau}_k=\bar{y}_{..k}-\bar{y}_{...}$

Example 4.1

An oil company tested four different blends of gasoline for fuel efficiency according to a Latin square design in order to control for the variability of four different drivers (1, 2, 3, 4) and four different models of cars (I, II, III, IV). Fuel efficiency was measured in miles per gallon (mpg) after driving cars over a standard course. The results for the fuel efficiencies (mpg) for 4 blends of gasoline (A, B, C, D) are given

Fuel Efficiencies (mpg) For 4 Blends of Gasoline
(Latin Square Design: Blends Indicated by Letters A-D)

	Car Model			
Driver	**I**	**II**	**III**	**IV**
1	**D** 15.5	**B** 33.9	**C** 13.2	**A** 29.1
2	**B** 16.3	**C** 26.6	**A** 19.4	**D** 22.8
3	**C** 10.8	**A** 31.1	**D** 17.1	**B** 30.3
4	**A** 14.7	**D** 34.0	**B** 19.7	**C** 21.6

a. Carry out the analysis of variance. Are there any differences between the blends in regards to fuel efficiency?

b. Would you conclude that there is an effect due to the driver? Why?
c. Would you conclude that there is an effect due to model of the car? Why?

	Car Model				
Driver	**I**	**II**	**III**	**IV**	$y_{i..}$
1	**D** 15.5	**B** 33.9	**C** 13.2	**A** 29.1	91.7
2	**B** 16.3	**C** 26.6	**A** 19.4	**D** 22.8	85.1
3	**C** 10.8	**A** 31.1	**D** 17.1	**B** 30.3	89.3
4	**A** 14.7	**D** 34.0	**B** 19.7	**C** 21.6	90
$y_{.j.}$	57.3	125.6	69.4	103.8	
$y_{...}$					356.1

Treatment	**Total**
A	94.3
B	100.2
C	72.2
D	89.4

a-$H_0: \tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ $\quad H_a: \tau_k \neq 0$ for at least one k $\quad, k = 1,2,3,4$

2-Test statistic

$$F_{tr} = \frac{MSTr}{MSE}$$

3-Rejection region:

We reject H_0 if $F_{tr} > F_{\alpha,(r-1),(r-1)(r-2)}$ or, p-value < α.

4-Calculations.

$$SST = \sum_{i=1}^{r}\sum_{j=1}^{r} y_{ijk}^2 - \frac{y_{...}^2}{N} = 8801.05 - \frac{356.1^2}{16} = 875.599$$

$$SSTr = \frac{1}{r}\sum_{k=1}^{r} y_{..k}^2 - \frac{y_{...}^2}{N} = \frac{1}{4}[94.3^2 + 100.2^2 + 72.2^2 + 89.4^2] - \frac{3221^2}{16}$$
$$= 108.982$$

$$SSR = \frac{1}{r}\sum_{i=1}^{r} y_{i..}^2 - \frac{y_{...}^2}{N} = \frac{1}{4}[91.7^2 + 85.1^2 + 89.3^2 + 90^2] - \frac{356.1^2}{16} = 5.897$$

$$SSC = \frac{1}{r}\sum_{j=1}^{r} y_{.j.}^2 - \frac{y_{...}^2}{N} = \frac{1}{4}[57.3^2 + 125.6^2 + 69.4^2 + 103.8^2] - \frac{356.1^2}{16}$$
$$= 736.912$$

and the error sum of squares is obtained by subtraction as

$$SSE = SST - SSTr - SSR - SSC = 875.599 - 108.982 - 5.897 - 736.912$$
$$= 23.808$$

Analysis of variance for a Latin Square Design

Source of variation	**Sum of squares**	**Degrees of freedom**	**Mean square**	**F**
Treatments	108.982	3	36.327	9.15
Rows	5.897	3	1.966	0.50
Columns	736.912	3	245.637	61.90
Error	23.808	6	3.968	
Total	875.599	15		

$$F_{\alpha,(r-1),(r-1)(r-2)} = F_{0.05,3,6} = 4.76$$

Since $F = 9.15 > 4.76$,we reject H_0

5-Conclusion

We conclude that the blends are significantly different at the 5% level of significance

b-$H_0: \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ $\quad H_a: \alpha_i \neq 0$ for at least one i $\quad , i = 1,2,3,4$

2-Test statistic

$$F_R = \frac{MSR}{MSE}$$

3-Rejection region:

We reject H_0 if $F_R > F_{\alpha,(r-1),(r-1)(r-2)}$

Since $F = 0.5 \ngtr 4.76$,we don't reject H_0

We conclude that there are not any significant differences among our Drivers. (In the general population some drivers are easier on fuel than others; perhaps the drivers for this study have been carefully trained so that their driving styles are unusually similar.)

c-$H_0: \beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$ $\quad H_a: \beta_j \neq 0$ for at least one j $\quad , j = 1,2,3,4$

2-Test statistic

$$F_C = \frac{MSC}{MSE}$$

3-Rejection region:

We reject H_0 if $F_C > F_{\alpha,(r-1),(r-1)(r-2)}$

Since $F = 61.90 > 4.76$,we reject H_0

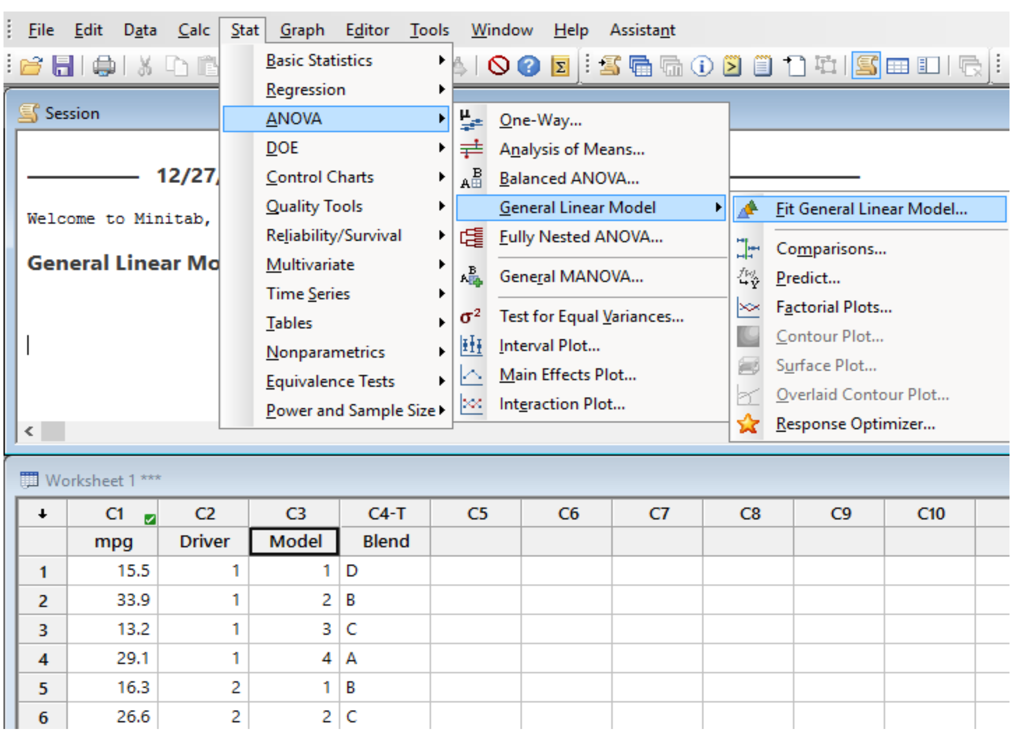
We conclude that there is an effect due to model of the car.(It is not surprising that the fuel efficiencies vary among various models of cars (which might range from small compacts to large RVs).

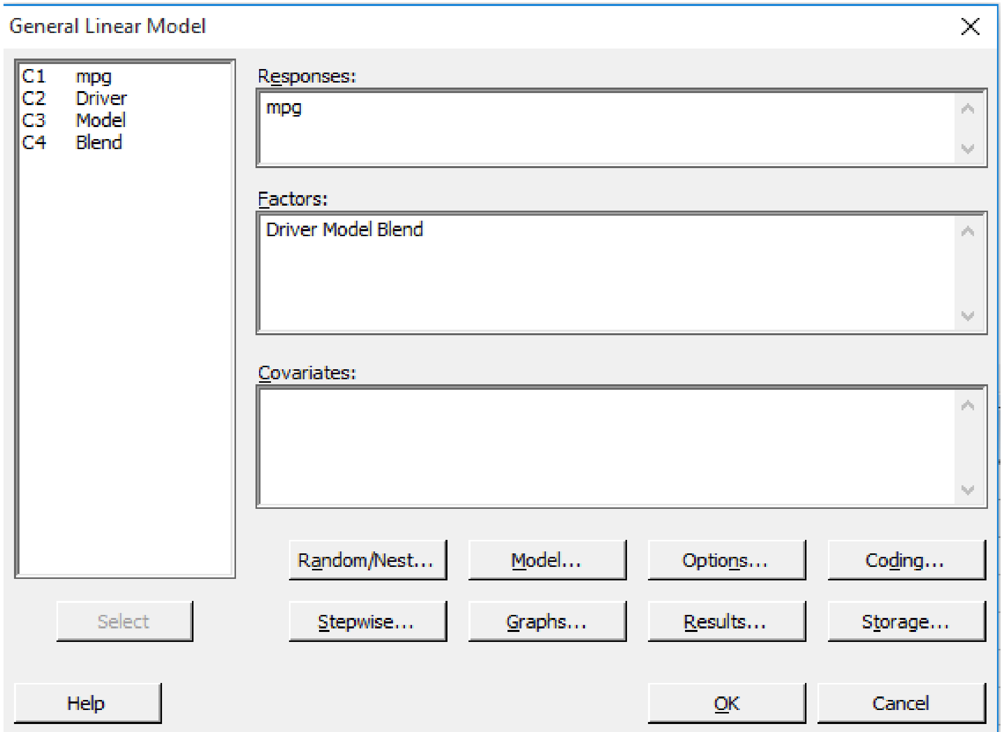
Using Minitab

Worksheet 1 ***

↓	C1	C2	C3	C4-T
	mpg	Driver	Model	Blend
1	15.5	1	1	D
2	33.9	1	2	B
3	13.2	1	3	C
4	29.1	1	4	A
5	16.3	2	1	B
6	26.6	2	2	C
7	19.4	2	3	A
8	22.8	2	4	D
9	10.8	3	1	C
10	31.1	3	2	A
11	17.1	3	3	D
12	30.3	3	4	B
13	14.7	4	1	A
14	34.0	4	2	D
15	19.7	4	3	B
16	21.6	4	4	C

Choose Stat>ANOVA>General Linear Model>Fit General Linear Model





General Linear Model: mpg versus Driver, Model, Blend

Factor Information

Factor	Type	Levels	Values
Driver	Fixed	4	1, 2, 3, 4
Model	Fixed	4	1, 2, 3, 4
Blend	Fixed	4	A, B, C, D

Analysis of Variance

Source	DF	Adj SS	Adj MS	F-Value	P-Value
Driver	3	5.897	1.966	0.50	0.699
Model	3	736.912	245.637	61.90	0.000
Blend	3	108.982	36.327	9.15	0.012
Error	6	23.809	3.968		
Total	15	875.599			

Model Summary

S	R-sq	R-sq(adj)	R-sq(pred)
1.99202	97.28%	93.20%	80.66%